EE2008/E ~ Circuits (Part 1)

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Course Outlines

Electrical Engineering Circuits

EE2008 module is to introduce basic electrical circuit components & analyses. Emphasis is on the **fundamental methodologies** and **mathematical tools** for solving and analysing some basic electrical circuits.

Roughly, the first part of this course focuses on the approaches in the time domain (ordinary differential equations, state-space equations), while the second half deals more with the frequency-domain methods (Laplace transformations, etc.).

Textbook

A. B. Carlson, *Circuits*, PWS Publishing Company, New York, 1999.

References

C. C. Ko & B. M. Chen, *Basic Circuit Analysis*, Prentice Hall Asia, Singapore, 2nd Edition, 1998.

C. K. Alexander & M. N. O. Sadiku, *Electric Circuits*, McGraw Hill, New York, 2000.

Lectures

Attendance is essential.

Ask any question at any time during the lecture.

Tutorials

The tutorials will start on Week 4 of the semester.

You should make an effort to attempt each question before the tutorial.

Some of the questions are straightforward, but quite a few are difficult and meant to serve as a platform for the introduction of new concepts.

ASK your tutor any question related to the tutorials and the course.

Outline of the Course

1. Circuit Elements

Capacitors. Inductors. V-I Relationships. Energy Storage.

2. Transient Analysis

First-order Transients. Switched DC Transients. Switched AC Transients. Second-order Natural Response (Overdamped, Underdamped and Critically Damped Responses). Second-Order Transients.

3. State Variable Analysis

Review of Laplace and Inverse Laplace Transforms. State Variables. State and Output Equations. Transform Solutions to State Equations. Zero-Input Response. Complete Response.

Mid-term Test, Homework and Examination

There will be a mid-term test and two homework assignments for the first part. The mid-term test will be held in a **tutorial session**. The exact date will be made known in due course. The test and homework assignments will be marked counted as 30% (10% for homework assignments and 20% for the test) of your final grade for Part 1, i.e., your final grade for Part 1 will be computed as follows:

Your Part 1 Grade = 30% of Your Test and HW Marks (max. = 50)

+ 70% of Your Exam Marks for Part 1 (max.=50)

0. Preliminary Materials

0.1 Operations of Complex Numbers

Coordinates: Cartesian Coordinate and Polar Coordinate

$$12 + j5 = 13 \; e^{j0.39} = \sqrt{12^2 + 5^2} \; e^{j \tan^{-1}\left(\frac{5}{12}\right)}$$

real part imaginary part magnitude argument

Euler's Formula:

$$\boxed{e^{j\theta} = \cos(\theta) + j \sin(\theta)}$$

Additions: It is easy to do additions (subtractions) in Cartesian coordinate.

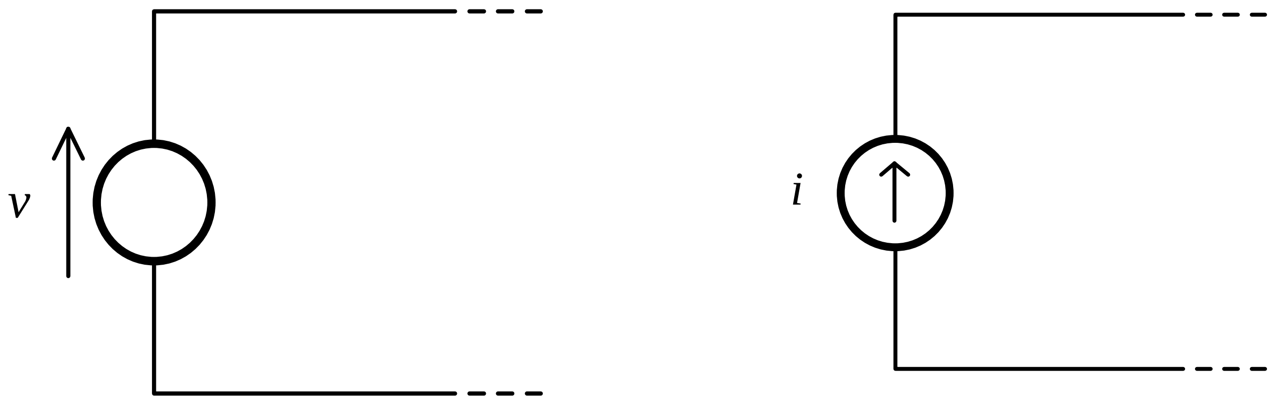
$$\boxed{(a + jb) + (v + jw) = (a + v) + j(b + w)}$$

Multiplication's: It is easy to do multiplication's (divisions) in Polar coordinate.

$$\boxed{re^{j\theta} \cdot ue^{j\omega} = (ru)e^{j(\theta+\omega)}} \qquad \boxed{\frac{re^{j\theta}}{ue^{j\omega}} = \frac{r}{u} e^{j(\theta-\omega)}}$$

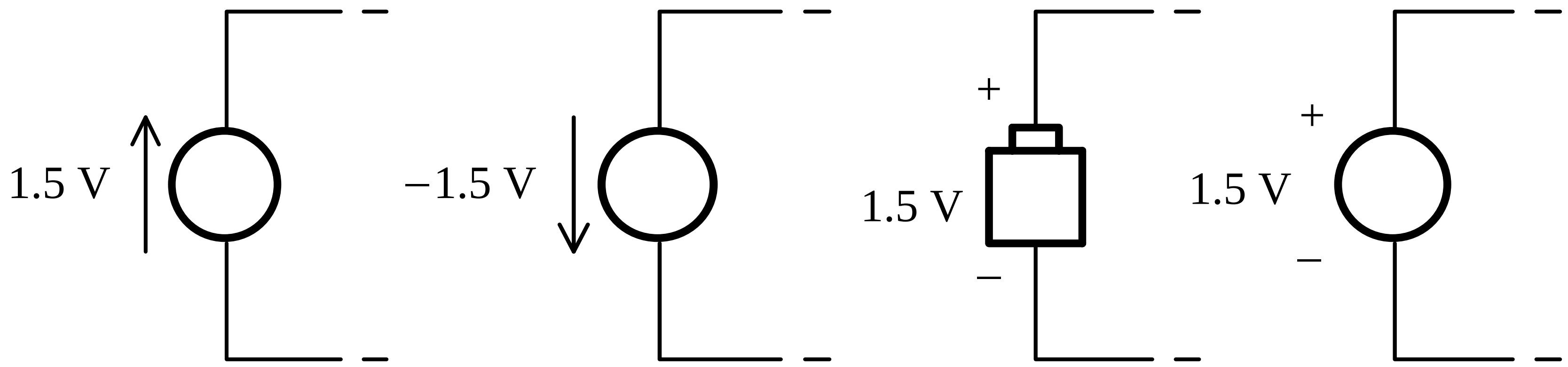
0.2 Symbols of Voltage and Current Sources

The circuit symbols of voltage and current sources (either DC or AC) used in this part of the course are:



Basically, the arrow and the value in the voltage source signifies that the top terminal has a potential of v (could be either positive or negative) with respect to the bottom terminal regardless of what has been connected to it. Similarly, the arrow and the value of the current source signifies that there is a current i (could be either positive or negative) flowing upwards.

Remark: The following symbols for the voltage source are identical:



Note that on its own, the arrow does not correspond to the positive terminal. Instead, the positive terminal depends on both the arrow and the sign of the voltage which may be negative.

0.3 Phasor

Euler's Formula: $e^{j\omega} = \cos(\omega) + j\sin(\omega) \quad \Rightarrow \quad \cos(\omega) = \mathrm{Re}[e^{j\omega}]$

A sinusoidal voltage/current is represented using complex number format:

$$v(t) = \sqrt{2}r\cos(\omega t + \theta) = \sqrt{2}r\,\mathrm{Re}\left[e^{j(\omega t + \theta)}\right] = \mathrm{Re}\left[\left(re^{j\theta}\right)\left(\sqrt{2}e^{j\omega t}\right)\right]$$

The advantage of this can be seen if, say, we have to add 2 sinusoidal voltages given by:

$$v_1(t) = 3\sqrt{2}\cos\left(\omega t + \frac{\pi}{6}\right) \qquad v_2(t) = 5\sqrt{2}\cos\left(\omega t - \frac{\pi}{4}\right)$$

$$v_1(t) = 3\sqrt{2}\cos\left(\omega t + \frac{\pi}{6}\right) = \mathrm{Re}\left[\left(3e^{j\frac{\pi}{6}}\right)\left(\sqrt{2}e^{j\omega t}\right)\right] \qquad v_2(t) = 5\sqrt{2}\cos\left(\omega - \frac{\pi}{4}\right) = \mathrm{Re}\left[\left(5e^{-j\frac{\pi}{4}}\right)\left(\sqrt{2}e^{j\omega t}\right)\right]$$

$$v_1(t) + v_2(t) = \mathrm{Re}\left[\left(3e^{j\frac{\pi}{6}} + 5e^{-j\frac{\pi}{4}}\right)\left(\sqrt{2}e^{j\omega t}\right)\right] = \mathrm{Re}\left[\left(6.47e^{-j0.32}\right)\left(\sqrt{2}e^{j\omega t}\right)\right] = 6.47\sqrt{2}\cos(\omega t - 0.32)$$

Note that the complex time factor $\sqrt{2}e^{j\omega t}$ appears in all the expressions.

If we represent $v_1(t)$ and $v_2(t)$ by the complex numbers or ***phasors***:

$$V_1 = 3e^{j\frac{\pi}{6}} \text{ representing } v_1(t) = 3\sqrt{2}\cos\left(\omega t + \frac{\pi}{6}\right)$$

$$V_2 = 5e^{-j\frac{\pi}{4}} \text{ representing } v_2(t) = 5\sqrt{2}\cos\left(\omega t - \frac{\pi}{4}\right)$$

then the phasor representation for $v_1(t) + v_2(t)$ will be

$$V_1 + V_2 = 3e^{j\frac{\pi}{6}} + 5e^{-j\frac{\pi}{4}} = 6.47e^{-j0.32} \quad \text{representing } v_1(t) + v_2(t) = 6.47\sqrt{2}\cos(\omega t - 0.32)$$

$$3e^{j\frac{\pi}{6}} + 5e^{-j\frac{\pi}{4}} = 3\left(\cos\left(\frac{\pi}{6}\right) + j\sin\left(\frac{\pi}{6}\right)\right) + 5\left(\cos\left(-\frac{\pi}{4}\right) + j\sin\left(-\frac{\pi}{4}\right)\right) = 6.14 - j2.03 = 6.47e^{-j0.32}$$

Euler's Formula: $e^{j\omega} = \cos(\omega) + j\sin(\omega)$

By using phasors, a time-varying ac voltage

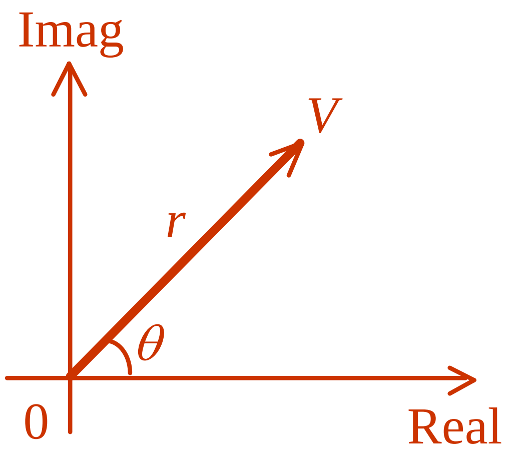
$$v(t) = \sqrt{2} r \cos(\omega t + \theta) = \mathrm{Re}[(re^{j\theta})(\sqrt{2} e^{j\omega t})]$$

becomes a simple complex time-invariant number/voltage $V = re^{j\theta} = r\angle\theta$

$r = |V| =$ magnitude/modulus of $V =$ r.m.s. value of $v(t)$

$\theta = \mathrm{Arg}[V] =$ phase of V

Graphically, on a phasor diagram:



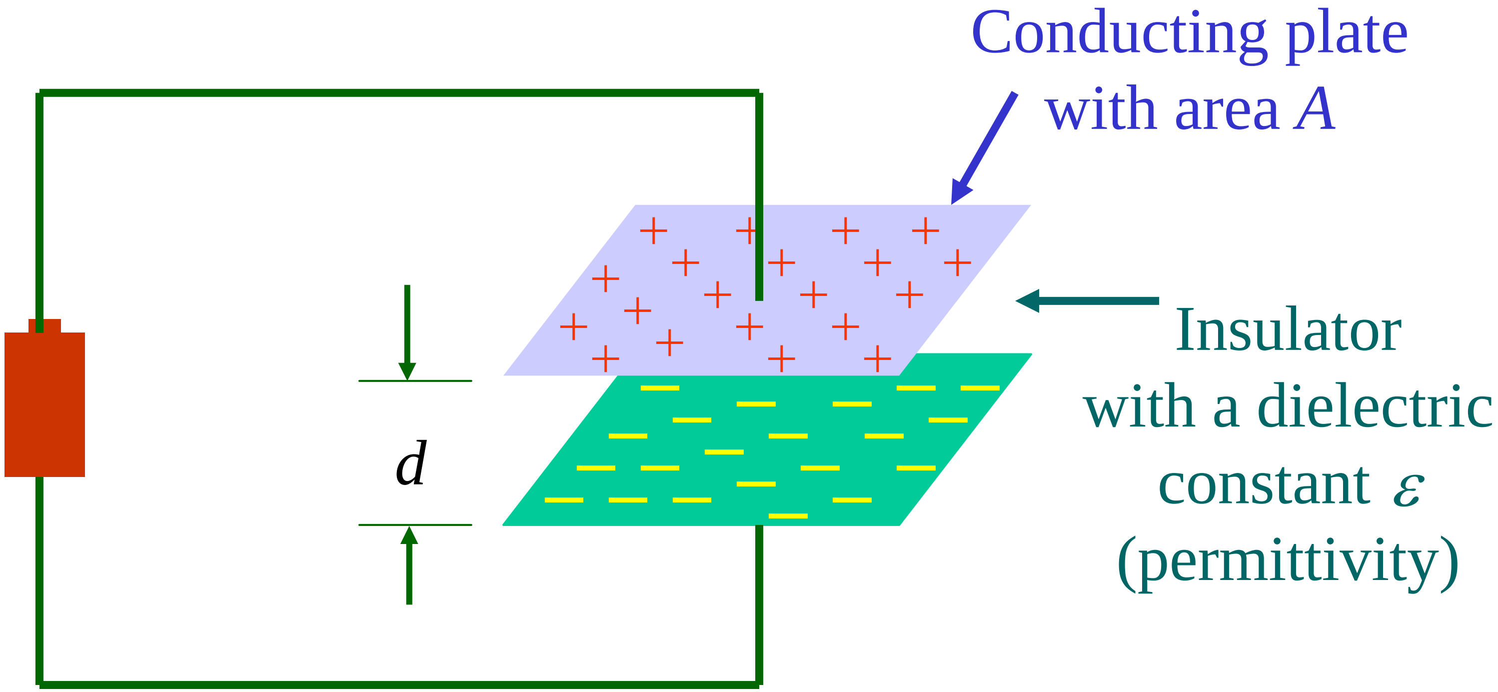
Complex Plane

Using phasors, all time-varying ac quantities become complex dc quantities and all dc circuit analysis techniques can be employed for ac circuit with virtually no modification.

1. Circuit Elements

1.1 Capacitor

A ***capacitor*** consists of parallel metal plates for storing electric charges.

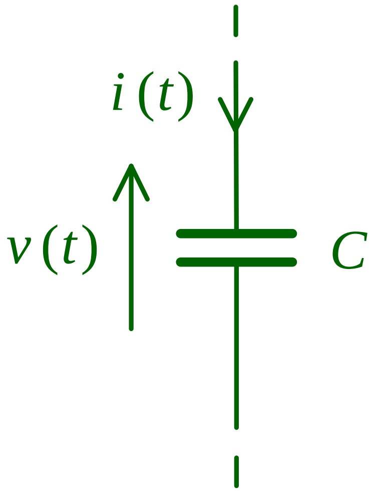


The capacitance of the capacitor is given by $C = \varepsilon \dfrac{A}{d}$ F or Farad

Area of metal plates required to produce a 1F capacitor in the free space if $d = 0.1$ mm is

$$A = \frac{Cd}{\varepsilon} = \frac{1\text{F} \times 0.0001\text{m}}{8.85 \times 10^{-12}\ \text{F/m}} = 11.3\ (\text{km})^2$$

The circuit symbol for an ideal capacitor is:



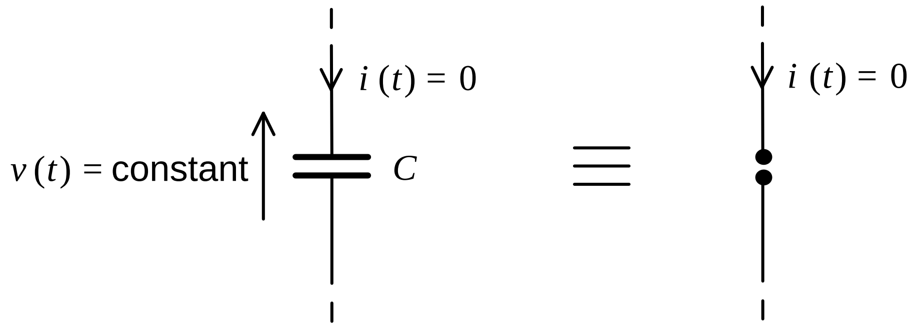
Provided that the voltage and current arrows are in opposite directions, the voltage-current relationship is:

$$i(t) = C\frac{dv(t)}{dt}$$

For dc circuits:

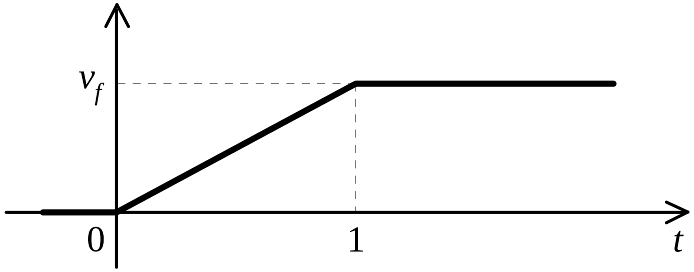
$$v(t) = \text{constant} \Rightarrow \frac{dv(t)}{dt} = 0 \quad \Rightarrow \quad i(t) = 0$$

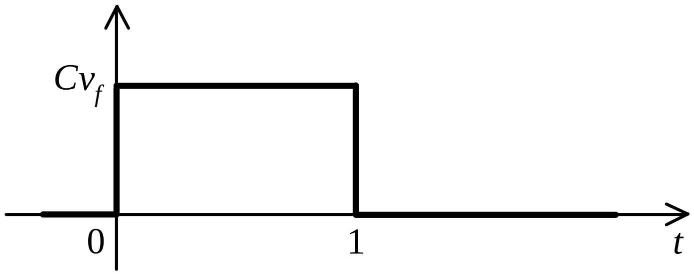
and the capacitor is equivalent to an open circuit:



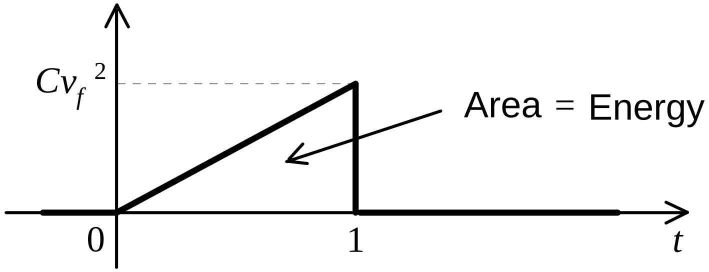
This is why we don't consider the capacitor in DC circuits.

Consider the change in voltage, current and power supplied to the capacitor as indicated below:





$p(t) = v(t)\, i(t) =$ Instantaneous power consumed



Area = Energy stored = $\dfrac{Cv_f^{\,2}}{2}$

In general, the total **energy stored** in the electric field established by the charges on the capacitor plates at time is

$$e(t) = \frac{Cv^2(t)}{2}$$

Proof.

$$e(t) = \int_{-\infty}^{t} p(x)dx = \int_{-\infty}^{t} v(x)i(x)dx$$

$$= \int_{-\infty}^{t} v(x)C\frac{dv(x)}{dx}dx$$

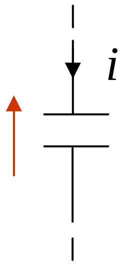
$$= C\int_{-\infty}^{t} v(x)dv(x) = \frac{C}{2}v^2(x)\Big|_{-\infty}^{t}$$

$$= \frac{C}{2}\left[v^2(t) - v^2(-\infty)\right]$$

$$= \frac{Cv^2(t)}{2}, \qquad \text{if} \quad v(-\infty) = 0.$$

Now consider the operation of a capacitor in an ac circuit:

$$v(t) = r_v \sqrt{2}\cos(\omega t + \theta_v)$$

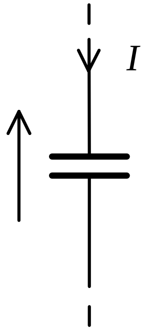


$$i(t) = C\frac{dv(t)}{dt} = -\omega C r_v \sqrt{2}\sin(\omega t + \theta_v)$$

$$= \omega C r_v \sqrt{2}\cos(\omega t + \theta_v + \frac{\pi}{2})$$

In phasor format:

$$V = r_v e^{j\theta_v}$$

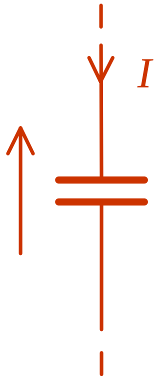


C

$$I = \omega C r_v e^{j\theta_v} e^{j\frac{\pi}{2}} = j\omega C r_v e^{j\theta_v} = j\omega C V$$

$$\Rightarrow \quad \frac{V}{I} = \frac{1}{j\omega C} \quad \Rightarrow$$

V



I

$$\frac{1}{j\omega C}$$

With phasor representation, the capacitor behaves as if it is a resistor with a "complex resistance" or an **impedance** of

$$Z_C = \frac{1}{j\omega C} \quad \Rightarrow \quad p_{av} = \mathrm{Re}\left[I^* V\right] = \mathrm{Re}\left[I^* I Z_C\right] = \mathrm{Re}\left[\frac{|I|^2}{j\omega C}\right] = 0$$

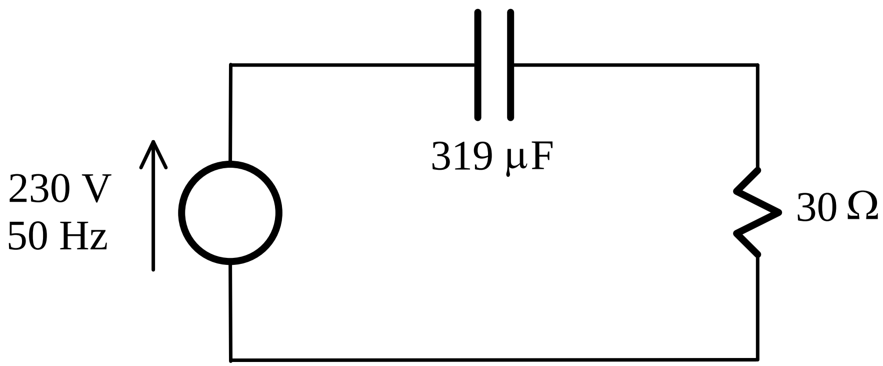
An ideal capacitor is a non-dissipative but energy-storing device.

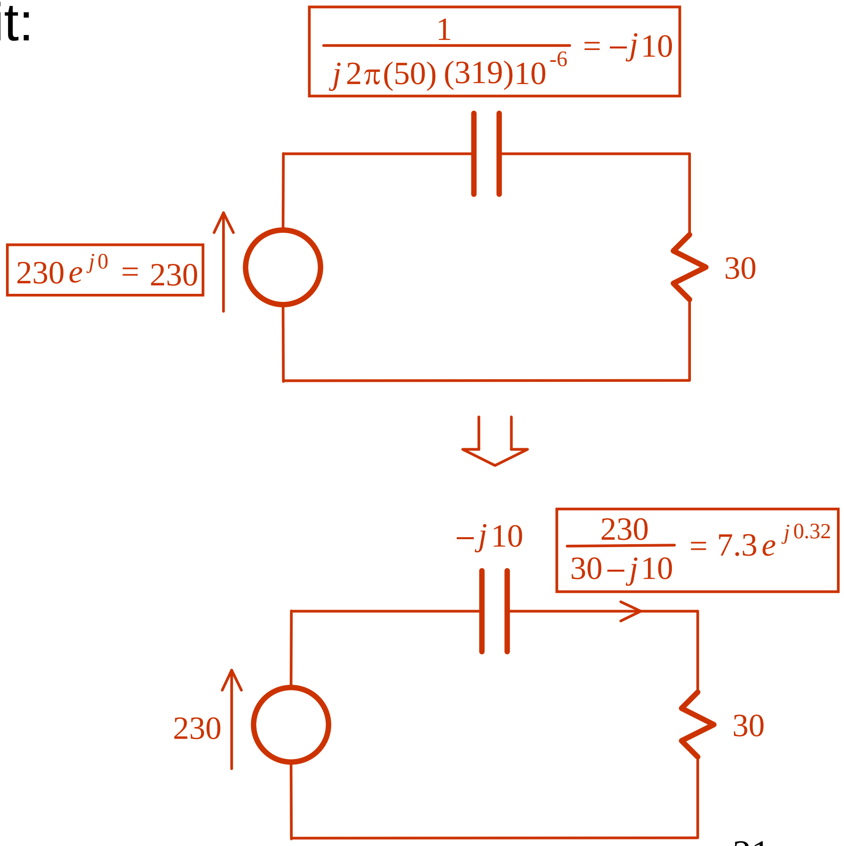
Since the phase of I relative to V that of is

$$\text{Arg}[I]-\text{Arg}[V]=\text{Arg}\left[\frac{I}{V}\right]=\text{Arg}\left[\frac{1}{Z_C}\right]=\text{Arg}[j\omega\ \ C]=90^0$$

the ac current $i(t)$ of the capacitor leads the voltage $v(t)$ by 90°.

Example: Consider the following ac circuit:





In phasor notation (taking the source to have a reference phase of 0):

Total circuit impedance	$Z = (30 - j10)\,\Omega$
Total circuit *reactance*	$X = \mathrm{Im}[Z] = \mathrm{Im}[30 - j10] = -10\,\Omega$
Total circuit resistance	$R = \mathrm{Re}[Z] = \mathrm{Re}[30 - j10] = 30\,\Omega$
Current (rms)	$\mid I \mid = 7.3\mathrm{A}$
Current (peak)	$\sqrt{2}\lvert I \rvert = 7.3\sqrt{2} = 10\mathrm{A}$
Source *V*-*I* phase relationship	I leads by $0.32\mathrm{rad}$
Power factor of entire circuit	$\cos(0.32) = 0.95$ leading
Power supplied by source	$\mathrm{Re}\left[(230)^{*}\left(7.3e^{j0.32}\right)\right] = (230)(7.3)\cos(0.32) = 1.6\ \mathrm{kW}$
Power consumed by resistor	$\mathrm{Re}\left[\left(7.3e^{j0.32}\right)^{*}\left(30 \times 7.3e^{j0.32}\right)\right] = (7.3)^{2}\,30 = 1.6\ \mathrm{kW}$

Impedance, **Resistance,** **Reactance,** } Relations?

Admittance, **Conductance,** and **Susceptance**

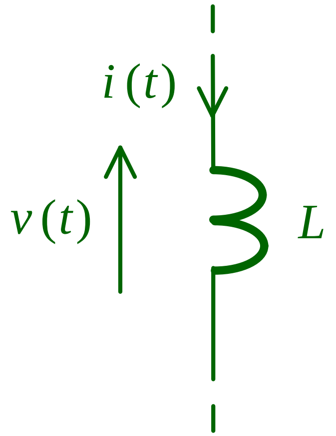
Impedance: $Z = R + jX$

Admittance:
$$Y = \frac{1}{Z} = \frac{1}{R + jX} = \frac{R - jX}{(R + jX)(R - jX)}$$
$$= \frac{R - jX}{R^2 + X^2} = \frac{R}{R^2 + X^2} + j\frac{-X}{R^2 + X^2}$$
$$= G + jB$$

Conductance **Susceptance**

1.2 Inductor

An **inductor** consists of a coil of wires for establishing a magnetic field. The circuit symbol for an ideal inductor is:



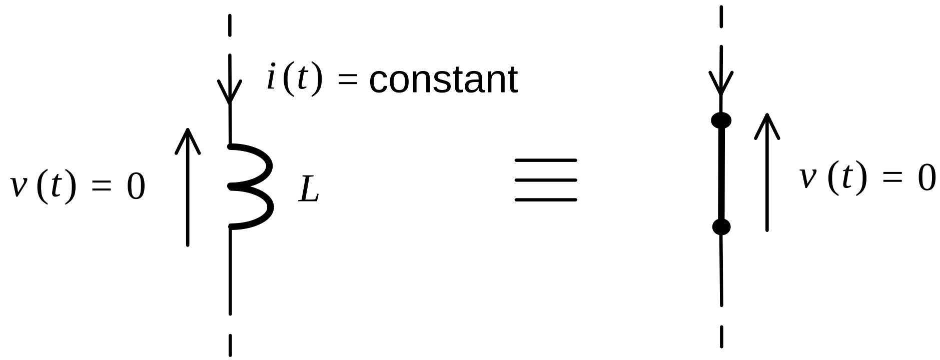
Provided that the voltage and current arrows are in opposite directions, the voltage-current relationship is:

$$\boxed{v(t) = L\frac{di(t)}{dt}}$$

For dc circuits:

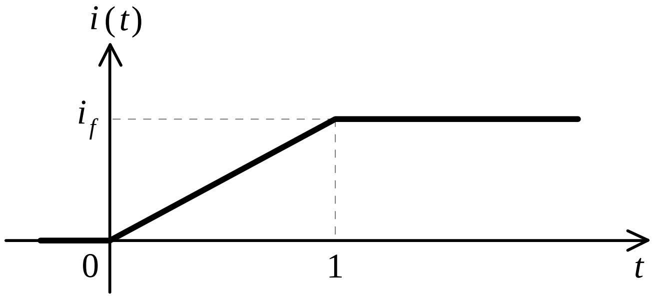
$$i(t) = \text{constant} \Rightarrow \frac{di(t)}{dt} = 0 \Rightarrow v(t) = 0$$

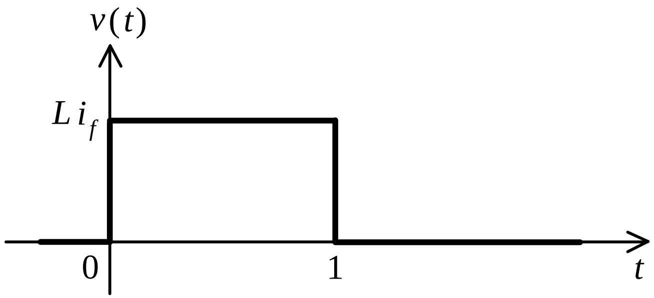
and the inductor is equivalent to a short circuit:



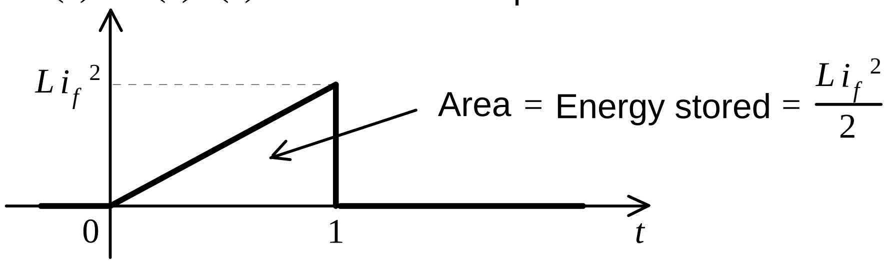
That is why there is nothing interesting about the inductor in DC circuits.

Consider the change in voltage, current and power supplied to the inductor as indicated below:





$p(t) = v(t)\,i(t)$ =Instantaneous power consumed



In general, the total energy stored in the magnetic field established by the current $i(t)$ in the inductor at time t is given by

$$e(t) = \frac{Li^2(t)}{2}$$

$$e(t) = \int_{-\infty}^{t} p(x)dx = \int_{-\infty}^{t} v(x)i(x)dx$$

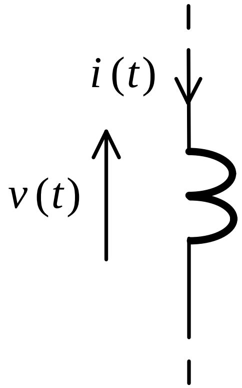
$$= \int_{-\infty}^{t} i(x)L\frac{di(x)}{dx}dx$$

$$= L\int_{-\infty}^{t} i(x)di(x) = \frac{L}{2}i^2(x)\Big|_{-\infty}^{t}$$

$$= \frac{L}{2}\left[i^2(t) - i^2(-\infty)\right]$$

$$= \frac{Li^2(t)}{2}, \qquad \text{if} \;\; i(-\infty) = 0.$$

Now consider the operation of an inductor in an ac circuit:

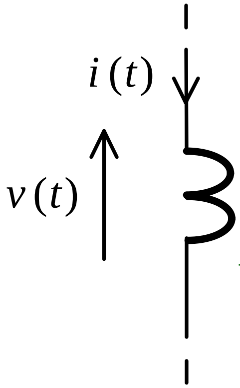


$$i(t) = r_i\sqrt{2}\cos(\omega t + \theta_i)$$

$$v(t) = L\frac{di(t)}{dt} = -\omega L r_i\sqrt{2}\sin(\omega t + \theta_i)$$

$$= \omega L r_i\sqrt{2}\cos(\omega t + \theta_i + \frac{\pi}{2})$$

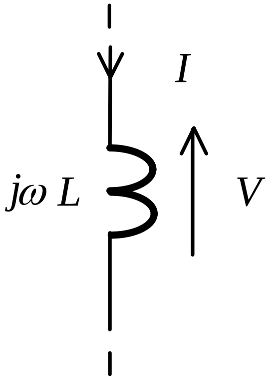
In phasor:



$$I = r_i e^{j\theta_i}$$

$$V = \omega L r_i e^{j\theta_i} e^{j\pi/2} = j\omega L r_i e^{j\theta_i} = (j\omega L)I$$

$\Longrightarrow$



$$\boxed{Z_L = \frac{V}{I} = j\omega L}$$

Z_L is the impedance of the inductor. The ave. power absorbed by the inductor:

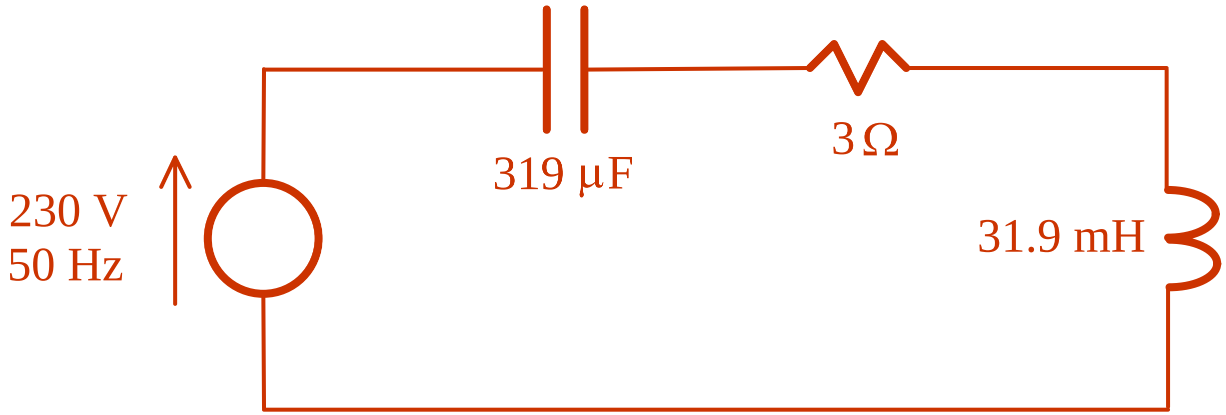
$$p_{av} = \mathrm{Re}\left[I^* V\right] = \mathrm{Re}\left[I^* Z_L I\right] = \mathrm{Re}\left[j\omega L I^* I\right] = \mathrm{Re}\left[j\omega L |I|^2\right] = 0$$

Since the phase of I relative to that of V is

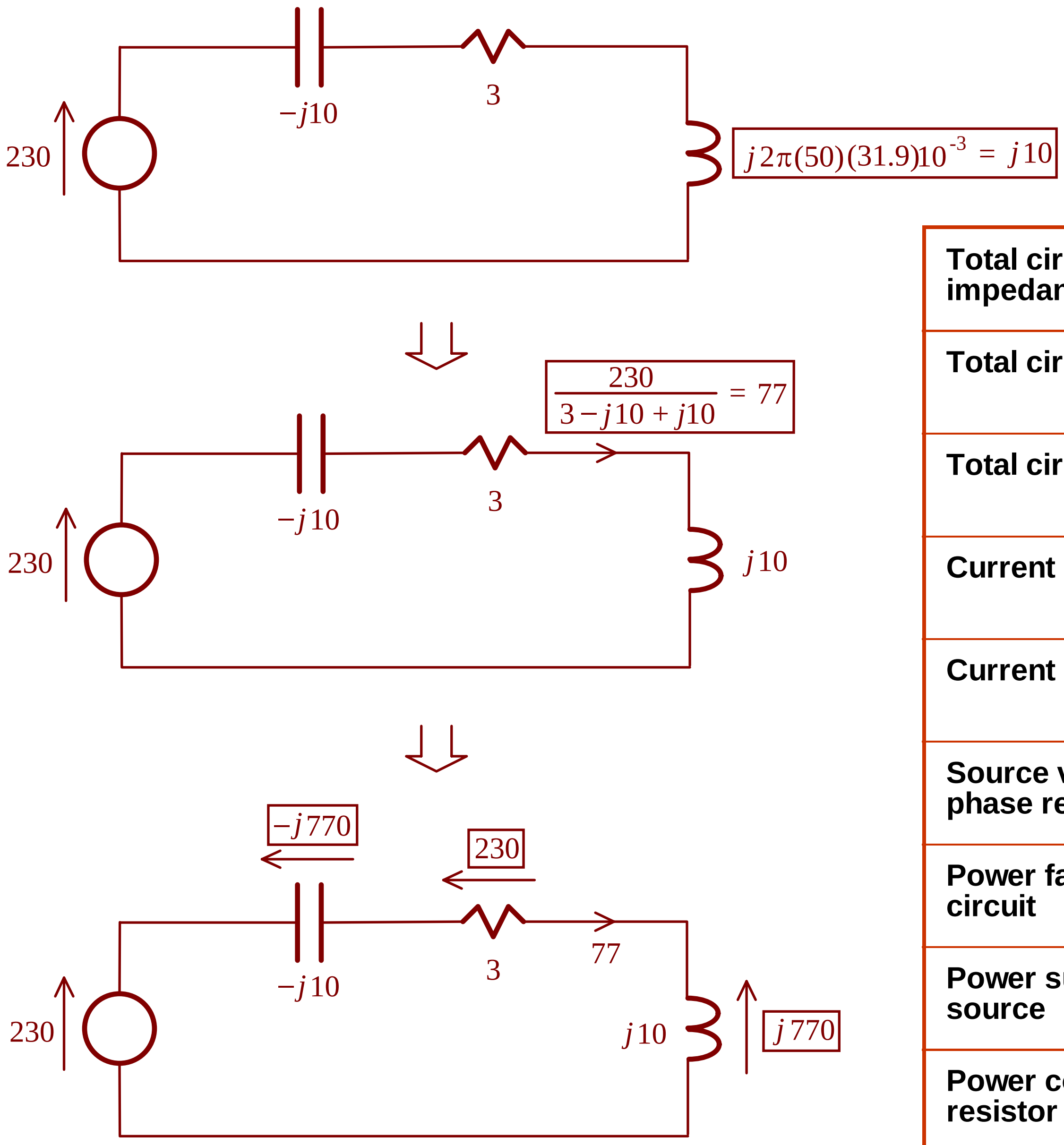
$$\text{Arg}[I] - \text{Arg}[V] = \text{Arg}\left[\frac{I}{V}\right] = \text{Arg}\left[\frac{1}{Z_L}\right] = \text{Arg}\left[\frac{1}{j\omega L}\right] = -90^0$$

the ac current $i(t)$ lags the voltage $v(t)$ by 90º.

As an example, consider the following series ac circuit:



We can use the phasor representation to convert this ac circuit to a 'DC' circuit with complex voltage and resistance.



Summary of the circuit:

Total circuit impedance	$Z = 3 - j10 + j10 = 3\Omega$
Total circuit reactance	$X = \mathrm{Im}[Z] = \mathrm{Im}[3] = 0\,\Omega$
Total circuit resistance	$R = \mathrm{Re}[Z] = \mathrm{Re}[3] = 3\Omega$
Current (rms)	$\lvert I \rvert = 77\mathrm{A}$
Current (peak)	$\sqrt{2}\lvert I \rvert = 77\sqrt{2} = 108\,\mathrm{A}$
Source voltage-current phase relationship	$0\,(\text{in phase})$
Power factor of entire circuit	$\cos(0) = 1$
Power supplied by source	$\mathrm{Re}\left[(77)^*(230)\right] = 18\,\mathrm{kW}$
Power consumed by resistor	$\mathrm{Re}\left[(77)^*(3 \times 77)\right] = 18\,\mathrm{kW}$

2. Transient Response

2.0 Linear Differential Equations

General solution:

nth order linear differential equation	$\frac{d^n x(t)}{dt^n} + a_{n-1}\frac{d^{n-1}x(t)}{dt^{n-1}} + \cdots + a_0 x(t) = u(t)$
General solution	$x(t) = x_{ss}(t) + x_{tr}(t)$
Steady state response with no arbitrary constant	$x_{ss}(t) =$ particular integral obtained from assuming solution to have the same form as $u(t)$
Transient response with n arbitrary constants	$x_{tr}(t) =$ general solution of homogeneous equation $\frac{d^n x_{tr}(t)}{dt^n} + a_{n-1}\frac{d^{n-1}x_{tr}(t)}{dt^{n-1}} + \cdots + a_0 x_{tr}(t) = 0$

General solution of homogeneous equation:

n th order linear homogeneous equation	$\frac{d^n x_{tr}(t)}{dt^n} + a_{n-1} \frac{d^{n-1} x_{tr}(t)}{dt^{n-1}} + \cdots + a_0 x_{tr}(t) = 0$
Roots of polynomial from homogeneous equation	Roots : $z_1, \cdots, z_n$ given by $(z - z_1) \cdots (z - z_n) = z^n + a_{n-1} z^{n-1} + \cdots + a_0$
General solution (distinct roots)	$x_{tr}(t) = k_1 e^{z_1 t} + \cdots + k_n e^{z_n t}$
General solution (non-distinct roots)	$x_{tr}(t) = (k_1 + k_2 t + k_3 t^2) e^{13t} + (k_4 + k_5 t) e^{22t} + k_6 e^{31t} + k_7 e^{41t}$ if roots are $13, 13, 13, 22, 22, 31, 41$

Particular integral:

$x_{ss}(t)$	Any specific solution (with no arbitrary constant) of $\frac{d^n x(t)}{dt^n} + a_{n-1}\frac{d^{n-1}x(t)}{dt^{n-1}} + \cdots + a_0 x(t) = u(t)$
Method to determine $x_{ss}(t)$	Trial and error approach: assume $x_{ss}(t)$ to have the same form as $u(t)$ and substitute into differential equation
Example to find $x_{ss}(t)$ for $\frac{dx(t)}{dt} + 2x(t) = e^{3t}$	Try a solution of he^{3t} $\frac{dx(t)}{dt} + 2x(t) = e^{3t} \Rightarrow 3he^{3t} + 2he^{3t} = e^{3t} \Rightarrow h = 0.2$ $x_{ss}(t) = 0.2e^{3t}$
Standard trial solutions	$u(t)$ — trial solution for $x_{ss}(t)$ $e^{\alpha t}$ — $he^{\alpha t}$ t — ht $te^{\alpha t}$ — $(h_1 + h_2 t)e^{\alpha t}$ $a\cos(\omega t) + b\sin(\omega t)$ — $h_1\cos(\omega t) + h_2\sin(\omega t)$

2.1 What is Transient Analysis?

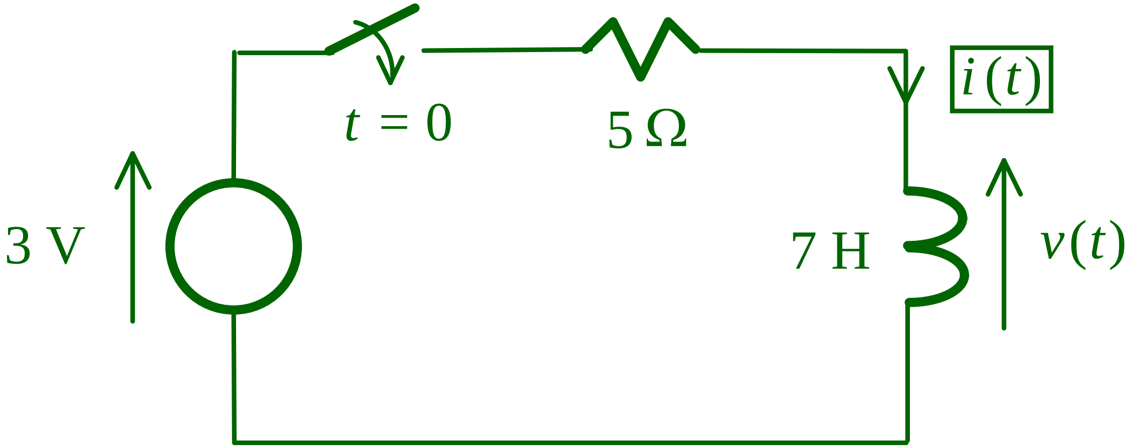
DC and AC circuit analyses using the frequency domain approach are often called ***steady state*** analysis, as signals are assumed to exist at all time.

In order for the results obtained from these analyses to be valid, it is necessary for the circuit to have been working for a considerable period of time. This will ensure that all the ***transients*** caused by, say, the switching on of the sources have died out, the circuit is working in the steady state, and all the voltages and currents are as if they exist from all time.

However, when the circuit is first switched on, the circuit will not be in the steady state and it will be necessary to go back to first principle to determine the ***behavior*** of the system. This process is then called the ***transient analysis*****.**

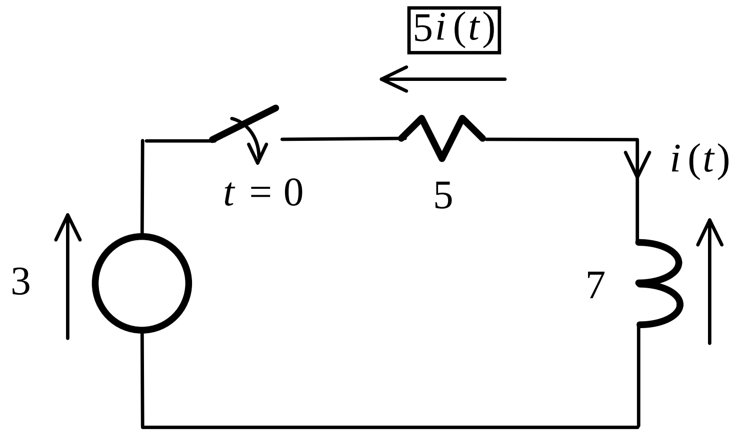
2.2 RL Circuit and Governing Differential Equation

Consider determining $i(t)$ in the following series RL circuit:

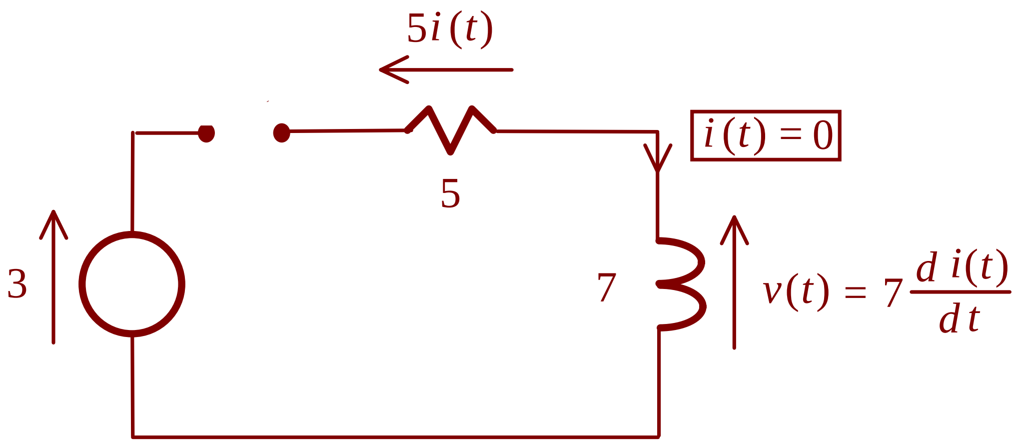


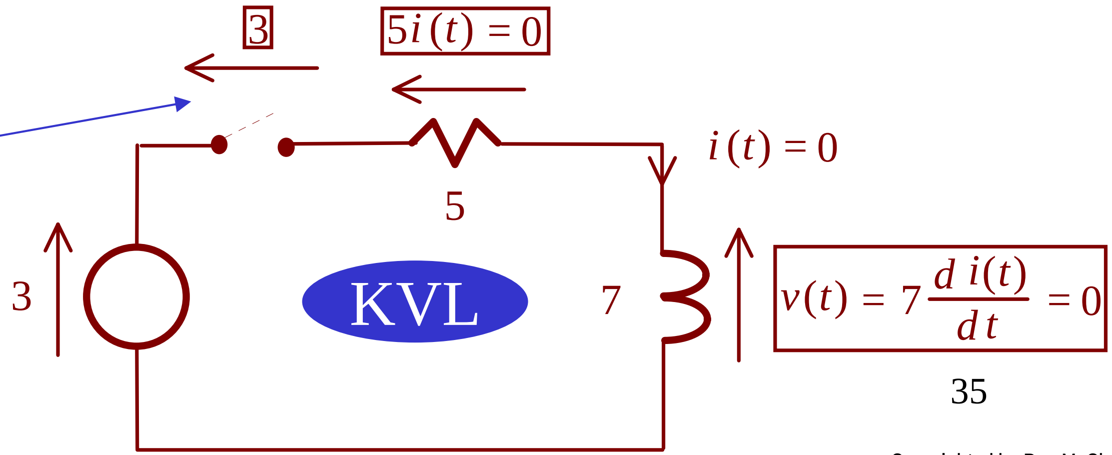
where the switch is open for $t < 0$ and is closed for $t \geq 0$.

Since $i(t)$ and $v(t)$ will not be equal to constants or sinusoids for all time, these cannot be represented as constants or phasors. Instead, the basic general voltage-current relationships for the resistor and inductor have to be used:

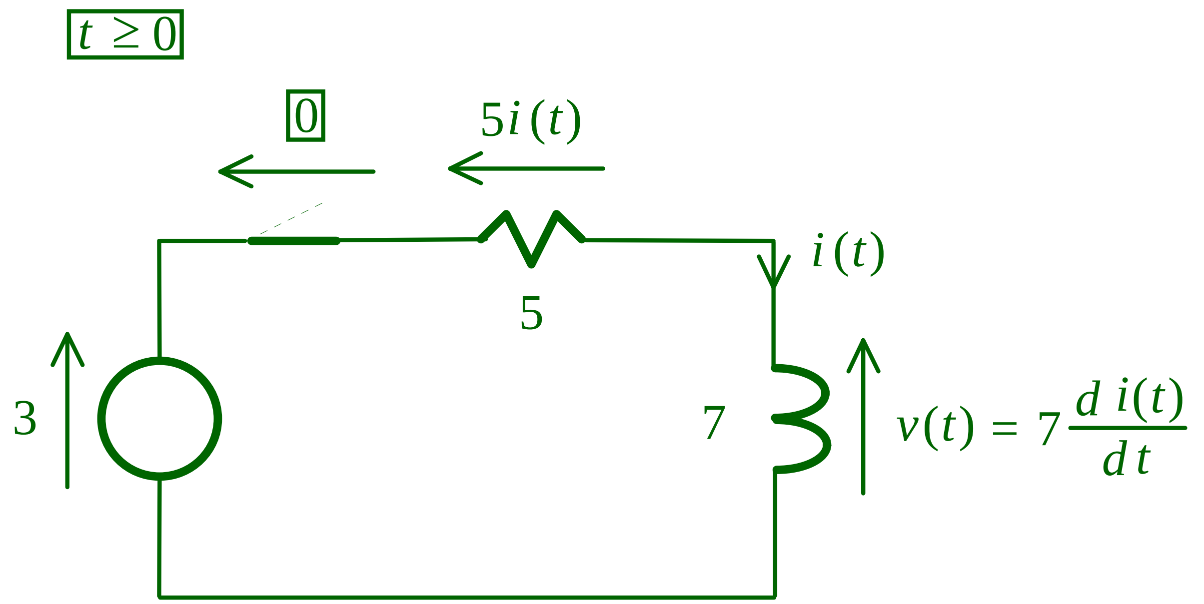


For t < 0





voltage cross over the switch



Applying KVL:

$$7\frac{di(t)}{dt}+5i(t)=3, \quad t\geq 0$$

and $i(t)$ can be found from determining the ***general solution*** to this first order linear differential equation (d.e.) which governs the behavior of the circuit for $t \geq 0$.

Mathematically, the above d.e. is often written as

$$7\frac{di(t)}{dt}+5i(t)=u(t),\ t\geq 0$$

where the r.h.s. is $u(t)=3,\ t\geq 0$

and corresponds to the dc source or excitation in this example.

2.3 Steady State Response

Since the r.h.s. of the governing d.e.

$$7\frac{di(t)}{dt}+5i(t)=u(t)=3,\ \ t\geq 0$$

Let us try a steady state solution of

$$i_{ss}(t)=k, \quad t\geq 0$$

which has the same form as $u(t)$, as a possible solution.

$$7\frac{di_{ss}(t)}{dt}+5i_{ss}(t)=3$$

$$\Rightarrow 7(0)+5(k)=3$$

$$\Rightarrow k=\frac{3}{5}$$

$$\Downarrow$$

$$i_{ss}(t)=\frac{3}{5}, \quad t\geq 0$$

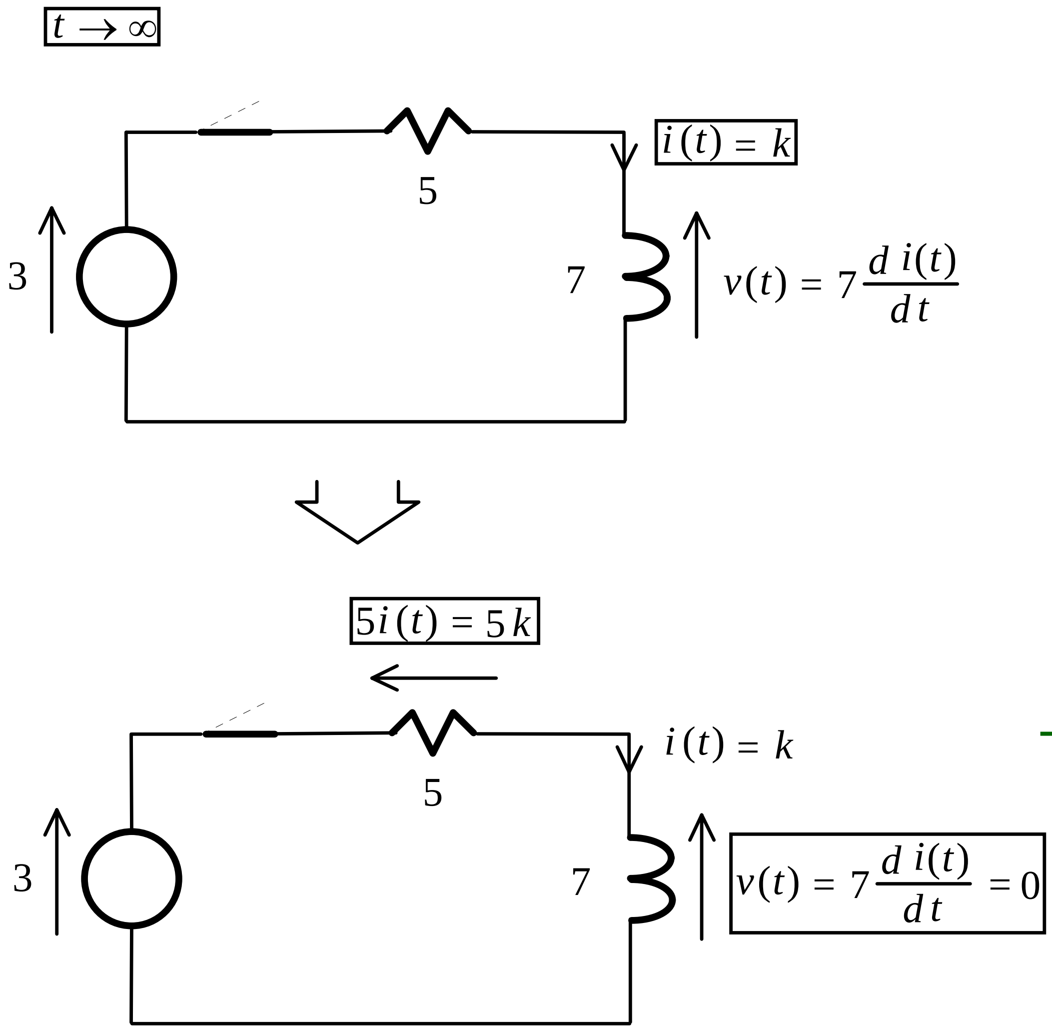
$$\Downarrow$$

$$7\frac{di_{ss}(t)}{dt}+5i_{ss}(t)=7\frac{d}{dt}\left(\frac{3}{5}\right)+5\left(\frac{3}{5}\right)=3, \quad t\geq 0$$

and is a solution of the governing d.e.

In mathematics, the above solution is called the ***particular integral*** or solution and is found from letting the answer to have the same form as $u(t)$. The word "particular" is used as the solution is only one possible function that satisfy the d.e.

In circuit analysis, the derivation of $i_{ss}(t)$ by letting the answer to have the same form as $u(t)$ can be shown to give the ***steady state response*** of the circuit as $t \to \infty$.



Using KVL, the steady state response is

$$3 = 0 + 5k + 0 = 5k$$

$$\Rightarrow k = \frac{3}{5}$$

$$\Rightarrow i(t) = \frac{3}{5}, \quad t \to \infty$$

This is the same as $i_{ss}(t)$.

2.4 Transient Response

To determine $i(t)$ for all t, it is necessary to find the complete solution of the governing d.e.

$$7\frac{di(t)}{dt}+5i(t)=u(t)=3,\ \ t\geq 0$$

From mathematics, the complete solution can be obtained from summing a particular solution, say, $i_{ss}(t)$, with $i_{tr}(t)$: $i(t)=i_{ss}(t)+i_{tr}(t), \quad t\geq 0$

where $i_{tr}(t)$ is the general solution of the ***homogeneous*** equation

$$7\frac{di(t)}{dt}+5i(t)=0,\ \ t\geq 0$$

$$7\frac{di_{tr}(t)}{dt}+5i_{tr}(t)\Big|_{\frac{di_{tr}(t)}{dt}\ \text{replaced by}\ z}$$

$$=7z^1+5z^0=7z+5$$

$$\Longrightarrow \quad z_1=-\frac{5}{7}$$

$$\Longrightarrow \quad i_{tr}(t)=k_1e^{z_1t}=k_1e^{-\frac{5}{7}t},\ t\geq 0$$

where k_1 is a constant (unknown now).

$$i_{tr}(t)=k_1e^{-\frac{5}{7}t}\rightarrow 0,\ t\rightarrow\infty$$

Thus, it is called ***transient response***.

2.5 Complete Response

To see that summing $i_{ss}(t)$ and $i_{tr}(t)$ gives the general solution of the governing d.e.

$$7\frac{di(t)}{dt}+5i(t)=3,\ \ t\geq 0$$

note that

$$i_{ss}(t)=\frac{3}{5},\quad t\geq 0 \quad \text{satisfies} \quad 7\frac{d}{dt}\left(\frac{3}{5}\right)+5\left(\frac{3}{5}\right)=3,\quad t\geq 0$$

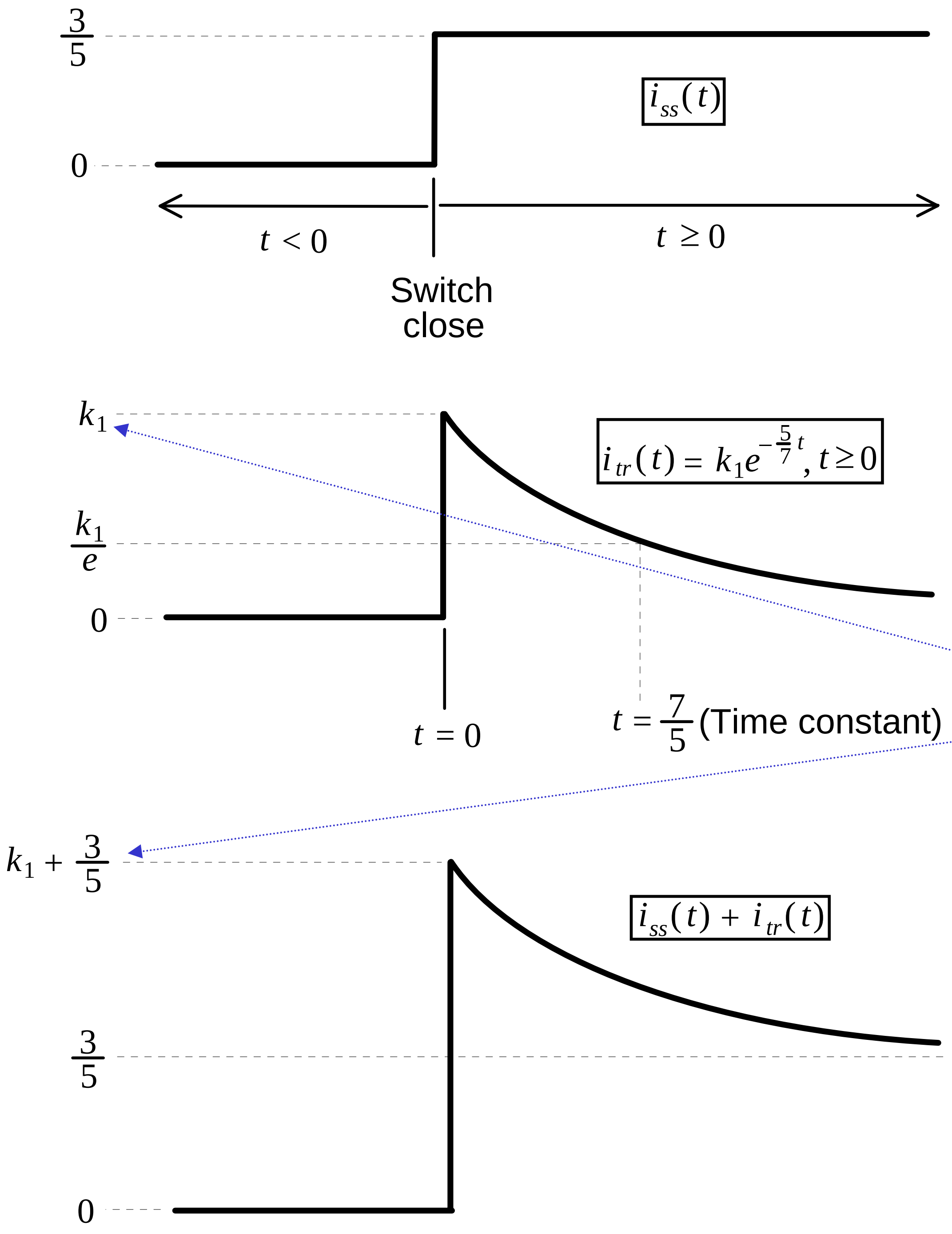
$$i_{tr}(t)=k_1e^{-\frac{5}{7}t},\quad t\geq 0 \quad \text{satisfies} \quad 7\frac{d}{dt}\left(k_1e^{-\frac{5}{7}t}\right)+5\left(k_1e^{-\frac{5}{7}t}\right)=0,\quad t\geq 0$$

$$\Downarrow$$

$$i_{ss}(t)+i_{tr}(t)=\frac{3}{5}+k_1e^{-\frac{5}{7}t},\quad t\geq 0 \quad \text{satisfies} \quad 7\frac{d}{dt}\left(\frac{3}{5}+k_1e^{-\frac{5}{7}t}\right)+5\left(\frac{3}{5}+k_1e^{-\frac{5}{7}t}\right)=3$$

$$\Rightarrow$$

$$\boxed{i(t)=i_{ss}(t)+i_{tr}(t)=\frac{3}{5}+k_1e^{-\frac{5}{7}t},\quad t\geq 0}$$ is the general solution of the d.e.



Complete response

Steady State Response

Transient Response

k_1 is to be determined later

Complete Response

Note that the time it takes for the transient or zero-input response $i_{\mathrm{tr}}(t)$ to decay to $1/e$ of its initial value is

Time taken for $i_{\mathrm{tr}}(t)$ to decay to $1/e$ of initial value $= \dfrac{7}{5}$

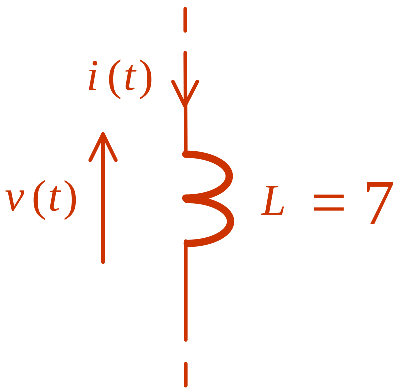
and is called the ***time constant*** of the response or system.

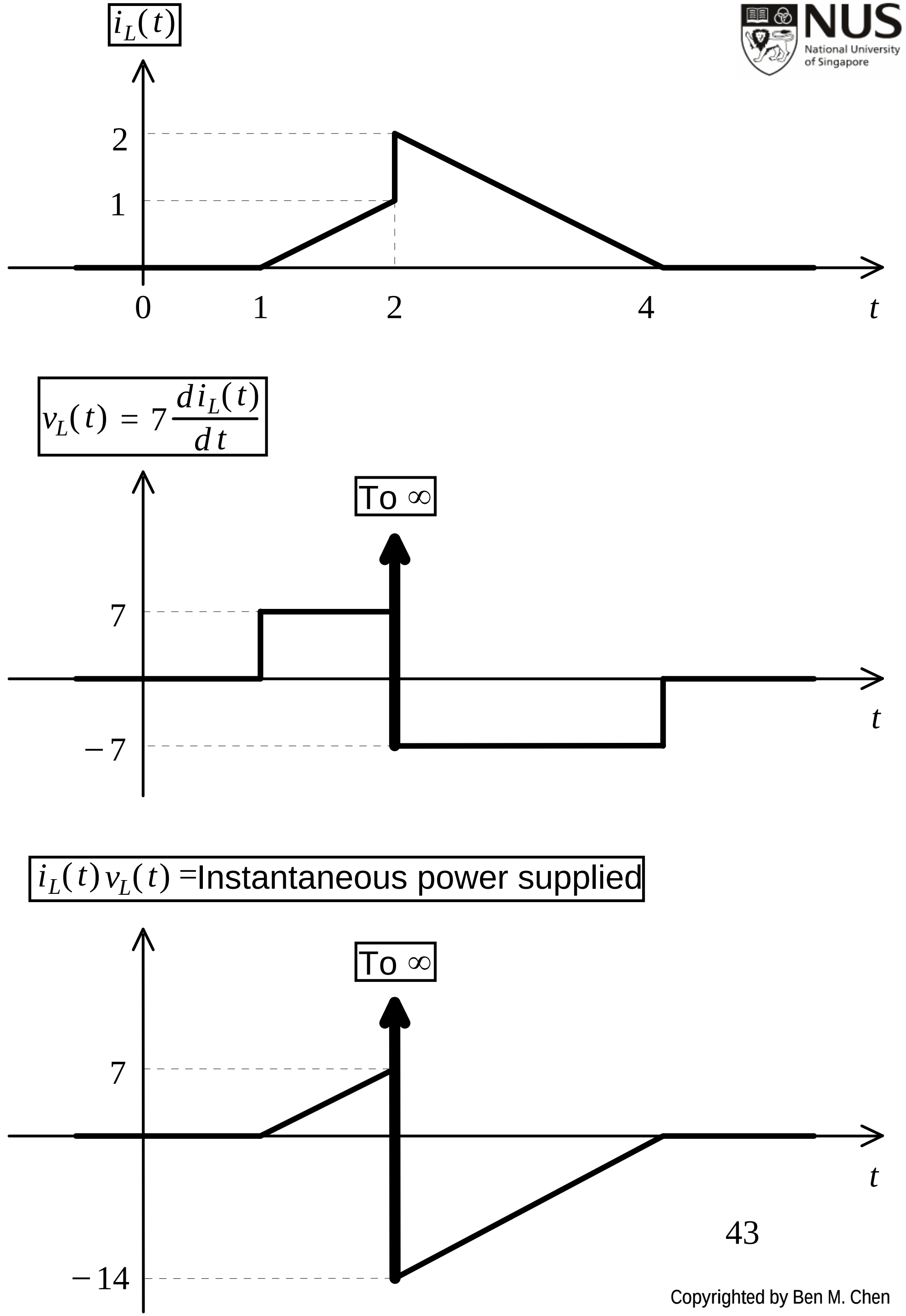
We can take the transient response to have died out after a few time constants.

2.6 Current Continuity for Inductor

To determine the constant $\boldsymbol{k_1}$ in the transient response of the RL circuit, the concept of ***current continuity for an inductor*** has to be used.

Consider the following example:

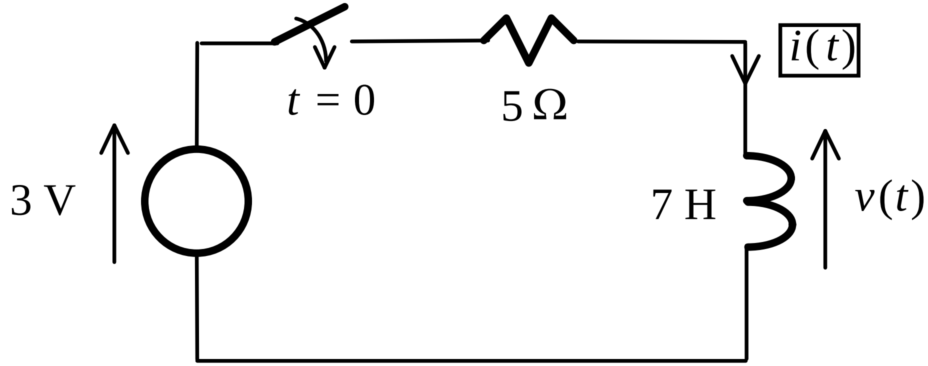




Due to the step change or discontinuity in $i_L(t)$ at $t = 2$, and the power supplied to the inductor at $t = 2$ will go to infinity. Since it is impossible for any system to deliver an infinite amount of power at any time, it is impossible for $i_L(t)$ to change in the manner shown.

In general, the **current through an inductor must be a continuous function of time** and cannot change in a step manner.

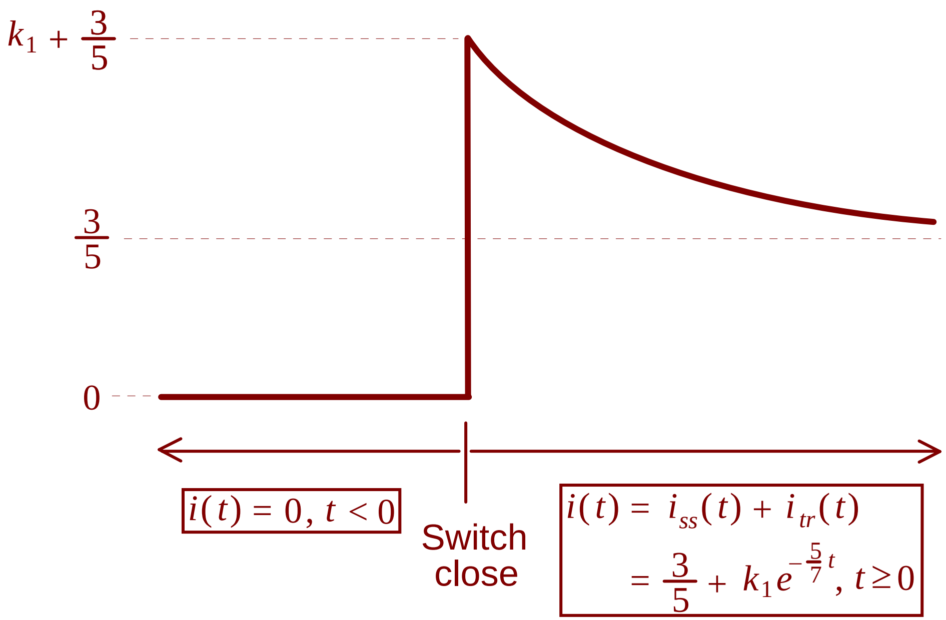
Now back to our RL Circuit:

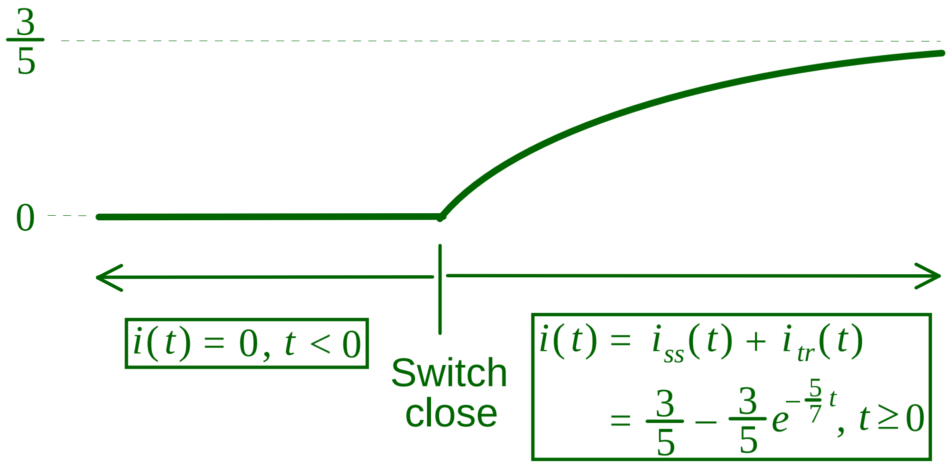


Using current continuity for an inductor at t = 0:

$$i(t=0) = \frac{3}{5} + k_1 = 0 \quad \Rightarrow \quad k_1 = -\frac{3}{5}$$

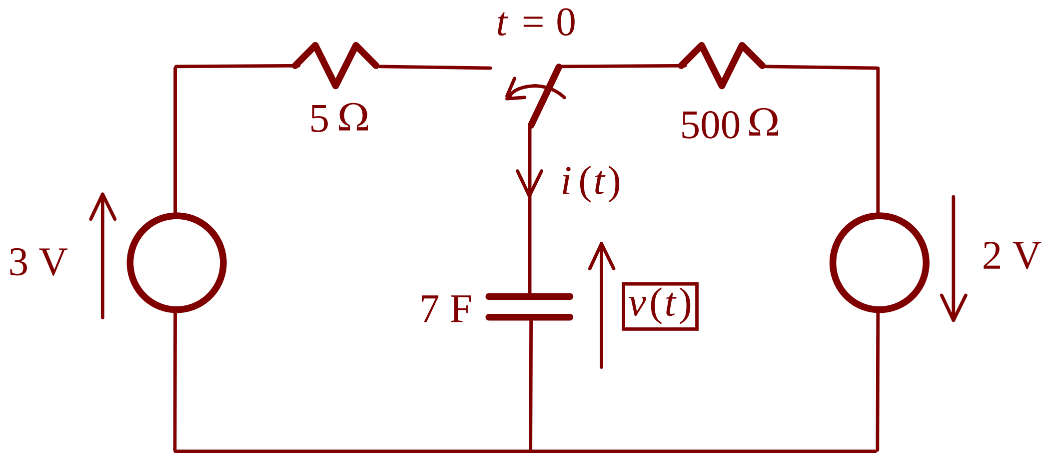
$$i(t) = \begin{cases} 0, & t < 0 \\ \dfrac{3}{5} - \dfrac{3}{5} e^{-\frac{5}{7}t}, & t \geq 0 \end{cases}$$



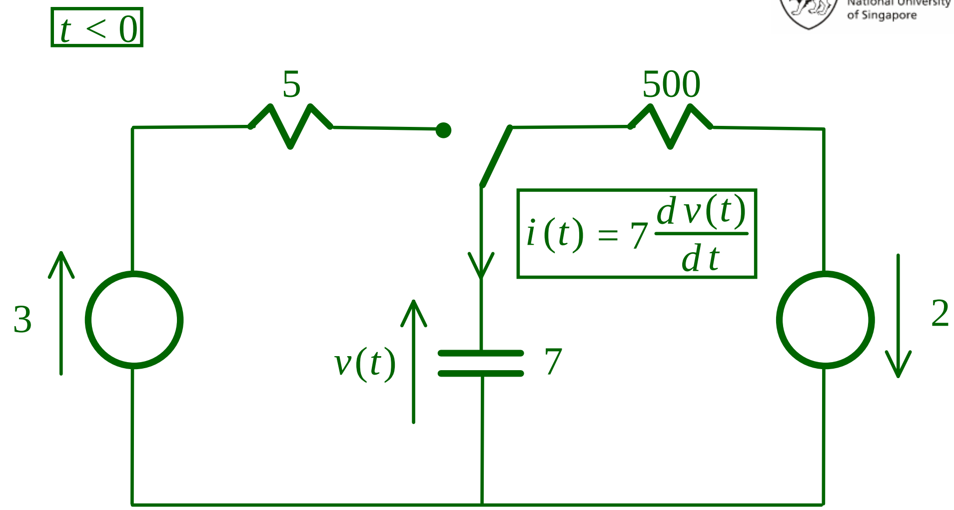


2.7 RC Circuit

Consider finding $v(t)$ in the following RC circuit:



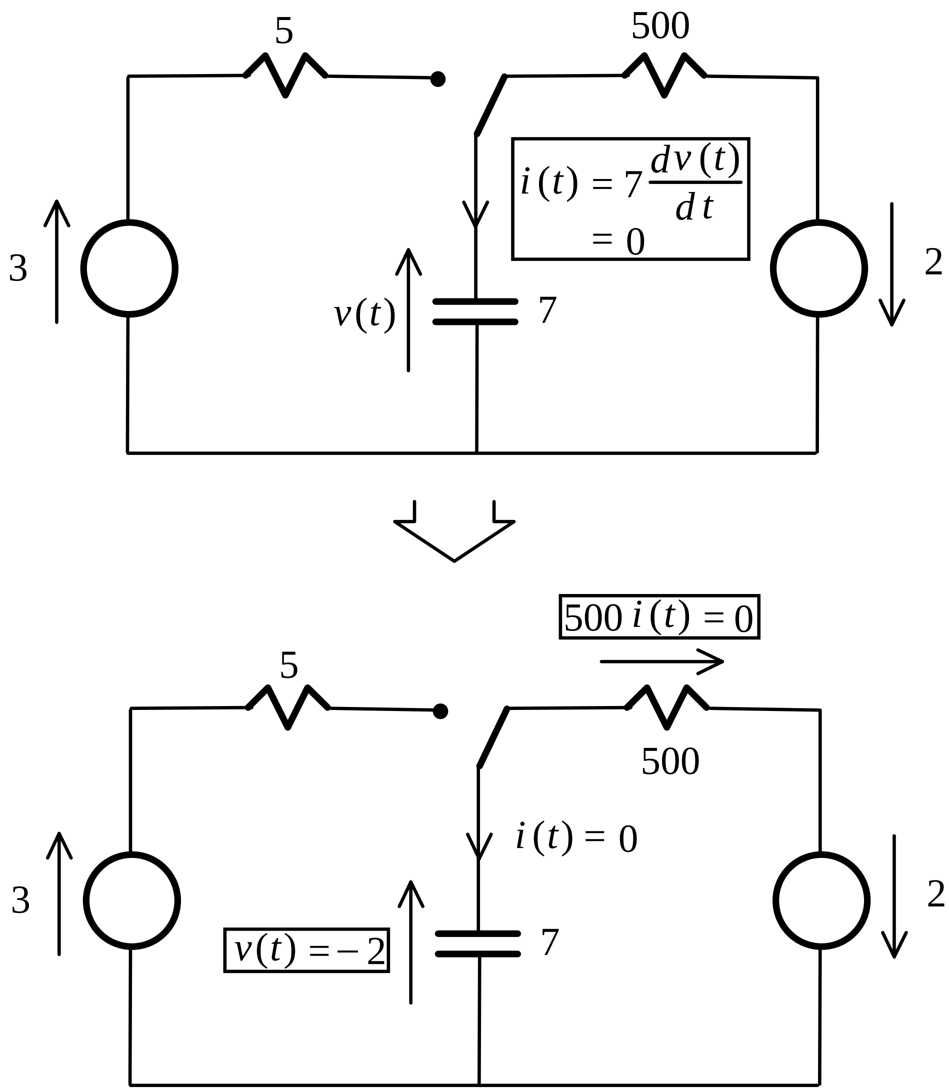
where the switch is in the position shown for $t < 0$ and is in the other position for $t \geq 0$.



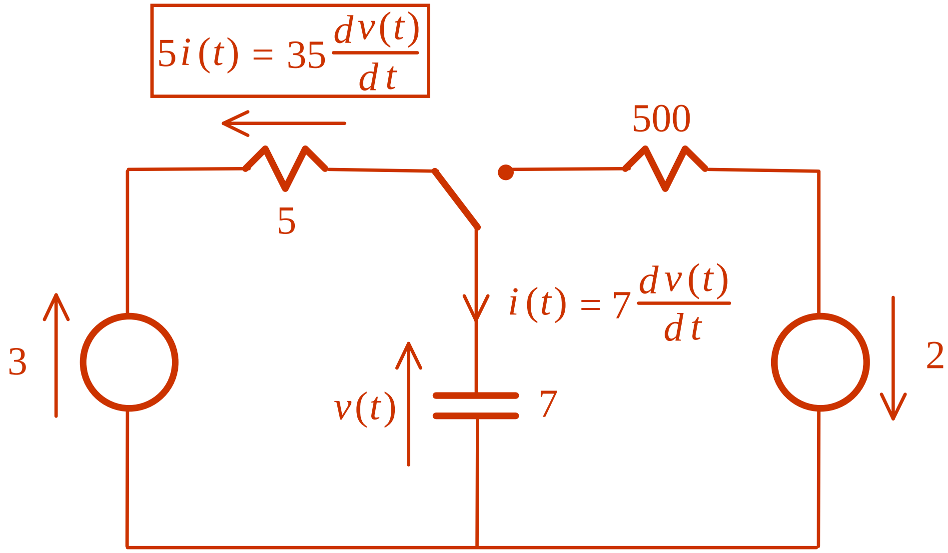
Taking the switch to be in this position starting from $t = -\infty$, the voltages and currents will have settled down to constant values for practically all $t < 0$.

$$i(t) = 7\frac{dv(t)}{dt} = 7\frac{d(\text{constant})}{dt} = 0, \quad t < 0$$

$t < 0$



$\boxed{t \ge 0}$



Applying KVL:

$$35\frac{dv(t)}{dt} + v(t) = u(t) = 3, \quad t \ge 0$$

which has a solution

$$v(t) = v_{ss}(t) + v_{tr}(t), \quad t \ge 0$$

(1) Steady State Response

$$u(t) = 3, \quad t \ge 0$$

$$v_{ss}(t) = k, \quad t \ge 0$$

$$35\frac{dv_{ss}(t)}{dt} + v_{ss}(t) = 3$$

$$\Rightarrow 0 + k = 3 \Rightarrow k = 3$$

$$v_{ss}(t) = 3, \quad t \ge 0$$

(2) Transient Response

$$35\frac{dv_{tr}(t)}{dt} + v_{tr}(t) = 0, \quad t \ge 0$$

$$35\frac{dv_{tr}(t)}{dt} + v_{tr}(t) \Bigg|_{\frac{dv_{tr}(t)}{dt} \text{ replaced by } z}$$

$$= 35z^1 + z^0 = 35z + 1$$

$$z_1 = -\frac{1}{35}$$

$$v_{tr}(t) = k_1 e^{z_1 t} = k_1 e^{-\frac{t}{35}}, \quad t \ge 0$$

$$v(t) = \begin{cases} -2, & t < 0 \\ v_{ss}(t) + v_{tr}(t), & t \ge 0 \end{cases} = \begin{cases} -2, & t < 0 \\ 3 + k_1 e^{-\frac{t}{35}}, & t \ge 0 \end{cases}$$

Complete Response

2.8 Voltage Continuity for Capacitor

To determine k_1 in the transient response of the RC circuit, the concept of ***voltage continuity for a capacitor*** has to be used.

Similar to current continuity for an inductor, the voltage $v(t)$ across a capacitor C must be continuous and cannot change in a step manner.

Thus, for the RC circuit we consider, the complete solution was derived as:

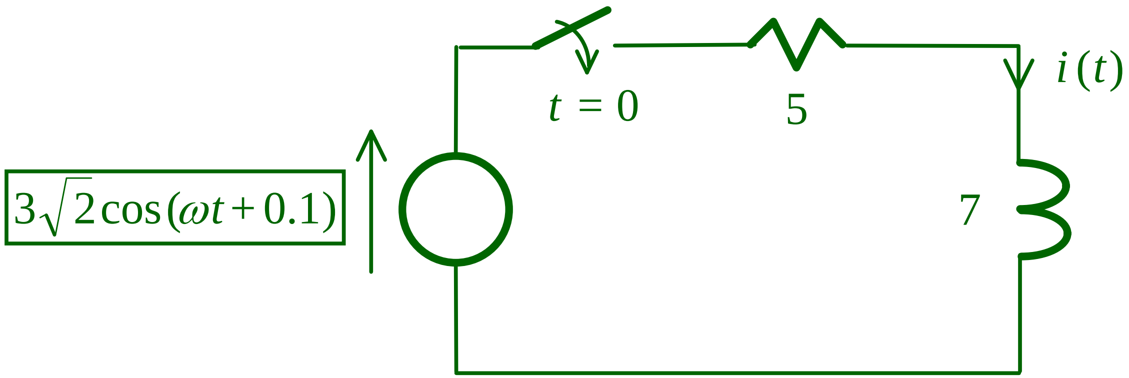
$$v(t) = \begin{cases} -2, & t < 0 \\ v_{ss}(t) + v_{tr}(t), & t \geq 0 \end{cases} = \begin{cases} -2, & t < 0 \\ 3 + k_1 e^{-\frac{t}{35}}, & t \geq 0 \end{cases}$$

At $t = 0$,

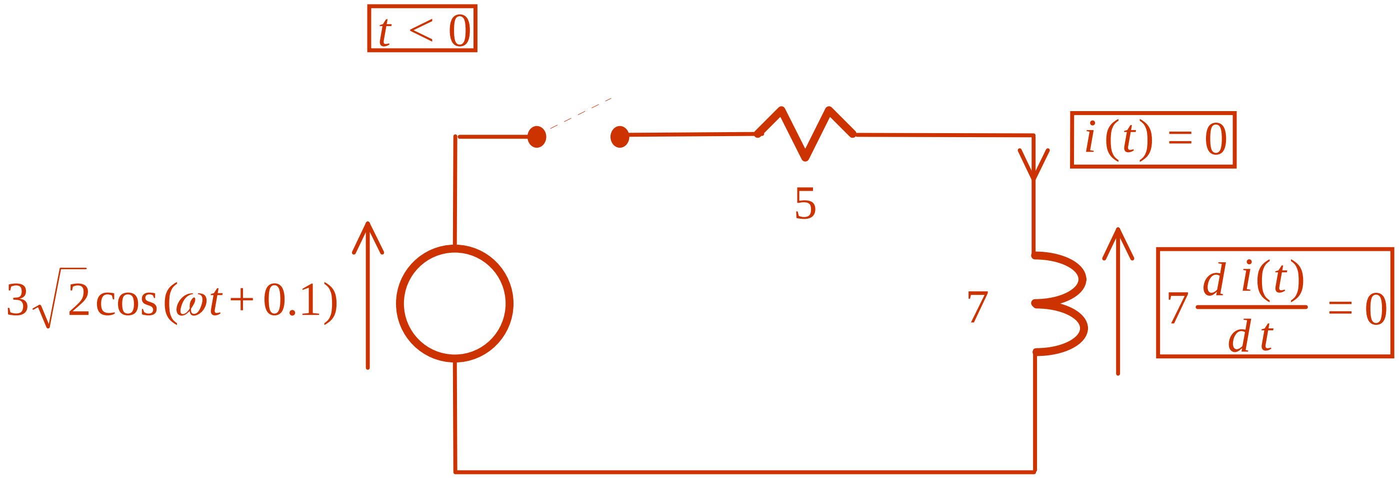
$$v(0) = 3 + k_1 = -2 \;\Rightarrow\; k_1 = -5 \quad\Longrightarrow\quad v(t) = \begin{cases} -2, & t < 0 \\ 3 - 5e^{-\frac{t}{35}}, & t \geq 0 \end{cases}$$

2.9 Transient with Sinusoidal Source

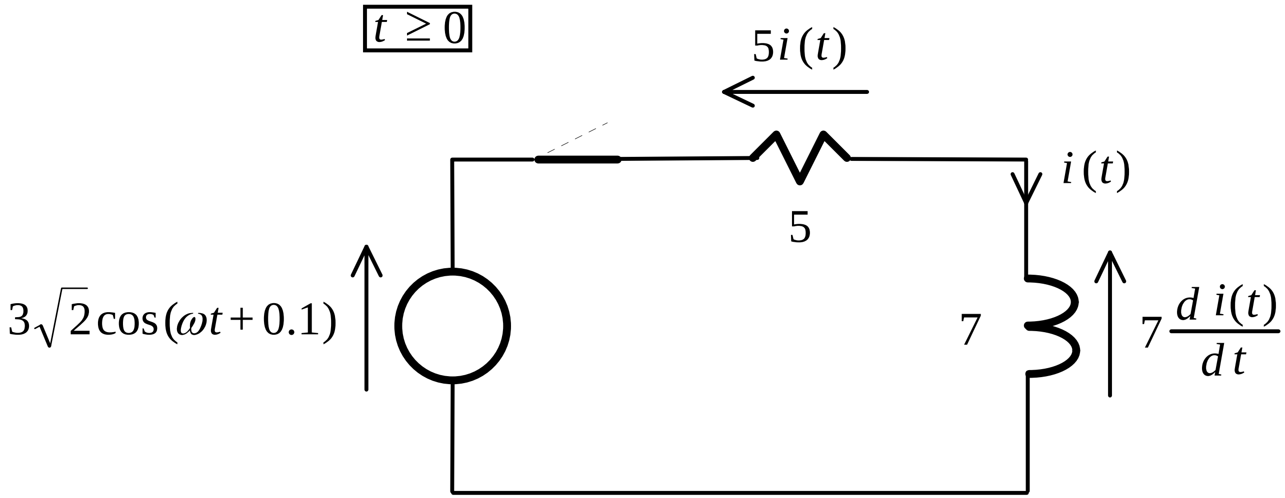
Consider the RL circuit with the dc source changed to a sinusoidal one:



For $t < 0$ when the switch is open:



For $t \geq 0$ when the switch is closed:



The governing d.e. is

$$7\frac{d\,i(t)}{dt} + 5\,i(t) = u(t), \quad t \geq 0$$

Looking for general solution

$$i(t) = i_{ss}(t) + i_{tr}(t), \;\; t \geq 0$$

with

$$u(t) = 3\sqrt{2}\cos(\omega t + 0.1) = \mathrm{Re}\left[3\sqrt{2}e^{j(\omega\, t + 0.1)}\right] = \mathrm{Re}\left[\left(3e^{j0.1}\right)\left(\sqrt{2}e^{j\omega t}\right)\right], \quad t \geq 0$$

Since $u(t)$ is sinusoidal in nature, a trial solution for the steady state response or particular integral $i_{ss}(t)$ may be

$$i_{ss}(t) = r\sqrt{2}\cos(\omega t + \theta) = \mathrm{Re}\left[\left(re^{j\theta}\right)\left(\sqrt{2}e^{j\omega t}\right)\right], \quad t \geq 0$$

$$\Longrightarrow 7\frac{di_{ss}(t)}{dt} + 5i_{ss}(t) = 7\frac{d\,\mathrm{Re}\left[\left(re^{j\theta}\right)\left(\sqrt{2}e^{j\omega t}\right)\right]}{dt} + 5\,\mathrm{Re}\left[\left(re^{j\theta}\right)\left(\sqrt{2}e^{j\omega t}\right)\right]$$

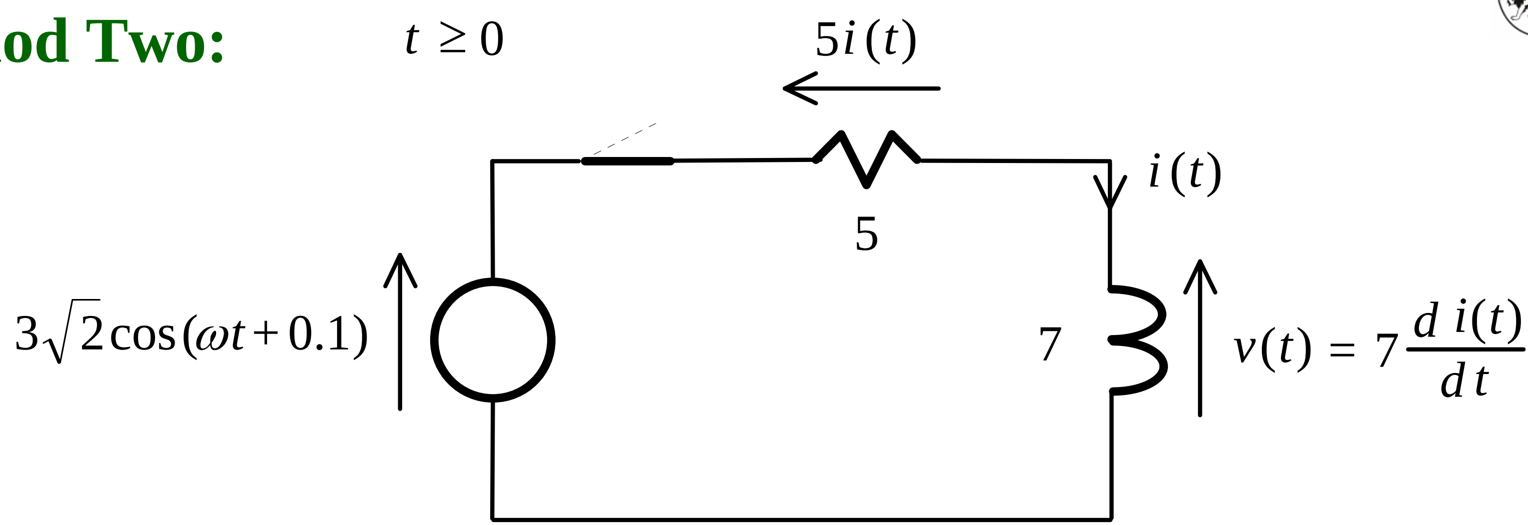
$$= 7\,\mathrm{Re}\left[\left(re^{j\theta}\right)(j\omega)\left(\sqrt{2}e^{j\omega t}\right)\right] + 5\,\mathrm{Re}\left[\left(re^{j\theta}\right)\left(\sqrt{2}e^{j\omega t}\right)\right]$$

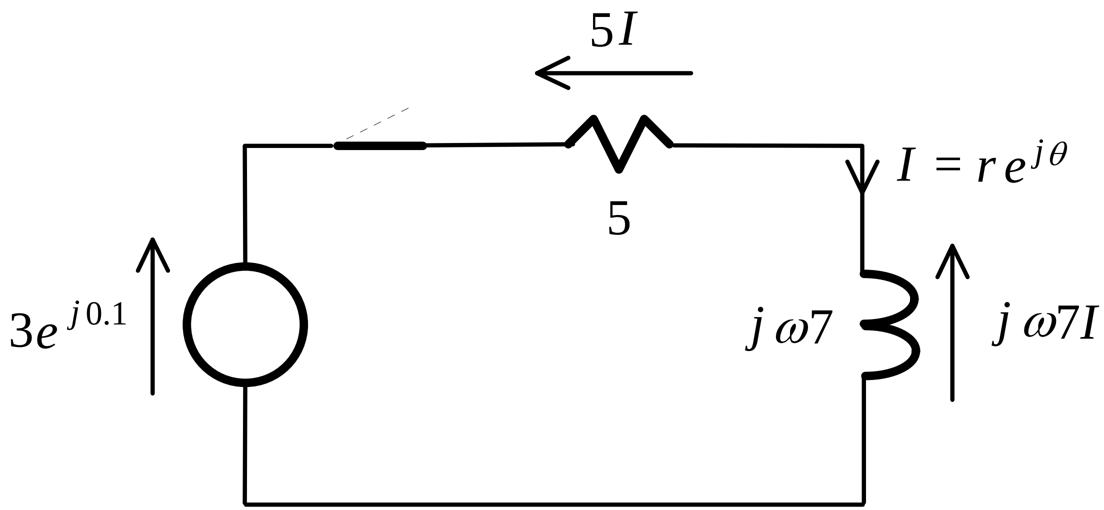
$$= \mathrm{Re}\left[\left(re^{j\theta}\right)(j\omega 7 + 5)\left(\sqrt{2}e^{j\omega t}\right)\right]$$

$$= \mathrm{Re}\left[\left(3e^{j0.1}\right)\left(\sqrt{2}e^{j\omega t}\right)\right] = u(t)$$

This is Method One: $\Longrightarrow$ $\boxed{(j\omega 7+5)\,re^{j\theta}=3e^{j0.1}}$

Method Two:





$$\boxed{(j\omega 7+5)I = (j\omega 7+5)r e^{j\theta} = 3e^{j0.1}}$$

(1) Steady State Response

$$\left(j\omega 7+5\right) r e^{j\theta} = 3e^{j0.1}$$

$$\Rightarrow \quad r e^{j\theta} = \frac{3e^{j0.1}}{5+j\omega 7}$$

$$\Rightarrow \quad r = \frac{\left|3e^{j0.1}\right|}{\left|5+j\omega 7\right|} = \frac{3}{\sqrt{5^2+7^2\omega^2}}$$

$$\theta = \operatorname{Arg}\left[e^{j0.1}\right] - \operatorname{Arg}\left[5+j\omega 7\right]$$

$$= 0.1 - \tan^{-1}\left(\frac{7\omega}{5}\right)$$

(2) Transient Response

$$7\frac{di_{tr}(t)}{dt} + 5i_{tr}(t) = 0, \quad t \ge 0$$

$i_{tr}(t)$ will have the same form as the dc source case:

$$i_{tr}(t) = k_1 e^{-\frac{5}{7}t}, \quad t \ge 0$$

Complete Response

$$i_{ss}(t) = r\sqrt{2}\cos(\omega t+\theta) = \frac{3\sqrt{2}}{\sqrt{25+49\omega^2}}\cos\left[\omega t + 0.1 - \tan^{-1}\left(\frac{7\omega}{5}\right)\right], \quad t \ge 0$$

Complete Response

$$i(t) = i_{ss}(t) + i_{tr}(t), \quad t \ge 0$$

$$= \frac{3\sqrt{2}}{\sqrt{25+49\omega^2}} \cos\left[\omega t + 0.1 - \tan^{-1}\left(\frac{7\omega}{5}\right)\right] + k_1 e^{-\frac{5}{7}t}, \quad t \ge 0$$

To determine k_1, the continuity of $i(t)$, the current through the inductor, can be used.

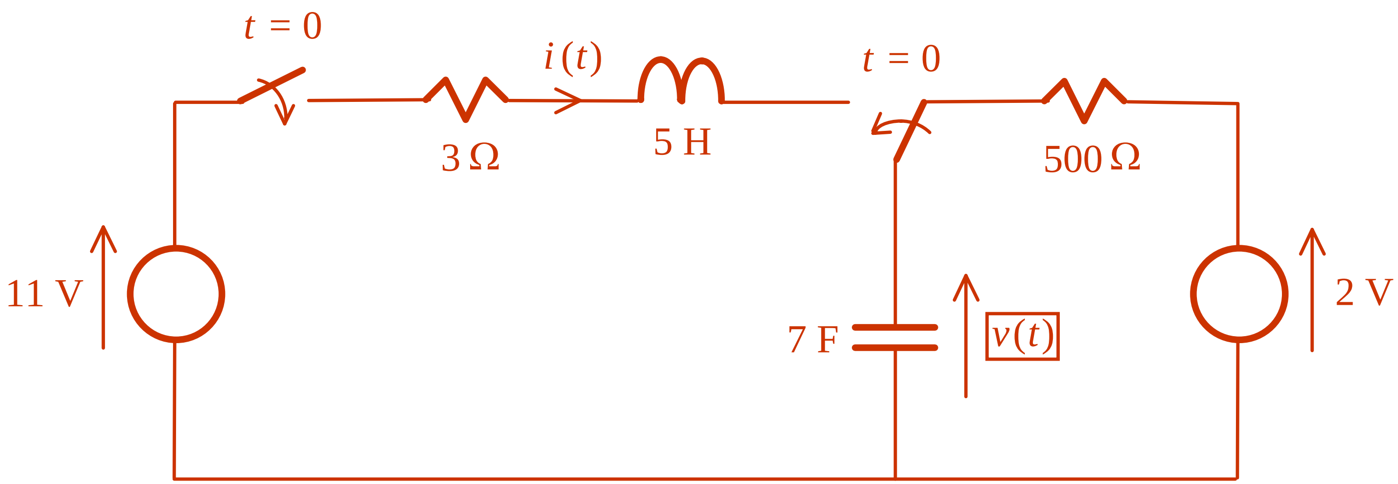
$$i(t) = 0, \quad t < 0 \quad \Rightarrow \quad i(0) = i_{ss}(0) + i_{tr}(0) = \frac{3\sqrt{2}}{\sqrt{25+49\omega^2}} \cos\left[0.1 - \tan^{-1}\left(\frac{7\omega}{5}\right)\right] + k_1$$

$$\Rightarrow \quad k_1 = -\frac{3\sqrt{2}}{\sqrt{25+49\omega^2}} \cos\left[0.1 - \tan^{-1}\left(\frac{7\omega}{5}\right)\right] \quad \Rightarrow$$

$$i(t) = \begin{cases} 0, & t < 0 \\ \dfrac{3\sqrt{2}}{\sqrt{25+49\omega^2}} \left\{ \cos\left[\omega t + 0.1 - \tan^{-1}\left(\frac{7\omega}{5}\right)\right] - \cos\left[0.1 - \tan^{-1}\left(\frac{7\omega}{5}\right)\right] e^{-\frac{5}{7}t} \right\}, & t \ge 0 \end{cases}$$

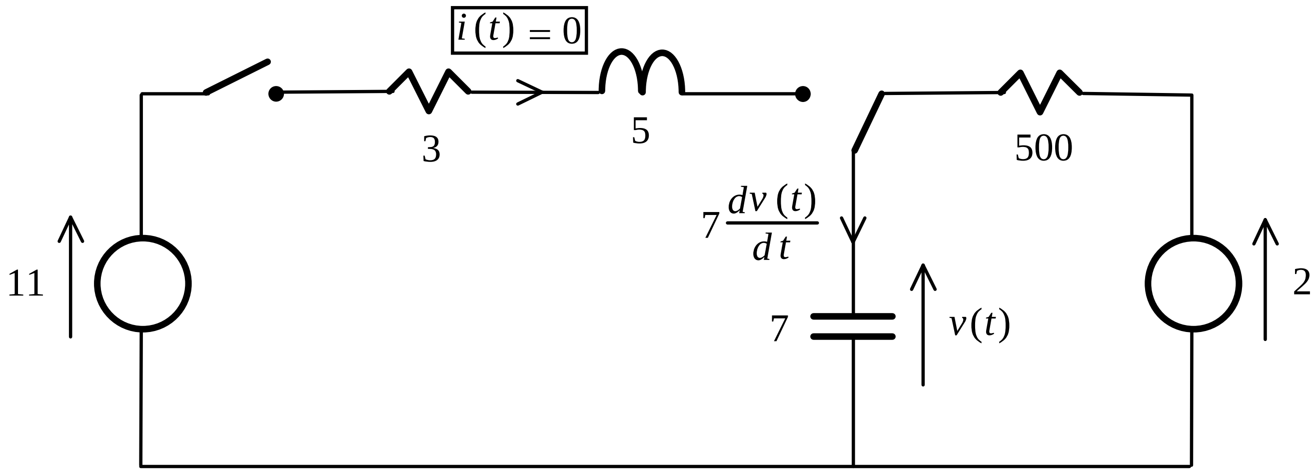
2.10 Second Order RLC Circuit

Consider determining $v(t)$ in the following series RLC circuit:

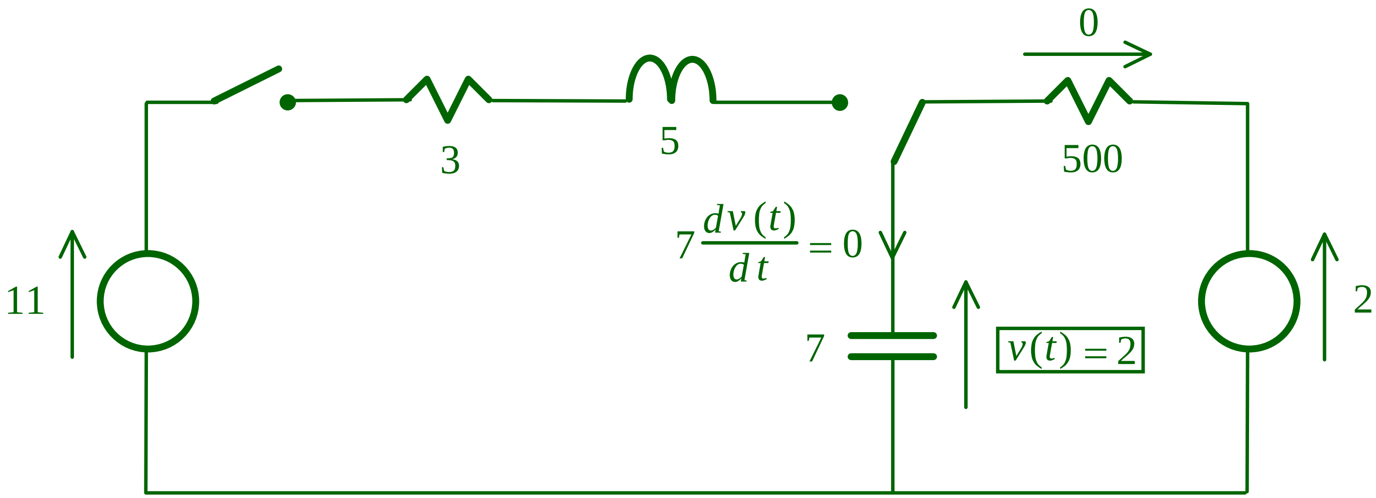


Both switches are in the position shown for $t < 0$ & are in the other positions for $t \geq 0$.

For $t < 0$



Taking the switches to be in the positions shown starting from $t = -\infty$, the voltages and currents will have settled down to constant values for practically all $t < 0$ and the important voltages and currents are given by:

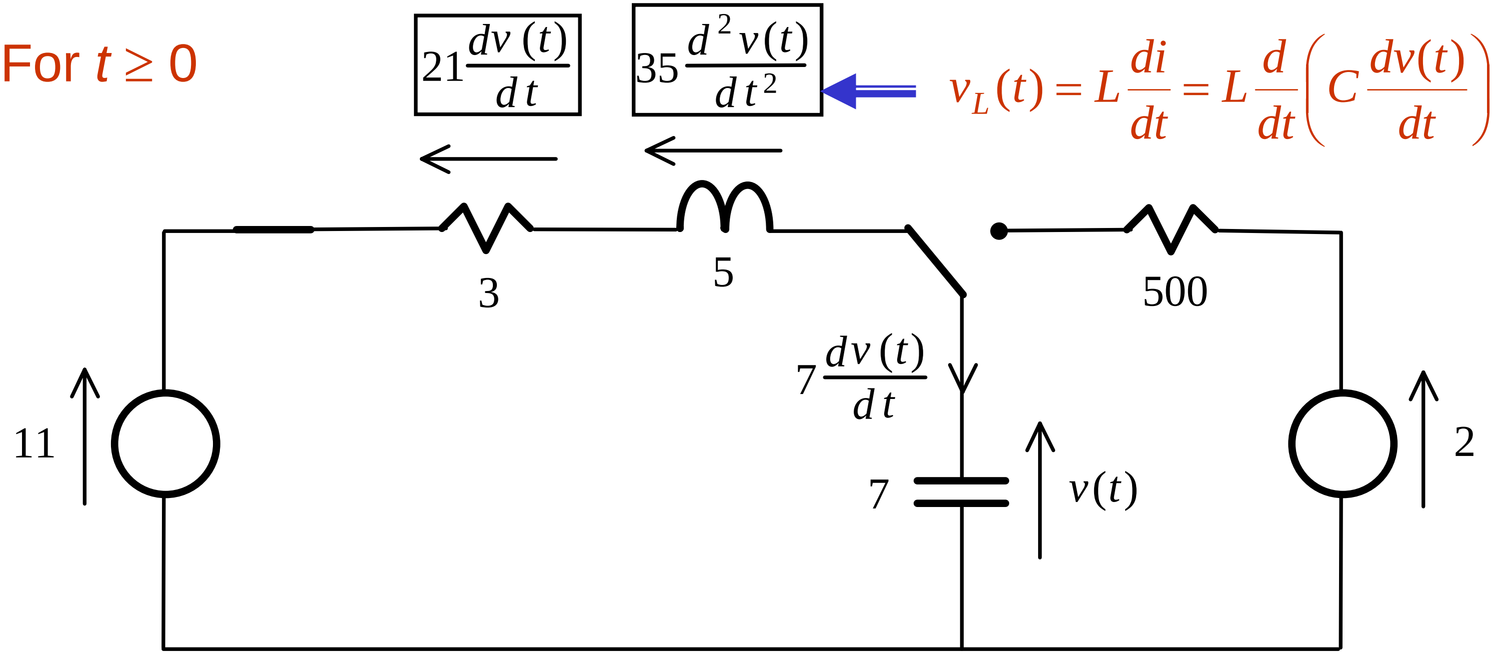


Mathematically:

$$v(t) = 2, \quad t < 0 \qquad \& \qquad i(t) = 0, \quad t < 0$$

For $t \geq 0$

$$v_L(t) = L\frac{di}{dt} = L\frac{d}{dt}\left(C\frac{dv(t)}{dt}\right) = LC\frac{d^2v(t)}{dt^2}$$



Applying KVL:

$$35\frac{d^2v(t)}{dt^2} + 21\frac{dv(t)}{dt} + v(t) = u(t) = 11, \quad t \geq 0$$

Due to the presence of 2 energy storage elements, the governing d.e. is a second order one and the general solution is

$$v(t) = v_{ss}(t) + v_{tr}(t), \quad t \geq 0$$

(1) Steady State Response

$$u(t) = 11, \quad t \geq 0 \quad \Longrightarrow \quad v_{ss}(t) = k, \quad t \geq 0 \quad \Longrightarrow$$

$$35\frac{d^2 v_{ss}(t)}{dt^2} + 21\frac{dv_{ss}(t)}{dt} + v_{ss}(t) = 0 + 0 + k = 11 \quad \Longrightarrow \quad \boxed{v_{ss}(t) = 11, \quad t \geq 0}$$

(2) Transient Response

$$35\frac{d^2 v_{tr}(t)}{dt^2} + 21\frac{dv_{tr}(t)}{dt} + v_{tr}(t) = 0, \quad t \geq 0 \quad \Longrightarrow$$

$$\left. 35\frac{d^2 v_{tr}(t)}{dt^2} + 21\frac{dv_{tr}(t)}{dt} + v_{tr}(t) \right|_{\frac{dv_{tr}(t)}{dt} \text{ replaced by } z} = 35z^2 + 21z^1 + z^0 = 35z^2 + 21z + 1$$

$$\Longrightarrow \quad z_1,\, z_2 = \frac{-21 \pm \sqrt{21^2 - 4(35)(1)}}{2(35)} = \frac{-21 \pm 17}{2(35)} = -0.548,\, -0.052$$

$$\boxed{v_{tr}(t) = k_1 e^{z_1 t} + k_2 e^{z_2 t} = k_1 e^{-0.548t} + k_2 e^{-0.052t}, \quad t \geq 0} \quad \Longleftarrow$$

Complete Solution (Response)

To be determined

$$v(t) = \begin{cases} 2, & t < 0 \\ v_{ss}(t) + v_{tr}(t), & t \ge 0 \end{cases} = \begin{cases} 2, & t < 0 \\ 11 + k_1 e^{-0.548t} + k_2 e^{-0.052t}, & t \ge 0 \end{cases}$$

$$\Longrightarrow \quad i(t) = 7\frac{dv(t)}{dt} = \begin{cases} 0, & t < 0 \\ 7(-0.548 k_1 e^{-0.548t} - 0.052\, k_2 e^{-0.052t}), & t \ge 0 \end{cases}$$

To determine k_1 and k_2, voltage continuity for the capacitor and current continuity for the inductor have to be used.

The voltage across the capacitor at $t = 0$:

$$v(0) = 11 + k_1 + k_2 = 2 \quad \Rightarrow \quad k_1 + k_2 = -9$$

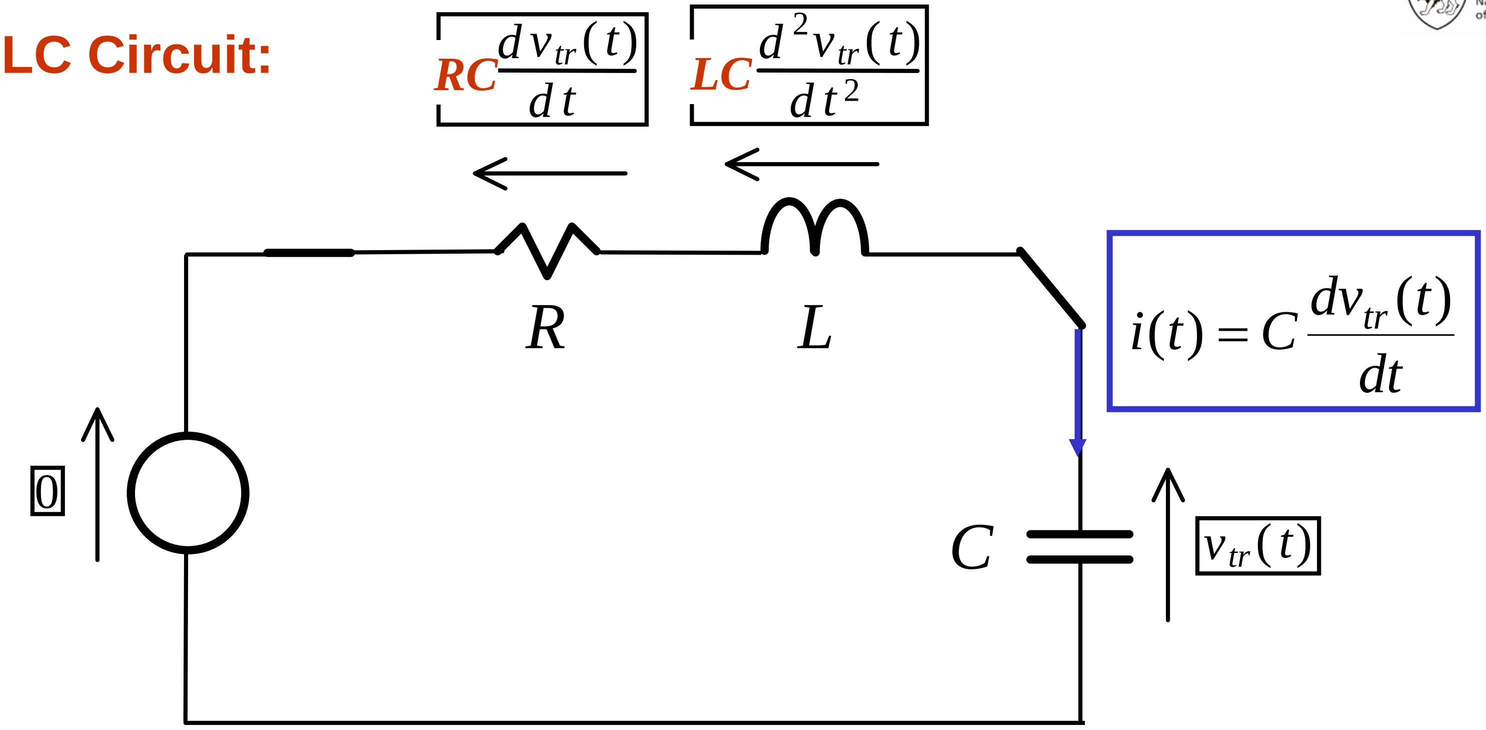
The current passing through the inductor at $t = 0$:

$$i(0) = -0.548k_1 - 0.052k_2 = 0 \quad \Rightarrow \quad 0.548k_1 + 0.052k_2 = 0$$

$$\boxed{k_1 = 0.94 \qquad k_2 = -9.94}$$

General RLC Circuit:

for t ≥ 0



By KVL:

$$LC\frac{d^2v_{tr}(t)}{dt^2}+RC\frac{dv_{tr}(t)}{dt}+v_{tr}(t)=0,\quad t\ge 0$$

$$\left. LC\frac{d^2v_{tr}(t)}{dt^2}+RC\frac{dv_{tr}(t)}{dt}+v_{tr}(t)\right|_{\frac{dv_{tr}(t)}{dt}\text{ replaced by } z}=LCz^2+RCz^1+1=0$$

$$\Rightarrow\quad z_1,\, z_2=\frac{-RC\pm\sqrt{(RC)^2-4LC}}{2LC}$$

Recall that for RLC circuit, the Q factor is defined as

$$Q = \frac{2\pi f_0 L}{R} = \frac{2\pi L}{R}\frac{1}{2\pi\sqrt{LC}} = \frac{L}{R\sqrt{LC}} = \frac{\sqrt{LC}}{RC}$$

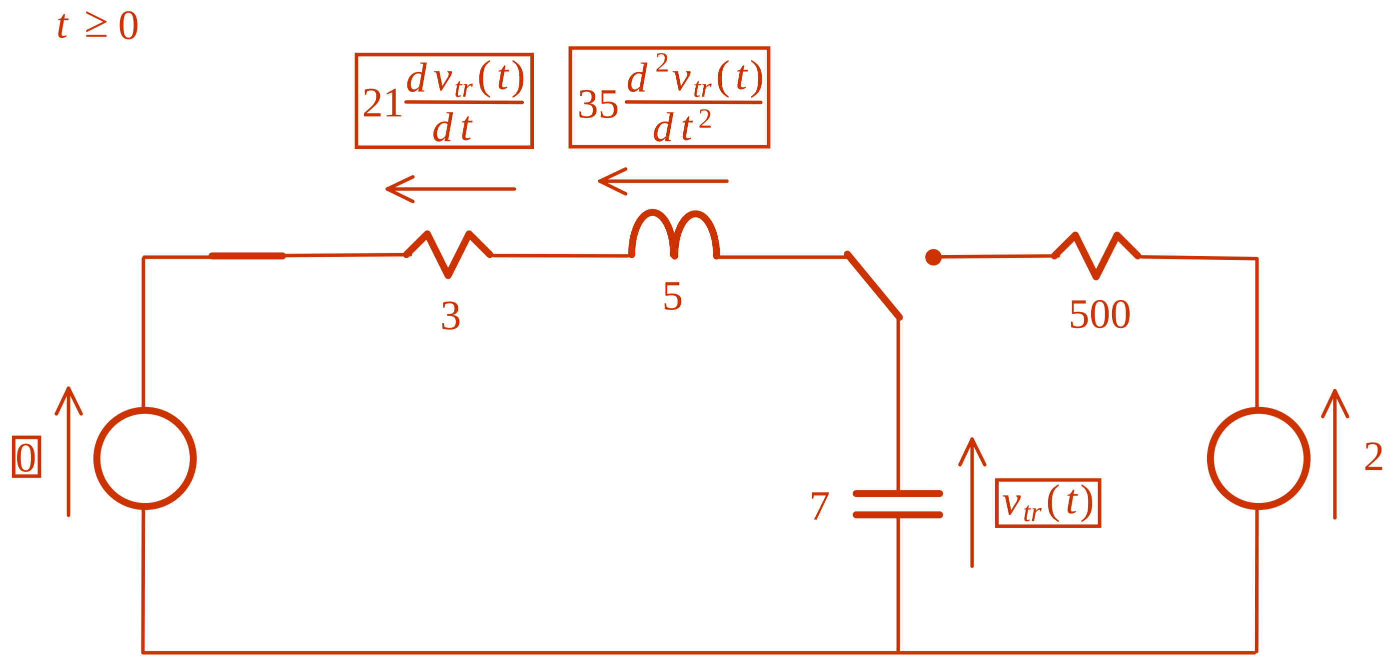
Thus,

$$z_1, z_2 = \frac{-RC \pm \sqrt{(RC)^2 - 4LC}}{2LC} = \frac{-RC \pm RC\sqrt{1 - 4\dfrac{LC}{(RC)^2}}}{2LC} = \frac{-R \pm R\sqrt{1-4Q^2}}{2L}$$

$$= \begin{cases} \text{two real roots if } 1-4Q^2 > 0 \text{ or } Q^2 < 1/4 \text{ or } Q < 1/2 \\ \text{two complex conjugate roots if } 1-4Q^2 < 0 \text{ or } Q > 1/2 \\ \text{two identical roots if } 1-4Q^2 = 0 \text{ or } Q = 1/2 \end{cases}$$

2.11 Overdamped Response

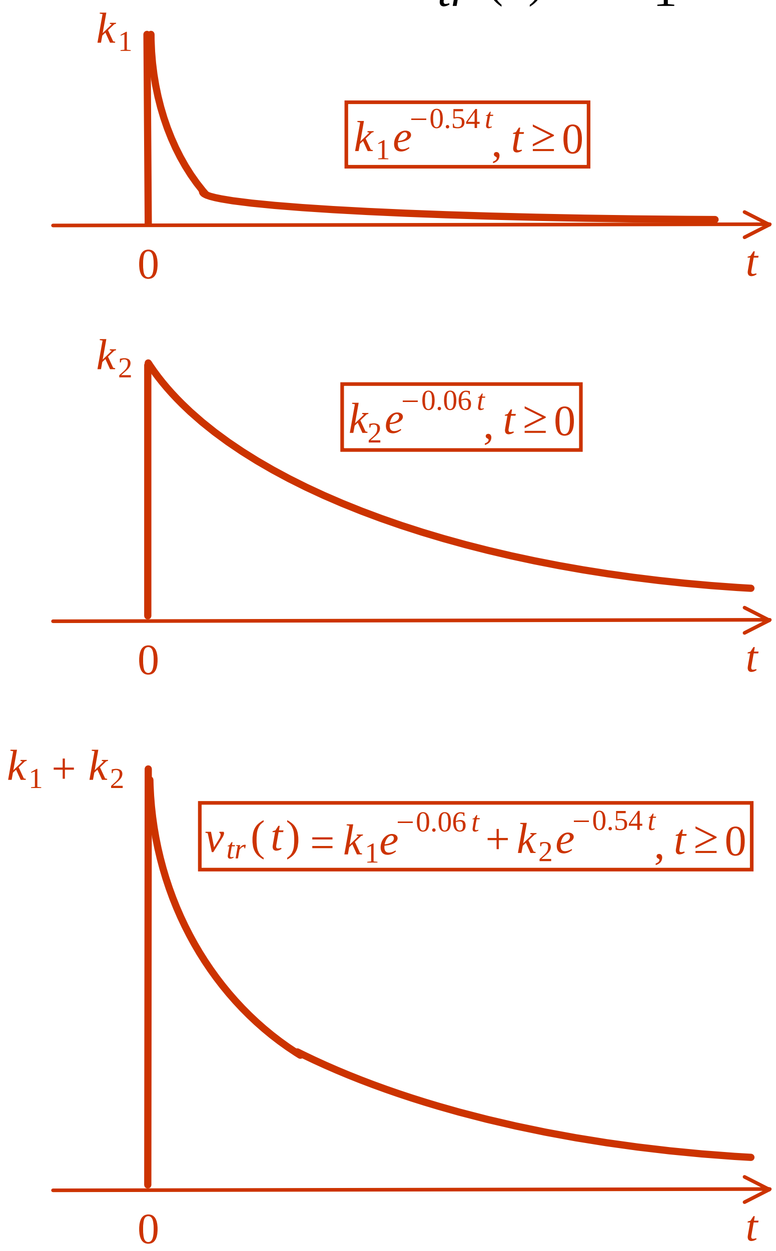
Reconsider the previous RLC example, i.e.,



$$35\frac{d^2v_{tr}(t)}{dt^2}+21\frac{dv_{tr}(t)}{dt}+v_{tr}(t)=0,\quad t\geq 0 \quad \Longrightarrow \quad Q=\frac{\sqrt{LC}}{RC}=\frac{\sqrt{35}}{21}=0.2817<\tfrac{1}{2}$$

$$z_1,\, z_2=\frac{-21\pm\sqrt{21^2-4(35)(1)}}{2(35)}=\frac{-21\pm 17}{2(35)}=-0.54,\,-0.06$$

$$v_{tr}(t) = k_1 e^{z_1 t} + k_2 e^{z_2 t} = k_1 e^{-0.54t} + k_2 e^{-0.06t}, \quad t \geq 0$$

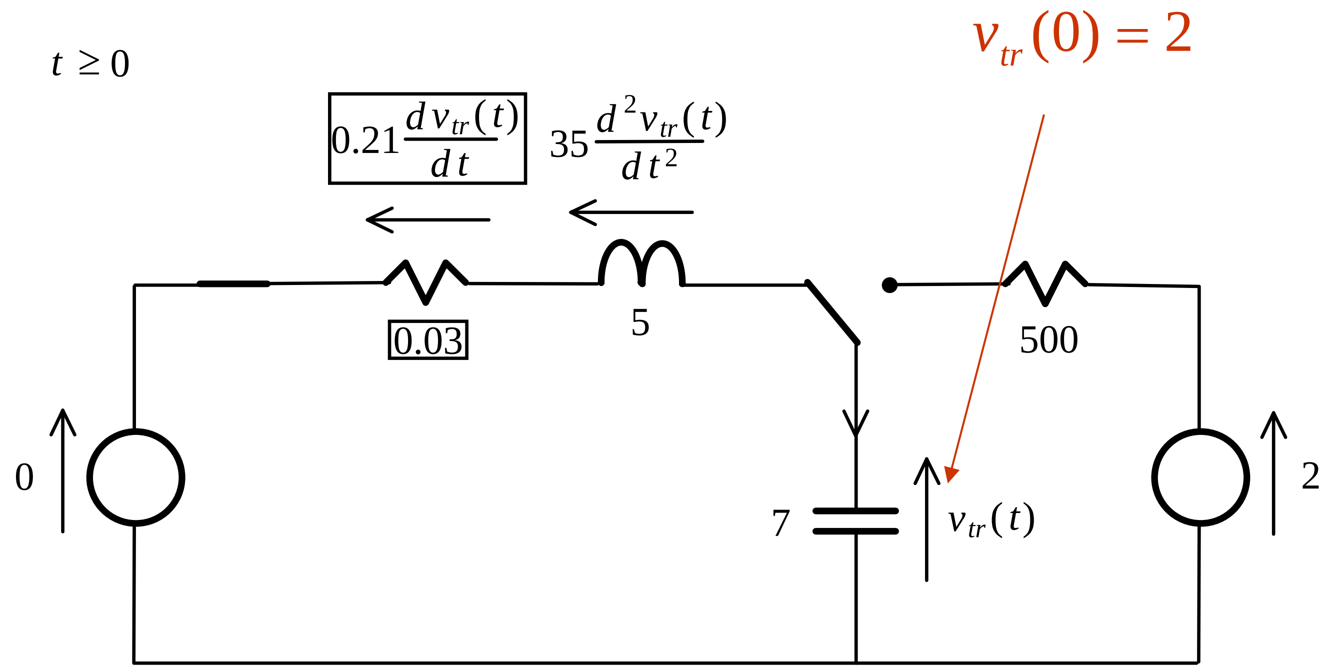


Due to its exponentially decaying nature, the response $i_{tr}(t)$ and the RLC circuit are said to be **overdamped**.

Typically, when an external input is suddenly applied to an overdamped system, the system will take a long time to move in an exponentially decaying manner to the steady state position.

The response is slow and sluggish, and the Q factor is small.

2.12 Underdamped Response



$$Q = \frac{\sqrt{5}}{0.03\sqrt{7}} = 28 > \tfrac{1}{2}$$

$$35\frac{d^2v_{tr}(t)}{dt^2} + 0.21\frac{dv_{tr}(t)}{dt} + v_{tr}(t) = 0, \quad t \geq 0$$

$$z_1, z_2 = \frac{-0.21 \pm \sqrt{0.21^2 - 4(35)(1)}}{2(35)} = \frac{-0.21 \pm \sqrt{-139.96}}{2(35)} = -0.003 \pm j0.17$$

$$v_{tr}(t) = k_1 e^{z_1 t} + k_2 e^{z_2 t} = k_1 e^{(-0.003 + j0.17)t} + k_2 e^{(-0.003 - j0.17)t}$$

$$\Longrightarrow \quad v_{tr}(0) = k_1 + k_2 = 2$$

$$i_{tr}(t) = 7k_1(-0.003 + j0.17)e^{(-0.003 + j0.17)t} + 7k_2(-0.003 - j0.17)e^{(-0.003 - j0.17)t}$$

$$\begin{aligned} i_{tr}(0) &= 7k_1(-0.003 + j0.17) + 7k_2(-0.003 - j0.17) \\ &= -0.021(k_1 + k_2) + j1.19(k_1 - k_2) \\ &= -0.042 + j1.19(k_1 - k_2) = 0 \end{aligned}$$

$$\Longrightarrow \quad k_1 - k_2 = -j0.0353$$

$$v_{tr}(t)=k_1e^{z_1 t}+k_2e^{z_2 t}=k_1e^{(-0.003+j0.17)t}+k_2e^{(-0.003-j0.17)t}$$

$$=e^{-0.003t}\left(k_1e^{j0.17t}+k_2e^{-j0.17t}\right)$$

$$=e^{-0.003t}\{k_1[\cos(0.17t)+j\sin(0.17t)]+k_2[\cos(0.17t)-j\sin(0.17t)]\}$$

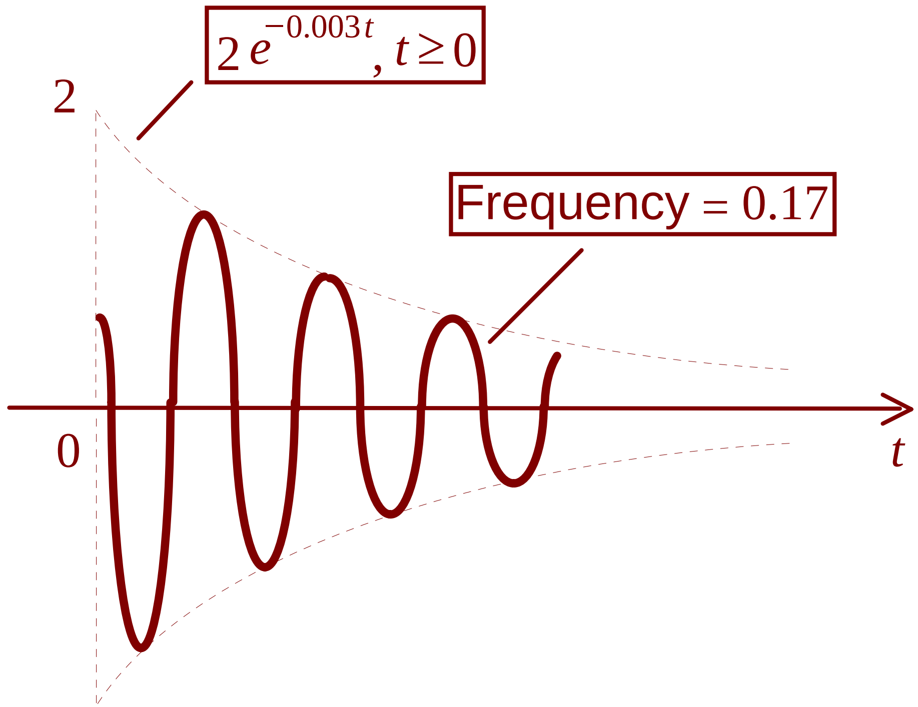
$$=e^{-0.003t}[(k_1+k_2)\cos(0.17t)+j(k_1-k_2)\sin(0.17t)]$$

$$=e^{-0.003t}[2\cos(0.17t)+0.0353\sin(0.17t)],\quad t\geq 0$$

$$=e^{-0.003t}\sqrt{2^2+0.0353^2}\left[\frac{2}{\sqrt{2^2+0.0353^2}}\cos(0.17t)+\frac{0.0353}{\sqrt{2^2+0.0353^2}}\sin(0.17t)\right]$$

$$=2e^{-0.003t}\left[\cos 1^\circ\cos(0.17t)+\sin 1^\circ\sin(0.17t)\right],\quad t\geq 0$$

$$=2e^{-0.003t}\cos\left(0.17t-1^\circ\right),\quad t\geq 0$$



$$v_{tr}(t) = 2\,e^{-0.003t}\cos(0.17t - 1^{o}),\ t \geq 0$$

Since this is an exponentially decaying sinusoid, the response $v_{tr}(t)$ and the RLC circuit are said to be **underdamped**.

When an external input is applied to an underdamped system, the system will oscillate. The oscillation will decay exponentially but it may take some time for the system to reach its steady state position.

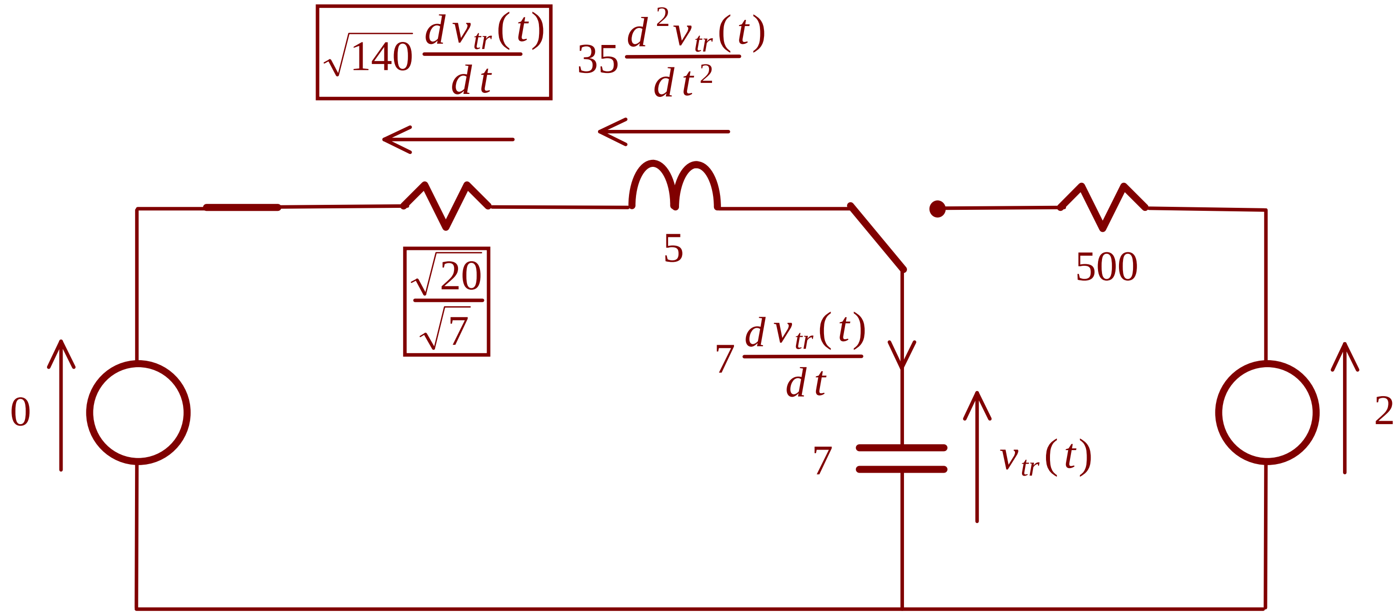
Underdamped systems have large *Q* factors and are used in systems such as tune circuit. However, they will be not be suitable in situations such as car suspensions or instruments with moving pointers.

It will take too long for the pointer to oscillate and settle down to its final position if the damping system for the pointer is highly underdamped in nature.

2.13 Critically Damped Response

$t \ge 0$

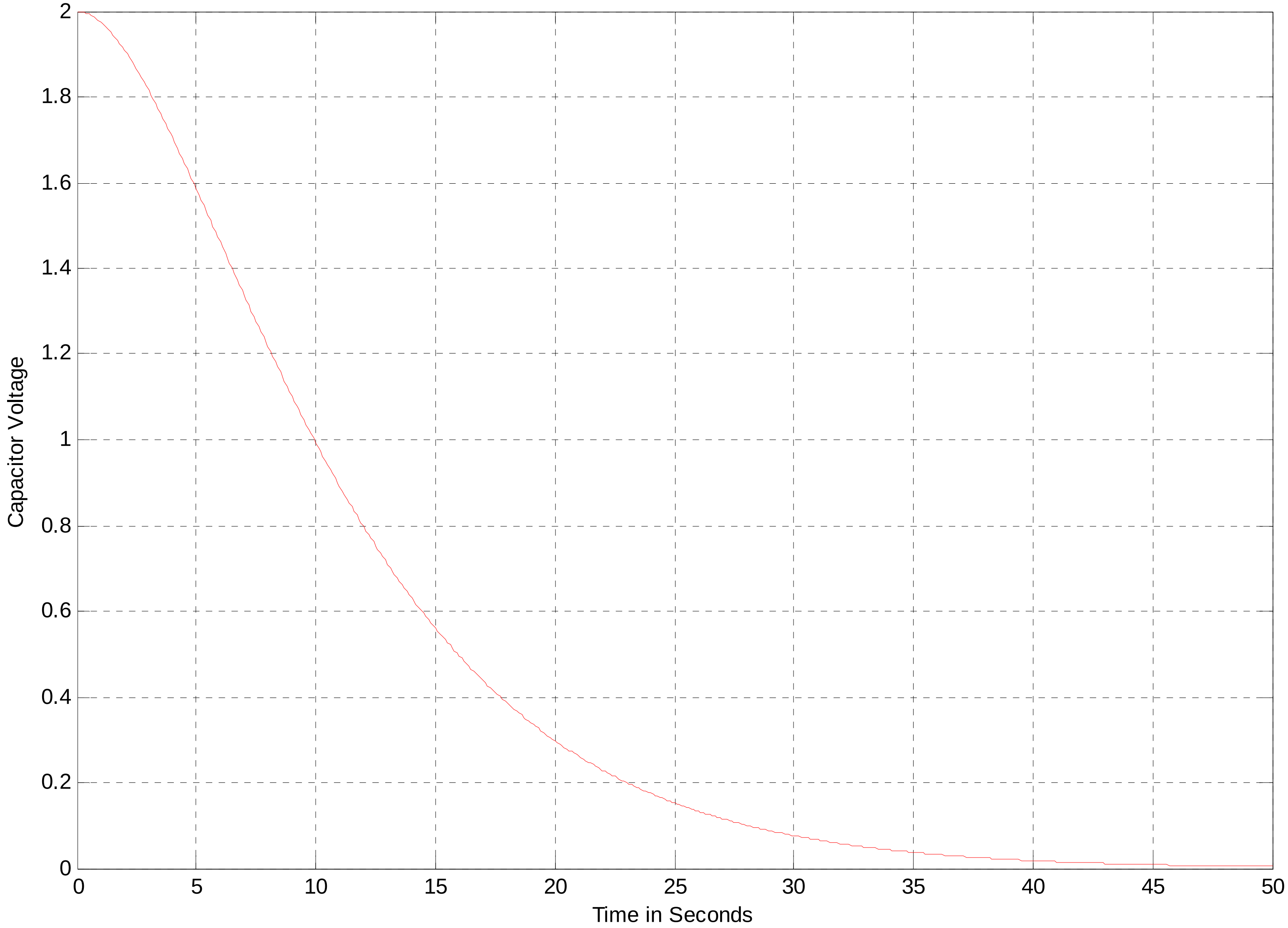
$$Q = \frac{\sqrt{5}}{\sqrt{20/7}\sqrt{7}} = \frac{1}{2}$$



$$35\frac{d^2 v_{tr}(t)}{dt^2} + \sqrt{140}\frac{dv_{tr}(t)}{dt} + v_{tr}(t) = 0, \quad t \ge 0 \qquad z_1, z_2 = -\frac{1}{\sqrt{35}}$$

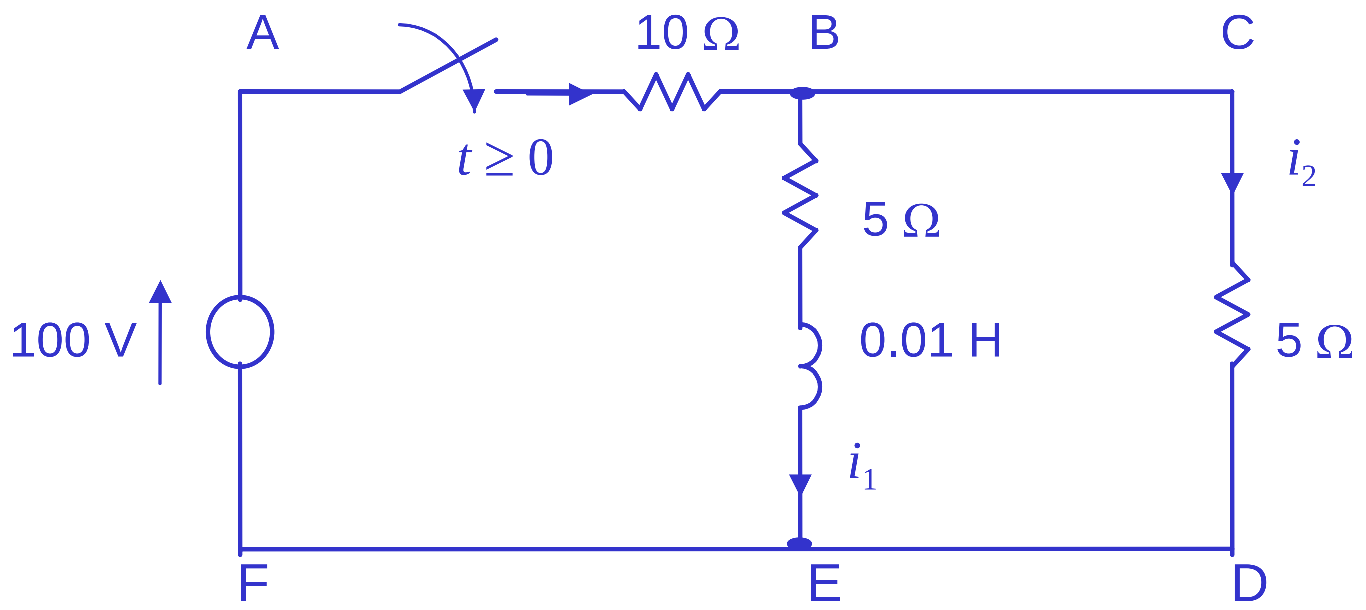
$$v_{tr}(t) = (k_1 + k_2 t)e^{z_1 t} = (k_1 + k_2 t)e^{-\frac{t}{\sqrt{35}}} \quad \Rightarrow \quad v_{tr}(0) = k_1 = 2$$

$$\frac{i_{tr}(0)}{7} = \frac{d}{dt}(k_1 + k_2 t)e^{-\frac{t}{\sqrt{35}}}\Bigg|_{t=0} = \left(k_2 - \frac{k_1}{\sqrt{35}} - \frac{k_2 t}{\sqrt{35}}\right)e^{-\frac{t}{\sqrt{35}}}\Bigg|_{t=0} = 0 \Rightarrow k_2 = \frac{2}{\sqrt{35}}$$



Capacitor Voltage
2
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
0
5
10
15
20
25
30
35
40
45
50
Time in Seconds

Example: The switch in the circuit shown in the following circuit is closed at time $t = 0$. Obtain the current $i_2(t)$ for $t > 0$.



After the switch is closed, the current passing through the source or the 10Ω resistor is $i_1 + i_2$. Applying the KVL to the loops, ABEFA and ABCDEFA, respectively, we obtain

$$10(i_1 + i_2) + 5i_1 + 0.01\frac{di_1}{dt} = 100$$

$$10(i_1 + i_2) + 5i_2 = 100 \Longrightarrow i_2 = (100 - 10i_1)/15$$

$$\Downarrow$$

$$\frac{di_1}{dt} + 833i_1 = 3333$$

$$\Longrightarrow \quad i_1(\infty) = 3333/833 = 4.0\text{A} \Longrightarrow i_1(t) = \alpha e^{-833t} + 4.0$$

$$i_1(0) = 0 \Longrightarrow \alpha = -4.0 \Longrightarrow i_1(t) = 4.0(1 - e^{-833t})$$

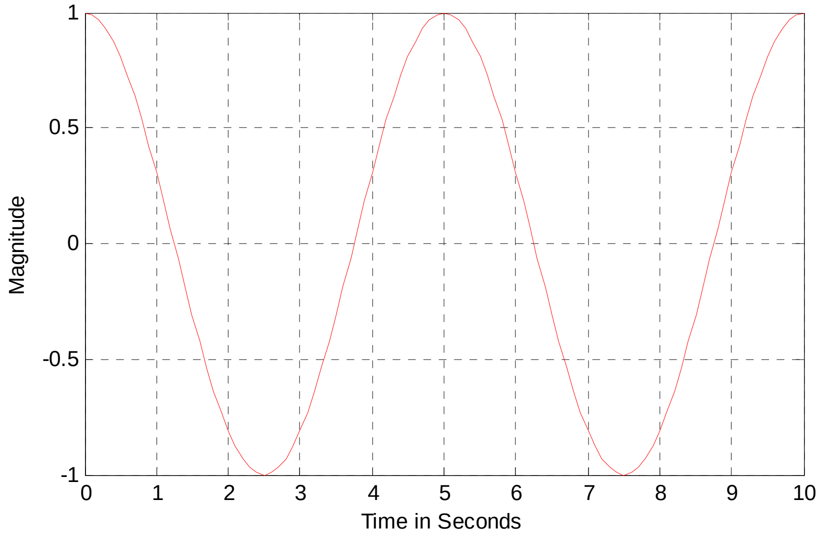
$$\Downarrow$$

$$i_2(t) = 4.0 + 2.67e^{-833t}$$

3. Review of Laplace Transforms

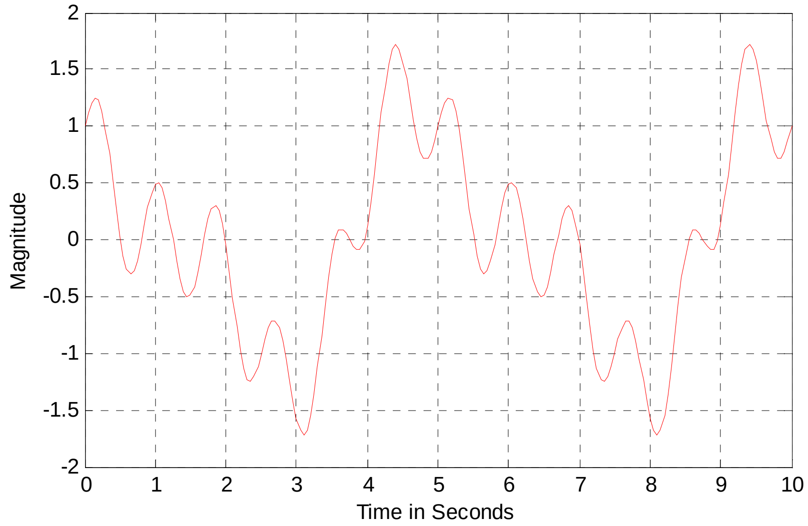
3.1 Introduction

Let us first examine the following time-domain functions:



A cosine function with a frequency $f = 0.2$ Hz.

Note that it has a period $T = 5$ seconds.



$x(t) = \cos(0.4\pi t) + \sin(0.8\pi t)\cos(1.6\pi t)$

What are frequencies of this function?

Laplace transform is a tool to convert time-domain functions into a frequency-domain ones in which information about frequencies of the function can be captured. It is often much easier to solve problems in frequency-domain with the help of Laplace transform.

3.2 Laplace Transform

Given a time-domain function $f(t)$, its Laplace transform is defined as follows:

$$F(s) = L\{f(t)\} = \int_0^{\infty} f(t)e^{-st}dt$$

Example 1: Find the Laplace transform of a constant function $f(t) = 1$.

$$F(s) = \int_0^{\infty} f(t)e^{-st}dt = \int_0^{\infty} e^{-st}dt = -\frac{1}{s}e^{-st}\Big|_0^{\infty} = -\frac{1}{s}e^{-\infty} - \left(-\frac{1}{s}e^{0}\right) = -\frac{1}{s}\cdot 0 - \left(-\frac{1}{s}\cdot 1\right) = \frac{1}{s}$$

Example 2: Find the Laplace transform of an exponential function $f(t) = e^{-at}$.

$$F(s) = \int_0^{\infty} f(t)e^{-st}dt = \int_0^{\infty} e^{-at}e^{-st}dt = \int_0^{\infty} e^{-(s+a)t}dt = -\frac{1}{s+a}e^{-(s+a)t}\Big|_0^{\infty} = \frac{1}{s+a}$$

3.3 Inverse Laplace Transform

Given a frequency-domain function $F(s)$, the inverse Laplace transform is to convert it back to its original time-domain function $f(t)$.

Here are some very useful Laplace and inverse Laplace transform pairs:

$f(t)$	$\Leftrightarrow$	$F(s)$
1	$\Leftrightarrow$	$\dfrac{1}{s}$
t	$\Leftrightarrow$	$\dfrac{1}{s^2}$
e^{-at}	$\Leftrightarrow$	$\dfrac{1}{s+a}$
te^{-at}	$\Leftrightarrow$	$\dfrac{1}{(s+a)^2}$

$f(t)$	$\Leftrightarrow$	$F(s)$
$\sin at$	$\Leftrightarrow$	$\dfrac{a}{s^2+a^2}$
$\cos at$	$\Leftrightarrow$	$\dfrac{s}{s^2+a^2}$
$e^{-at}\sin bt$	$\Leftrightarrow$	$\dfrac{b}{(s+a)^2+b^2}$
$e^{-at}\cos bt$	$\Leftrightarrow$	$\dfrac{s+a}{(s+a)^2+b^2}$

Some useful properties of Laplace transform:

1. Superposition:

$$L\{a_1 f_1(t) + a_2 f_2(t)\} = a_1 L\{f_1(t)\} + a_2 L\{f_2(t)\} = a_1 F_1(s) + a_2 F_2(s)$$

2. Differentiation:

$$L\left\{\frac{df(t)}{dt}\right\} = L\{\dot{f}(t)\} = sL\{f(t)\} - f(0) = sF(s) - f_0(0)$$

$$L\left\{\frac{d^2 f(t)}{dt^2}\right\} = L\{\ddot{f}(t)\} = s^2 L\{f(t)\} - sf(0) - f'(0) = s^2 F(s) - sf(0) - f'(0)$$

3. Integration:

$$L\left\{\int_0^t f(\zeta) d\zeta\right\} = \frac{1}{s} L\{f(t)\} = \frac{1}{s} F(s)$$

4. State Variable Analysis

4.0 Matrix Algebra

Addition (2 × 2): $$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} a+1 & b+2 \\ c+3 & d+4 \end{bmatrix}$$

Multiplication (2 × 2): $$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1a+3b \\ 1c+3d \end{bmatrix}, \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1a+3b & 2a+4b \\ 1c+3d & 2c+4d \end{bmatrix}$$

Determinant (2 × 2): $$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Inverse (2 × 2): $$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} = \frac{\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}{ad - bc}$$

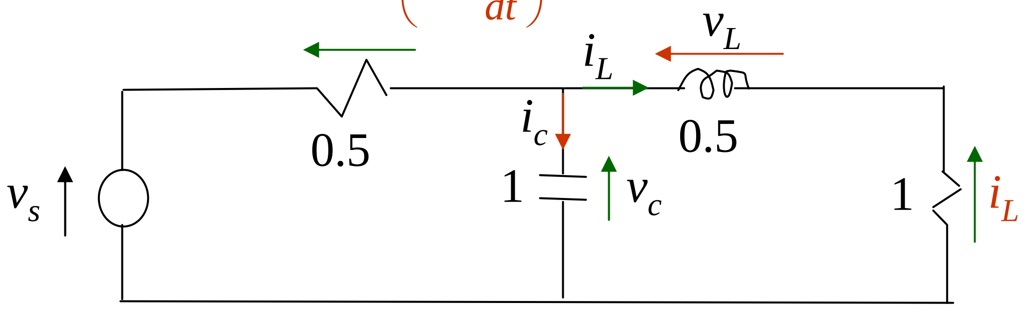
Linear Equation (2 × 2): $$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \frac{\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}}{ad - bc} = \frac{\begin{bmatrix} 1d - 2b \\ -1c + 2a \end{bmatrix}}{ad - bc}$$

4.1 Introduction to State Variables

State variable technique is to convert circuit analysis problems into some first order ordinary differential equations (in a matrix form), which many advanced matrix theories and computational tools can be applied to. Thus, circuit analysis using the state variable technique can be done in a very systematic fashion and many well-developed commercial software tools such as MATLAB can be readily utilized. Solutions to transient response, steady state response and complete response can be solved uniformly regardless the order of the circuits.

There are two important equations associated with such a technique, the state equation and the output equation, which completely characterized the properties of any linear circuits. To be specific, let us consider the following RLC circuit:

$$0.5(i_L + i_c) = 0.5\left(i_L + \frac{dv_c}{dt}\right)$$



Step 2: For the capacitor, compute its current as

$$i_c(t) = C\frac{dv_c(t)}{dt} = 1\cdot\frac{dv_c(t)}{dt} = \frac{dv_c}{dt}$$

For the inductor, compute its voltage as

$$v_L(t) = L\frac{di_L(t)}{dt} = 0.5\cdot\frac{di_L(t)}{dt} = 0.5\frac{di_L}{dt}$$

Step 3: Compute the currents and voltages for the rest of circuit in terms of state variables. For this example, these are the currents and voltages for the 0.5Ω and 1Ω resistors.

Step 1: For every capacitor, assign a voltage to it, and every inductor, assign a current. These are the state variables of the circuit. Generally, the total number of state variables are the number of capacitors & inductors.

Step 4: Apply KCL and/or KVL to obtain state equations: Using KVL to the left loop, we obtain,

$$v_c + 0.5\left(i_L + \frac{dv_c}{dt}\right) = v_s$$

Using KVL to the right loop, we have

$$i_L + v_L - v_c = i_L + 0.5\frac{di_L}{dt} - v_c = 0$$

Step 5: Rewrite the state equations obtained in Step 4 in the following format:

$$v_c + 0.5\left(i_L + \frac{dv_c}{dt}\right) = v_s \quad \Rightarrow \quad 2v_c + i_L + \frac{dv_c}{dt} = 2v_s \quad \Rightarrow \quad \frac{dv_c}{dt} = -2v_c - i_L + 2v_s$$

$$i_L + 0.5\frac{di_L}{dt} - v_c = 0 \quad \Rightarrow \quad \frac{di_L}{dt} = 2v_c - 2i_L$$

Step 6: (State Equation) Rewrite these equation in a compact matrix form,

$$\begin{bmatrix} \dot{v}_c \\ \dot{i}_L \end{bmatrix} = \begin{bmatrix} -2 & -1 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} v_c \\ i_L \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} v_s \quad \Rightarrow \quad \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$

This is the so-called *matrix state equation* of the circuit. v_c and i_L are called the *state variables*, and v_s is called the input (the source that provide energy to the circuit).

Step 7: (Output Equation) Lastly, express the variables of interest as a linear combinations of state variables. For example, if you want to study the voltages of the capacitor and the 1 Ω resistor, we have an output equation:

$$\begin{pmatrix} v_c \\ v_{1\Omega} \end{pmatrix} = \begin{pmatrix} v_c \\ i_L \end{pmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{pmatrix} v_c \\ i_L \end{pmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} v_s \quad \Rightarrow \quad \mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$$

The advantages of using state variable or state space representation of the circuit is as follows:

1. No matter how complicated the circuit is, the final expression for the circuit is always in the form,

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}, \qquad \mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$$

   Only the dimensions of the matrices and vectors in the above equations are different.

2. As such, the solutions to these state space equations can be unified into a single framework. Actually, it is nothing more than solving a first order linear differential equation (in a matrix form). Many commercial software tools such as MATLAB and MATHEMATICA can be utilized to solve these equations without additional efforts.

4.2 Transform Solution of State Equations

The state equation and output equation of the circuit are:

$$\frac{d\mathbf{x}}{dt} = \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}\,, \qquad \mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$$

Let the initial value of the state variable be $\mathbf{x}(0) = \mathbf{x}_0$ and let us take Laplace transform on these equations. We have

$$L\left\{\frac{d\mathbf{x}}{dt}\right\} = L\{\mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}\} \quad \Rightarrow \quad s\mathbf{X}(s) - \mathbf{x}_0 = \mathbf{A}\mathbf{X}(s) + \mathbf{B}\mathbf{U}(s)$$

$$\Rightarrow \quad (s\mathbf{I} - \mathbf{A})\mathbf{X}(s) = \mathbf{B}\mathbf{U}(s) + \mathbf{x}_0 \quad \Rightarrow \quad \mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\mathbf{U}(s) + (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{x}_0$$

$$L\{\mathbf{y}\} = L\{\mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}\} \quad \Rightarrow \quad \mathbf{Y}(s) = \mathbf{C}\mathbf{X}(s) + \mathbf{D}\mathbf{U}(s)$$

$$= \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\mathbf{U}(s) + \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{x}_0 + \mathbf{D}\mathbf{U}(s)$$

$$\Rightarrow \quad \mathbf{Y}(s) = \underbrace{[\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}]\mathbf{U}(s)}_{\text{due to external force or input.}} + \underbrace{\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{x}_0}_{\text{due to initial condition.}}$$

Let us get back to our example (assuming that the initial conditions: $v_c(0) = 1$ and $i_L(0) = 1$). We have

$$\mathbf{A} = \begin{bmatrix} -2 & -1 \\ 2 & -2 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \mathbf{x}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\Rightarrow \quad s\mathbf{I} - \mathbf{A} = s\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -2 & -1 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} s+2 & 1 \\ -2 & s+2 \end{bmatrix}$$

$$\Rightarrow \quad \det\,(s\mathbf{I} - \mathbf{A}) = \det\begin{bmatrix} s+2 & 1 \\ -2 & s+2 \end{bmatrix} = (s+2)^2 + 2 = s^2 + 4s + 6$$

$$\Rightarrow \quad (s\mathbf{I} - \mathbf{A})^{-1} = \frac{\begin{bmatrix} s+2 & -1 \\ 2 & s+2 \end{bmatrix}}{s^2 + 4s + 6} = \begin{bmatrix} \dfrac{s+2}{s^2+4s+6} & \dfrac{-1}{s^2+4s+6} \\ \dfrac{2}{s^2+4s+6} & \dfrac{s+2}{s^2+4s+6} \end{bmatrix}$$

$$\Rightarrow \quad \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \dfrac{s+2}{s^2+4s+6} & \dfrac{-1}{s^2+4s+6} \\ \dfrac{2}{s^2+4s+6} & \dfrac{s+2}{s^2+4s+6} \end{bmatrix} = \begin{bmatrix} \dfrac{s+2}{s^2+4s+6} & \dfrac{-1}{s^2+4s+6} \\ \dfrac{2}{s^2+4s+6} & \dfrac{s+2}{s^2+4s+6} \end{bmatrix}$$

$$\Rightarrow \quad \mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}+\mathbf{D} = \begin{bmatrix} \dfrac{s+2}{s^2+4s+6} & \dfrac{-1}{s^2+4s+6} \\ \dfrac{2}{s^2+4s+6} & \dfrac{s+2}{s^2+4s+6} \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \dfrac{2(s+2)}{s^2+4s+6} \\ \dfrac{4}{s^2+4s+6} \end{bmatrix}$$

$$\Rightarrow \quad \mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{x}_0 = \begin{bmatrix} \dfrac{s+2}{s^2+4s+6} & \dfrac{-1}{s^2+4s+6} \\ \dfrac{2}{s^2+4s+6} & \dfrac{s+2}{s^2+4s+6} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \dfrac{s+1}{s^2+4s+6} \\ \dfrac{s+4}{s^2+4s+6} \end{bmatrix}$$

$$\Rightarrow \quad \mathbf{Y}(s) = \begin{bmatrix} \mathbf{V}_c(s) \\ \mathbf{I}_L(s) \end{bmatrix} = [\mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}+\mathbf{D}]\mathbf{U}(s) + \mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{x}_0$$

$$= \begin{bmatrix} \dfrac{2(s+2)}{s^2+4s+6} \\ \dfrac{4}{s^2+4s+6} \end{bmatrix} \mathbf{U}(s) + \begin{bmatrix} \dfrac{s+1}{s^2+4s+6} \\ \dfrac{s+4}{s^2+4s+6} \end{bmatrix}$$

Solutions to the circuit can be obtained by taking inverse Laplace transform of the above expression of $\mathbf{Y}(s)$.

4.3 Zero-Input Response

Zero-input response is the response of the system without an external input, i.e., $\mathbf{u} = \mathbf{0}$. Thus, it is merely the response due to initial condition of the state variables. Recall that

$$\mathbf{Y}(s) = [\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}]\mathbf{U}(s) + \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{x}_0$$

If the input is zero, $\mathbf{Y}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{x}_0$ and the output response of the circuit is simply given by

$$\boxed{\mathbf{y}(t) = L^{-1}\left\{\mathbf{Y}(s)\right\} = L^{-1}\left\{\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{x}_0\right\}}$$

The problem of finding for the transient response becomes solving the inverse Laplace transform problems. The Laplace and inverse Laplace transform tables given in the previous topic can thus be used to find the solutions.

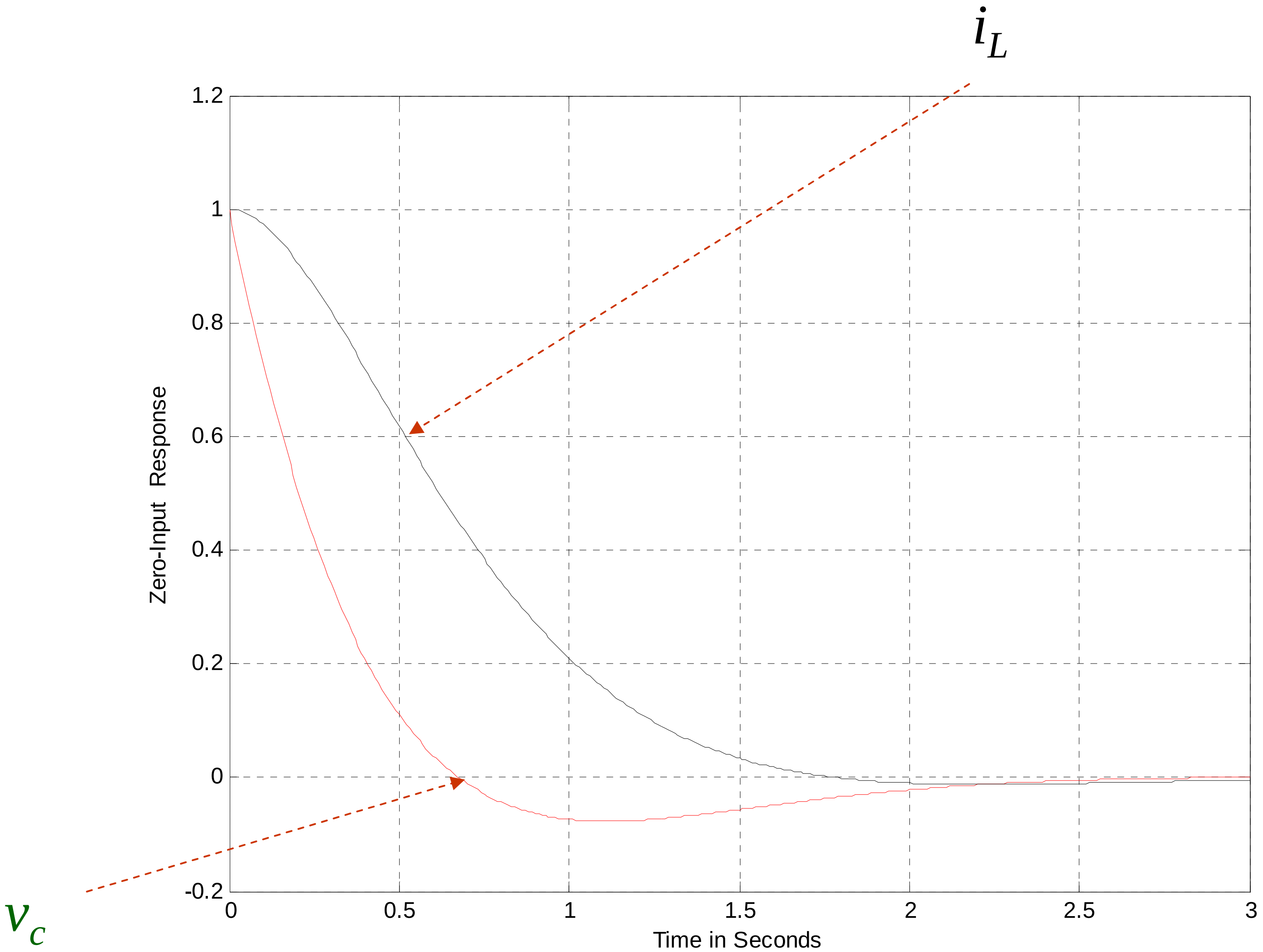
Again, let us get back to our example. We have obtained,

$$\mathbf{Y}(s)=\begin{bmatrix}\mathbf{V}_c(s)\\ \mathbf{I}_L(s)\end{bmatrix}=\begin{bmatrix}\dfrac{2(s+2)}{s^2+4s+6}\\ \dfrac{4}{s^2+4s+6}\end{bmatrix}\mathbf{U}(s)+\begin{bmatrix}\dfrac{s+1}{s^2+4s+6}\\ \dfrac{s+4}{s^2+4s+6}\end{bmatrix}$$

The zero-input response can then be computed as,

$$\mathbf{y}(t)=L^{-1}\{\mathbf{Y}(s)\}=\begin{bmatrix}v_c(t)\\ i_L(t)\end{bmatrix}=L^{-1}\left\{\begin{bmatrix}\mathbf{V}_c(s)\\ \mathbf{I}_L(s)\end{bmatrix}\right\}=L^{-1}\left\{\begin{bmatrix}\dfrac{s+1}{s^2+4s+6}\\ \dfrac{s+4}{s^2+4s+6}\end{bmatrix}\right\}=\begin{bmatrix}L^{-1}\left\{\dfrac{s+1}{s^2+4s+6}\right\}\\ L^{-1}\left\{\dfrac{s+4}{s^2+4s+6}\right\}\end{bmatrix}$$

$$=\begin{bmatrix}L^{-1}\left\{\dfrac{s+2}{(s+2)^2+\left(\sqrt{2}\right)^2}-\dfrac{\frac{1}{\sqrt{2}}\sqrt{2}}{(s+2)^2+\left(\sqrt{2}\right)^2}\right\}\\ L^{-1}\left\{\dfrac{s+2}{(s+2)^2+\left(\sqrt{2}\right)^2}+\dfrac{\sqrt{2}\sqrt{2}}{(s+2)^2+\left(\sqrt{2}\right)^2}\right\}\end{bmatrix}=\begin{bmatrix}e^{-2t}\left(\cos\sqrt{2}t-\dfrac{1}{\sqrt{2}}\sin\sqrt{2}t\right)\\ e^{-2t}\left(\cos\sqrt{2}t+\sqrt{2}\sin\sqrt{2}t\right)\end{bmatrix}$$



i_L
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
Zero-Input Response
0
0.5
1
1.5
2
2.5
3
Time in Seconds
v_c

4.4 Complete Response

Complete response is the response of the circuit due to both initial condition and external input. It can be computed by taking an inverse Laplace transform of the following term,

$$\mathbf{Y}(s) = [\mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}+\mathbf{D}]\mathbf{U}(s) + \mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{x}_0$$

i.e.,

$$\boxed{\mathbf{y}(t) = L^{-1}\left\{\mathbf{Y}(s)\right\} = L^{-1}\left\{[\mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}+\mathbf{D}]\mathbf{U}(s)\right\} + L^{-1}\left\{\mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{x}_0\right\}}$$

For our example, assume that the voltage source is DC with a constant voltage of 2V, i.e,

$$\mathbf{u} = v_s = 2 \quad \Rightarrow \quad \mathbf{U}(s) = L\left\{\mathbf{u}\right\} = \frac{2}{s} \quad \Rightarrow$$

$$\mathbf{Y}(s) = \begin{bmatrix} \mathbf{V}_c(s) \\ \mathbf{I}_L(s) \end{bmatrix} = \begin{bmatrix} \dfrac{2(s+2)}{s^2+4s+6} \\ \dfrac{4}{s^2+4s+6} \end{bmatrix} \mathbf{U}(s) + \begin{bmatrix} \dfrac{s+1}{s^2+4s+6} \\ \dfrac{s+4}{s^2+4s+6} \end{bmatrix} = \begin{bmatrix} \dfrac{2(s+2)}{s^2+4s+6} \cdot \dfrac{2}{s} \\ \dfrac{4}{s^2+4s+6} \cdot \dfrac{2}{s} \end{bmatrix} + \begin{bmatrix} \dfrac{s+1}{s^2+4s+6} \\ \dfrac{s+4}{s^2+4s+6} \end{bmatrix}$$

$$\mathbf{y}(t)=\begin{bmatrix} v_c(t)\\ i_L(t)\end{bmatrix}=L^{-1}\left\{\mathbf{Y}(s)\right\}=L^{-1}\left\{\begin{bmatrix}\dfrac{2(s+2)}{s^2+4s+6}\cdot\dfrac{2}{s}\\ \dfrac{4}{s^2+4s+6}\cdot\dfrac{2}{s}\end{bmatrix}\right\}+L^{-1}\left\{\begin{bmatrix}\dfrac{s+1}{s^2+4s+6}\\ \dfrac{s+4}{s^2+4s+6}\end{bmatrix}\right\}$$

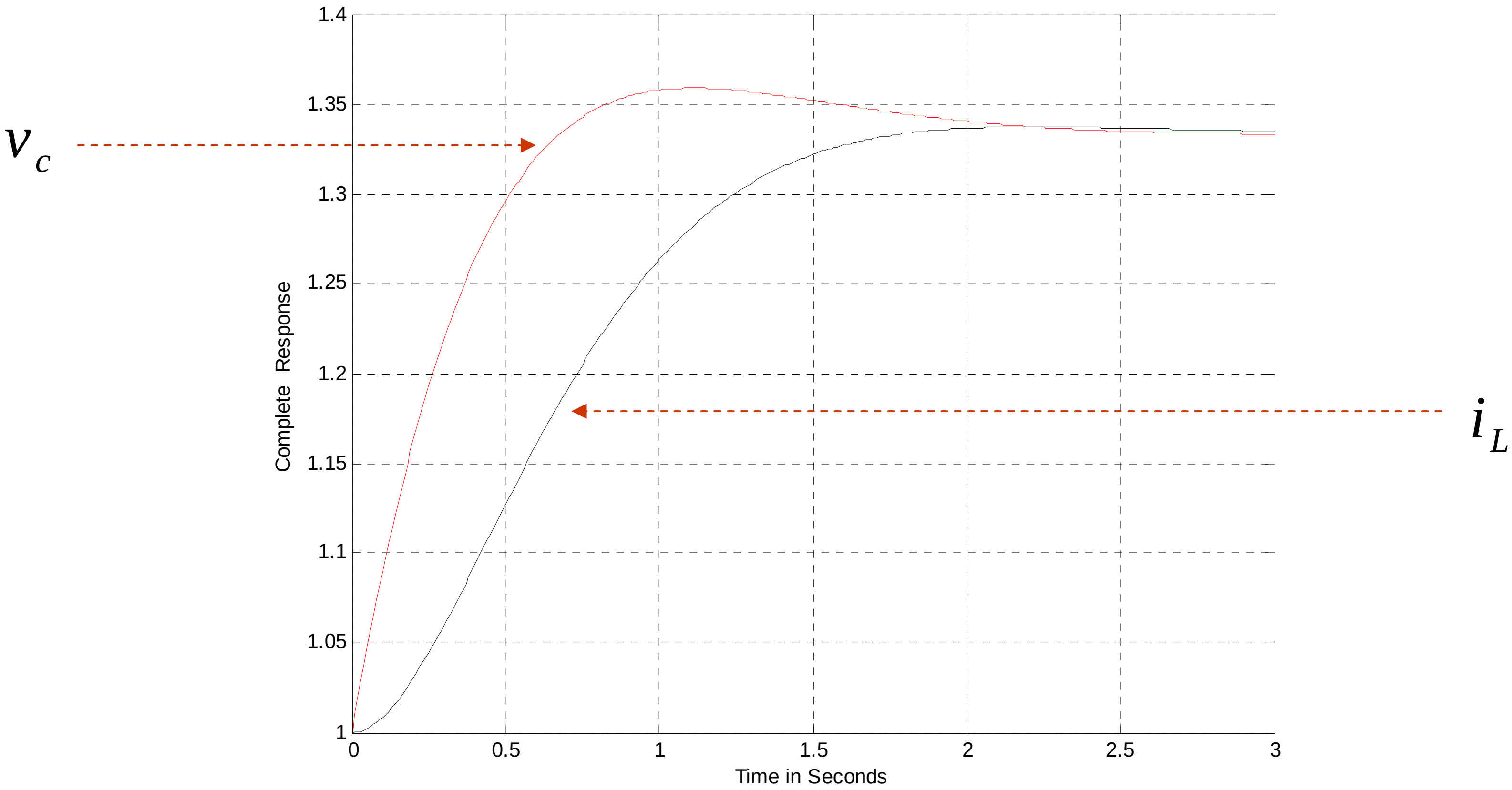
$$=L^{-1}\left\{\begin{bmatrix}\dfrac{-\frac{4}{3}(s+1)}{s^2+4s+6}+\dfrac{\frac{4}{3}}{s}\\ \dfrac{-\frac{4}{3}(s+4)}{s^2+4s+6}+\dfrac{\frac{4}{3}}{s}\end{bmatrix}\right\}+\begin{bmatrix}e^{-2t}\left(\cos\sqrt{2}t-\dfrac{1}{\sqrt{2}}\sin\sqrt{2}t\right)\\ e^{-2t}\left(\cos\sqrt{2}t+\sqrt{2}\sin\sqrt{2}t\right)\end{bmatrix}$$

$$=\begin{bmatrix}L^{-1}\left\{\dfrac{-\frac{4}{3}(s+1)}{s^2+4s+6}\right\}+L^{-1}\left\{\dfrac{\frac{4}{3}}{s}\right\}\\ L^{-1}\left\{\dfrac{-\frac{4}{3}(s+4)}{s^2+4s+6}\right\}+L^{-1}\left\{\dfrac{\frac{4}{3}}{s}\right\}\end{bmatrix}+\begin{bmatrix}e^{-2t}\left(\cos\sqrt{2}t-\dfrac{1}{\sqrt{2}}\sin\sqrt{2}t\right)\\ e^{-2t}\left(\cos\sqrt{2}t+\sqrt{2}\sin\sqrt{2}t\right)\end{bmatrix}$$

$$=\begin{bmatrix}-\dfrac{4}{3}e^{-2t}\left(\cos\sqrt{2}t-\dfrac{1}{\sqrt{2}}\sin\sqrt{2}t\right)+\dfrac{4}{3}\\ -\dfrac{4}{3}e^{-2t}\left(\cos\sqrt{2}t+\sqrt{2}\sin\sqrt{2}t\right)+\dfrac{4}{3}\end{bmatrix}+\begin{bmatrix}e^{-2t}\left(\cos\sqrt{2}t-\dfrac{1}{\sqrt{2}}\sin\sqrt{2}t\right)\\ e^{-2t}\left(\cos\sqrt{2}t+\sqrt{2}\sin\sqrt{2}t\right)\end{bmatrix}$$

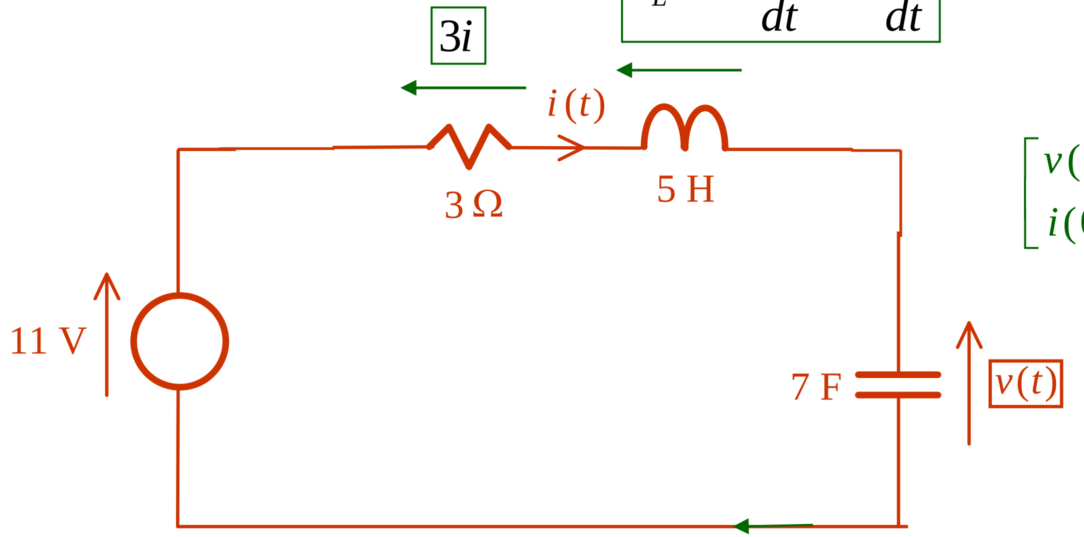
The complete response of the circuit is thus given by

$$\mathbf{y}(t) = \begin{bmatrix} v_c(t) \\ i_L(t) \end{bmatrix} = \begin{bmatrix} \dfrac{4}{3} - \dfrac{1}{3} e^{-2t} \left(\cos \sqrt{2}t - \dfrac{1}{\sqrt{2}} \sin \sqrt{2}t \right) \\ \dfrac{4}{3} - \dfrac{1}{3} e^{-2t} \left(\cos \sqrt{2}t + \sqrt{2} \sin \sqrt{2}t \right) \end{bmatrix}$$



4.5 More Example: Consider the RLC circuit in Section 2.10, i.e.,

$$\boxed{v_L = L\frac{di}{dt} = 5\frac{di}{dt}}$$



$$\begin{bmatrix} v(0) \\ i(0) \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

By KVL:

$$\boxed{i = C\frac{dv}{dt} = 7\frac{dv}{dt} \quad \Rightarrow \quad v' = \frac{dv}{dt} = \frac{1}{7}i}$$

$$v + v_L + 3i = 5\frac{di}{dt} + 3i + v = v_s \quad \Rightarrow \quad i' = \frac{di}{dt} = -\frac{1}{5}v - \frac{3}{5}i + \frac{1}{5}v_s$$

$$\Rightarrow \begin{bmatrix} \dot{v} \\ \dot{i} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1/7 \\ -1/5 & -3/5 \end{bmatrix}}_{\mathbf{A}} \begin{bmatrix} v \\ i \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 1/5 \end{bmatrix}}_{\mathbf{B}} v_s \qquad v = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{\mathbf{C}} \begin{bmatrix} v \\ i \end{bmatrix} + \underbrace{0}_{\mathbf{D}}\, v_s$$

$$\mathbf{Y}(s) = [\mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}+\mathbf{D}]\mathbf{U}(s) + \mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{x}_0$$

$$= \left\{ \begin{bmatrix} 1 & 0 \end{bmatrix} \left(s\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1/7 \\ -1/5 & -3/5 \end{bmatrix} \right)^{-1} \begin{bmatrix} 0 \\ 1/5 \end{bmatrix} + 0 \right\} \cdot \frac{11}{s}$$

$$+ \begin{bmatrix} 1 & 0 \end{bmatrix} \left(s\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1/7 \\ -1/5 & -3/5 \end{bmatrix} \right)^{-1} \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s & -1/7 \\ 1/5 & s+3/5 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1/5 \end{bmatrix} \cdot \frac{11}{s} + \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s & -1/7 \\ 1/5 & s+3/5 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \end{bmatrix} \frac{\begin{bmatrix} s+3/5 & 1/7 \\ -1/5 & s \end{bmatrix}}{s^2 + 3/5\, s + 1/35} \begin{bmatrix} 0 \\ 1/5 \end{bmatrix} \cdot \frac{11}{s} + \begin{bmatrix} 1 & 0 \end{bmatrix} \frac{\begin{bmatrix} s+3/5 & 1/7 \\ -1/5 & s \end{bmatrix}}{s^2 + 3/5\, s + 1/35} \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$$= \frac{1/35}{s^2 + 3/5\, s + 1/35} \cdot \frac{11}{s} + \frac{2\left(s + 3/5\right)}{s^2 + 3/5\, s + 1/35}$$

$$v(t) = L^{-1}\{\mathbf{Y}(s)\} = L^{-1}\left\{ \frac{1/35}{s^2 + 3/5\, s + 1/35} \cdot \frac{11}{s} + \frac{2\left(s + 3/5\right)}{s^2 + 3/5\, s + 1/35} \right\}$$

$$= L^{-1}\left\{ \frac{0.0285}{(s+0.548)(s+0.052)} \cdot \frac{11}{s} + \frac{2\left(s+0.6\right)}{(s+0.548)(s+0.052)} \right\}$$

$$= L^{-1}\left\{ \frac{1.15}{s+0.548} - \frac{12.15}{s+0.052} + \frac{11}{s} - \frac{0.2}{s+0.548} + \frac{2.2}{s+0.052} \right\}$$

$$= L^{-1}\left\{ \frac{1.15}{s+0.548} - \frac{12.15}{s+0.052} + \frac{11}{s} - \frac{0.21}{s+0.548} + \frac{2.21}{s+0.052} \right\}$$

$$= L^{-1}\left\{ \frac{0.94}{s+0.548} - \frac{9.94}{s+0.052} + \frac{11}{s} \right\}$$

$$= 0.94\, e^{-0.548\, t} - 9.94\, e^{-0.052\, t} + 11, \qquad t \geq 0$$

The above answer is exactly the same as that in Section 2.10. Actually, as stated earlier, the power of state space representation of the circuit is it unifies all related problems under a single framework.

4.6 Transfer Function

The state equation and output equation of the circuit or system are:

$$\frac{d\mathbf{x}}{dt} = \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}, \quad \mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$$

Assume the initial condition $x(0) = x_0 = 0$, then we have

$$\mathbf{Y}(s) = [\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}]\mathbf{U}(s) + \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{x}_0 = [\underbrace{\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}}_{\mathbf{H}(s)}]\mathbf{U}(s)$$

$\mathbf{H}(s)$ in general is a rational function of s and is called the transfer function of the circuit of system, which completely characterized the properties of the circuit with a zero initial condition. The roots of the denominator of $\mathbf{H}(s)$ are called the poles of the system while the roots of the numerator of $\mathbf{H}(s)$ are called the zeros of the system.

Example: We consider again the example in Section 4.5 in which we have

$$\begin{bmatrix} \dot{v} \\ \dot{i} \end{bmatrix} = \begin{bmatrix} 0 & 1/7 \\ -1/5 & -3/5 \end{bmatrix} \begin{bmatrix} v \\ i \end{bmatrix} + \begin{bmatrix} 0 \\ 1/5 \end{bmatrix} v_s, \quad v = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} v \\ i \end{bmatrix} + 0 \ v_s$$

The transfer function of the above system

$$\mathbf{Y}(s) = [\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}]\mathbf{U}(s) = \left\{ \begin{bmatrix} 1 & 0 \end{bmatrix} \left(s \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1/7 \\ -1/5 & -3/5 \end{bmatrix} \right)^{-1} \begin{bmatrix} 0 \\ 1/5 \end{bmatrix} + 0 \right\} \cdot \mathbf{U}(s)$$

$$= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s & -1/7 \\ 1/5 & s + 3/5 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1/5 \end{bmatrix} \cdot \mathbf{U}(s) = \begin{bmatrix} 1 & 0 \end{bmatrix} \frac{\begin{bmatrix} s + 3/5 & 1/7 \\ -1/5 & s \end{bmatrix}}{s^2 + 3/5\, s + 1/35} \begin{bmatrix} 0 \\ 1/5 \end{bmatrix} \cdot \mathbf{U}(s)$$

$$= \frac{1/35}{s^2 + 3/5\, s + 1/35} \cdot \mathbf{U}(s) \qquad \Rightarrow \qquad \boxed{\mathbf{H}(s) = \frac{1/35}{s^2 + 3/5\, s + 1/35}}$$

The system has no zero but have two poles respectively at – 0.548 & – 0.052.

4.7 System Stability

Example 1: Consider a circuit with a transfer function,

$$\mathbf{U}(s)=1 \longrightarrow \boxed{\mathbf{H}(s)=\frac{1}{s^2-1}} \longrightarrow \mathbf{Y}(s)$$

$$e^{-at} \Leftrightarrow \frac{1}{s+a}$$

$$0.5e^{-t} \Leftrightarrow \frac{0.5}{s+1}$$

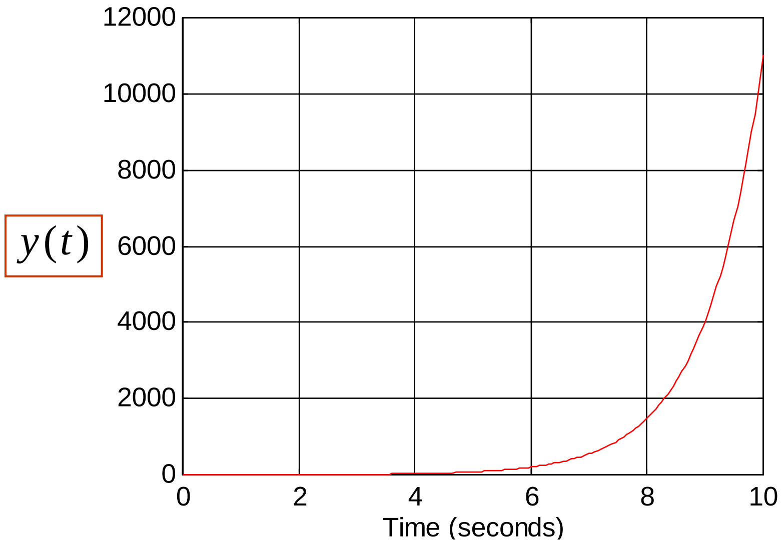
&

$$0.5e^{t} \Leftrightarrow \frac{0.5}{s-1}$$

We have

$$\mathbf{Y}(s)=\mathbf{H}(s)\mathbf{U}(s)=\frac{1}{s^2-1}=\frac{1}{(s+1)(s-1)}=\frac{0.5}{s-1}-\frac{0.5}{s+1}$$

From the Laplace transform table, we obtain $y(t)=0.5(e^{t}-e^{-t})$



This system is said to be *unstable* because the output response $y(t)$ goes to infinity as time t is getting larger and large. This happens because the denominator of $\mathbf{H}(s)$ has one positive root at $s=1$.

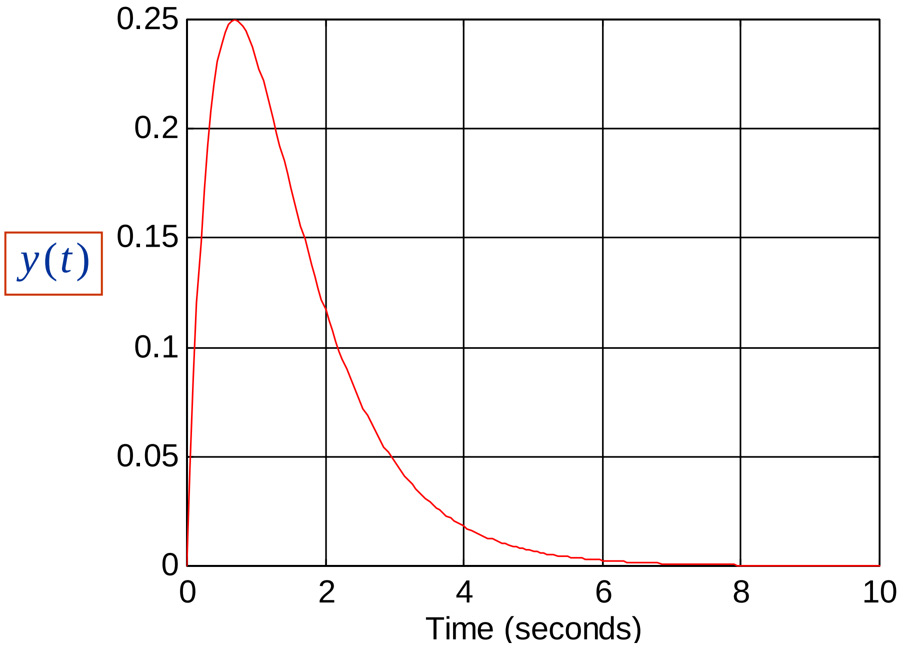
Example 2: Consider a closed-loop system with,

$$U(s)=1 \longrightarrow \boxed{H(s)=\frac{1}{s^2+3s+2}} \longrightarrow Y(s)$$

We have

$$\mathbf{Y}(s)=\mathbf{H}(s)\mathbf{U}(s)=\frac{1}{s^2+3s+2}=\frac{1}{(s+1)(s+2)}=\frac{1}{s+1}-\frac{1}{s+2}$$

From the Laplace transform table, we obtain $y(t)=e^{-t}-e^{-2t}$



This system is said to be ***stable*** because the output response $y(t)$ goes to 0 as time t is getting larger and large. This happens because the denominator of $\mathbf{H}(s)$ has no positive roots.

We consider a general system,

$$\mathbf{U}(s) \longrightarrow \boxed{\mathbf{H}(s) = \frac{\mathbf{N}(s)}{\mathbf{D}(s)}} \longrightarrow \mathbf{Y}(s)$$

The system is stable if the denominator, i.e., $\mathbf{D}(s)$, has no positive roots. It is unstable if it has positive roots. In particular,