ENGG 5403

Linear System Theory and Design / Part 2: Control

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Recap: What are we going to learn in this class?

How to describe a physical system by mathematical equations? (Dynamic modeling) ①

How does it behave? (System responses) ②

Is the system stable? (Stability issues) ③

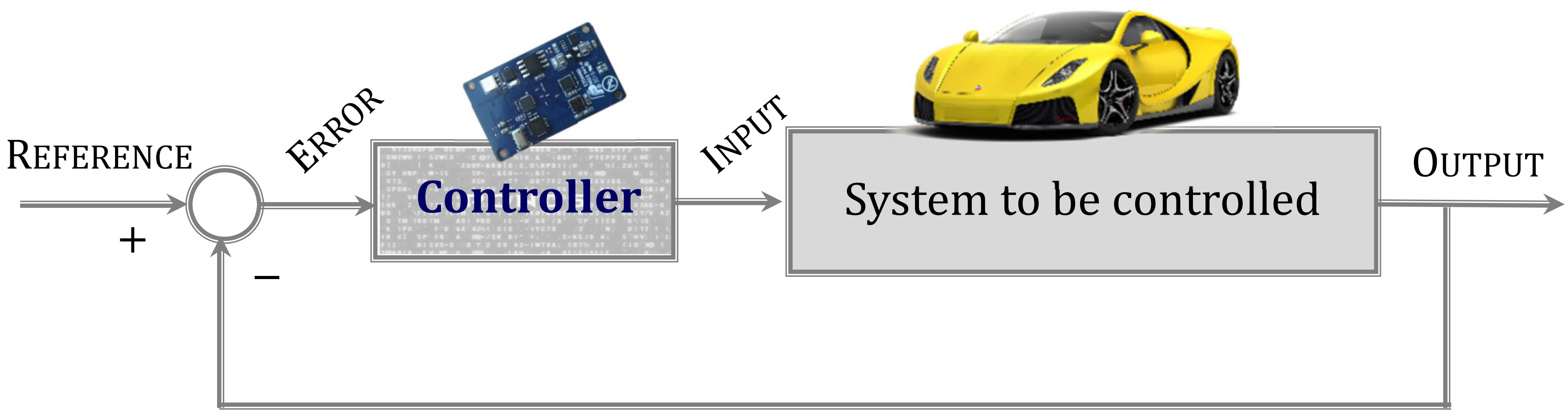
Can the system be stabilized? (Controllability & observability) ④

Is it a good system? (System structural properties) ⑤

How to control the physical system? (Control system design) ⑥

This last topic is to be learned in the second part…

Typical structure of a control system (revisit)…

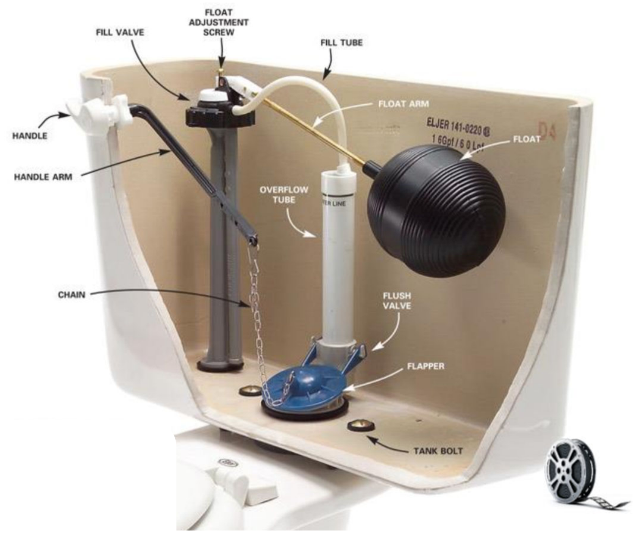


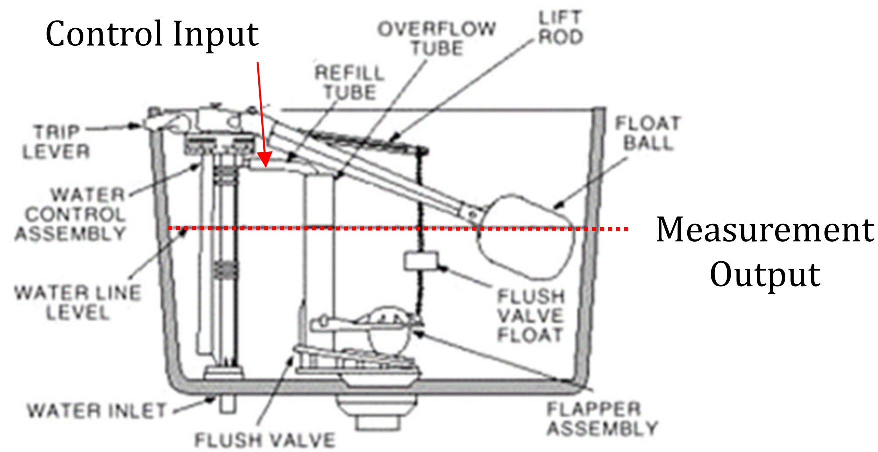
Objective: To make the system OUTPUT and the desired REFERENCE as close as possible, i.e., to make the ERROR as small as possible.

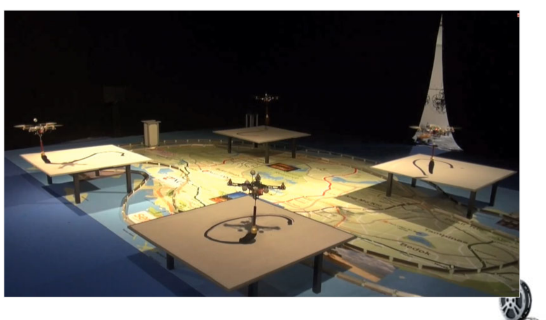
Key issues: (1) How to describe systems to be controlled? (Done in Part 1)

(2) How to design control laws? (To be done in Part 2)

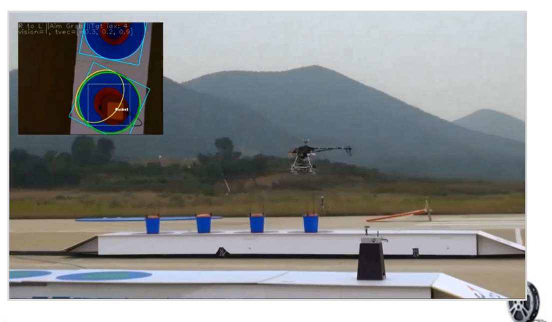
Examples of classical and advanced control







Optimal control



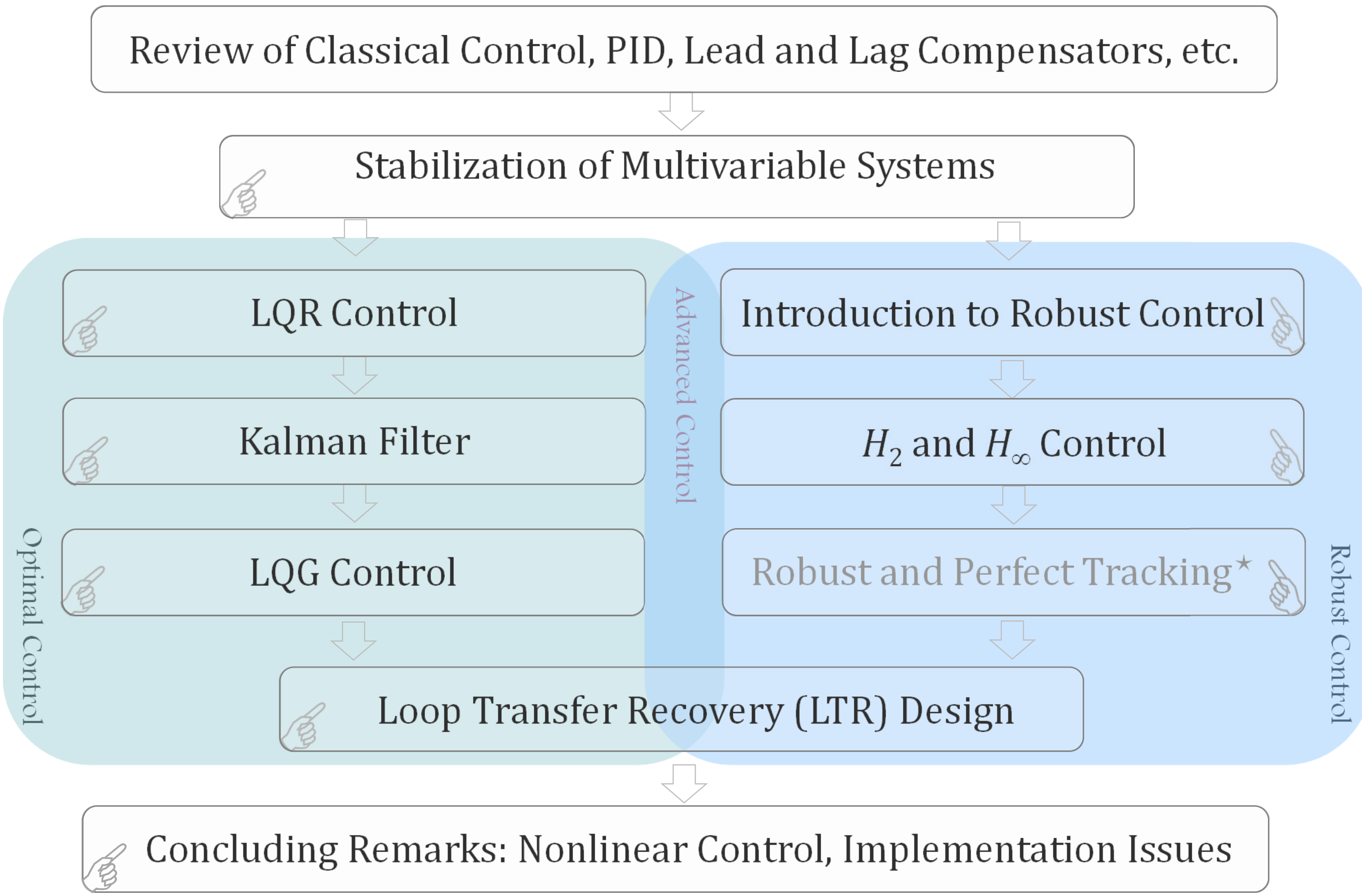
Robust control

More than just control

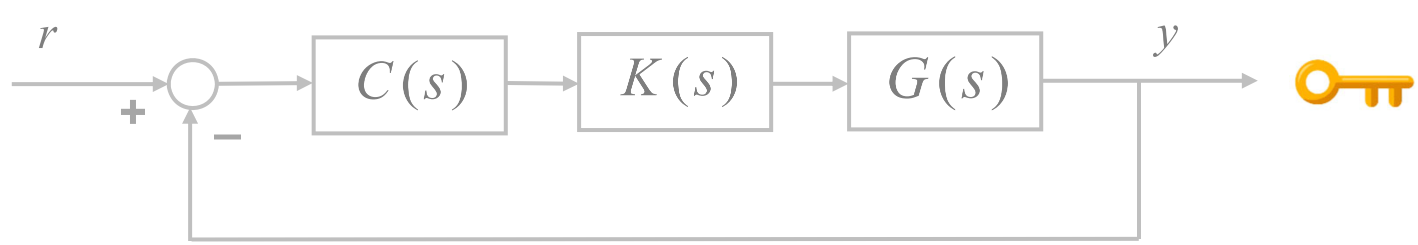
Outline for Part 2

- Revisit of classical control design methods for SISO systems.
- Stablization of multivariable systems.
- Linear quadratic regulation (LQR) control and its properties; returned differences; guaranteed gain and phase margins; Kalman filter; linear quadratic Gaussian (LQG) design.
- Introduction to modern control system design; H_2 and H_∞ optimal control; solutions to regular and singular H_2 and H_∞ optimal control problems; solutions to some robust control problems.
- Loop transfer recovery (LTR) design technique.
- Robust and perfect tracking (RPT) control and composite nonlinear feedback (CNF) control techniques (if time permits).

Material Flow of Part 2: Design…



Review of Classical Control Techniques



*The first real PID-type controller developed by Elmer Sperry in **1911**. The first theoretical analysis of a PID controller was published by Nicolas Minorsky in **1922**. His observations grew out of efforts to design automatic steering systems for the U.S. Navy.*

——PID Control History and Advancements by Jim Cahill

Elmer A. Sperry, Sr.
American Inventor and Entrepreneur
1860–1930

Nicolas Minorsky
Russian American Mathematician & Control Engineer
1885–1970

John Ziegler
American Engineer
1909–1997

Nathaniel Nichols
American Engineer
1914–1997

1940-1990 50th
50 Years of Automatic Control

Modern Control Started with Ziegler-Nichols Tuning

CONTROL ENGINEERING
2ND OCTOBER 1990

GEORGE J. BLICKLEY, CONTROL ENGINEERING

When two engineers at Taylor Instrument Co. decided to document the work they had done in finding ways to tune process controllers, they changed the whole control industry.

Up until 1940, most tuning of process controllers was an art conducted by seat-of-the-pants methods on controllers that were a hodge-podge of techniques or add-on components that defied any rigid rules that could be universally applied.

One of the engineers at Taylor was John G. Ziegler, the practical one of the pair with a lot of experience in process applications, and who performed all the simulator tests that led to the methods they were seeking. The other was Nathaniel B. Nichols who was the mathematician and who reduced all of the math to a few simple relationships that could be understood by technicians and operators.

The result was the now famous "Ziegler-Nichols" method of tuning controllers—a method that survived the slings and arrows of its early detractors, withstood the test of time, and works just as well as many of the later, more sophisticated optimizing forms on a great majority of process applications. Most of the work was done in 1940, a paper entitled "Optimum Settings for Automatic Controllers" was formulated and presented in December 1941 at the annual meeting of the American Society of Mechanical Engineers.

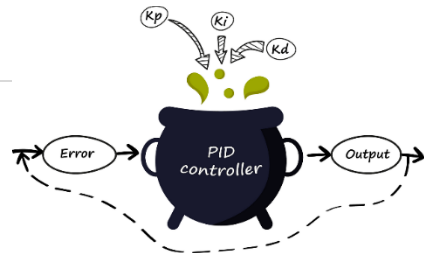
It must be remembered that all of this was done before the theory of servo-

"I was a very poor mathematician, so any sinusoidal oscillation was way beyond me mathematically."

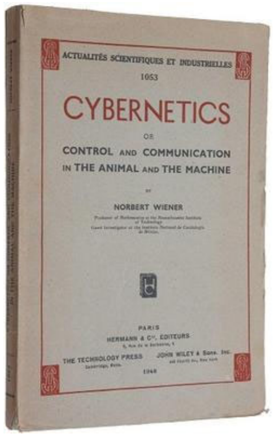
"When we gave that paper, there was a great hue and cry. The preprints came out late, when the old timers got it and read it, they said it was heresy and we were damned to the deepest hell because we did not know what we were talking about."

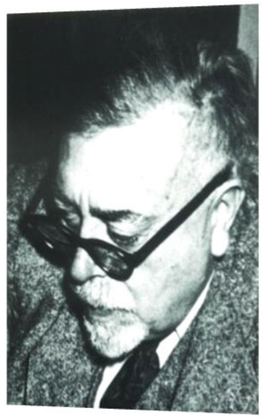
"All of these things were developed in the 'golden years of control' from 1935 to 1940. It was an interesting time and I'm glad I was in on it."

"We did not know how to set this new controller and I realized that we had to get some way of determining the controller settings rather than cut-and-try."



PID Tuning





Norbert Wiener
American Mathematician and Philosopher
1894–1964

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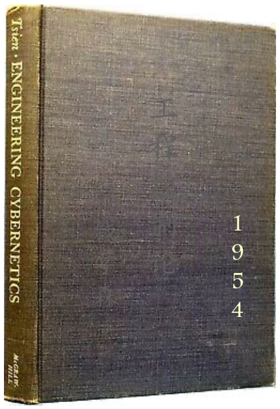
It is the first public usage of the term **cybernetics** to refer to self-regulating mechanisms. The book laid the theoretical foundation for servomechanisms (whether electrical or mechanical), automatic navigation, analog computing, artificial intelligence, neuroscience, and reliable communications.

Feedback and Oscillation: This chapter lays down the foundations for the mathematical treatment of negative feedback in automated control systems. The opening passage illustrates the effect of faulty feedback mechanisms by the example of patients with various forms of ataxia. He then discusses railway signaling, the operation of a thermostat, and a steam engine centrifugal governor. The rest of the chapter is mostly taken up with the development of a mathematical formulation of the operation of the principles underlying all of these processes. More complex systems are then discussed such as automated navigation, and the control of nonlinear situations such as steering on an icy road. He concludes with a reference to the homeostatic processes in living organisms.



Jet Propulsion Laboratory
California Institute of Technology

Hsue-shen Tsien
Chinese Mathematician and Aerospace Engineer
1911–2009

U.S. Army Colonel in World War II

ENGINEERING CYBERNETICS

H. S. TSIEN

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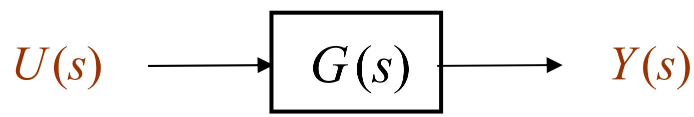
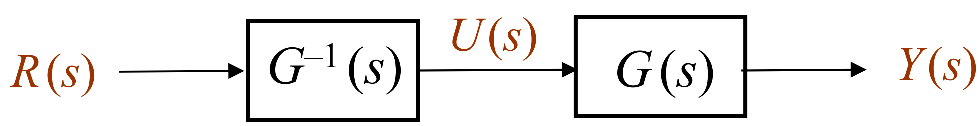
He was influenced by the methods of American engineering education, especially its focus on experimentation. This was in contrast to the contemporary approach practiced by many Chinese scientists, which emphasized theoretical elements rather than ***hands-on*** *experience…*

Recall that the main objectives in control system design are: (1) to stabilize the given system; and (2) to track certain desired references. As illustrated in Part 1, if we consider a SISO system

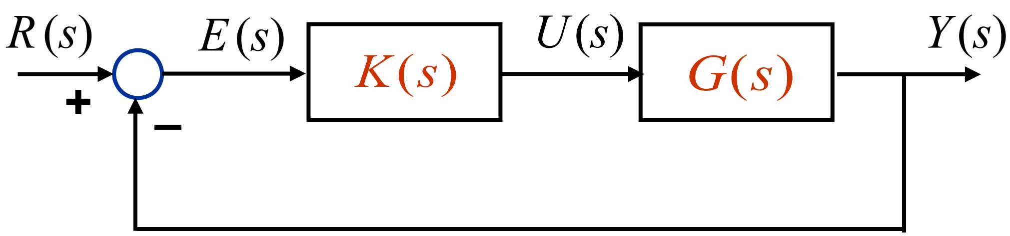


and if we want the output to track a reference *r*, the simplest solution is to design a control law of the following form



Besides the issue on **unstable pole-zero cancellations** as explained in Part 1, the above open-loop control strategy is not robust with respect to uncertainties in unmodeled system dynamics and external disturbances. As such, such an open-loop control system has never been adopted for practical uses!

Instead, we adopt the following feedback control scheme:



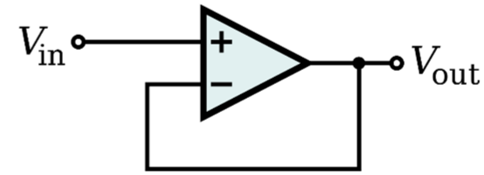
Recall that the objective of control system design is trying to match the output $Y(s)$ to the reference $R(s)$. Thus, it is important to find the relationship between them. Recall that

$$G(s) = \frac{Y(s)}{U(s)} \quad \Rightarrow \quad Y(s) = G(s)U(s)$$

Similarly, we have $U(s) = K(s)E(s)$, and $E(s) = R(s) - Y(s)$. Thus,

$$\begin{aligned} Y(s) = G(s)U(s) &= G(s)K(s)E(s) \\ &= G(s)K(s)\left[R(s) - Y(s)\right]. \end{aligned}$$

Similar Idea… Op Amp



Harold S. Black (1898–1983) was an American electrical engineer, who revolutionized the field of applied electronics by discovering the negative feedback amplifier in 1927.

To some, his discovery is considered the most important breakthrough of the twentieth century in the field of electronics, as it has a wide area of application.

He published a famous paper, *Stabilized Feedback Amplifiers*, in 1934.

$$Y(s) = G(s)K(s)R(s) - G(s)K(s)Y(s)$$

$$\Rightarrow \quad \left[1 + G(s)K(s)\right]Y(s) = G(s)K(s)R(s)$$

$$\Rightarrow \quad H(s) = \frac{Y(s)}{R(s)} = \frac{G(s)K(s)}{1 + G(s)K(s)}$$

$$R(s) \longrightarrow \boxed{H(s)} \longrightarrow Y(s)$$

which is the closed-loop transfer function from the reference input R to the system output Y.

Classical control techniques are focusing on designing an appropriate controller $K(s)$ such that the resulting closed-loop transfer function $H(s)$ is stable and meets given design specifications, such as settling time and overshoot in time domain and gain and phase margins in frequency domain.

Classical Control System Design Philosophy…

It is to select an appropriate controller such that when it is applied to the given plant, the resulting closed-loop system $H(s)$ meets the time domain specifications (such as **rise time**, **settling time** and **overshoot**, etc.).

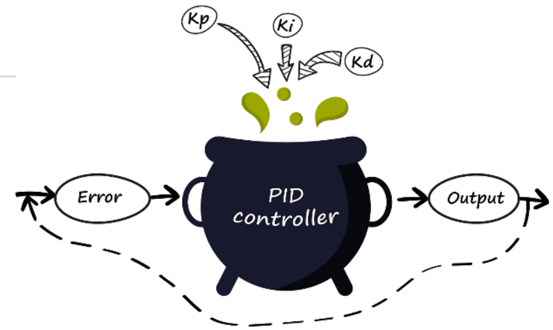
We observe that the best choice is to have an overall closed-loop system

$$R(s) \longrightarrow \boxed{H(s) = 1} \longrightarrow Y(s)$$

Unfortunately, having a unity transfer function is practically impossible, we would thus try to make $H(s) \approx 1$, instead. More specifically, we will try to make $H(s)$ to be as close to 1 as possible within the operating frequency range (working bandwidth) of the system.

In almost all classical control system designs, we are trying to match the closed-loop system to 1 at one particular frequency point, i.e., $s = 0$. We always carry out to design a controller such that the resulting $H(0) = 1$, a unity **DC gain**.

Typical choices of $K(s)$ in the classical design are as follows…



- **P** (proportional) control:

$$u(t) = k_p (r - y) = k_p e(t) \quad \Leftrightarrow \quad U(s) = k_p\, E(s)$$

- **PI** (proportional-integral) control,

$$u(t) = k_p e(t) + k_i \int_0^t e(\tau) d\tau \quad \Leftrightarrow \quad U(s) = \left(k_p + \frac{k_i}{s} \right) E(s)$$

- **PD** (proportional-derivative) control,

$$u(t) = k_p e(t) + k_d \frac{de(t)}{dt} \quad \Leftrightarrow \quad U(s) = \left(k_p + k_d s \right) E(s)$$

- **PID** (proportional-integral-derivative) control:

$$u(t) = k_p e(t) + k_i \int_0^t e(\tau) d\tau + k_d \frac{de(t)}{dt} \quad \Leftrightarrow \quad U(s) = \left(k_p + \frac{k_i}{s} + k_d s \right) E(s)$$

Proportional Control: $k_p e(t)$ $\Leftrightarrow$ *Action based on signal in the present…*

The proportional action depends only on the error. Generally, increasing the proportional gain will increase the speed of the control system response. However, if the proportional gain is too large, the process variable will begin to oscillate. If k_p is increased further, the oscillations will become larger and the system will become unstable.

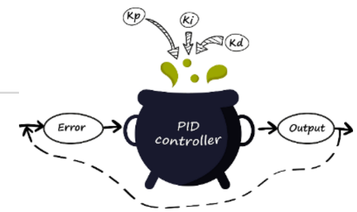
Integral Control: $k_i \int_0^t e(\tau) d\tau$ $\Leftrightarrow$ *Action based on signal in the past…*

The integral action sums the error term over time. The integral response will continually increase over time unless the error is zero, so the effect is to drive the steady-state error to zero. A phenomenon called integral windup results when integral action saturates a controller without the controller driving the error signal toward zero.

Derivative Control: $k_d \dfrac{de(t)}{dt}$ $\Leftrightarrow$ *Action based on signal in the 'future' …*

Increasing k_d will cause the control system to react more strongly to changes in the error and will increase the speed of the overall control system response. Generally, a small k_d is used, because its response is highly sensitive to noise in the process. If the sensor feedback signal is noisy, the derivative action can make the overall system out of control.

Tuning of PID Gains…



The gains of a PID controller can be obtained by **trial and error method**. Once a designer understands the significance of each gain parameter, this method becomes relatively easy.

P In most of the process control problems, the **I** and **D** terms are set to zero first and the proportional gain is increased until the output of the loop oscillates. As one increases k_p, the system becomes faster, but care must be taken not make the system unstable.

I Once **P** has been set to obtain a desired fast response, the integral term is increased to stop the oscillations. The integral term reduces the steady state error, but increases overshoot. The integral term is tweaked to achieve a minimal steady state error.

D Once the **P** and **I** have been set to get the desired fast control system with minimal steady state error, the derivative term is increased until the loop is acceptably quick to its set point. Increasing derivative term decreases overshoot and yields higher gain with stability but would cause the system to be highly sensitive to noise.

★ Designers need to **tradeoff** one characteristic of a control system for another to better meet their requirements. There are many auto-tuning methods available for process control in the literature…

In what follows, we are recall some simplest method in designing a classical controller, i.e., to choose a suitable controller with appropriate gains such that the resulting closed-loop transfer function $H(s)$ is dominated by a 2nd order system. We then compared it with the behaviors of a typical 2nd order prototype whose properties are well studied and documented.

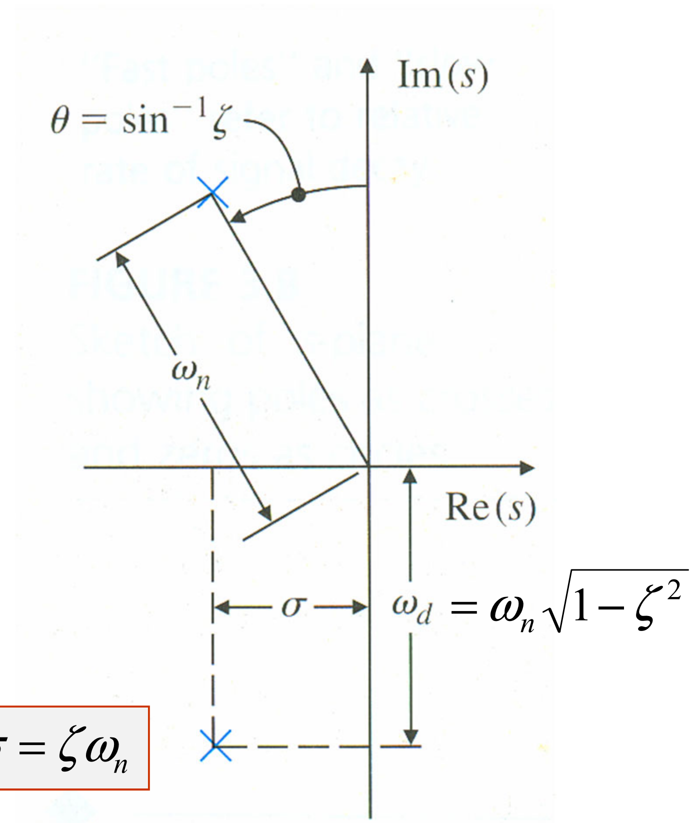
A 2nd Order Prototype:

The following is a commonly used prototype and important benchmark for classical control system design:

$$H_{\text{prototype}}(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

ζ is called the **damping ratio** of the system

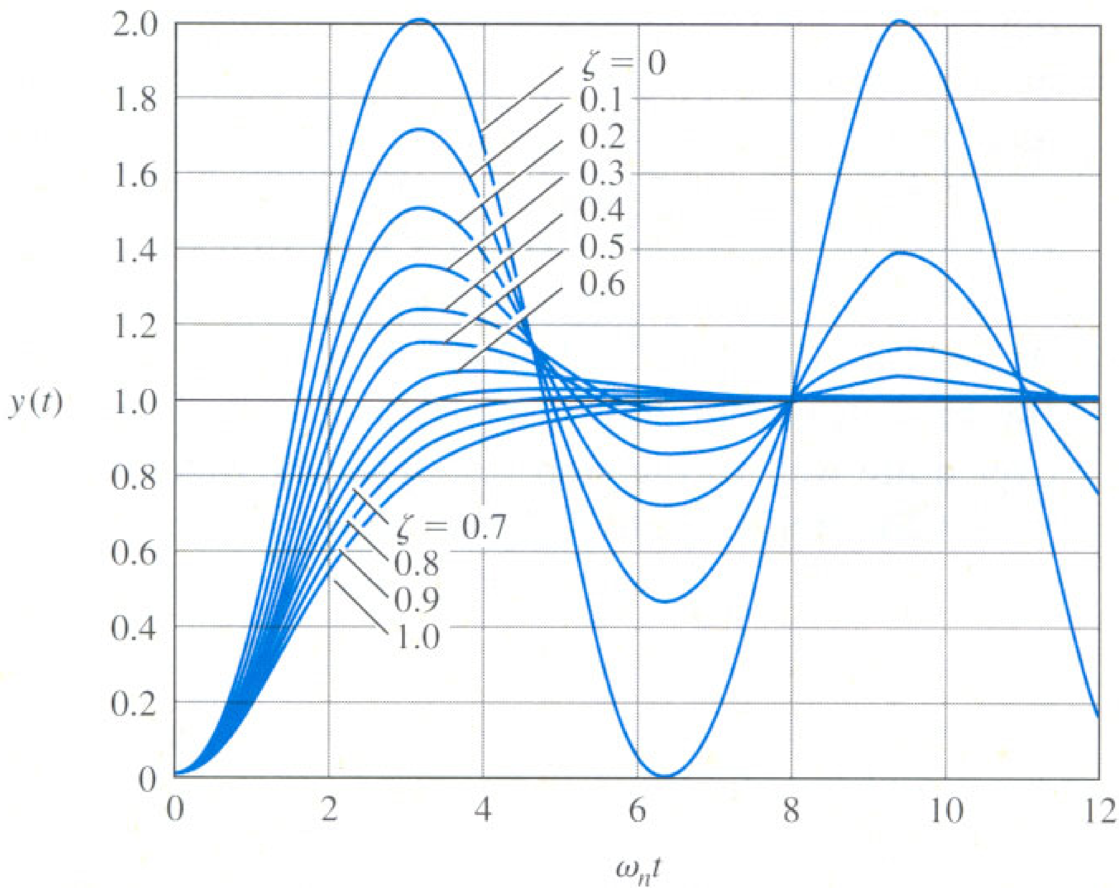
ω_n is called the **natural frequency**



$$\boxed{\sigma = \zeta\omega_n}$$

Unit step response of

$$H_{\text{prototype}}(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$



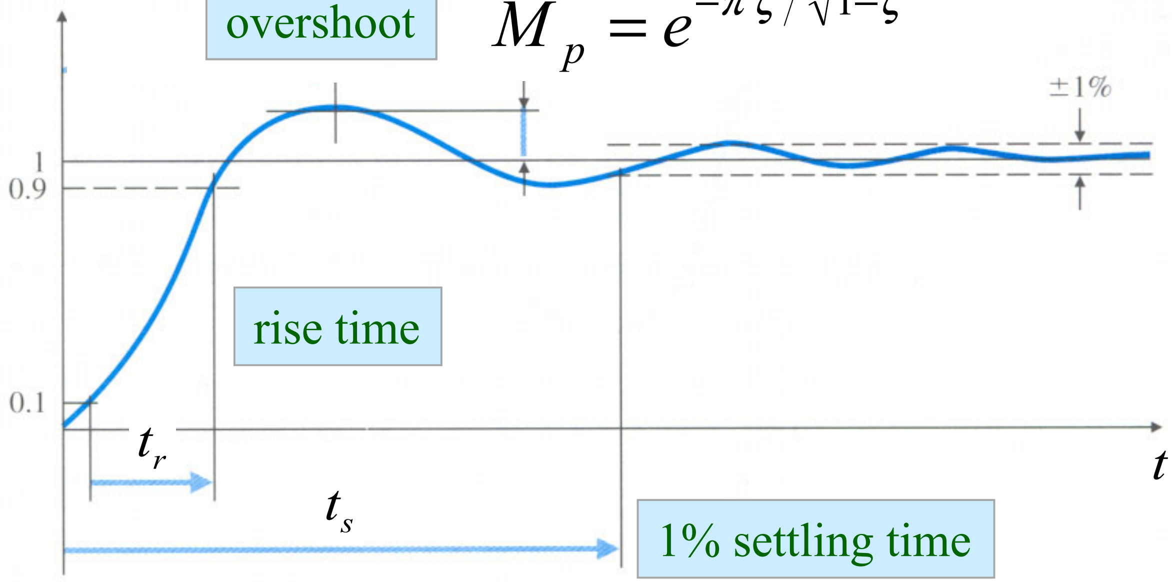
The behavior of the system is fully characterized by ζ, the *damping ratio*, and ω_n, the *natural frequency*.

Further zoom in to the unit step response of the 2nd order prototype…

$$H_{\text{prototype}}(s)=\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$

overshoot

$$M_p = e^{-\pi\zeta/\sqrt{1-\zeta^2}}$$



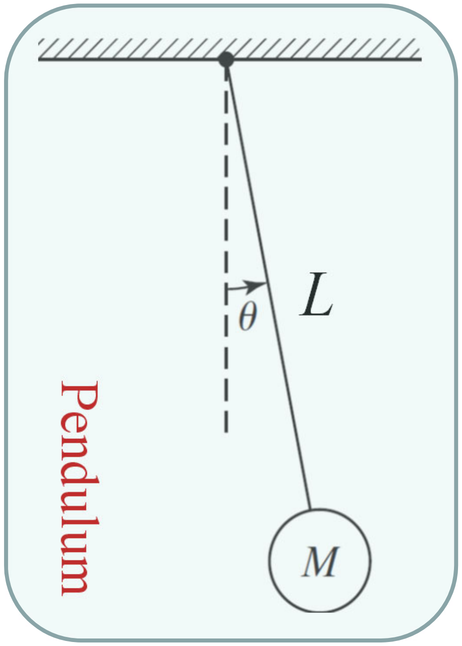
rise time

1% settling time

$$t_r \cong \frac{1.8}{\omega_n}$$

$$t_s \cong \frac{4.6}{\zeta\omega_n}$$

Example: Recall that the linear model of the pendulum system around $\theta_0 = 0$ is



$$\begin{pmatrix} \dot{\theta} \\ \ddot{\theta} \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ -\dfrac{g}{L} & 0 \end{bmatrix} \begin{pmatrix} \theta \\ \dot{\theta} \end{pmatrix} + \begin{bmatrix} 0 \\ \dfrac{1}{ML^2} \end{bmatrix} u, \quad y = [1 \quad 0] \begin{pmatrix} \theta \\ \dot{\theta} \end{pmatrix}$$

For simplicity, we assume

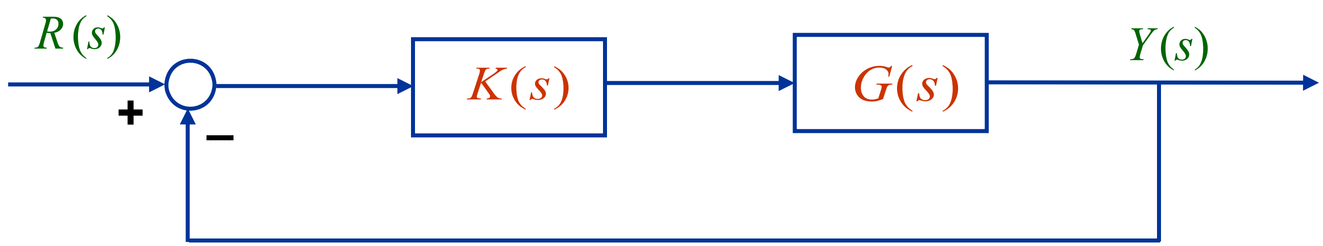
$$\frac{g}{L} = 1, \quad \frac{1}{ML^2} = 1$$

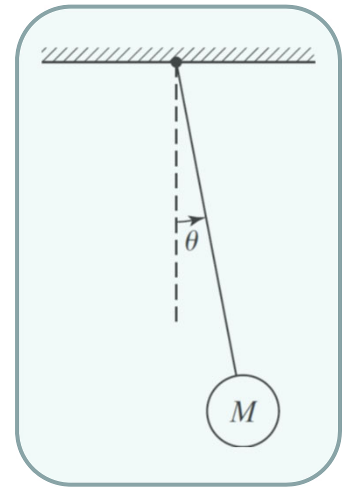
The above system can thus be expressed as

$$\begin{pmatrix} \dot{\theta} \\ \ddot{\theta} \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{pmatrix} \theta \\ \dot{\theta} \end{pmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \quad y = [1 \quad 0] \begin{pmatrix} \theta \\ \dot{\theta} \end{pmatrix}$$

It has a transfer function $G(s) = \dfrac{1}{s^2 + 1}$. We wish to design a PD controller to yield a settling time of 1 sec. and an overshoot less than 10% for a step response.

The problem can be formulated as follows…





$$G(s) = \frac{1}{s^2 + 1}$$

design a PD control law

$$K(s) = k_p + k_d s$$

such that the closed-loop system response due to a unit step input has a settling time $t_s = 1$ second and overshoot less than 10%.

The resulting closed-loop system is given by

Cannot do anything with this term…

$$H(s) = \frac{G(s)K(s)}{1+G(s)K(s)} = \frac{k_d s + k_p}{s^2 + k_d s + 1 + k_p}$$

Compare this with the standard 2nd order system:

$$H_{\text{prototype}}(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$\boxed{\begin{aligned} k_d &= 2\zeta\omega_n \\ k_p &= \omega_n^2 - 1 \end{aligned}}$$

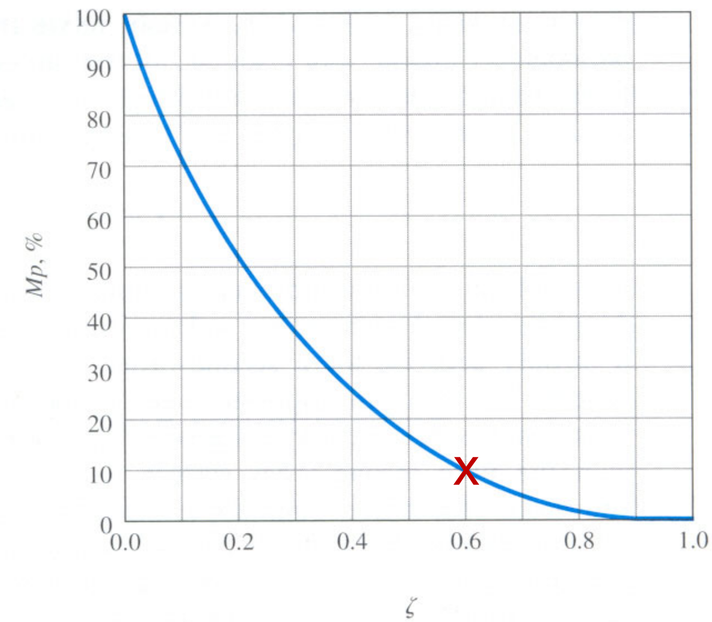
The key issue now is to choose parameters k_p and k_d such that the above resulting system has desired properties, such as prescribed settling time and overshoot. We should note that the numerator cannot be exactly matched no matter what and the resulting DC gain is always equal to $k_p/(1+k_p)$. We can add an additional feedforward constant gain to make the DC gain unity.

To achieve an overshoot less than 10%, we obtain $\zeta > 0.6$ from the figure on the right that

To be safe, we choose $\zeta = 0.8$.

To achieve a settling time of 1 second, we use

$$t_s = \frac{4.6}{\zeta\omega_n} = 1 \quad \Rightarrow \quad \omega_n = \frac{4.6}{\zeta t_s} = \frac{4.6}{0.8} = 5.75$$

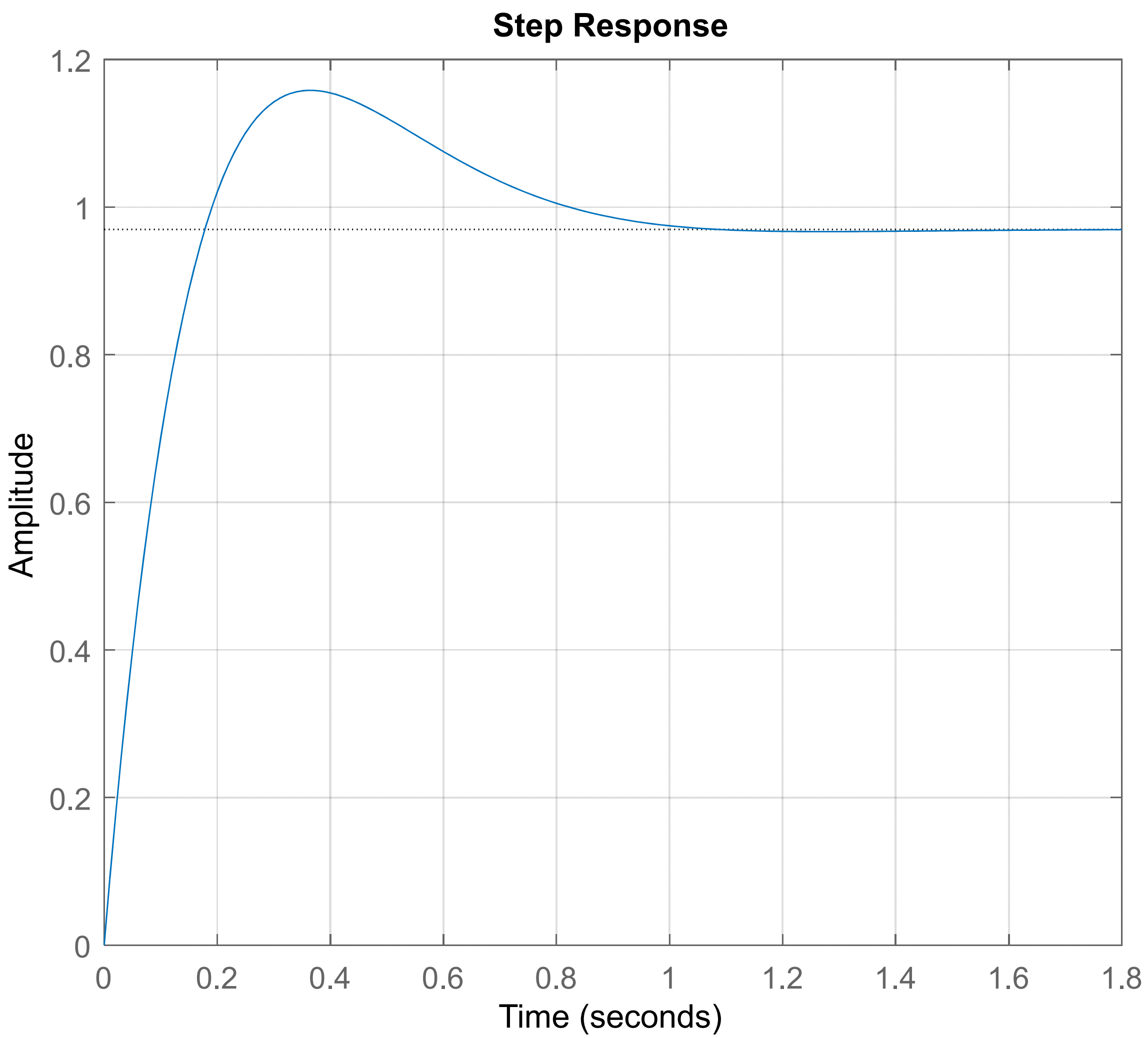


$$M_p = e^{-\pi\zeta/\sqrt{1-\zeta^2}}$$

$$\boxed{\begin{aligned} k_d &= 2\zeta\omega_n \\ k_p &= \omega_n^2 - 1 \end{aligned}}$$

$$k_d = 9.2$$
$$k_p = 32.1$$

Simulation Result:

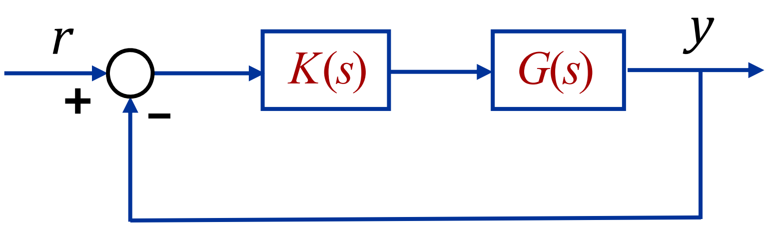


The resulting overshoot is about 18% and the settling time is about 1 sec. Also, there is a steady state error. Thus, our design goal is only partially achieved. We need to resign the controller.

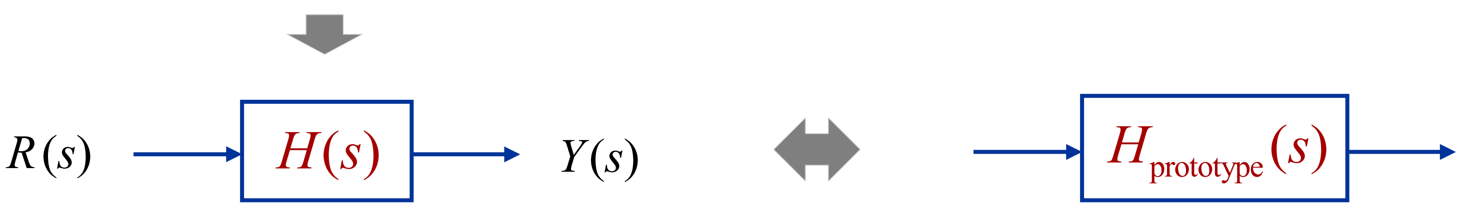
Summary for designing $K(s)$…

Step 1: Given a plant, $G(s)$, to be controlled and given design specifications (e.g., the required on settling time, overshoot, etc.), determine an appropriate 2nd order prototype $H_{\text{prototype}}(s)$, which meets the requirements.

Step 2: Choose an appropriate (P/PI/PD/PID) controller, $K(s)$, and work out its closed-loop transfer function $H(s)$.



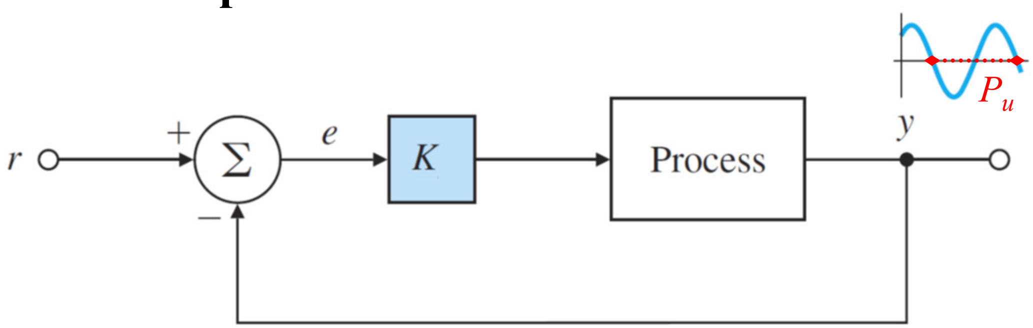
Step 3: Determine the required gain parameters in $K(s)$ by matching $H(s)$ in **Step 2** and $H_{\text{prototype}}(s)$ in **Step 1**, either exactly or approximately (usually we would only be able to match them loosely).

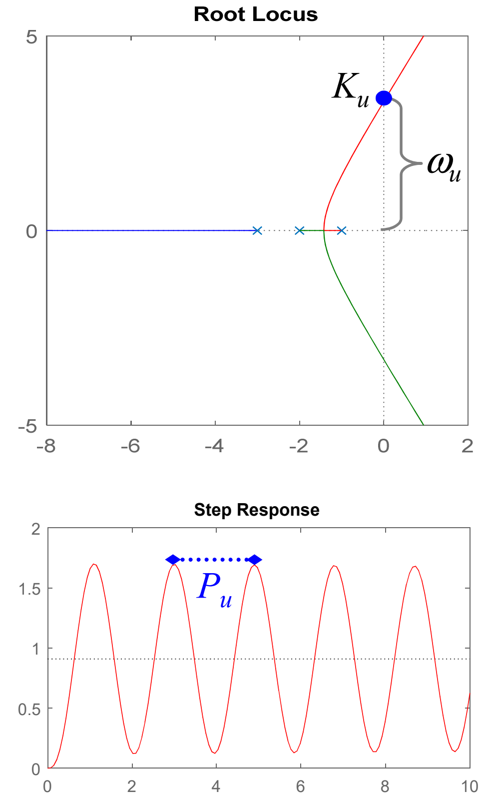


Step 4: Simulate the above design to verify the result. Repeat **Step 2** and **Step 3** until a satisfactory result is obtained.

♠ Ziegler-Nichols tuning method for designing a process controller…

Step 1: In the configuration below, the proportional gain is increased until the closed-loop system becomes marginally stable (i.e., the closed-loop system has simple poles on the imaginary axis, say $j\omega_u$, in the complex plane). Such a gain, K_u, is called the **ultimate gain**. The corresponding period of oscillation, $P_u = 2\pi/\omega_u$, is called the **ultimate period**.





Step 2: Ziegler-Nichols tuning parameters are set as…

Type of Controller	Optimum Gain
P	$k_p = 0.5\,K_u$
PI	$k_p = 0.45\,K_u,\quad k_i = 0.54\,K_u / P_u$
PID	$k_p = 1.6\,K_u,\quad k_i = 3.2\,K_u / P_u,\quad k_d = 0.2\,K_u P_u$

What is next?

Even though we might be happy with the time-domain performance, the frequency domain properties (such as **gain and phase margins** as well as **sensitivity function** specifications) are equally important in real-life applications. These frequency-domain specifications guarantee the robustness of the overall closed-loop system in face of uncertainties and disturbances.

In order to elaborate the concept of frequency-domain specifications, we need to recall the **Bode plot** and **Nyquist plot** of a transfer function that we have learned in the elementary introduction course to feedback control in our undergraduate studies.

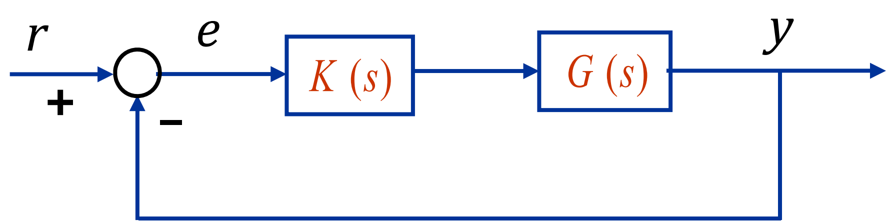
Hendrik W. Bode
American Engineer
1905–1982

In **1945** H. W. Bode presented a system for analyzing the stability of feedback systems by using graphical methods. Until this time, feedback analysis was done by multiplication and division, so calculation of transfer functions was a time consuming and laborious task. Remember, engineers did not have calculators or computers until the 1970s. Bode presented a log technique that transformed the intensely mathematical process of calculating a feedback system's stability into graphical analysis that was simple and perceptive. Feedback system design was still complicated, but it no longer was an art dominated by a few electrical engineers kept in a small dark room. Any electrical engineer could use Bode's methods to find the stability of a feedback circuit. —— *Ron Mancini*

Frequency responses

$$z = x + iy = re^{j\theta}$$

Consider the following feedback control system,



Frequency response of the open-loop transfer function, i.e., $K(s)G(s)$, are the key in examining the robustness properties of the closed-loop system.

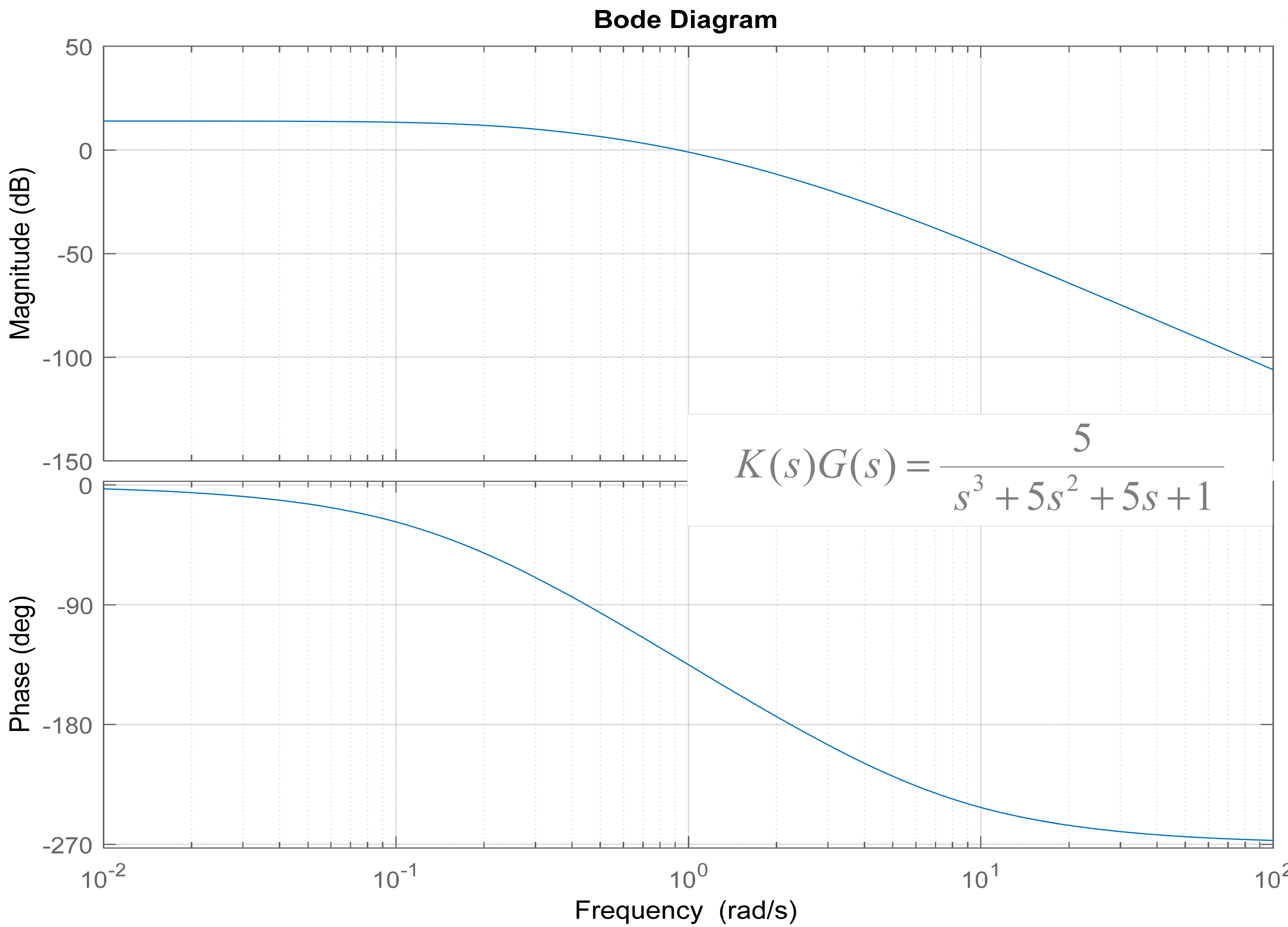
For example, suppose a control system has an open-loop transfer function

$$K(s)G(s)\Big|_{s=j\omega} = \left.\frac{5}{s^3+5s^2+5s+1}\right|_{s=j\omega} = \frac{5(1-5\omega^2)+j5(\omega^3-5\omega)}{(1-5\omega^2)^2+(5\omega-\omega^3)^2}$$

Nyquist plot

$$\underbrace{\left|K(j\omega)G(j\omega)\right| = \frac{5}{\sqrt{(1-5\omega^2)^2+(\omega^3-5\omega)^2}}\ , \quad \angle K(j\omega)G(j\omega) = \tan^{-1}\left(\frac{\omega^3-5\omega}{1-5\omega^2}\right)}_{\text{Bode plot}}$$

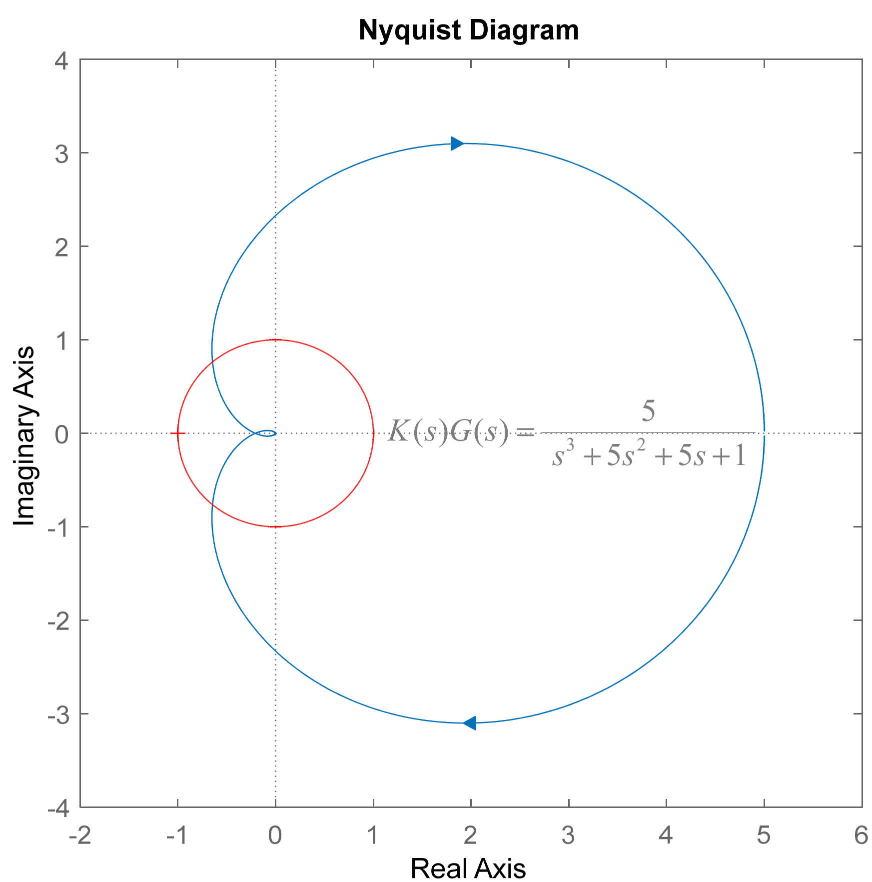
bode
ex2021



$$K(s)G(s)=\frac{5}{s^3+5s^2+5s+1}$$

Nyquist plot

Nyquist plot maps the open-loop transfer function $K(j\omega)\,G(j\omega)$ directly onto a complex plane. For the previous example, its Nyquist plot is as follows…

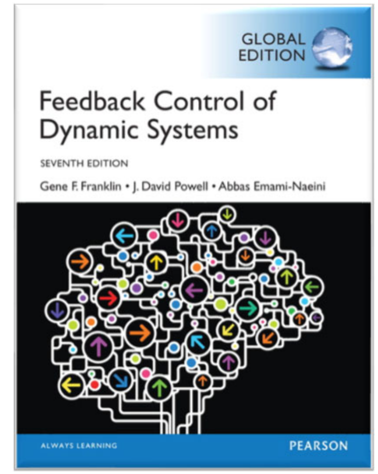


nyquist

Harry Nyquist
Swedish
American Engineer
1889–1976

Nyquist stability criterion, independently discovered by the German electrical engineer **Felix Strecker** at Siemens in **1930** and the Swedish-American electrical engineer Harry Nyquist at Bell Telephone Lab in **1932**, is a graphical technique for determining the stability of a dynamical system. Because it only looks at the **Nyquist plot** of the open loop systems, it can be applied without explicitly computing the poles and zeros of either the closed-loop or open-loop system. As a result, it can be applied to systems defined by non-rational functions, such as systems with delays.

Nyquist stability criterion



Recall the closed-loop transfer function of feedback system…

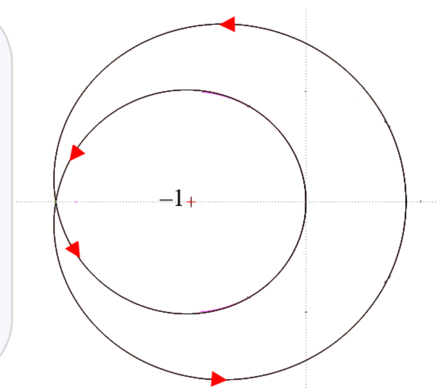
$$H(s) = \frac{K(s)G(s)}{1+K(s)G(s)}$$

The closed-loop characteristic polynomial is given by

$$1+K(s)G(s)=0 \quad \Rightarrow \quad K(s)G(s) = -1 = -1 + j0$$

Clearly, zeros of $1+K(s)\,G(s)$ are the closed-loop system poles.

Let Z be the number of zeros of $1+K(s)\,G(s)$ in the right half plane (i.e., the unstable closed-loop poles), P the number of unstable poles of the open-loop transfer function $K(s)\,G(s)$. Then, the Nyquist plot of $K(s)\,G(s)$ shall encircle the point $-1+j0$ (clockwise) $N=Z-P$ times (or $Z=N+P$).



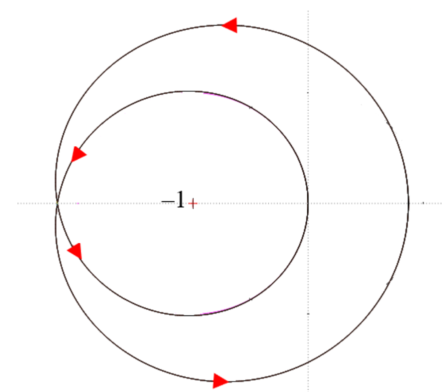
Note: The above result can be utilized to determine the stability of the closed-loop system. It can also be used to determine how far the system is from instability.

Procedure for determining Nyquist stability

1. Plot $KG(s)$ for $-j\infty \leq s \leq +j\infty$. Do this by first evaluating $KG(j\omega)$ for $\omega = 0$ to ω_h, where ω_h is so large that the magnitude of $KG(j\omega)$ is negligibly small for $\omega > \omega_h$, then reflecting the image about the real axis and adding it to the preceding image. The magnitude of $KG(j\omega)$ will be small at high frequencies for any physical system. The Nyquist plot will always be symmetric with respect to the real axis. The plot is normally created by the NYQUIST Matlab m-file.
2. Evaluate the number of clockwise encirclements of -1, and call that number N. Do this by drawing a straight line in any direction from -1 to ∞. Then count the net number of left-to-right crossings of the straight line by $KG(s)$. If encirclements are in the counterclockwise direction, N is negative.
3. Determine the number of unstable (RHP) poles of $G(s)$, and call that number P.
4. Calculate the number of unstable closed-loop roots Z:

$$Z = N + P. \tag{6.28}$$

Gene F. Franklin
Stanford University
1927–2012

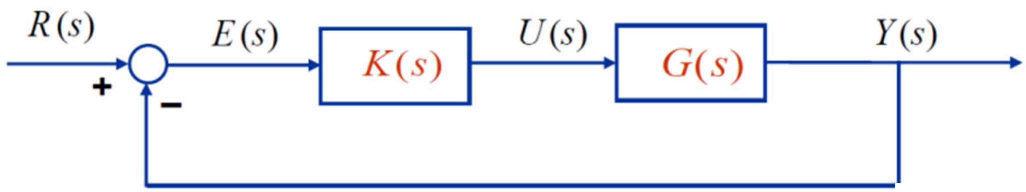


$N = -2$

For the case when the open-loop system is stable ($P = 0$), Thus, the closed-loop is stable iff the Nyquist plot has no encirclement of -1. If the open-loop system has two unstable pole ($P = 2$), then the closed-loop is stable iff $N = -2$, i.e., the Nyquist plot should encircle -1 anti-clockwise twice…

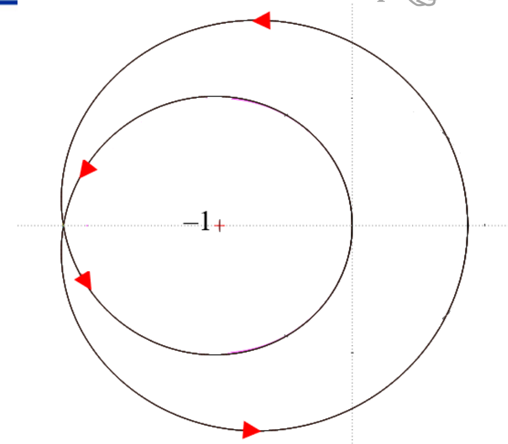
Side note…

First of all, the key idea of Nyquist stability criterion is to use **open-loop transfer function** to determine the **closed-loop stability**. This can be done because under the unity feedback framework



The closed-loop transfer function is given as

$$H(s) = \frac{Y(s)}{R(s)} = \frac{G(s)K(s)}{1+G(s)K(s)}$$



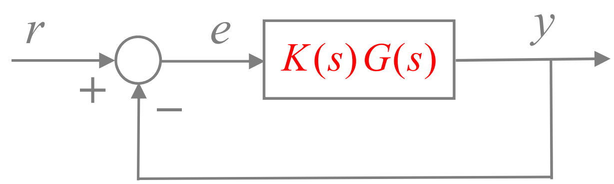
It can be seen from the above expression that the closed-loop stability is determined by the characteristic polynomial of $1 + G(s)K(s) = 0$, which is equivalent to $K(s)G(s) = -1 = -1 + j0$.

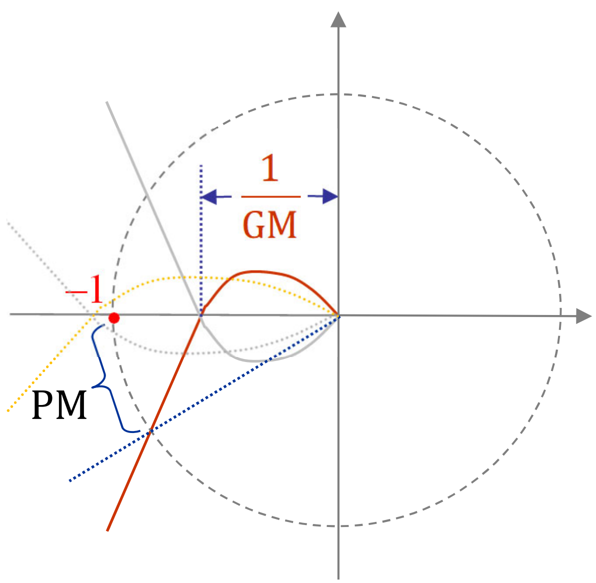
It together with the argument principle in complex analysis give Nyquist stability criterion.

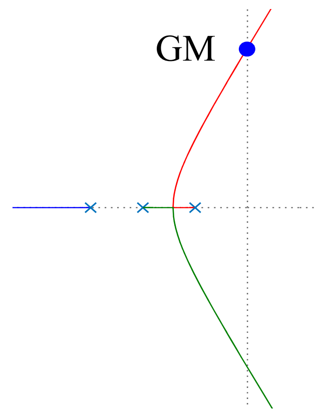
If we have other feedback structure, for example, if there is an additional controller in the feedback loop, you will have to modify the Nyquist stability criterion. The key point is to express a closed-loop characteristic polynomial as $1+T(s)=0$ and then study the open-loop property of $T(s)$.

Gain and phase margins

The gain margin and phase margin can be found from the Nyquist plot by zooming in the region in the neighbourhood of the origin.







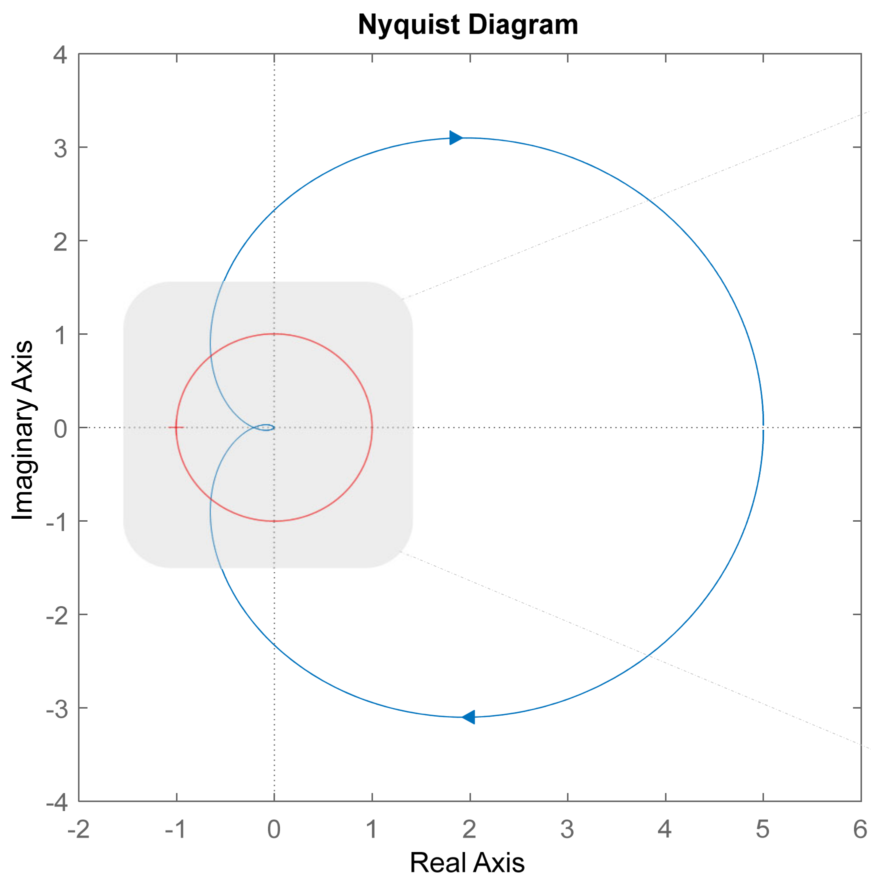
Gain margin is the additional gain that can be tolerated in $KG(s)$ (or gain uncertainties in $G(s)$) such that the resulting closed-loop system would still remain stable. Similarly, phase margin is the additional phase that can be tolerated in $KG(s)$ (or phase uncertainties in $G(s)$, such as input delay) such that the corresponding closed-loop would still be stable.

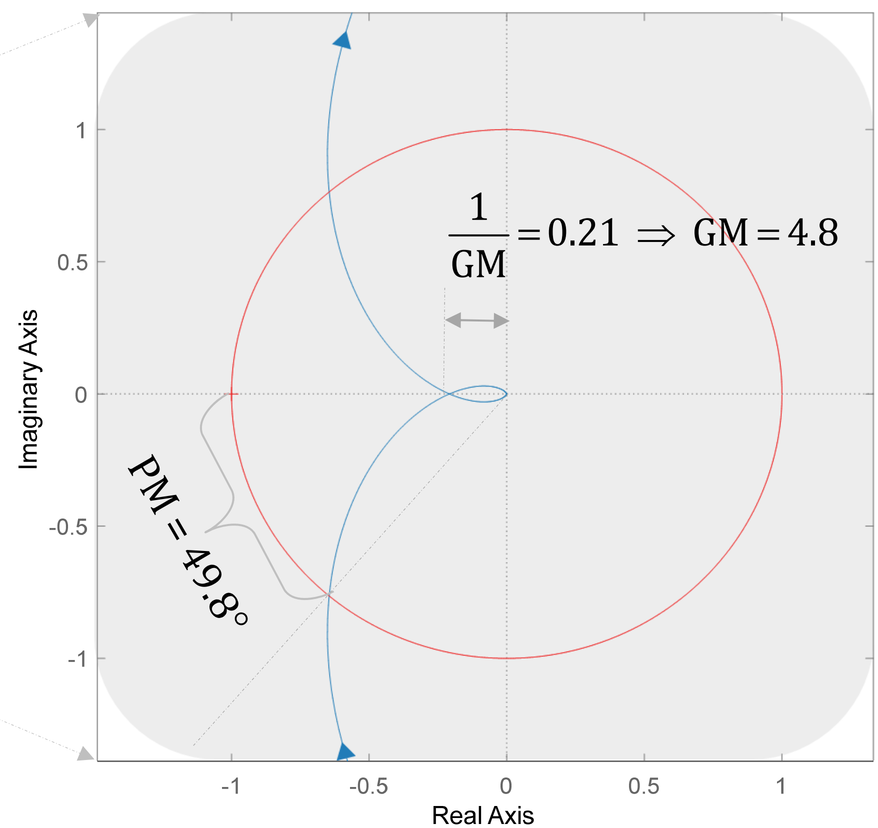
$$\text{PM} = \angle K(j\omega_g)G(j\omega_g) + 180^\circ, \quad \text{where } \omega_g \text{ is such that } \left|K(j\omega_g)G(j\omega_g)\right| = 1$$

$$\text{GM} = \left|K(j\omega_p)G(j\omega_p)\right|^{-1}, \quad \text{where } \omega_p \text{ is such that } \angle K(j\omega_p)G(j\omega_p) = -180^\circ$$

Gain and phase margins in Nyquist plot

Example: $K(s)G(s)=\dfrac{5}{s^3+5s^2+5s+1} \quad\Rightarrow\quad P=0,\ N=0 \quad\Rightarrow\quad Z=P+N=0$



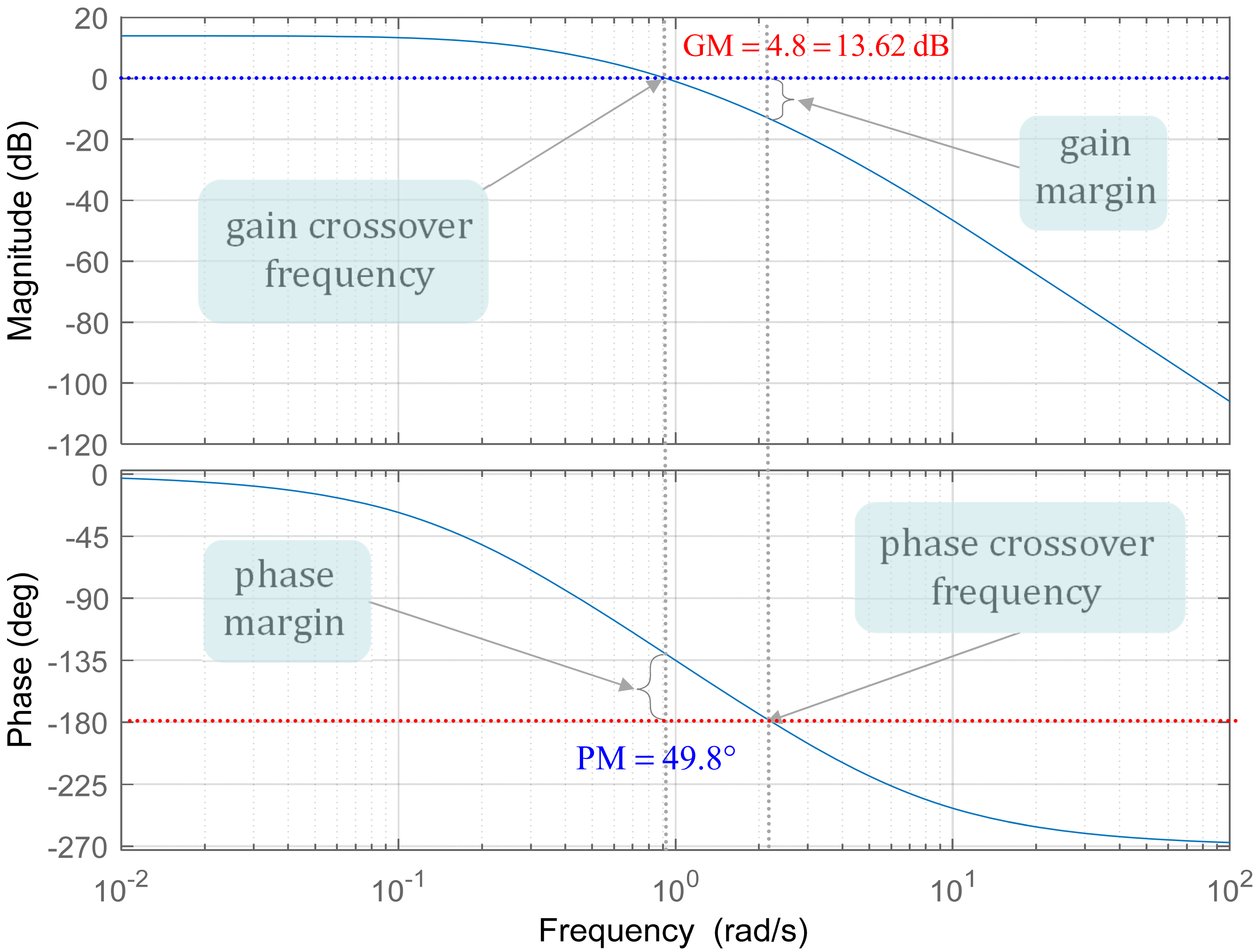


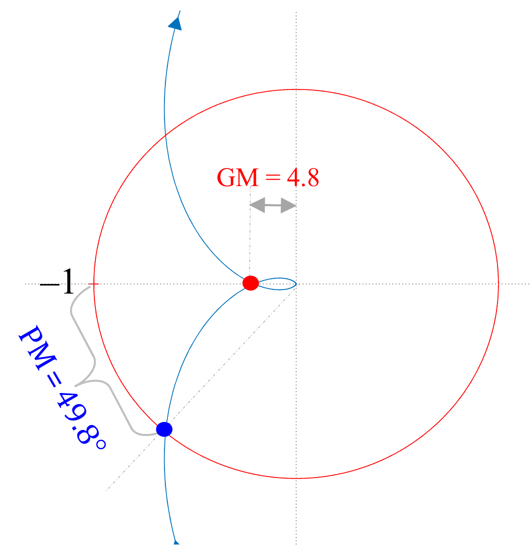
Gain and phase margins in Bode plot

Bode plot

$$K(s)G(s)=\frac{5}{s^3+5s^2+5s+1}$$

$$z=x+iy=re^{j\theta}$$



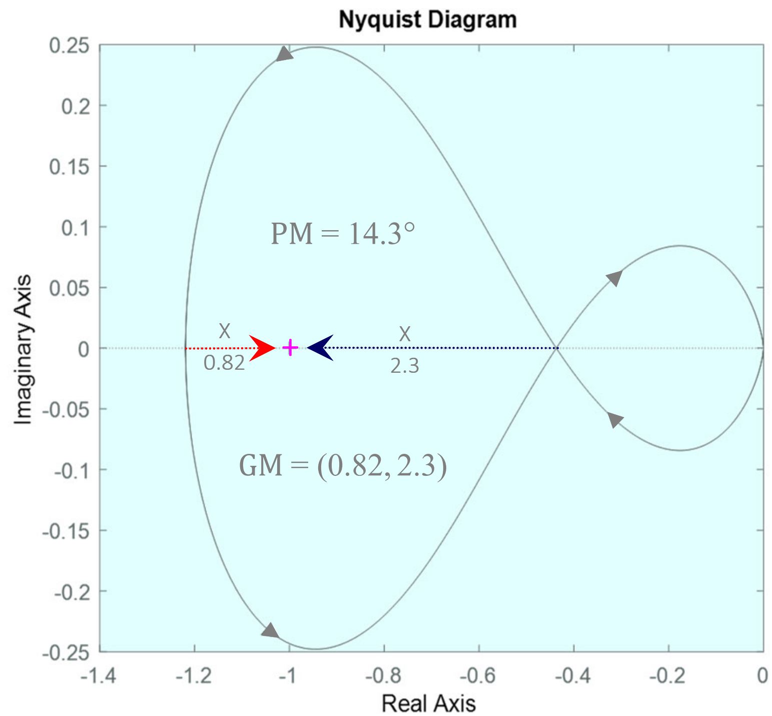


Nyquist plot

margin GMdemo

Gain margins for unstable open-loop systems

When the open-loop system is unstable, its Nyquist plot must encircle -1 point (counter clockwise) to ensure the closed-loop stability. The following is such an example (the open-loop system has one unstable pole)…



- If we increase the open-loop gain by more than 2.3, the right circle will encircle -1 point clockwise instead. By Nyquist stability criterion, the resulting closed-loop system has 2 unstable poles. Thus, we cannot increase the open-loop gain more than 2.3, which the upper gain limit.

- On the other hand, if we decrease the open-loop gain by a factor less than 0.82, there will be no encirclement of -1 point. By Nyquist stability criterion, the resulting closed-loop system one unstable pole. Thus, we cannot decrease the open-loop gain by less than 0.82, which is the lower gain limit.

- The closed-loop system will remain stable so long as the open-loop gain is perturbed within (0.82, 2.3), which is the gain margin for this example.

Sensitivity and complementary sensitivity functions

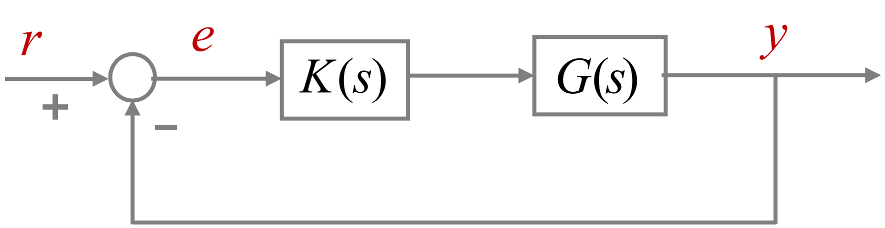
Sensitivity and complementary sensitivity functions are two other measures for a good control system design. The **sensitivity function** is defined as the closed-loop transfer function from the reference signal, r, to the tracking error, e, and is given by

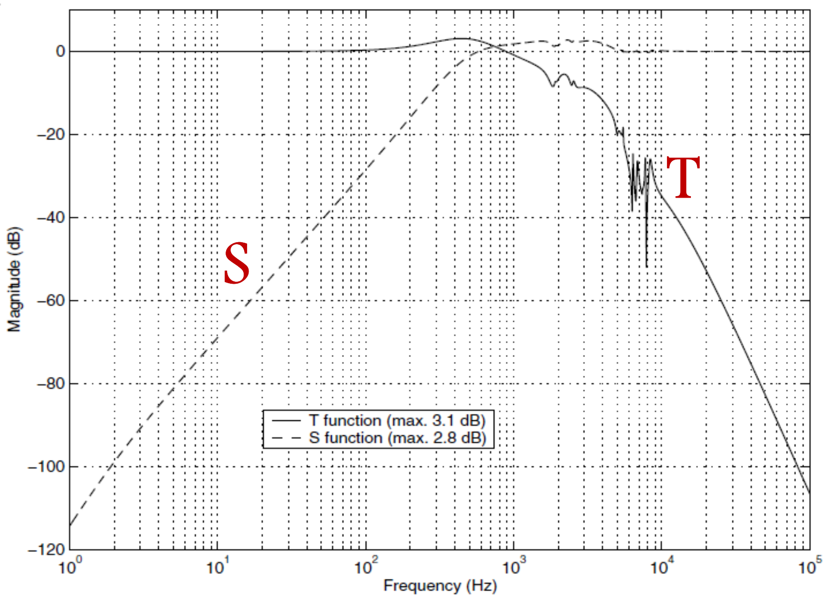
$$S(s) = \frac{1}{1 + K(s)G(s)}$$

The **complementary sensitivity function** is defined as the closed-loop transfer function between the reference, r, and the system output, y, i.e.,

$$T(s) = \frac{K(s)G(s)}{1 + K(s)G(s)}$$

Clearly, we have $S(s) + T(s) \equiv 1$.





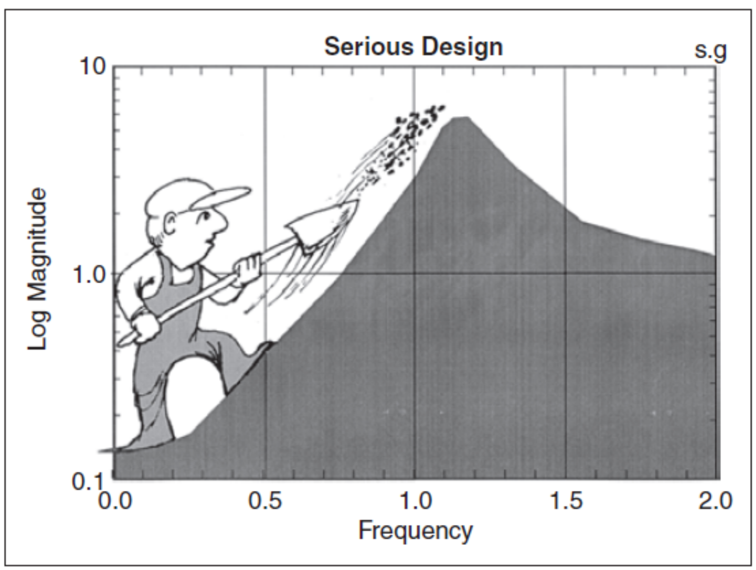
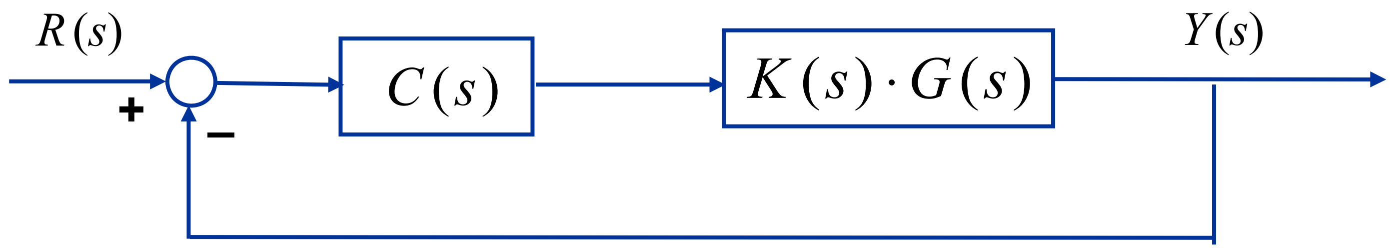


Figure 3. *Sensitivity reduction at low frequency unavoidably leads to sensitivity increase at higher frequencies.*

A good control system design should have a sensitivity function that is small at low frequencies for good tracking performance and disturbance rejection and is equal to unity at high frequencies. On the other hand, the complementary sensitivity function should be made unity at low frequencies. It must roll off at high frequencies to possess good attenuation of high-frequency noise.

Gunter Stein
Honeywell, USA

Now, recall the block diagram of control system:



in which as usual $G(s)$ is the plant to be controlled, $K(s)$ is the controller one has designed to meet the time-domain specifications, and controller (commonly called compensator) $C(s)$ is to be meet the frequency-domain specifications.

The common choices for $C(s)$ are either a lead or a lag compensator:

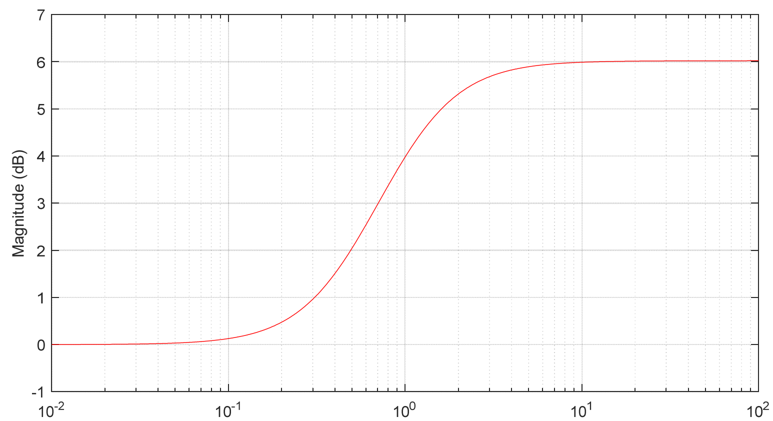
$$C(s) = \frac{Ts+1}{\alpha Ts+1}, \quad 0 < \alpha < 1 \qquad\qquad C(s) = \frac{Ts+1}{\alpha Ts+1}, \quad \alpha > 1$$

LEAD COMPENSATOR $\qquad C(0) \equiv 1 \cdots ? \cdots !$ $\qquad$ LAG COMPENSATOR

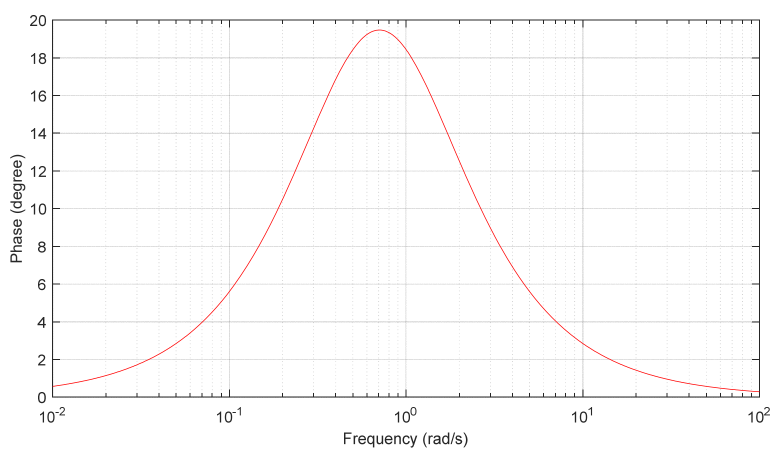
The key idea in designing these lead and lag compensators is rather simple – it tries to shift the frequency response to have desired gain and phase margins…

Frequency responses of lead and lag compensators

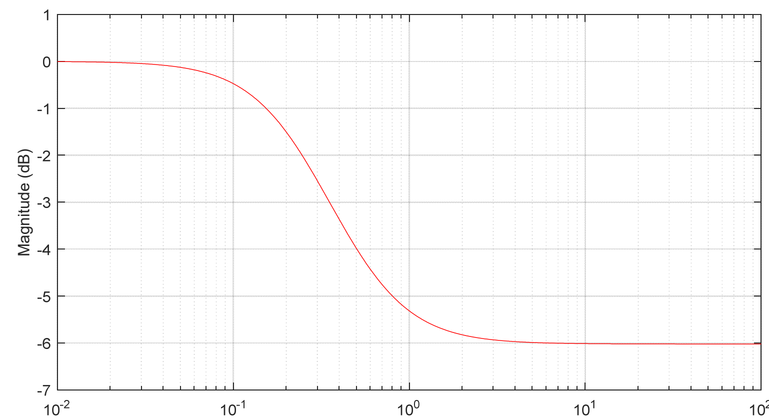
$$C(s)=\frac{Ts+1}{\alpha Ts+1},\quad 0<\alpha<1$$



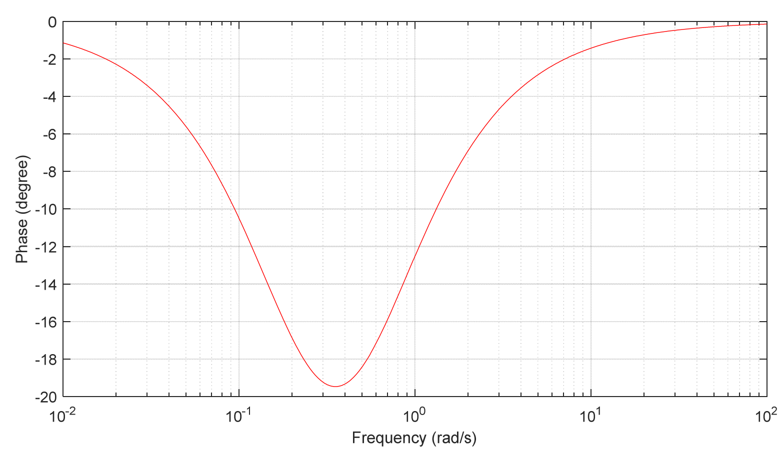
a lead compensator



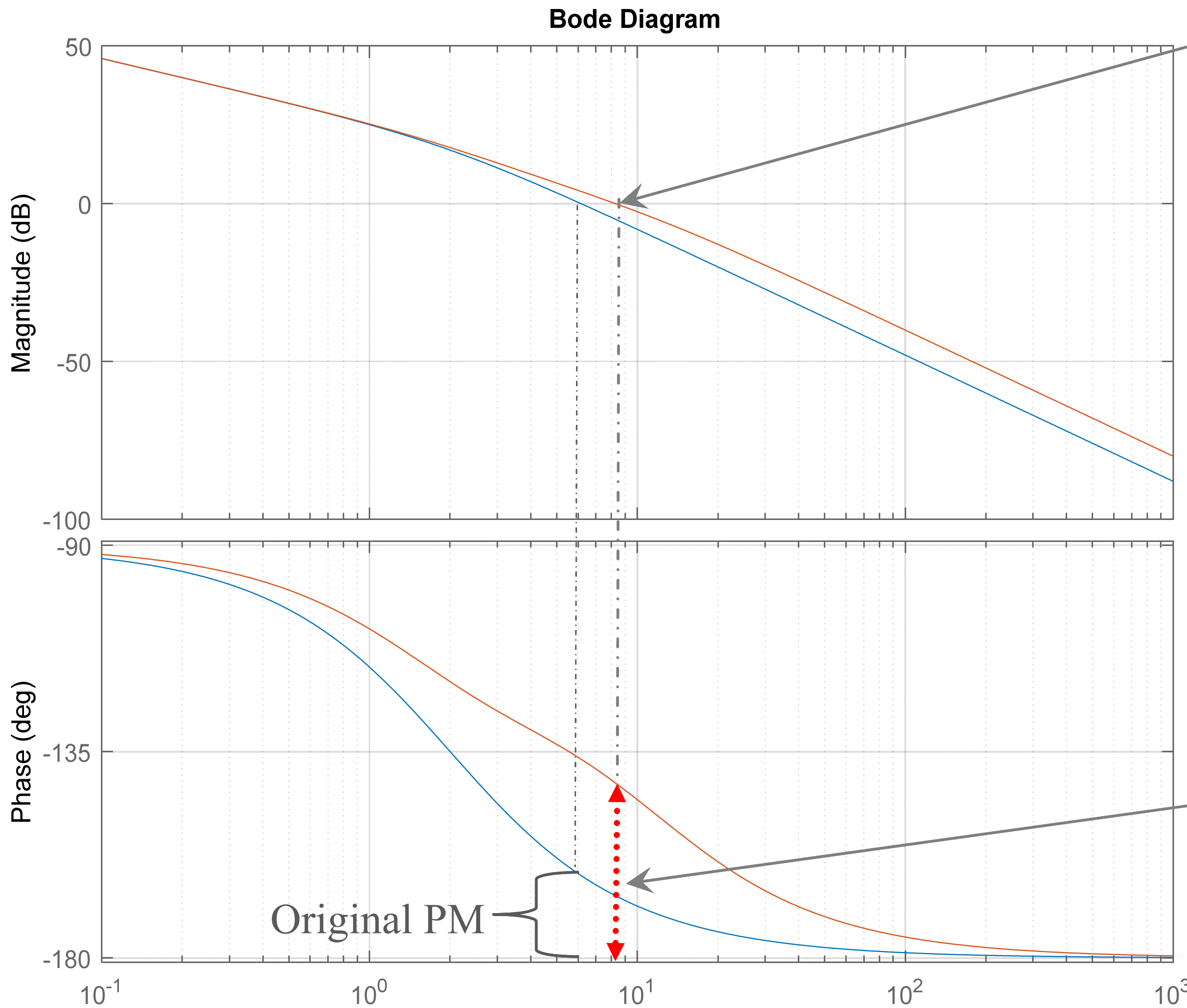
$$C(s)=\frac{Ts+1}{\alpha Ts+1},\quad \alpha>1$$



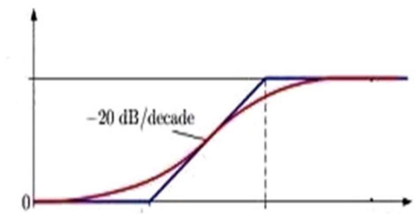
a lag compensator



Idea of adding a lead compensator



Gain cross-over frequency gets shifted after compensation

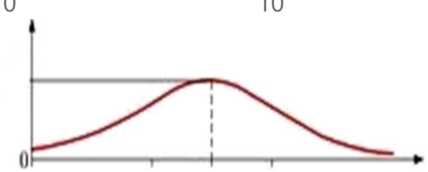


Gain of lead compensator

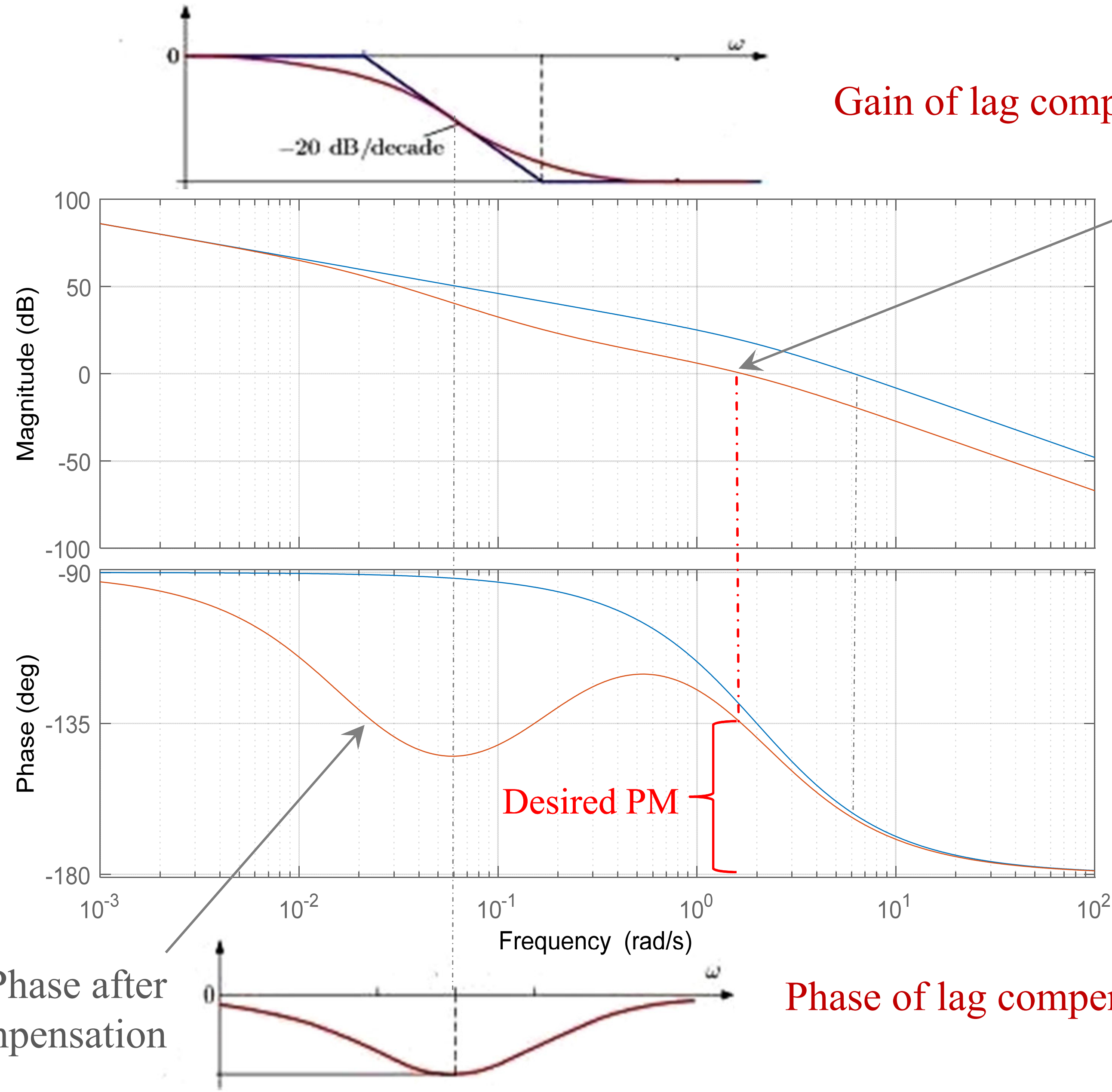
Phase margin after compensation

Generally, it requires many iterations to have a good result…

Phase of lead compensator



Idea of adding a lag compensator



Gain of lag compensator

Gain cross-over frequency shifted to lower frequency after compensation

Again, it requires many iterations to have a good result…

Phase after compensation

Phase of lag compensator

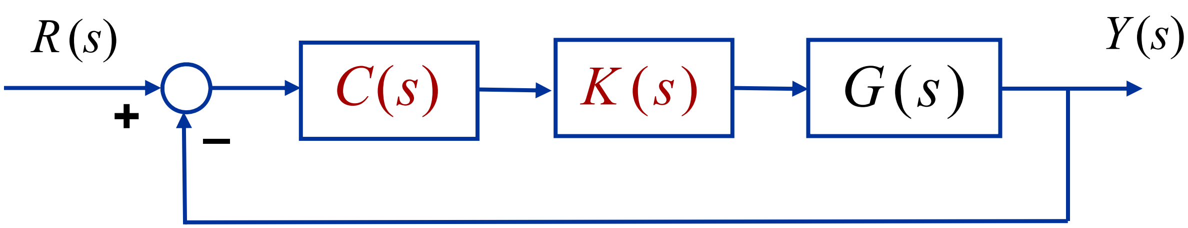
Summary for designing $C(s)$ and the overall controller…

Step 1: Given a plant, $G(s)$, together with a pre-designed controller, $K(s)$, which meets the time-domain design specifications (e.g., overshoot, settling time, etc.), we are to design an appropriate compensator to meet frequency-domain specifications (e.g., required gain and phase margins).

Step 2: Choose an appropriate compensator (either lead or lag compensator), $C(s)$, and work out the required design parameters.

Step 3: Simulate the above design to verify the result. Repeat **Step 2** until a satisfactory result is obtained.

Step 4: Perform simulation for the over design consisting of both $K(s)$ and $C(s)$. Check if all the design goals are achieved…



Step 5: Repeat the design processes of $K(s)$ and $C(s)$ all over again, if necessary ☹

Advantages and drawbacks of classical control

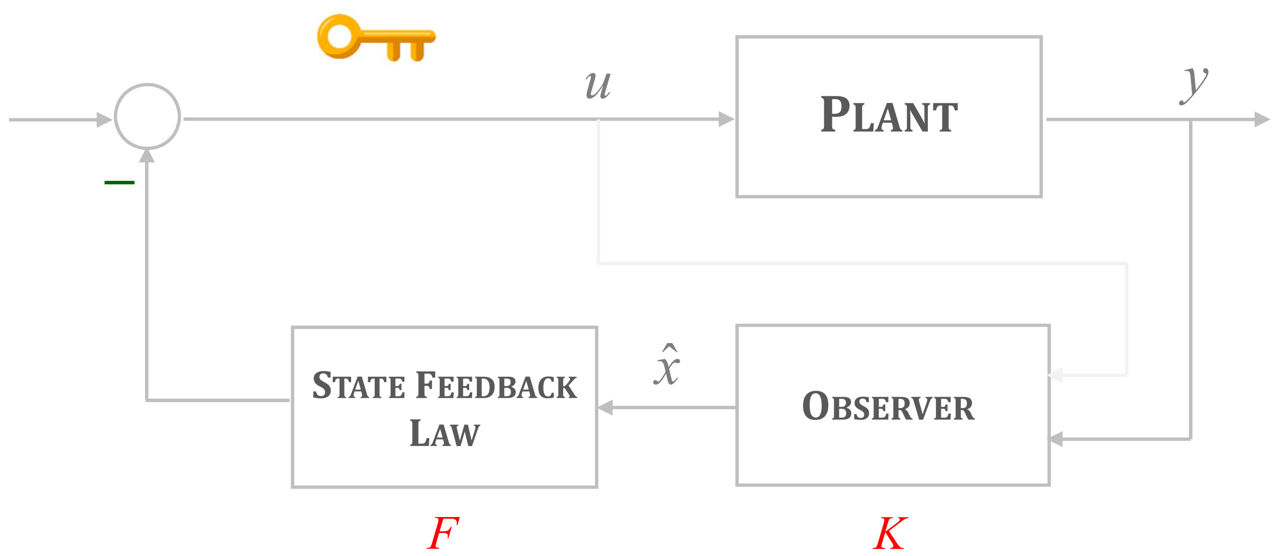
The advantages of the classical or PID control are:

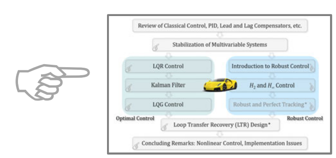
- It is structurally simple and it is easier to tune controller gains.
- It links directly to the time- and frequency-domain specifications.
- It can be applied to plants whose dynamic model is unknown.

The drawbacks are also very obvious:

- All specs are approximately met through out the design process. Many iterations are required.
- It can only be used to control certain classes of SISO plants. For instance, a PID controller cannot even stabilize a triple integrator plant.
- It only takes the error signal e for feedback rather than r and y independently, which limits the overall control performance.
- It is not feasible to control MIMO systems directly. We need to decouple a MIMO system first before utilizing PID control.

Stabilization of Multivariable Systems



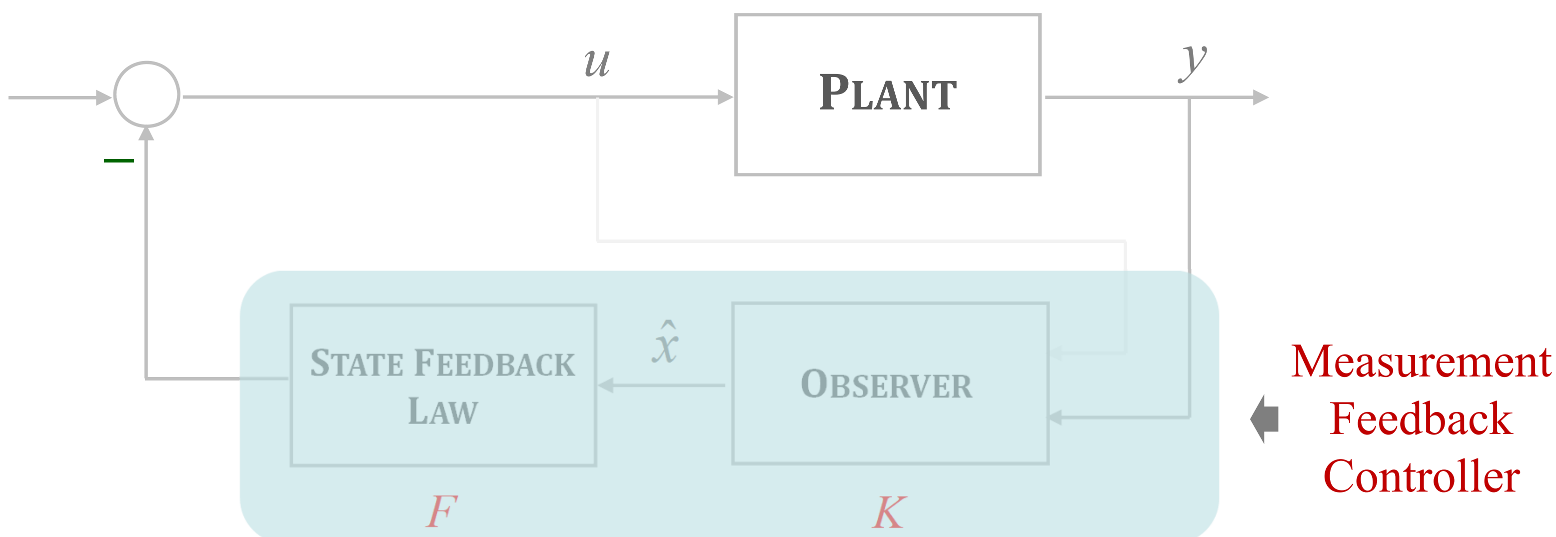


Design philosophy and procedure…

Given a general LTI system (plant) characterized by

$$\begin{cases} \dot{x} = A\,x + B\,u, & x \in \mathbb{R}^n,\ u \in \mathbb{R}^m \\ y = C\,x + D\,u, & y \in \mathbb{R}^p \end{cases}$$

with (A, B) being stabilizable and (A, C) detectable. We would like to design a measurement feedback controller to make it asymptotically stable.



Stability, more specifically the internal stability, is of the utmost importance in control systems design. It is meaningless to discuss control system performance without internal stability.

In this section, we focus on a systematic methodology in designing the necessary controller…

1. Assuming that all the state variables of the system, i.e., x, are available, we design a state feedback control law to stabilize an unstable system provided that it is stabilizable. This step would also allow us to re-locate the closed-loop poles of a controllable pair to any desired locations for a stable given system. More specifically, we can use any technique, such as the well-known pole placement method, to design an state feedback law $u=Fx$ such that when it is applied to

$$\dot{x} = A\,x + B\,u$$

the resulting $A+BF$ has desired poles that are prespecified.

2. Since we only know the measurement information from the system, i.e.,

$$y = Cx + Du$$

we then design an observer or estimator to estimate the state variables of the given plant, i.e., to obtain an $\hat{x} \rightarrow x$, provided that the given system is detectable. That is to design a gain K such that the resulting $A+KC$ is asymptotically stable. This is basically a dual version of the procedure in Item 1.

3. Replace x in Step 1 with $\hat{x}$ in Step 2 to form a so-called observer-based controller for the stabilization of a general multivariable LTI system with measurement feedback.

We will prove later in this section that the control law obtained above works. In fact, such a property is known as the so-called separation principle in the multivariable control theory…

State feedback control law

We first consider a SISO system characterized by

$$\dot{x} = Ax + Bu, \quad x \in \mathbb{R}^n, u \in \mathbb{R}$$

with (A, B) being controllable. It follows from Theorem 4.4.1 (the controllability structural decomposition or the Brunovsky canonical form) of Part 1 that there exist nonsingular state and input transformation T_{s} and T_{i} such that

$$x = T_{\mathrm{s}}\,\tilde{x}, \quad u = T_{\mathrm{i}}\,\tilde{u}$$

and

$$\tilde{A} = T_{\mathrm{s}}^{-1} A T_{\mathrm{s}} = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \ddots & 0 \\ \vdots & \vdots & \ddots & 1 \\ \Delta_1 & \Delta_2 & \cdots & \Delta_n \end{bmatrix}, \quad \tilde{B} = T_{\mathrm{s}}^{-1} B T_{\mathrm{i}} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

where Δ_i, $i = 1, 2, \cdots, n$, are some constants of no interest.

Mathematical background material from Part 1…

We can show that

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix}$$

has a characteristic polynomial of

$$\chi(\lambda) = \det(\lambda I - A) = \begin{vmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ a_3 & a_2 & \lambda + a_1 \end{vmatrix} = \lambda^3 + a_1\lambda^2 + a_2\lambda + a_3$$

Arthur Cayley
1821–1895
British Mathematician

Generally, we can show that

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_2 & -a_1 \end{bmatrix}$$

This result is particularly useful for pole placement… ☞

$$\Rightarrow \quad \chi(\lambda) = \lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n$$

William R. Hamilton
1805–1865
Irish Mathematician

We wish to design a state feedback law $u=Fx$ such that when it is applied to the given system, the resulting closed-loop system has poles are desired locations, say at s_1, s_2, …, s_n, respectively. We write the corresponding closed-loop characteristic equation as

$$\chi(\lambda) = (\lambda - s_1)(\lambda - s_2)\cdots(\lambda - s_n) = \lambda^n + a_1\lambda^{n-1} + \cdots + a_n$$

Then, the required gain matrix is

Replace the rubbish with the desired

$$F = T_\mathrm{i}\, \tilde{F}\, T_\mathrm{s}^{-1} = T_\mathrm{i}\Big(-\begin{bmatrix} \Delta_1 & \Delta_2 & \cdots & \Delta_n \end{bmatrix} - \begin{bmatrix} a_n & a_{n-1} & \cdots & a_1 \end{bmatrix} \Big) T_\mathrm{s}^{-1}$$

Let us examine the closed-loop system matrix $A + BF$,

$$A + BF = T_\mathrm{s}\tilde{A}T_\mathrm{s}^{-1} + T_\mathrm{s}\tilde{B}T_\mathrm{i}^{-1}T_\mathrm{i}\tilde{F}T_\mathrm{s}^{-1} = T_\mathrm{s}(\tilde{A} + \tilde{B}\tilde{F})T_\mathrm{s}^{-1} = T_\mathrm{s}\begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \ddots & 0 \\ \vdots & \vdots & \ddots & 1 \\ -a_n & -a_{n-1} & \cdots & -a_1 \end{bmatrix} T_\mathrm{s}^{-1}$$

which has a characteristic polynomial exactly matched the desired one.

The gain matrix F is uniquely determined for a SISO system if it is controllable!

Example: Consider a SISO system characterized by

$$\dot{x} = A\,x + B\,u = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} x + \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} u$$

Using CSD in the Linear Systems Toolkit, we obtain a set of transformations

$$T_{\mathrm{s}} = \begin{bmatrix} -1 & 0 & 2 \\ 1 & 3 & 1 \\ 1 & 0 & 1 \end{bmatrix}, \quad T_{\mathrm{i}} = 1 \quad \Rightarrow \quad \tilde{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 2 & 1 \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Let the desired closed-loop system poles be placed at $s_1 = -\alpha,\ s_{2,3} = -\sigma \pm j\omega$

$$\Rightarrow \quad \chi(s) = s^3 + (2\sigma + \alpha)s^2 + (\sigma^2 + \omega^2 + 2\alpha\sigma)s + \alpha(\sigma^2 + \omega^2)$$

and the desired state feedback gain is given by

$$F = -\frac{1}{3}\begin{bmatrix} \alpha(\sigma^2 + \omega^2) + 1 & \sigma^2 + \omega^2 + 2\alpha\sigma + 2 & 2\sigma + \alpha + 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 2 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix}$$

Just for fun…

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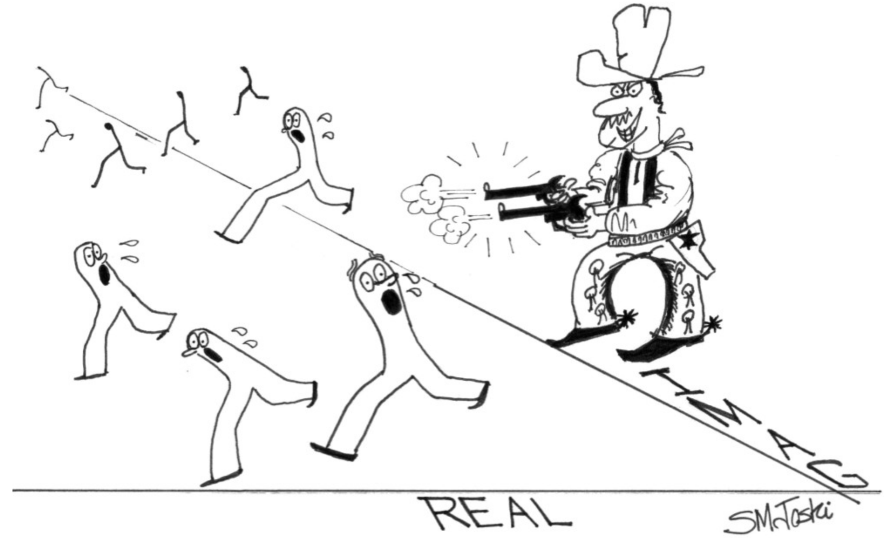
"Maybe I'm wrong- but in this case I wouldn't emphasize positive attitude!"



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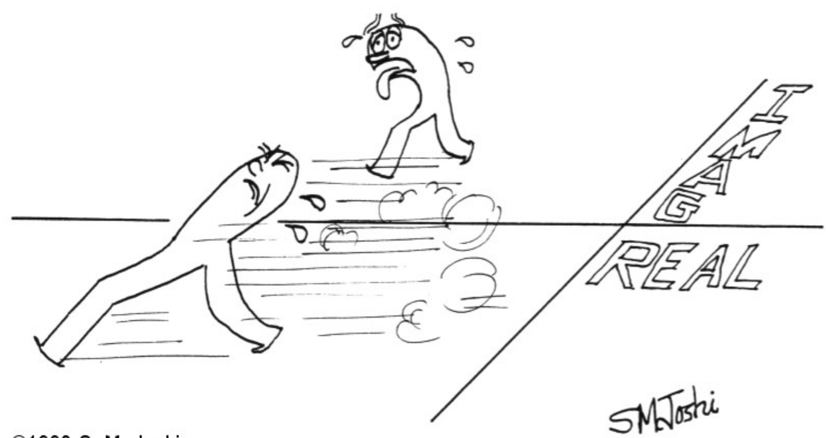
Lariat Logan applies state feedback ~~bang bang~~ control for eigenvalue placement.

Suresh M. Joshi
NASA Langley Research Center

Illustration of system stabilization using the pole placement through a state feedback control law…



"WHOA… WHOA… take it easy, man! Don't you know we're supposed to be conjugates and get there asymptotically?"

©1993 S. M. Joshi

We now consider a MIMO system characterized by

$$\dot{x} = Ax + Bu, \quad x \in \mathbb{R}^n, u \in \mathbb{R}^m$$

with (A, B) being stabilizable. By Theorem 4.4.1 of Part 1 that there exist nonsingular state and input transformation T_s and T_i such that

$$x = T_s \tilde{x}, \quad u = T_i \tilde{u}$$

and the transformed system $\dot{\tilde{x}} = \tilde{A}\tilde{x} + \tilde{B}\tilde{u}$ has the CSD form

uncontrollable modes

$$\tilde{A} = \begin{bmatrix} A_0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & I_{k_1-1} & \cdots & 0 & 0 \\ \star & \star & \star & \cdots & \star & \star \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & I_{k_m-1} \\ \star & \star & \star & \cdots & \star & \star \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} 0 & \cdots & 0 \\ 0 & \cdots & 0 \\ 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \\ 0 & \cdots & 1 \end{bmatrix}, \qquad (4.4.7)$$

controllable pairs

We can apply a pre-feedback again, say $\tilde{u} = \tilde{F}_0 \tilde{x}$, to clean up all the $\star$ terms first, i.e.,

$$\tilde{A} + \tilde{B}\tilde{F}_0 = \begin{bmatrix} A_0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & I_{k_1-1} & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & I_{k_m-1} \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} 0 & \cdots & 0 \\ 0 & \cdots & 0 \\ 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \\ 0 & \cdots & 1 \end{bmatrix}$$

For the uncontrollable modes in A_0, it cannot be changed by any state feedback law. The stabilizability of (A, B) implies A_0 is stable. We leave it as it.

For each controllable pair, we can use the result derived earlier for SISO systems to design an appropriate sub-gain matrix. A desired state feedback control law $u = F x$ can then be obtained by putting all these sub-gain matrices together. We omit the detailed procedure.

Note that one can also use m-function PLACE to obtain a desired gain F.

State observer or estimator

The state feedback control law given in the previous section requires all the state variables of the given system to be available for feedback, which usually not the case in real-life situations. More often, we would face problems in which the information of the system state variables is partially available.

In what follows, we proceed to design a so-called observer or estimator to estimate the state variables of the given system when there is only partial information available.

To be more specific, we consider an LTI system characterized by

$$\dot{x} = A\,x + B\,u, \quad x \in \mathbb{R}^n, \; u \in \mathbb{R}^m$$

$$y = C\,x + D\,u, \quad y \in \mathbb{R}^p, \; p < n$$

with the matrix pair (A, C) being detectable (which is necessary for designing an observer).

Let us try to be a copycat by duplicating the system dynamic equation

$$\dot{\hat{x}} = A\,\hat{x} + Bu$$

We note that u is the input to the system and is known. Define an error signal

$$e := x - \hat{x} \quad \Rightarrow \quad \dot{e} = \dot{x} - \dot{\hat{x}} = Ax + Bu - \left(A\hat{x} + Bu\right) = Ae$$

It follows from (3.2.1) in Part 1 that the solution to the above error dynamic equation is given by

$$e(t) = \mathrm{e}^{At} e_0, \quad e_0 = e(0) = x(0) - \hat{x}(0)$$

If $x(0)$ is known, we can choose the initial condition of the observer dynamics to match it exactly and thus $e_0 = 0$ gives perfect estimation $e(t) = 0$ for all t. Unfortunately, this can never be the case in real life, in which $x(0)$ is generally unknown. On the other hand, if A is a stable matrix,

$$\lim_{t\to\infty} e(t) = \lim_{t\to\infty} \mathrm{e}^{At} e_0 \to 0$$

$\hat{x}(t)$ would give us an asymptotic estimation of $x(t)$ if A is stable!

For the case when the system matrix A is unstable, the copycat estimator on the previous page does not work at all.

Instead, we introduce an output error feedback term to solve the problem, i.e.,

$$\begin{aligned}\dot{\hat{x}} &= A\,\hat{x} + Bu - K(y-\hat{y}) = A\,\hat{x} + Bu - K(y - C\,\hat{x} - Du) \\ &= A\,\hat{x} + Bu - K(C\,x + Du - C\,\hat{x} - Du) \\ &= A\,\hat{x} + Bu - KC(x-\hat{x})\end{aligned}$$

As usual, define the error signal $e(t) := x(t) - \hat{x}(t)$, we have

$$\dot{e} = \dot{x} - \dot{\hat{x}} = A\,x + B\,u - \left(A\,\hat{x} + B\,u\right) + KC(x-\hat{x}) = (A+KC)e$$

and

$$e(t) = \mathrm{e}^{(A+KC)t} e_0, \quad e_0 = x(0) - \hat{x}(0)$$

Obviously, if K is chosen such that $A+KC$ is stable, we would have an asymptotic estimation of $x(t)$, i.e.,

$$\lim_{t\to\infty} e(t) = \lim_{t\to\infty} \mathrm{e}^{(A+KC)t} e_0 \to 0, \quad \text{for any } e_0 = x(0) - \hat{x}(0)$$

The problem now becomes on how to choose K such that $A+KC$ is stable. It can be solved using the observability structure decomposition given in Theorem 4.3.1 of Part 1. Alternatively, we can define an auxiliary system

$$\dot{x} = A_{\text{aux}}\, x + B_{\text{aux}}\, u := A^{\top} x + C^{\top} u$$

It is straightforward to verify that (A, C) detectable implies $(A_{\text{aux}}, B_{\text{aux}})$ stabilizable.

Then, follow the procedure given in the **state feedback control law** section to design an appropriate state feedback gain F_{aux} such that $A_{\text{aux}}+B_{\text{aux}}F_{\text{aux}}$ is stable and has all its eigenvalues in the desired locations. The required observer gain matrix is therefore given as

$$K = (F_{\text{aux}})^{\top}$$

which gives

$$(A+KC)^{\top} = A^{\top} + C^{\top}K^{\top} = A_{\text{aux}} + B_{\text{aux}}F_{\text{aux}}$$

and $\lambda(A+KC) = \lambda(A_{\text{aux}} + B_{\text{aux}}F_{\text{aux}}) \subset \mathbb{C}^{-}$ (all are in the stale locations).

Example: Consider a SISO system characterized by

place
ex2050

$$\dot{x} = A\,x + B\,u = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} x + \begin{bmatrix} 9 \\ 8 \\ 5 \end{bmatrix} u, \qquad x_0 = x(0) = \begin{pmatrix} 3 \\ 6 \\ 9 \end{pmatrix}$$

$$y = C\,x + D\,u = \begin{bmatrix} 2 & 1 & 1 \end{bmatrix} x$$

Using m-function PLACE in MATLAB, we obtain an observer gain matrix

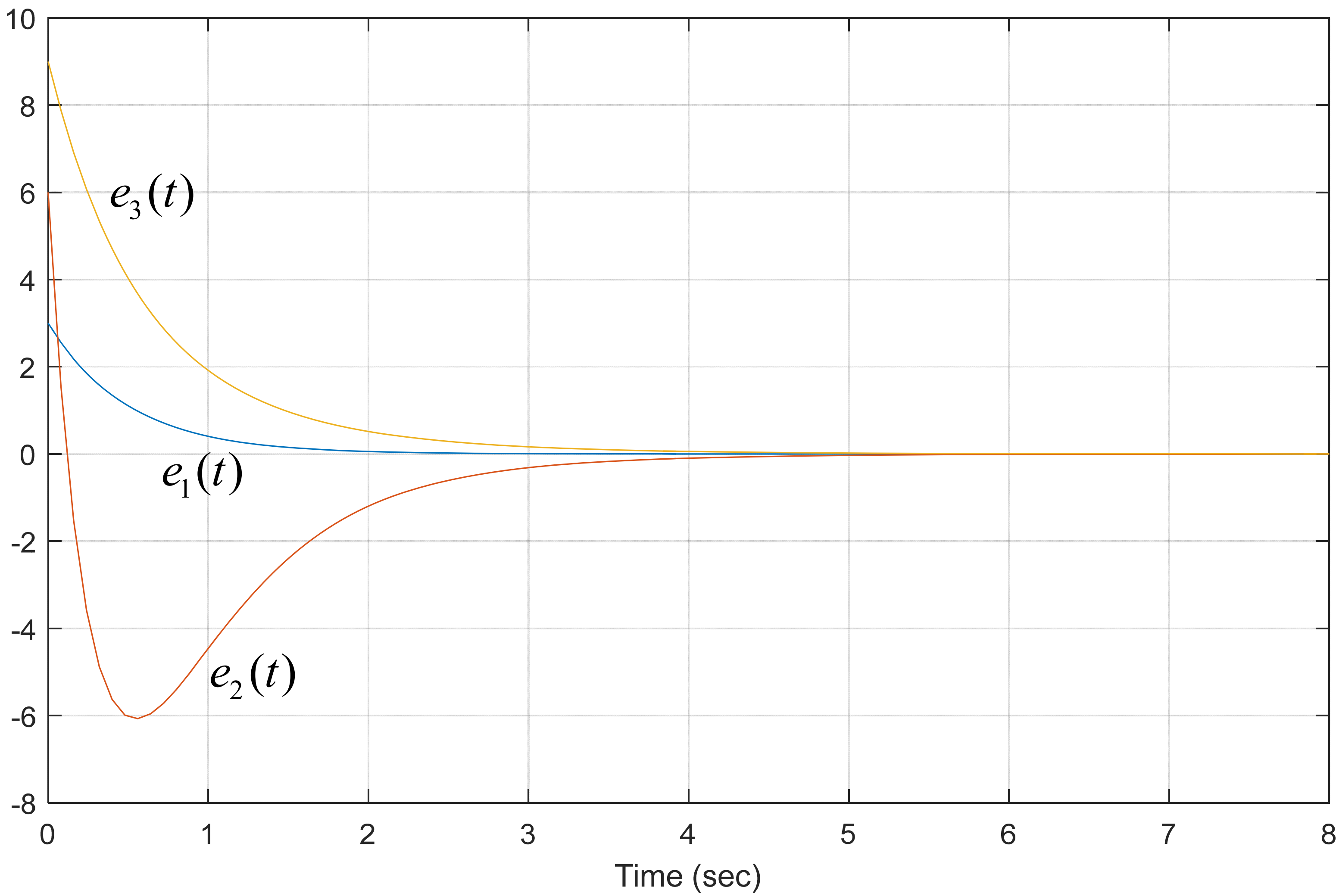
$$K = -\begin{pmatrix} 1 \\ 4 \\ 1 \end{pmatrix}$$

which places the eigenvalues of $A + K\,C$ at –1, –2 and –3, respectively. The simulation result shown on the next page verifies that the observer

$$\dot{\hat{x}} = A\,\hat{x} + B\,u - K(y - C\,\hat{x}) = (A + KC)\hat{x} + B\,u - K\,y, \quad \hat{x}(0) = 0$$

indeed gives an asymptotic estimation of the given plant.

Estimation error signals



Observer-based controller

We now consider the stabilization of a general LTI system with measurement feedback. Consider

$$\begin{cases} \dot{x} = A\,x + B\,u, & x \in \mathbb{R}^n,\ u \in \mathbb{R}^m \\ y = C\,x + D\,u, & y \in \mathbb{R}^p \end{cases}$$

with (A, B) being stabilizable and (A, C) detectable. We would like to design a proper dynamical measurement feedback control law of the following form

$$\begin{cases} \dot{x}_{\text{cmp}} = A_{\text{cmp}}\, x_{\text{cmp}} + B_{\text{cmp}}\, y \\ \quad u \ = C_{\text{cmp}}\, x_{\text{cmp}} + D_{\text{cmp}}\, y \end{cases}$$

such that when it is applied to the given plant, the overall closed-loop system is asymptotically stable.

The design procedure for the above problem turns out to be rather straightforward and systematic.

Step 1. Assume that all the state variables are available for feedback, we design a state feedback control law

$$u = F\,x \qquad (\star)$$

such that $A + BF$ is asymptotically stable.

Step 2. Design an observer

$$\dot{\hat{x}} = A\,\hat{x} + B\,u - K(y - C\,\hat{x} - Du) \qquad (\star\star)$$

such that $A + KC$ is asymptotically stable.

Step 3. Replace x in $(\star)$ with $\hat{x}$ in $(\star\star)$, we obtain a measurement feedback controller

$$\begin{cases} \dot{\hat{x}} = (A + BF + KC + KDF)\,\hat{x} - Ky \\ u = F\,\hat{x} \end{cases}$$

which will do the job for us.

$$\begin{cases} \dot{x}_{\text{cmp}} = A_{\text{cmp}}\,x_{\text{cmp}} + B_{\text{cmp}}\,y \\ u = C_{\text{cmp}}\,x_{\text{cmp}} + D_{\text{cmp}}\,y \end{cases}$$

Why? Let us recall

$$\dot{x} = A\,x + B\,u, \qquad y = C\,x + D\,u$$

and

$$\dot{\hat{x}} = (A + BF + KC + KDF)\,\hat{x} - Ky, \quad u = F\,\hat{x}$$

We have

$$\dot{x} = A\,x + B\,F\,\hat{x},$$

and

$$\begin{aligned}\dot{\hat{x}} &= (A + BF + KC + KDF)\,\hat{x} - K(Cx + Du) \\ &= (A + BF + KC + KDF)\,\hat{x} - K(Cx + DF\,\hat{x}) \\ &= (A + BF + KC)\,\hat{x} - KCx\end{aligned}$$

and the closed-loop dynamic equation

$$\begin{pmatrix} \dot{x} \\ \dot{\hat{x}} \end{pmatrix} = \begin{bmatrix} A & BF \\ -KC & A + BF + KC \end{bmatrix} \begin{pmatrix} x \\ \hat{x} \end{pmatrix}$$

Noting that

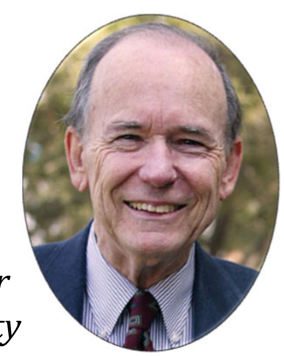
$$\begin{bmatrix} A & BF \\ -KC & A+BF+KC \end{bmatrix} = \begin{bmatrix} I & 0 \\ I & -I \end{bmatrix} \begin{bmatrix} A+BF & -BF \\ 0 & A+KC \end{bmatrix} \begin{bmatrix} I & 0 \\ I & -I \end{bmatrix}^{-1}$$

It is then clear that the closed-loop system is asymptotically stable provided that both $A+BF$ and $A+KC$ are asymptotically stable.

The beauty of this result shows that in order to stabilize the given plant, we can separate it into (i) designing a stabilizing state feedback gain, and (ii) designing a stabilizing observer gain. Such a result is commonly called the **separation principle** in the control literature and has been used heavily in deriving tons of new techniques including what we are to learn in this part. All advanced methods are to design specific F and K to meet specific requirements.

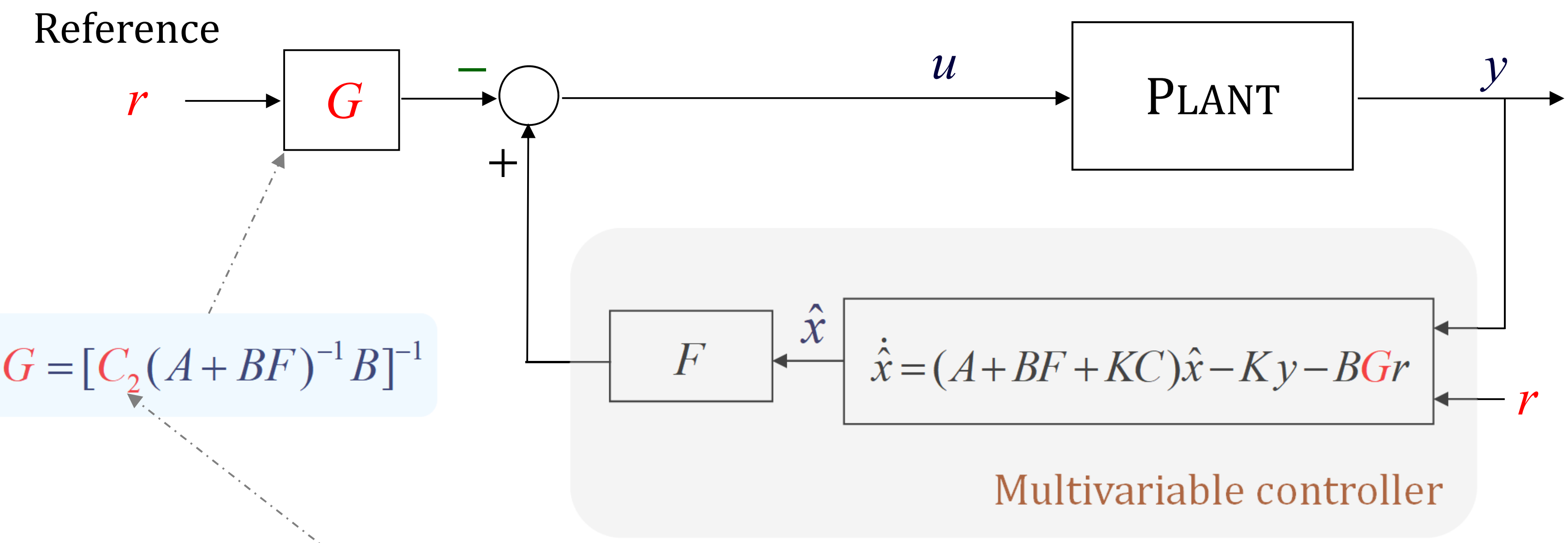
Finally, we should note that there are many types of observers studied in the literature, which include full order and reduced order types. The most general type of observer was given by Luenberger in 1966.


Observers for Multivariable Systems



David Luenberger
Stanford University

Implementation of a multivariable control law with a reference and $D=0$...



$$G = [C_2 (A + BF)^{-1} B]^{-1}$$

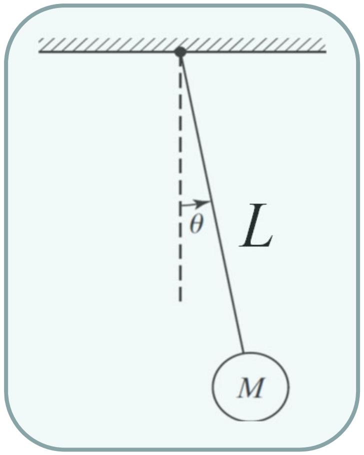
G is chosen such that the DC gain from r to z is unity.

Matrix C_2 is related to output variables of interest, say

$$z = C_2\, x$$

where z is to track the reference r.

Example: Consider the linear model of the pendulum system characterized by



$$\begin{pmatrix}\dot\theta\\ \ddot\theta\end{pmatrix}=\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}\begin{pmatrix}\theta\\ \dot\theta\end{pmatrix}+\begin{bmatrix}0\\ 1\end{bmatrix}u, \quad y=[1 \quad 0]\begin{pmatrix}\theta\\ \dot\theta\end{pmatrix}$$

with it being set at $\theta = 10°$ initially. Design an observer-based compensator to stabilize the system and to regulate the pendulum to $\theta = 0°$.

Step 1. Assume that all the state variables are available for feedback, we design a state feedback control law

$$u = F\,x = [f_1 \quad f_2]\begin{pmatrix}\theta\\ \dot\theta\end{pmatrix}$$

such that $A + BF$ has eigenvalues at $-1 \pm j$, i.e., the desired characteristic polynomial $s^2 + 2s + 2$ and

$$|sI - A - BF| = \begin{vmatrix} s & -1\\ 1-f_1 & s-f_2\end{vmatrix} = s^2 - f_2 s + (1-f_1) \quad \Rightarrow \quad F = -[1 \quad 2]$$

Step 2. Design an observer

$$\dot{\hat{x}} = A\,\hat{x} + Bu - K\left(y - C\,\hat{x}\right) = A\,\hat{x} + Bu - \begin{pmatrix} k_1 \\ k_2 \end{pmatrix}\left(y - C\,\hat{x}\right)$$

such that $A + KC$ has eigenvalues at $-1 \pm j$, i.e., the desired characteristic polynomial $s^2 + 2s + 2$ and

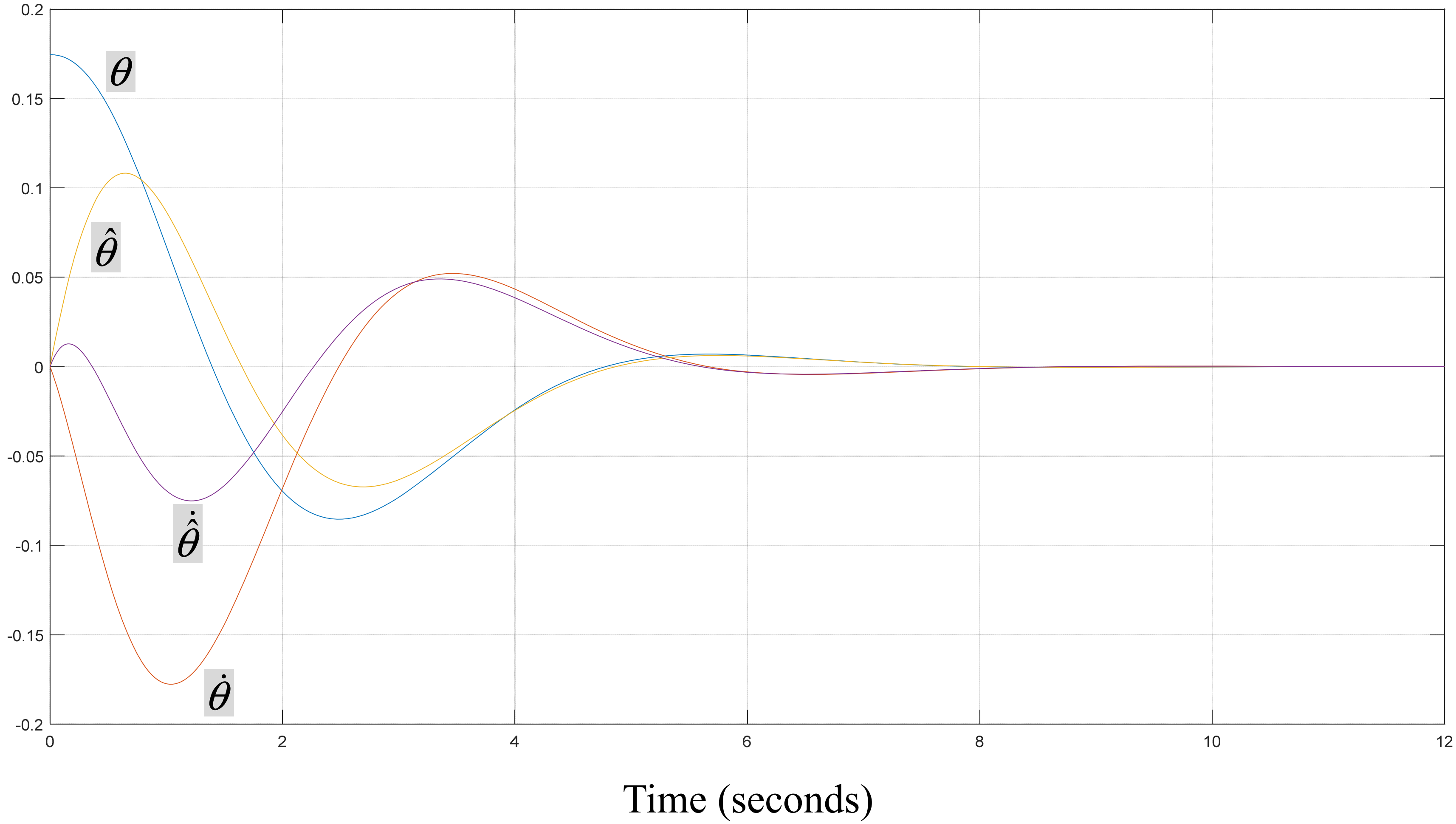
$$\left|sI - A - KC\right| = \begin{vmatrix} s - k_1 & -1 \\ 1 - k_2 & s \end{vmatrix} = s^2 - k_1 s + \left(1 - k_2\right) \quad \Rightarrow \quad K = -\begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Step 3. The measurement feedback controller with the reference $r = 0$ is given as

$$\begin{cases} \begin{pmatrix} \dot{\hat{\theta}} \\ \ddot{\hat{\theta}} \end{pmatrix} = (A + BF + KC + KDF)\begin{pmatrix} \hat{\theta} \\ \dot{\hat{\theta}} \end{pmatrix} - K\theta = \begin{bmatrix} -2 & 1 \\ -3 & -2 \end{bmatrix}\begin{pmatrix} \hat{\theta} \\ \dot{\hat{\theta}} \end{pmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix}\theta \\ u = F\begin{pmatrix} \hat{\theta} \\ \dot{\hat{\theta}} \end{pmatrix} = -\begin{bmatrix} 1 & 2 \end{bmatrix}\begin{pmatrix} \hat{\theta} \\ \dot{\hat{\theta}} \end{pmatrix} \end{cases}$$

$$\begin{cases} \dot{x}_{\mathrm{cmp}} = A_{\mathrm{cmp}}\, x_{\mathrm{cmp}} + B_{\mathrm{cmp}}\, y \\ u = C_{\mathrm{cmp}}\, x_{\mathrm{cmp}} + D_{\mathrm{cmp}}\, y \end{cases}$$

Closed-loop system response…



Exercise: Verify the above result using SIMULINK in MATLAB…

Homework Assignment 4

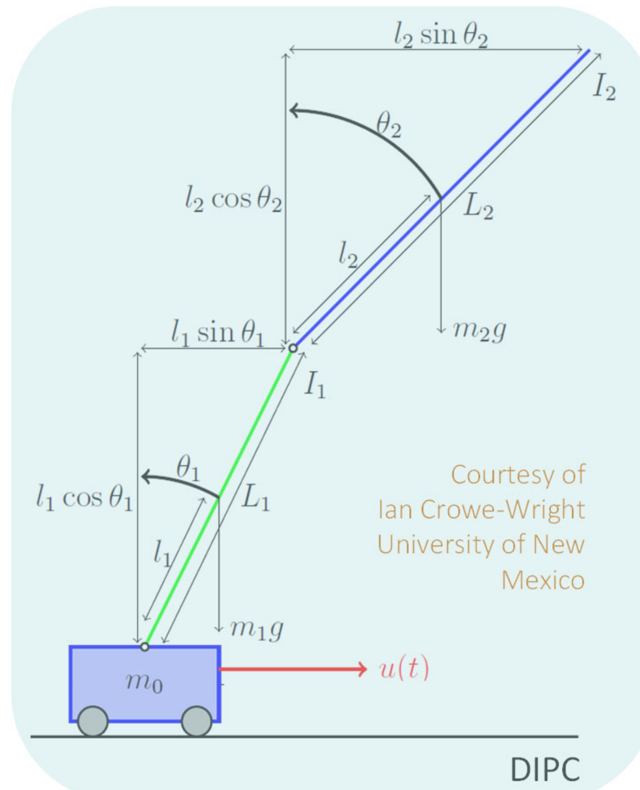
Consider the Double Inverted Pendulum on a Cart (DIPC) in Homework Assignment 2 (Part 1), which is characterized by

$$\dot{x} = A\,x + B\,u, \quad y = C\,x,$$

where

$$x = \begin{pmatrix} x_c \\ \theta_1 \\ \theta_2 \\ v_c \\ \dot{\theta}_1 \\ \dot{\theta}_2 \end{pmatrix}, \quad A = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -7.35 & 0.7875 & 0 & 0 & 0 \\ 0 & 73.5 & -33.075 & 0 & 0 & 0 \\ 0 & -58.8 & 51.1 & 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0.6071 \\ -1.5 \\ 0.2857 \end{bmatrix},$$

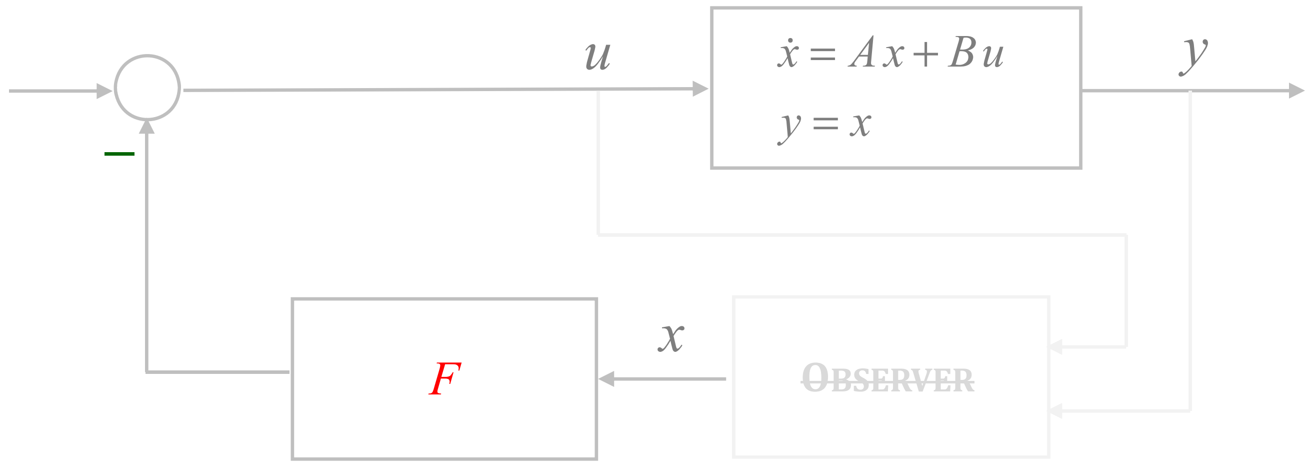
and where x_c and v_c are the displacement and velocity of the Cart, respectively, and $y = x_c$.



1. Design an observer-based controller to stabilize the system and to maintain the position of $\theta_1 = \theta_2 = 0$. Assume that the initial conditions of the state variables: $x_c = 0$, $\theta_1 = \theta_2 = 0.1$ rad, and $\dot{x}_c = 0$, $\dot{\theta}_1 = 0$, $\dot{\theta}_2 = 0$. Simulate your design in SIMULINK with a reference $r = 0$.

2. Comment if it is possible to control the system using a PID controller. Why or why not?

Linear Quadratic Regulator (LQR)



Background

The theory of optimal control is concerned with operating a dynamic system at minimum cost. The case where the system dynamics are described by a set of linear differential equations and the cost is described by a quadratic function is called the LQ problem. One of the main results in the theory is that the solution is provided by the linear quadratic regulator (LQR), a feedback controller utilizing all the information of the system state variables. LQR together Kalman filtering, which is commonly called LQG (linear quadratic Gaussian) form a corner stone in modern control theory.

Like the role of PID in the classical control, LQR (or LQG) plays even a more important role in modern multivariable control although there are tons of new control methods developed in the literature. To tackle a real-life problem, one should first try a PID controller if it is a SISO plant, or LQR control law if it is a MIMO system, before trying anything else.

Linear quadratic regulator (LQR)

Jacopo F. Riccati
Venetian Mathematician
1676–1754

Consider a linear system characterized by

$$\dot{x} = A\,x + B\,u$$

where (A, B) is stabilizable. We define a cost index

$$J(x,u,Q,R) = \int_0^\infty (x^\mathrm{T} Q x + u^\mathrm{T} R u)\,dt, \qquad Q \ge 0,\ \ R > 0$$

and $(A, Q^{1/2})$ is detectable. The linear quadratic regulation problem is to find a control law $u = -F x$ such that $A - BF$ is stable, and J is minimized. The solution is given by $F = R^{-1} B^\mathrm{T} P$, with P being a positive semi-definite solution of the following Riccati equation:

$$PA + A^\mathrm{T} P - PBR^{-1} B^\mathrm{T} P + Q = 0$$

lqr are

Nonetheless, LQR technique is a special way to design a state feedback law.

The derivation of the LQR result is rather involved. It is rooted from the general optimal control problem for a nonlinear time-varying plant characterized by the following dynamical equation:

$$\dot{x} = f(x, u, t)$$

where x is the state vector and u is the control vector, subject to the minimization of the cost function

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$$J(t_0) = \phi(x(T), T) + \int_{t_0}^{T} L(x(t), u(t), t)\,dt$$

with t_0 the initial time and T the final time of interest. The final-state weighting function $\phi(x(T), T)$ and weighting functions $L(x, u, t)$ are selected depending on the performance objectives.

The general optimal control problem is to determine a control input $u(t)$ that minimizes the cost function and also ensures some final state constraint.

The solution to the optimal control problem involves using Lagrange multipliers and the introduction of a costate variable and a Hamiltonian function. The original constrained problem can be reformulated into an optimization problem without constraints. For the LQR problem considered in this course, we are interested in the result for the linear time-invariant system

$$\dot{x} = f(x,u,t) = A\,x + B\,u$$

with a cost function

$$J = \int_0^\infty (x^\mathrm{T} Q x + u^\mathrm{T} R u)\,dt$$

with $Q \geq 0$ and $R > 0$, the optimal solution is

$$u(t) = -F\,x(t), \quad F = R^{-1} B^\mathrm{T} P$$

which gives a minimal cost $J_{\min} = x^\mathrm{T}(0)\,P\,x(0)$, where $P \geq 0$ is a solution of

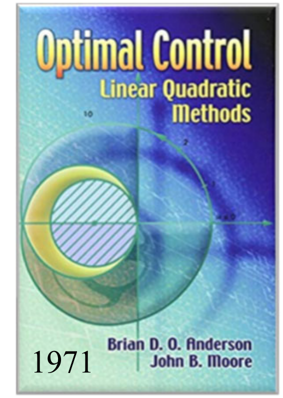
$$PA + A^\mathrm{T} P - PBR^{-1}B^\mathrm{T} P + Q = 0$$

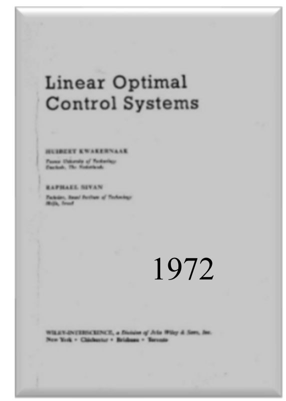
Brian Anderson
Australian National University

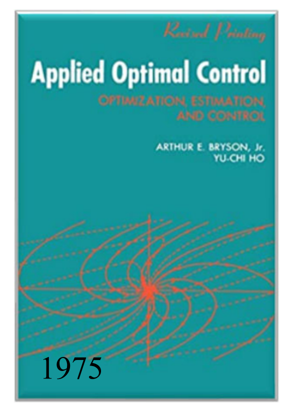
Huibert Kwakernaak
Univ. of Twente, The Netherlands

Arthur Bryson
Stanford University

Larry Y.C. Ho
Harvard University







The detailed derivations of all the optimal control problems can be found in the beautiful textbook by Lewis. In what follows, we should just concentrate on examining the properties of this remarkable LQR control.

We first prove that $A - BF$ is asymptotically stable. Noting that

$$PA + A^{\mathrm{T}} P - PBR^{-1}B^{\mathrm{T}}P + Q = 0$$

$$\Downarrow$$

$$PA - PBR^{-1}B^{\mathrm{T}}P + A^{\mathrm{T}}P - PBR^{-1}B^{\mathrm{T}}P + PBR^{-1}B^{\mathrm{T}}P + Q = 0$$

$$\Downarrow$$

$$P(A - BR^{-1}B^{\mathrm{T}}P) + (A - BR^{-1}B^{\mathrm{T}}P)^{\mathrm{T}}P + PBR^{-1}RR^{-1}B^{\mathrm{T}}P + Q = 0$$

$$\Downarrow$$

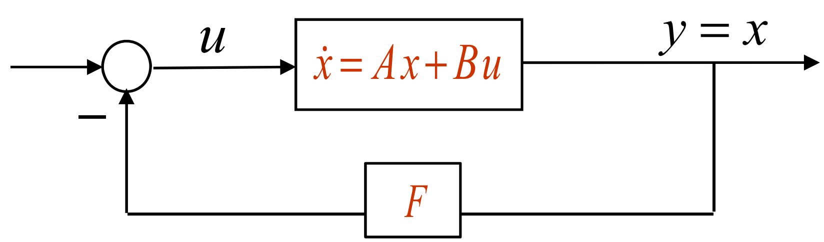
$$P(A - BF) + (A - BF)^{\mathrm{T}}P + F^{\mathrm{T}}RF + Q = 0$$

$$\Downarrow$$

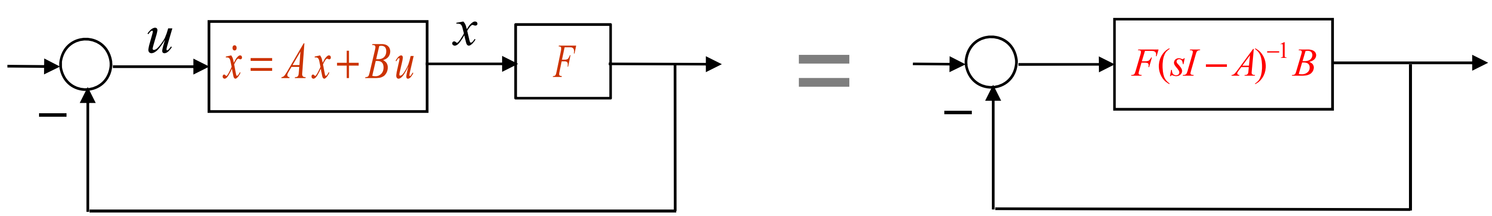
$$P(A - BF) + (A - BF)^{\mathrm{T}}P = -\tilde{Q}, \quad \text{where } \tilde{Q} = Q + F^{\mathrm{T}}RF \geq 0$$

By Lyapunov stability theorem, $A - BF$ is indeed asymptotically stable.

For the LQR design or the state feedback control in general, we assume that $y = x$. For such a case, the state feedback and output feedback are the same thing as the measurement output is the same as the state variable. Thus, we can arrange the state feedback control either as



or as the following to connect to the form linked to the Nyquist stability criterion



As such, the stability of the closed-loop system under the state feedback control law is fully determined by the its open-loop transfer function $T(s) = F(sI - A)^{-1}B$.

Return difference equality and inequality

Consider the LQR control law. The following so-called return difference equality holds:

$$R + B^{\top}(-j\omega I - A^{\top})^{-1} Q (j\omega I - A)^{-1} B = [I + B^{\top}(-j\omega I - A^{\top})^{-1} F^{\top}] R [I + F(-j\omega I - A)^{-1} B]$$

The following is called the return difference inequality:

$$[I + B^{\top}(-j\omega I - A^{\top})^{-1} F^{\top}] R [I + F(j\omega I - A)^{-1} B] \geq R$$

Proof. Recall that

$$F = R^{-1} B^{\top} P \qquad \& \qquad PA + A^{\top} P - PBR^{-1} B^{\top} P + Q = 0$$

Then, we have

$$-Pj\omega I + PA + Pj\omega I + A^{\top} P - \left(PBR^{-1}\right) R \left(R^{-1} B^{\top} P\right) + Q = 0$$

$$\Downarrow$$

$$P(j\omega I - A) + (-j\omega I - A^{\top}) P + F^{\top} R F = Q$$

Multiplying it on the left by $B^{\mathsf{T}}(-j\omega I - A^{\mathsf{T}})^{-1}$ and on the right by $(j\omega I - A)^{-1}B$,

$$B^{\mathsf{T}}(-j\omega I - A^{\mathsf{T}})^{-1}P(j\omega I - A)(j\omega I - A)^{-1}B + B^{\mathsf{T}}(-j\omega I - A^{\mathsf{T}})^{-1}(-j\omega I - A^{\mathsf{T}})P(j\omega I - A)^{-1}B$$
$$+ B^{\mathsf{T}}(-j\omega I - A^{\mathsf{T}})^{-1}F^{\mathsf{T}}RF(j\omega I - A)^{-1}B = B^{\mathsf{T}}(-j\omega I - A^{\mathsf{T}})^{-1}Q(j\omega I - A)^{-1}B$$

$$\Downarrow$$

$$B^{\mathsf{T}}(-j\omega I - A^{\mathsf{T}})^{-1}\underline{PB} + \underline{B^{\mathsf{T}}P}(j\omega I - A)^{-1}B + B^{\mathsf{T}}(-j\omega I - A^{\mathsf{T}})^{-1}F^{\mathsf{T}}RF(j\omega I - A)^{-1}B$$
$$= B^{\mathsf{T}}(-j\omega I - A^{\mathsf{T}})^{-1}Q(j\omega I - A)^{-1}B$$

Noting the fact that

$$\boxed{F = R^{-1}B^{\mathsf{T}}P \quad \Rightarrow \quad B^{\mathsf{T}}P = \underline{RF} \quad \& \quad PB = \underline{F^{\mathsf{T}}R}}$$

we have

$$R + B^{\mathsf{T}}(-j\omega I - A^{\mathsf{T}})^{-1}\underline{F^{\mathsf{T}}R} + \underline{RF}(j\omega I - A)^{-1}B + B^{\mathsf{T}}(-j\omega I - A^{\mathsf{T}})^{-1}F^{\mathsf{T}}RF(j\omega I - A)^{-1}B$$
$$= B^{\mathsf{T}}(-j\omega I - A^{\mathsf{T}})^{-1}Q(j\omega I - A)^{-1}B + R$$

$$\Downarrow$$

$$[I + B^{\mathsf{T}}(-j\omega I - A^{\mathsf{T}})^{-1}F^{\mathsf{T}}]R[I + F(-j\omega I - A)^{-1}B] = R + B^{\mathsf{T}}(-j\omega I - A^{\mathsf{T}})^{-1}Q(j\omega I - A)^{-1}B$$

Single input case

In the single input case, the transfer function

$$\text{Open loop transfer function} = f(sI - A)^{-1} b$$

is a scalar function. Then, the return difference equation is reduced to

$$r + b^{\mathsf{T}}(-j\omega I - A^{\mathsf{T}})^{-1} Q (j\omega I - A)^{-1} b = r[1 + b^{\mathsf{T}}(-j\omega I - A^{\mathsf{T}})^{-1} f^{\mathsf{T}}][1 + f(-j\omega I - A)^{-1} b]$$

$$\Downarrow$$

$$r + \alpha = r \left| 1 + f(j\omega I - A)^{-1} b \right|^2 \quad \text{where } \alpha \ge 0$$

$$\Downarrow$$

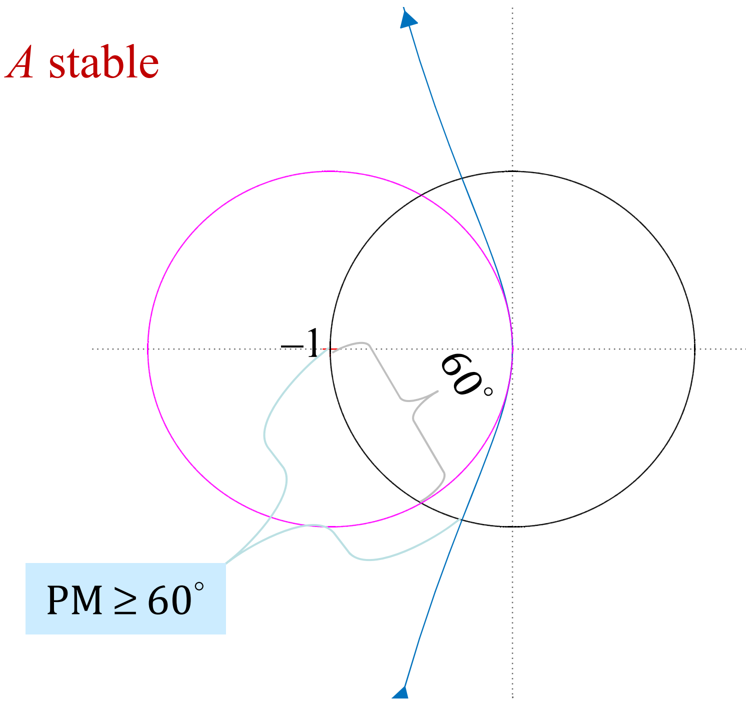
$$r \left| 1 + f(j\omega I - A)^{-1} b \right|^2 \ge r$$

$$\Downarrow$$

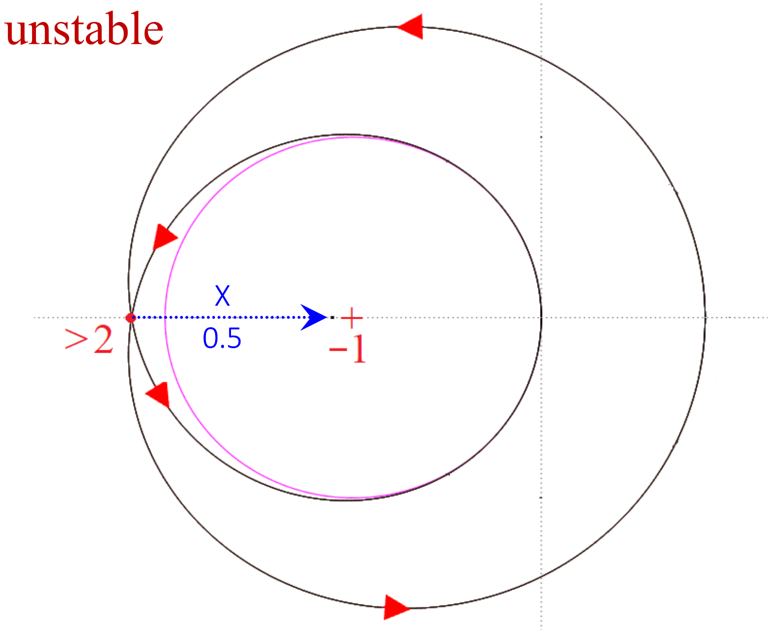
$$\boxed{\left| 1 + f(j\omega I - A)^{-1} b \right|^2 \ge 1 \quad \text{Return Difference Inequality...}}$$

Graphically, $\left|1+f(j\omega I-A)^{-1}b\right|^2 \geq 1 \;\Rightarrow\; \left|f(j\omega I-A)^{-1}b-(-1+j0)\right| \geq 1$ implies

A stable



A unstable



The phase margin resulting from the LQR design is at least 60 deg.

The gain margin is at least from $[0.5, \infty)$.

Let Z be the number of the unstable closed-loop poles, P the number of unstable open-loop poles. Then, the Nyquist plot of the open-loop transfer function shall encircle the point -1 (clock-wise) $N = Z - P$ times (i.e., $Z = N + P$).

* *R.E. Kalman, Contributions to the theory of optimal control, Boletín de la Sociedad Matemática Mexicana, Vol. 5, pp. 102–119, 1960…*

* *R.E. Kalman, When is a linear control system optimal? Journal of Basic Engineering, Trans of ASME, Series D, Vol. 86, pp. 51–60, 1964.*

ex2070

Example: Consider a given plant characterized by

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ a & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

which has a pole at $\frac{1}{2}\left(\sqrt{4a+1}-1\right)$, an unstable one if $a>0$. Solving an LQR problem, which minimizes the following cost function

$$J(x,u,Q,R) = \int_0^\infty (x^\top Q x + u^\top R u) dt, \quad Q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, R = 0.1$$

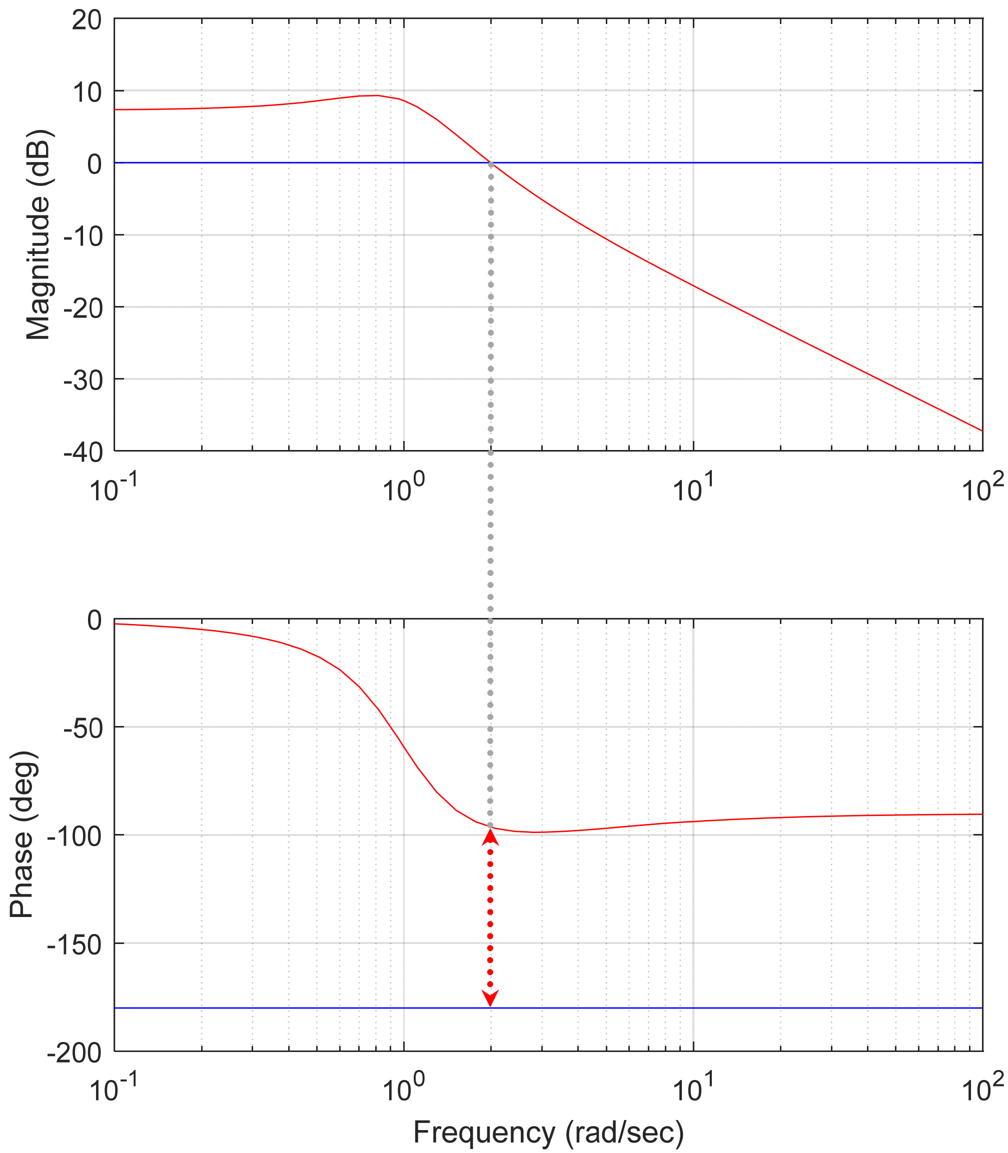
we obtain

$$P = \begin{bmatrix} p_1 & p_0 \\ p_0 & p_2 \end{bmatrix}, \; p_0 = \frac{a+\sqrt{a^2+10}}{10}, \; p_2 = \frac{\sqrt{20 p_0 + 1} - 1}{10}, \; p_1 = 10 p_0 p_2 + p_0 - a \cdot p_2$$

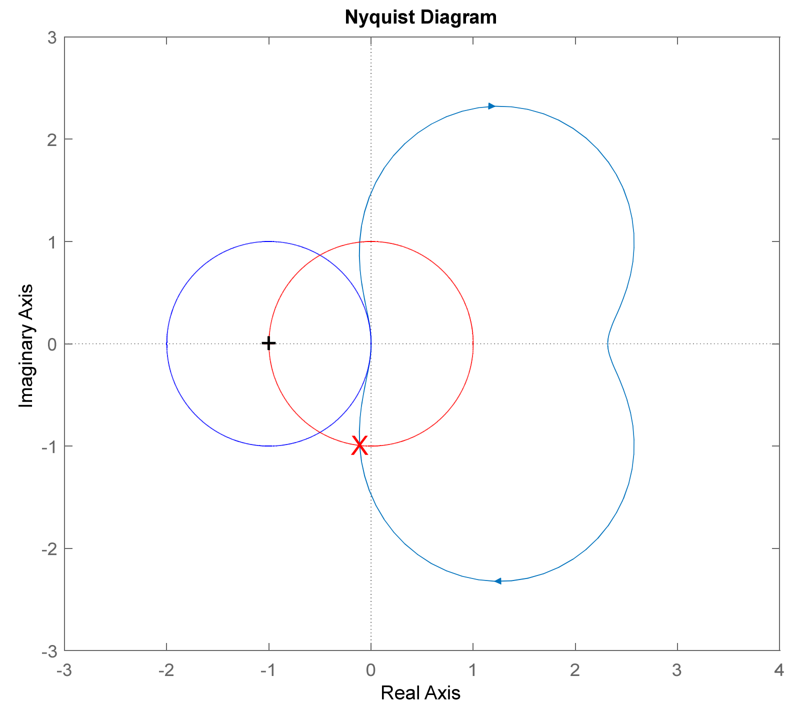
and

$$F = \begin{bmatrix} a+\sqrt{a^2+10} & \sqrt{2\left(a+\sqrt{a^2+10}\right)+1} - 1 \end{bmatrix}$$

- *For* $a = -1 \ldots$

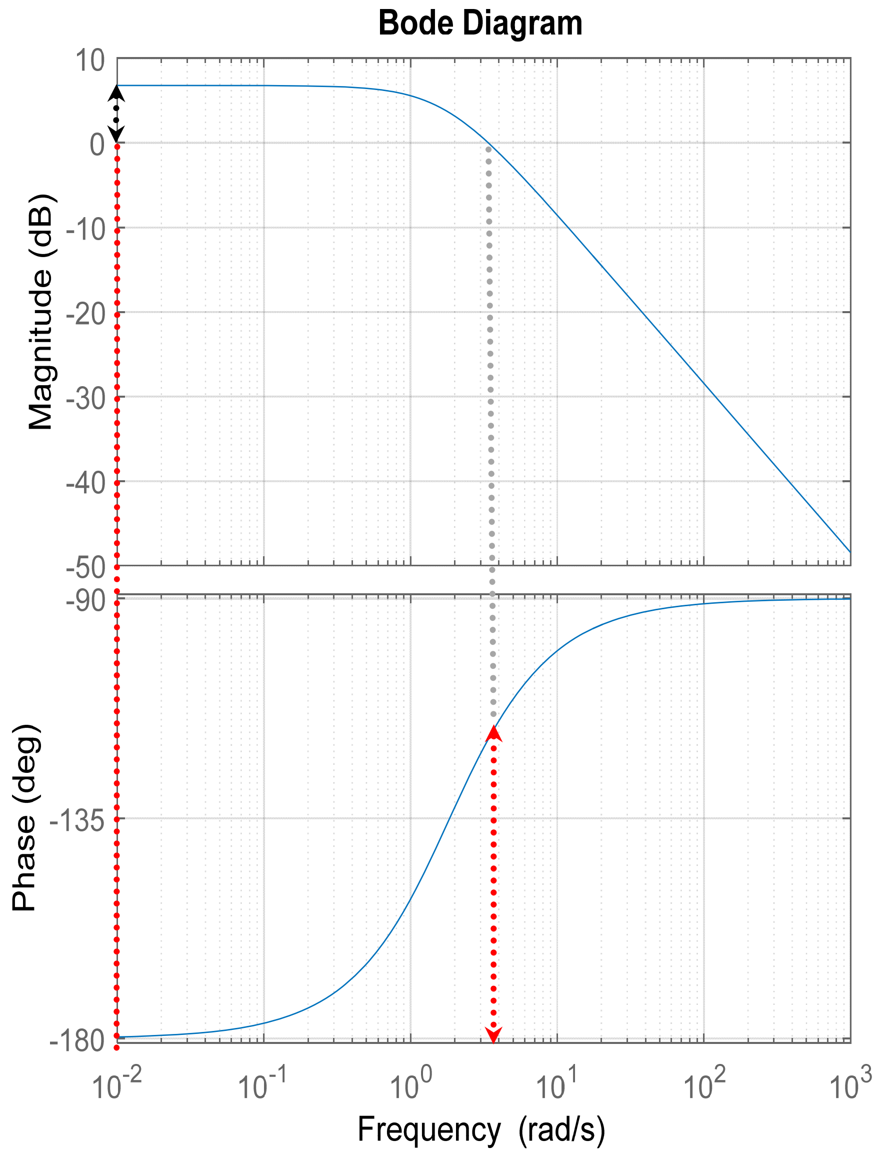


GM = ∞

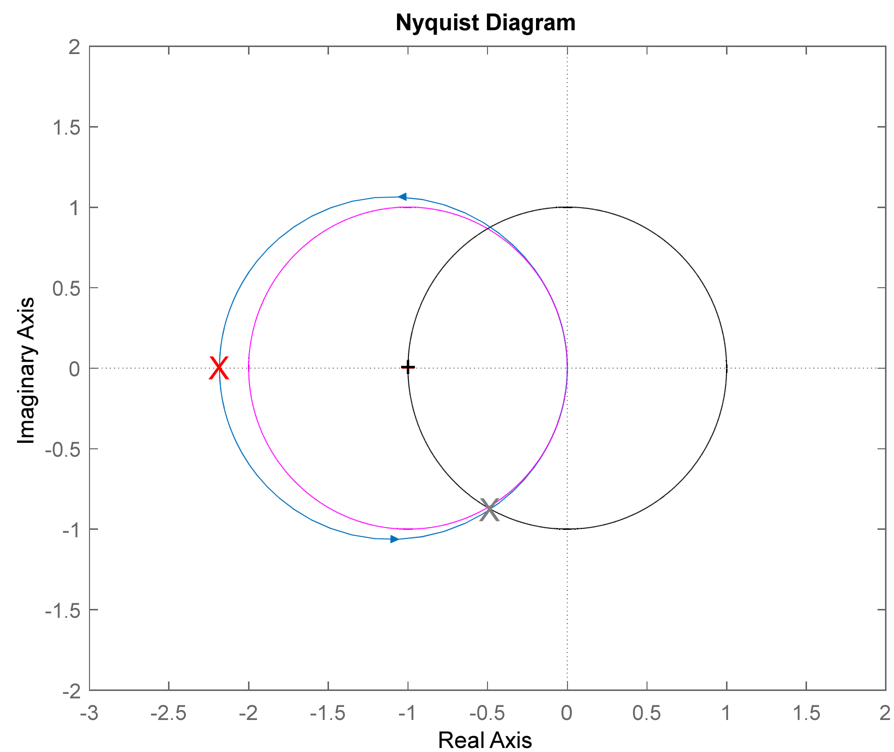


PM = 83.6°

- *For* $a = 5$…



$\text{GM} = (0.458, \infty)$



$\text{PM} = 61.2^\circ$

How to select *Q* and *R* in LQR design?

There is no universal rule in selecting the weighting matrices Q and R in the LQR design. In practice, one might try diagonal matrices first, i.e.,

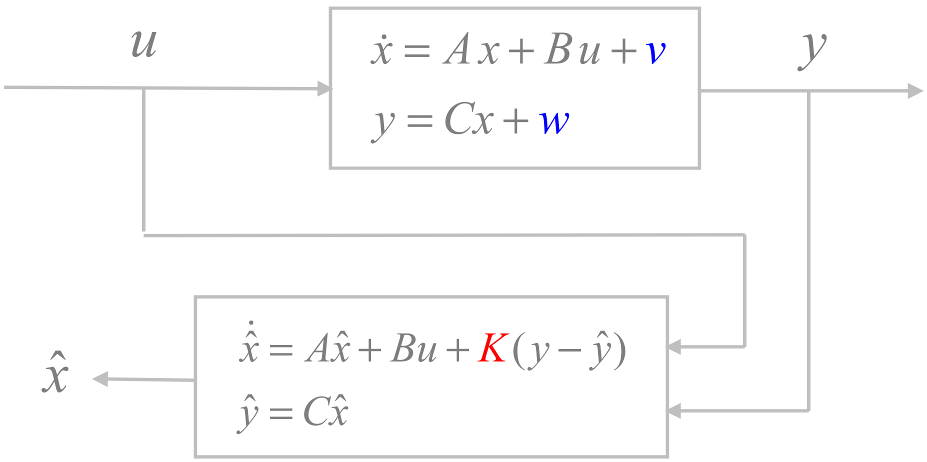
$$Q = \begin{bmatrix} q_{11} & & & \\ & q_{22} & & \\ & & \ddots & \\ & & & q_{nn} \end{bmatrix}, \quad R = \begin{bmatrix} r_{11} & & & \\ & r_{22} & & \\ & & \ddots & \\ & & & r_{mm} \end{bmatrix}$$

Then, the cost function can be written as

$$J(x,u,Q,R) = \int_0^\infty \left(x^\top Q x + u^\top R u \right) dt = \int_0^\infty \left(\sum_{i=1}^{n} q_{ii} x_i^2 + \sum_{i=1}^{m} r_{ii} u_i^2 \right) dt$$

We can then proceed to select the entries of Q and R in accordance with the properties of their associated state and input variables.

What is the shortfall with the LQR design?



Rudolf Kalman (1930–2016) & Richard Bucy (1935–2019)

Kalman Filter

The problem of Kalman filter (or Kalman–Bucy filter) is a special way to design an observer gain matrix K for a state estimator. What we have covered in this section is more related to the work done by Bucy and his co-workers…

* R. S. Bucy & P. D. Joseph (1968). *Filtering for Stochastic Processes with Applications to Guidance.* Interscience: New York.

Kalman–Bucy Filter?

Rudolf E. Kalman

Rudolf E. Kalman
1930–2016
Hungarian American

His passing not only brought about personal loss but also a sad reminder of the passing of a golden era in systems and control.

— Larry Y.C. Ho

Rudolf Emil Kalman was a Hungarian-born American electrical engineer, mathematician, and inventor. He earned both his bachelor's and master's degrees in electrical engineering from MIT, and completed his PhD at Columbia University. He was most noted for his co-invention and development of the Kalman filter, a mathematical algorithm that is widely used in signal processing, guidance, navigation and control systems. For this work, U.S. President Barack Obama awarded Kalman the National Medal of Science, 2009.

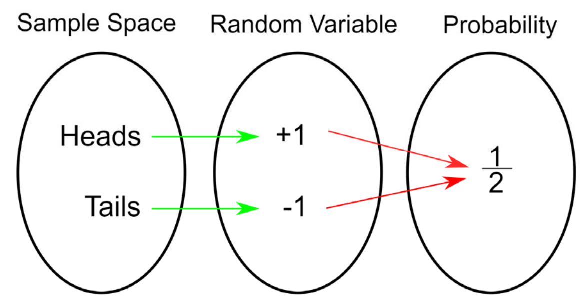
Kalman was a member of the U.S. National Academy of Sciences, the American National Academy of Engineering, and the American Academy of Arts and Sciences. He was a foreign member of the Hungarian, French, and Russian Academies of Science. In 2012 he became a Fellow of the American Mathematical Society.

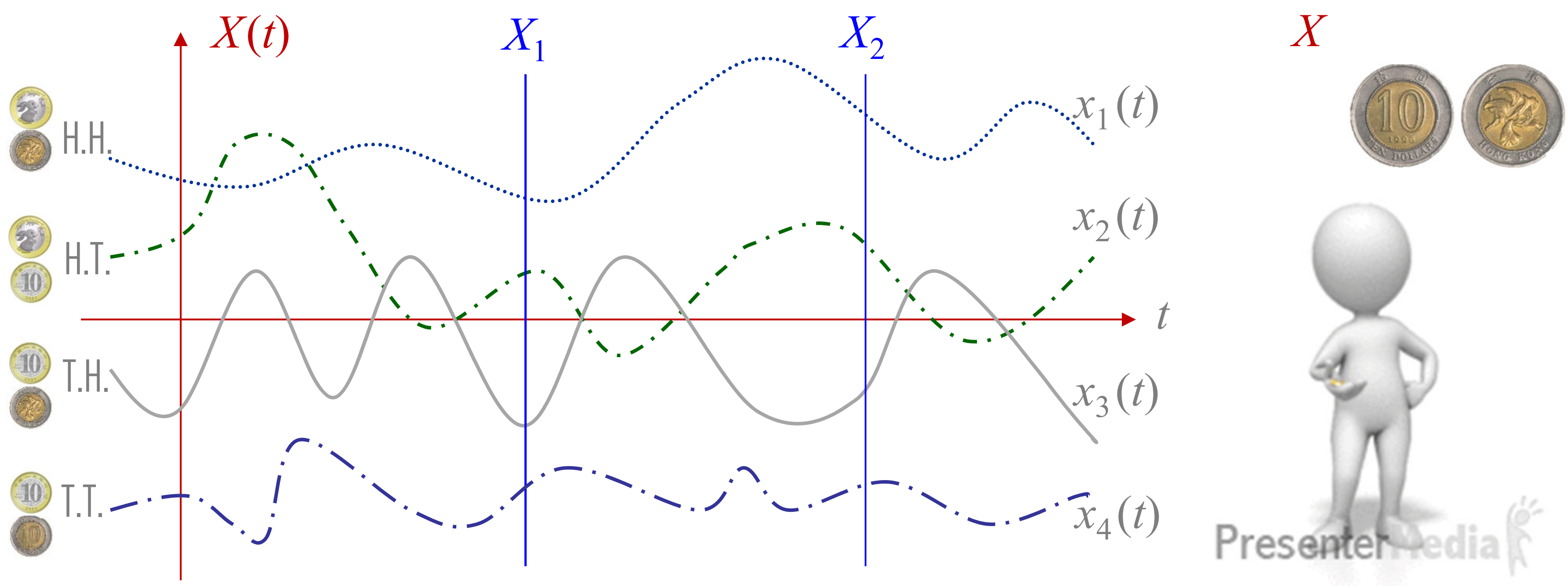
Kalman received the IEEE Medal of Honor in 1974, the IEEE Centennial Medal in 1984, Inamori Foundation's Kyoto Prize in Advanced Technology in 1985, Steele Prize of the American Mathematical Society in 1987, Richard E. Bellman Control Heritage Award in 1997, and National Academy of Engineering's Charles Stark Draper Prize in 2008.

★ R. E. Kalman, Y. C. Ho and K. S. Narendra, "Controllability of linear dynamical systems," *Contributions to Differential Equations*, vol. 1, no. 2, pp. 189–213, 1963.

Review of random processes

A **random variable** X is a mapping between the sample space and the real numbers. A **random process** (a.k.a **stochastic process**) is a mapping from the sample space into an ensemble of time functions (known as sample functions). To every member in the sample space, there corresponds a function of time (a sample function) $X(t)$.





Mean, moment, variance, covariance of random processes

Let $f(x,t)$ be the **probability density function** (p.d.f.) associated with a random process $X(t)$. If the p.d.f. is independent of time t, i.e., $f(x,t) = f(x)$, then the corresponding random process is said to be **stationary**. We will focus our attention only on this class of random processes in this course. For this type of random processes (RP), we define:

1) **mean** (or **expectation**):

$$m = E[X] = \int_{-\infty}^{\infty} x \cdot f(x) dx$$

2) **moment** (j-th order moment)

$$E\left[X^j\right] = \int_{-\infty}^{\infty} x^j \cdot f(x) dx$$

3) **variance**

$$\sigma^2 = E\left[(x-m)^2\right] = \int_{-\infty}^{\infty} (x-m)^2 f(x) dx$$

4) **covariance of two random processes**

$$\mathrm{con}(v,w) = E\left[(v - E[v])(w - E[w])\right]$$

Two RPs v and w are said to be **independent** if their joint p.d.f. $f(v,w) = g(v) \cdot h(w)$

$$\Rightarrow \quad E[vw] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} vw \cdot f(v,w) dv dw = \int_{-\infty}^{\infty} v \cdot g(v) dv \int_{-\infty}^{\infty} w \cdot h(w) dw = E[v]E[w]$$

Autocorrelation function and power spectrum

Autocorrelation function is used to describe the time domain property of a random process. Given a random process *v*, its **autocorrelation function** is defined as follows:

$$R_x(t_1,t_2) = E\left[v(t_1)v(t_2)\right]$$

If *v* is a **wide sense stationary** (WSS) process,

$$R_x(t_1,t_2) = R_x(t_2-t_1) = R_x(\tau) = R_x(t,t+\tau) = E\left[v(t)v(t+\tau)\right]$$

Note that $R_x(0)$ is the time average of the power or energy of the random process.

Power spectrum of a random process is the Fourier transform of its autocorrelation function. It is a frequency domain property of the random process. To be more specific, it is defined as

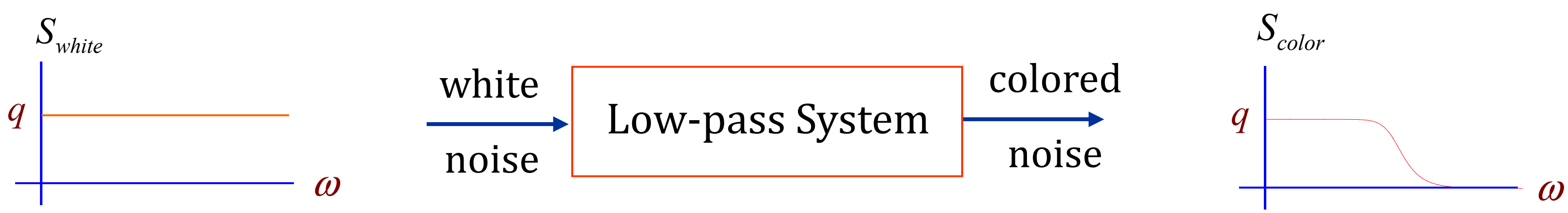
$$\boxed{S_x(\omega) = \int_{-\infty}^{\infty} R_x(\tau)\, e^{-j\omega\tau}\, d\tau}$$

White noise, colored noise and Gaussian random process

White Noise is a random process with a constant power spectrum, and an autocorrelation function

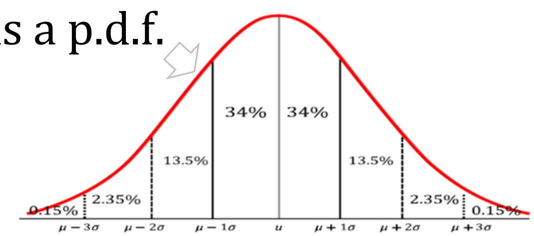
$$R_x(\tau) = q \cdot \delta(\tau) \quad \Rightarrow \quad S_x(\omega) = \int_{-\infty}^{\infty} R_x(\tau) e^{-j\omega\tau} d\tau = \int_{-\infty}^{\infty} q\delta(\tau) e^{-j\omega\tau} d\tau = q$$

which implies that a white noise has an infinite energy and thus it is nonexistent in real life. However, many noises (or the so-called colored noises, or noises with finite energy and finite frequency components) can be modeled as the outputs of low-pass linear systems with an injection of a white noise into their inputs, i.e., a **colored noise** can be generated by a white noise as follows



Finally, **Gaussian Process** v is also known as **normal process** has a p.d.f.

$$f(v) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(v-\mu)^2 / 2\sigma^2}, \quad \mu = \text{mean}, \ \sigma^2 = \text{variance}$$



Kalman filter for a linear time invariant (LTI) system

Consider an LTI system characterized by

$$\begin{cases} \dot{x} = Ax + Bu + v(t) & v \text{ is the input noise} \\ y = Cx + w(t) & w \text{ is the measurement noise} \end{cases}$$

Assume: (1) (A, C) is detectable

(2) $v(t)$ and $w(t)$ are independent white noises with the following properties

$$E[v(t)] = 0, \quad E[v(t)v^{\mathrm{T}}(\tau)] = Q_e \delta(t-\tau), \;\; Q_e = Q_e^{\mathrm{T}} \geq 0,$$

$$E[w(t)] = 0, \;\; E[w(t)w^{\mathrm{T}}(\tau)] = R_e \delta(t-\tau), \;\; R_e = R_e^{\mathrm{T}} > 0$$

(3) $\left(A,\ Q_e^{1/2}\right)$ is stabilizable (to guarantee closed-loop stability).

The problem of **Kalman Filter** is a special way to design a state estimator to estimate the state $x(t)$ by $\hat{x}(t)$ such that the estimation error covariance is minimized, i.e., the following index is minimized:

$$J_e = E[e^{\mathrm{T}}(t)e(t)], \quad e(t) \triangleq x(t) - \hat{x}(t)$$

Construction of steady state Kalman filter

Kalman filter is a state observer with a specially selected observer gain (or Kalman filter gain). It has the dynamic equation:

$$\dot{\hat{x}} = A\hat{x} + Bu + K_e(y - \hat{y}), \qquad \hat{x}(0) \text{ is given}$$
$$\hat{y} = C\hat{x}$$

lqe
ric

with the Kalman filter gain K_e being given as

$$K_e = P_e C^\top R_e^{-1}$$

where P_e is the positive semi-definite solution of the following Riccati equation,

$$P_e A^\top + A P_e - P_e C^\top R_e^{-1} C P_e + Q_e = 0$$

Let $e = x - \hat{x}$. We can show (see next) that such a Kalman filter has the following properties:

$$\lim_{t\to\infty} E\left[e(t)\right] = 0, \qquad \lim_{t\to\infty} J_e = \lim_{t\to\infty} E[\,e^\top(t)\; e(t)] = \text{trace } P_e$$

Kalman filter and LQR – They are dual

trace

Recall the optimal regulator problem,

$$\dot{x} = Ax + Bu, \qquad x(0) = x_0$$

$$J = \int_0^\infty \left(x^\top Q x + u^\top R u \right) dt, \quad Q = Q^\top \geq 0, \ \ R = R^\top > 0$$

The LQR problem is to find a state feedback law $u = -Fx$ such that J is minimized. It was shown that the solution to the above problem is given by

$$F = R^{-1}B^\top P \qquad \& \qquad PA + A^\top P - PBR^{-1}B^\top P + Q = 0, \quad P = P^\top \geq 0$$

and the optimal value of J is given by $J = x_0^\top P\, x_0$. Note that x_0 is arbitrary. Let us consider a special case when x_0 is a random vector with

$$E[x_0] = 0, \ \ E[x_0 x_0^\top] = I$$

Then, we have

$$E[J] = E[x_0^\top P x_0] = E\left[\sum_{i=1}^n \sum_{j=1}^n p_{ij} x_{0i} x_{0j} \right] = \sum_{i=1}^n \sum_{j=1}^n p_{ij} E[x_{0i} x_{0j}] = \sum_{i=1}^n p_{ii} = \text{trace } P$$

♣ Linear quadratic regulator

$$F = R^{-1} B^{\top} P$$

$$PA + A^{\top} P - PBR^{-1} B^{\top} P + Q = 0$$

$$J_{\text{optimal}} = \text{trace } P$$

♠ Kalman filter

$$K_e = P_e C^{\top} R^{-1}$$

$$P_e A^{\top} + A P_e - P_e C^{\top} R_e^{-1} C P_e + Q_e = 0$$

$$J_{\text{optimal}} = \text{trace } P_e$$

These two problems are equivalent (or dual) if we let

$$A^{\top} \longleftrightarrow A$$

$$B^{\top} \longleftrightarrow C$$

$$F^{\top} \longleftrightarrow K_e$$

$$P \longleftrightarrow P_e$$

Proof of properties of Kalman filter*

Recall that the dynamics of the given plant and Kalman filter, i.e.,

$$\dot{x} = Ax + Bu + v(t) \qquad \& \qquad \dot{\hat{x}} = A\hat{x} + Bu + K_e(y - \hat{y})$$
$$y = Cx + w(t) \qquad\qquad \hat{y} = C\hat{x}$$

We have

$$\dot{e} = \dot{x} - \dot{\hat{x}} = Ax + Bu + v(t) - A\hat{x} - Bu - K_e[Cx + w(t) - C\hat{x}]$$
$$= (A - K_eC)(x - \hat{x}) + v(t) - K_e w(t)$$
$$= \left(A - K_eC\right)e + \begin{bmatrix} I & -K_e \end{bmatrix}\begin{pmatrix} v \\ w \end{pmatrix} = \bar{A}e + d(t)$$

with

$$E[d(t)] = E\left[\begin{bmatrix} I & -K_e \end{bmatrix}\begin{pmatrix} v(t) \\ w(t) \end{pmatrix}\right] = \begin{bmatrix} I & -K_e \end{bmatrix}\begin{pmatrix} E[v(t)] \\ E[w(t)] \end{pmatrix} = \begin{bmatrix} I & -K_e \end{bmatrix}\begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0$$

Next, it is reasonable to assume that initial error $e(0)$ and $d(t)$ are independent, i.e.,

$$E[e(0)\, d^{\top}(t)] = E[e(0)] \cdot E[d^{\top}(t)] = 0$$

Furthermore,

$$E[d(t)d^{\mathsf{T}}(\tau)] = [I \quad -K_e]\begin{pmatrix} E[v(t)v^{\mathsf{T}}(\tau)] & E[v(t)w^{\mathsf{T}}(\tau)] \\ E[w(t)v^{\mathsf{T}}(\tau)] & E[w(t)w^{\mathsf{T}}(\tau)] \end{pmatrix}\begin{bmatrix} I \\ -K_e^{\mathsf{T}} \end{bmatrix}$$

$$= [I \quad -K_e]\begin{pmatrix} Q_e\delta(t-\tau) & 0 \\ 0 & R_e\delta(t-\tau) \end{pmatrix}\begin{bmatrix} I \\ -K_e^{\mathsf{T}} \end{bmatrix}$$

$$= \left(Q_e + K_e R_e K_e^{\mathsf{T}} \right)\delta(t-\tau)$$

$$= \nabla\,\delta(t-\tau)$$

where $\nabla = Q_e + K_e R_e K_e^{\mathsf{T}} \geq 0$

☞ We will next show $\bar{A} = A - K_e C$ is asymptotically stable and

$$\lim_{t\to\infty} E[\, e(t)\, e^{\mathsf{T}}(t)\,] = P_e$$

Recall that $K_e = P_e C^\top R_e^{-1}$ and $P_e A^\top + A P_e - P_e C^\top R_e^{-1} C P_e + Q_e = 0$. We have

$$P_e A^\top - P_e C^\top R_e^{-1} C P_e + A P_e - P_e C^\top R_e^{-1} C P_e + P_e C^\top R_e^{-1} C P_e + Q_e = 0$$

$$\Downarrow$$

$$P_e \left(A^\top - C^\top R_e^{-1} C P_e \right) + \left(A - P_e C^\top R_e^{-1} C \right) P_e + P_e C^\top R_e^{-1} C P_e + Q_e = 0$$

$$\Downarrow$$

$$P_e \left(A^\top - C^\top K_e^\top \right) + \left(A - K_e C \right) P_e + \left(P_e C^\top R_e^{-1} \right) R_e \left(R_e^{-1} C P_e \right) + Q_e = 0$$

$$\Downarrow$$

$$P_e \bar{A}^\top + \bar{A} P_e = -\left(K_e R_e K_e^\top + Q_e \right) = -\nabla \le 0 \qquad (\clubsuit)$$

Since $Q_e \ge 0$ and $\left(A, Q_e^{1/2} \right)$ is assumed to be stabilizable, it follows from Lyapunov stability theory (see Theorem 3.3.1 of Part 1) that matrix $\bar{A}$ is stable.

Recall also the solution to $\dot{e} = \bar{A} e + d(t)$, i.e.,

$$e(t) = e^{\bar{A} t} \cdot e(0) + \int_0^t e^{\bar{A}(t-\tau)} d(\tau) \cdot d\tau$$

Noting that $e^{\bar{A}t}$ is deterministic, we have

$$P(t) := E[e(t)e^{\mathsf{T}}(t)] = E\left[\left(e^{\bar{A}t}\cdot e(0)+\int_0^t e^{\bar{A}(t-\tau)}d(\tau)\cdot d\tau\right)\cdot\left(e^{\bar{A}t}\cdot e(0)+\int_0^t e^{\bar{A}(t-\tau)}d(\tau)\cdot d\tau\right)^{\mathsf{T}}\right]$$

$$= e^{\bar{A}t}E[e(0)e^{\mathsf{T}}(0)]e^{\bar{A}^{\mathsf{T}}t} + \int_0^t e^{\bar{A}(t-\tau)}\underbrace{E[d(\tau)e^{\mathsf{T}}(0)]}_{0}e^{\bar{A}^{\mathsf{T}}t}\cdot d\tau$$

$$+\int_0^t e^{\bar{A}t}\underbrace{E[e(0)d^{\mathsf{T}}(\tau)]}_{0}e^{\bar{A}^{\mathsf{T}}(t-\tau)}\cdot d\tau + \int_0^t e^{\bar{A}(t-\tau)}d\tau\int_0^t E[d(\tau)d^{\mathsf{T}}(\sigma)]e^{\bar{A}^{\mathsf{T}}(t-\sigma)}\cdot d\sigma$$

$$= e^{\bar{A}t}E[e(0)e^{\mathsf{T}}(0)]e^{\bar{A}^{\mathsf{T}}t} + \int_0^t e^{\bar{A}(t-\tau)}d\tau\int_0^t \nabla\delta(\tau-\sigma)e^{\bar{A}^{\mathsf{T}}(t-\sigma)}\cdot d\sigma$$

$$= e^{\bar{A}t}E[e(0)e^{\mathsf{T}}(0)]e^{\bar{A}^{\mathsf{T}}t} + \int_0^t e^{\bar{A}(t-\tau)}\nabla e^{\bar{A}^{\mathsf{T}}(t-\tau)}\cdot d\tau = e^{\bar{A}t}E[e(0)e^{\mathsf{T}}(0)]e^{\bar{A}^{\mathsf{T}}t} + \int_0^t e^{\bar{A}\eta}\nabla e^{\bar{A}^{\mathsf{T}}\eta}\cdot d\eta$$

Since $\bar{A}$ is stable, we have $e^{\bar{A}t}\to 0,\quad \text{as}\quad t\to\infty$. Thus,

$$\boxed{P(\infty) = \int_0^\infty e^{\bar{A}\eta}\nabla e^{\bar{A}^{\mathsf{T}}\eta}\cdot d\eta}$$

We next show that $P(\infty) = P_e$, i.e., the solution to the Kalman filter ARE. Let

$$\dot{z} = \bar{A}^\top z, \quad z(0) \text{ given} \quad \Rightarrow \quad z(t) = e^{\bar{A}^\top t} z(0), \quad z(\infty) = 0$$

In view of (♣), i.e., $P_e \bar{A}^\top + \bar{A} P_e = -\nabla$, we have

$$z^\top [P_e \bar{A}^\top + \bar{A} P_e] z = -z^\top \nabla z \quad \Rightarrow \quad z^\top P_e \bar{A}^\top z + z^\top \bar{A} P_e z = -z^\top \nabla z$$

$$\Rightarrow \quad z^\top P_e \dot{z} + \dot{z}^\top P_e z = -z^\top \nabla z \quad \Rightarrow \quad \frac{d}{dt}(z^\top P_e z) = -z^\top \nabla z$$

Next, we have

$$\int_0^\infty \frac{d}{dt}(z^\top P_e z) dt = z^\top(t) P_e z(t) \Big|_0^\infty = z^\top(\infty) P_e z(\infty) - z^\top(0) P_e z(0) = 0 - z^\top(0) P_e z(0)$$

$$-\int_0^\infty z^\top \nabla z dt = -\int_0^\infty z^\top(0) e^{\bar{A}t} \nabla e^{\bar{A}^\top t} z(0) dt = -z^\top(0) \left[\int_0^\infty e^{\bar{A}t} \nabla e^{\bar{A}^\top t} dt \right] z(0) = -z^\top(0) P(\infty) z(0)$$

Thus, we have for every given $z(0)$, $z^\top(0) P_e z(0) = z^\top(0) P(\infty) z(0)$, which implies

☞ $$\boxed{P_e = P(\infty) = \int_0^\infty e^{\bar{A}^\top \eta} \nabla e^{\bar{A}\eta} d\eta}$$

It is now simple to see from the definition of $P(t)=E[e(t)e^{\top}(t)]$ that

$$\lim_{t\to\infty} E[e(t)e^{\top}(t)] = P(\infty) = P_e \quad \Rightarrow \quad \lim_{t\to\infty} E[e^{\top}(t)e(t)] = \text{trace } P_e$$

Finally, we have

$$\lim_{t\to\infty} E[e(t)] = \lim_{t\to\infty}\left[e^{\bar{A}t}\cdot E[e(0)] + \int_0^t e^{\bar{A}(t-\tau)} E[d(\tau)]\cdot d\tau \right] = 0.$$

Example: Consider a given plant characterized by the following state space model,

$$\begin{cases} \dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + v(t), \quad E[v(t)v^{\top}(\tau)] = Q_e\delta(t-\tau) = \begin{bmatrix} 0.1 & 0 \\ 0 & 0 \end{bmatrix}\delta(t-\tau) \\ y = \begin{bmatrix} 1 & 0 \end{bmatrix} x + w(t), \quad E[w(t)w^{\top}(\tau)] = R_e\delta(t-\tau) = 0.2\delta(t-\tau) \end{cases}$$

Solving the Kalman filter ARE, we obtain

$$P_e = \begin{bmatrix} 0.0792 & -0.0343 \\ -0.0343 & 0.0314 \end{bmatrix}, \quad K_e = \begin{bmatrix} 0.3962 \\ -0.1715 \end{bmatrix}$$

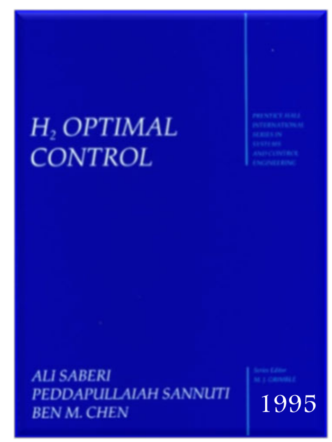
$$\begin{cases} \dot{\hat{x}} = A\hat{x} + Bu + K_e(y-\hat{y}) \\ \hat{y} = C\hat{x} \end{cases}$$

Related topics:

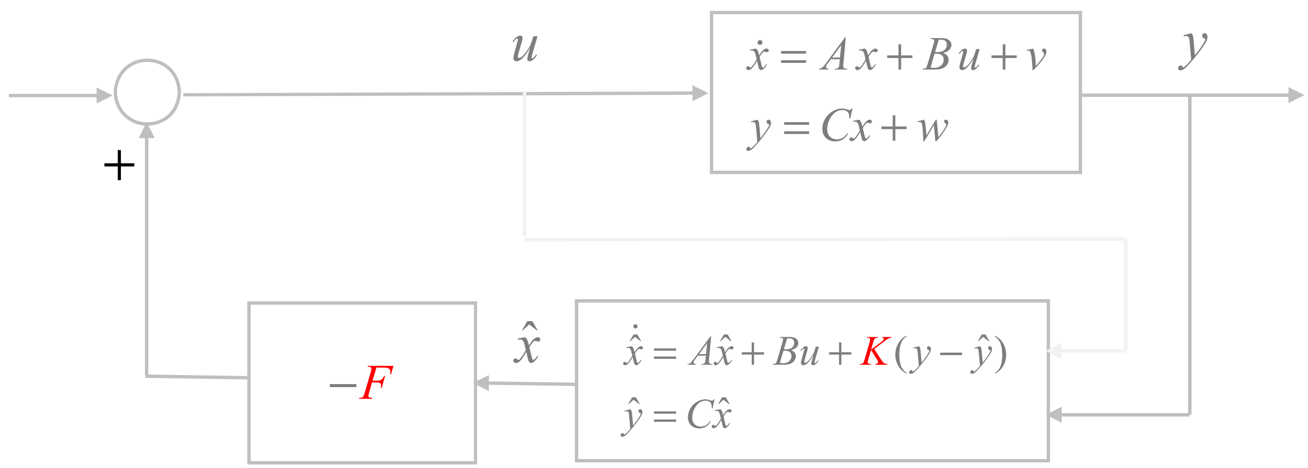
Extended Kalman filter (EKF): In estimation theory, the EKF is the nonlinear version of the Kalman filter, which linearizes about the current mean and covariance. The EKF has been considered the *de facto* standard in the theory of nonlinear state estimation, navigation systems and GPS.

Unscented Kalman filter (UKF): When the state transition and observation models, the predict and update functions are highly nonlinear, the extended Kalman filter can give particularly poor performance. UKF uses a deterministic sampling technique known as the unscented transform to pick a minimal set of sample points (called sigma points) around the mean, which more accurately captures the true mean and covariance.

Solutions to algebraic Riccati equations (AREs): Solutions to AREs are rather numerically involved. All solutions are closely associated with the eigenvectors of a so-called Hamiltonian matrix. A fairly completed compilation of the literature and results on solutions to AREs can be found in Chapter 4 of Saberi *et al*. (1995).



Linear Quadratic Gaussian (LQG) Control



Problem statement

It is very often in control system design for a real-life problem that one cannot measure all the state variables of the given plant. Thus, the linear quadratic regulator, although it has very impressive gain and phase margins (GM = ∞ and PM > 60 degrees), is impractical as it utilizes all state variables in the feedback, i.e., $u = -Fx$. In most of practical situations, only partial information of the state of the given plant is accessible or can be measured for feedback. The natural questions one would ask:

- Can we replace x the control law in LQR, i.e., $u = -Fx$, by the estimated state to carry out a meaningful control system design?

  The answer is yes. One of the solutions is called **LQG control**.

- Do we still have impressive properties associated with the LQG control?

  The answer is **NO**!

- Any solution? Yes. The technique is called a **loop transfer recovery (LTR).**

Linear quadratic Gaussian design

Consider a given plant characterized by

$$\begin{cases} \dot{x} = Ax + Bu + v(t) \\ y = Cx + w(t) \end{cases}$$

v is the input noise — (A, B) stabilizable

w is the measurement noise — (A, C) detectable

where $v(t)$ and $w(t)$ are white with zero means. $v(t)$, $w(t)$ and $x(0)$ are independent, and

$$E[v(t)v^{\top}(\tau)] = Q_e\delta(t-\tau), \;\; Q_e \geq 0, \;\; E[w(t)w^{\top}(\tau)] = R_e\delta(t-\tau), \;\; R_e > 0, \quad E[x(0)] = x_0$$

The performance index has to be modified as follows:

$$J = \lim_{T\to\infty} \frac{1}{T} E\left[\int_0^T (x^{\top}Qx + u^{\top}Ru)dt \right], \qquad Q \geq 0, \;\; R > 0$$

The **linear quadratic Gaussian** (LQG) control is to design a control law that only requires the measurable information such that when it is applied to the given plant, the overall system is stable and the performance index is minimized.

Solution to the LQG problem – the separation principle

Step 1. Design an LQR control law $u = -Fx$ which solves the following problem,

$$\dot{x} = A\,x + B\,u$$

i.e., compute

$$PA + A^{\mathrm{T}}P - PBR^{-1}B^{\mathrm{T}}P + Q = 0, \quad P \ge 0, \quad F = R^{-1}B^{\mathrm{T}}P.$$

Step 2. Design a Kalman filter for the given plant, i.e.,

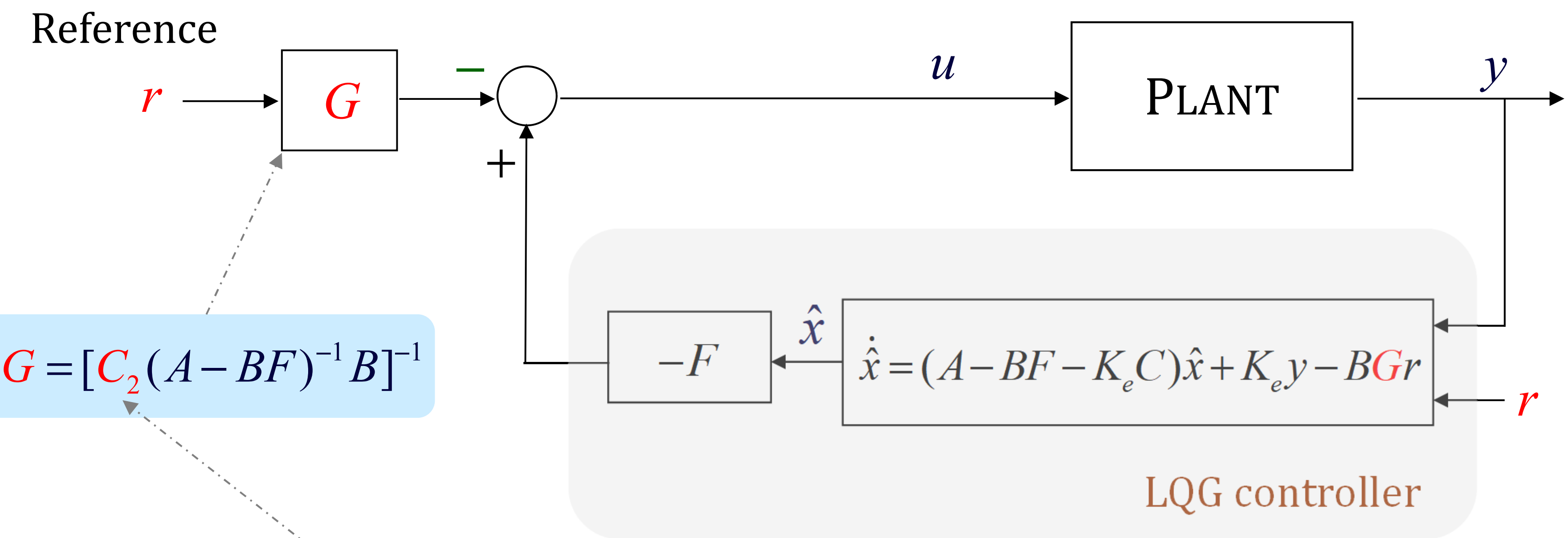
$$\dot{\hat{x}} = A\hat{x} + Bu + K_e(y - \hat{y}), \quad \hat{y} = C\hat{x}$$

where $K_e = P_e C^{\mathrm{T}} R_e^{-1}, \; P_e A^{\mathrm{T}} + AP_e - P_e C^{\mathrm{T}} R_e^{-1} C P_e + Q_e = 0, \quad P_e \ge 0.$

Step 3. The LQG control law is given by $u = -F\,\hat{x}$, i.e.,

$$\begin{cases} \dot{\hat{x}} = A\,\hat{x} + B\,u + K_e(y - C\,\hat{x}) \\ u = -F\,\hat{x} \end{cases} \Rightarrow \boxed{\begin{cases} \dot{\hat{x}} = (A - BF - K_e C)\,\hat{x} + K_e\,y \\ u = -F\,\hat{x} \end{cases}}$$

Block implementation of an LQG control law with a reference…



$$G = [C_2 (A - BF)^{-1} B]^{-1}$$

G is chosen such that the DC gain from r to z is unity.

Matrix C_2 is related to output variables of interest, say

$$z = C_2\, x$$

where z is to track the reference r.

Closed-loop dynamics of the given plant together with LQG controller

Recall the plant: $\begin{cases} \dot{x} = Ax + Bu + v(t) \\ y = Cx + w(t) \end{cases}$ and controller $\begin{cases} \dot{\hat{x}} = (A - BF - K_e C)\,\hat{x} - BGr + K_e\, y \\ u = -F\,\hat{x} - Gr \end{cases}$

We define a new variable $e = x - \hat{x}$ and thus

$$\dot{e} = \dot{x} - \dot{\hat{x}} = Ax - BF\hat{x} - BGr + v(t) - A\hat{x} + BF\hat{x} + K_e C\hat{x} + BGr - K_e Cx - K_e w(t)$$
$$= A(x - \hat{x}) - K_e C(x - \hat{x}) + v(t) - K_e w(t) = (A - K_e C)e + v(t) - K_e w(t)$$

and

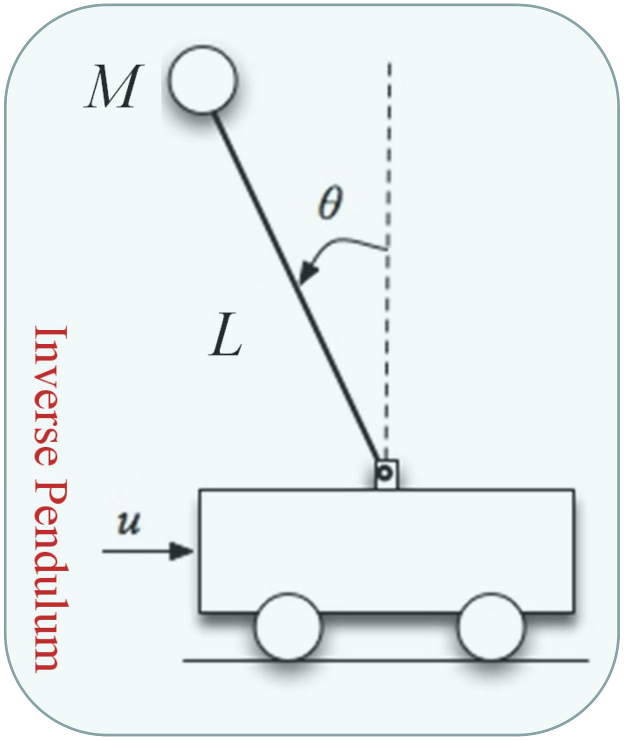
$$\dot{x} = Ax + Bu + v(t) = Ax - BF\hat{x} - BGr + v(t) = Ax - BF(x - e) - BGr + v(t)$$
$$= (A - BF)x + BFe - BGr + v(t)$$

Clearly, the closed-loop system is characterized by the following state space equation,

$$\begin{cases} \begin{pmatrix} \dot{x} \\ \dot{e} \end{pmatrix} = \begin{bmatrix} A - BF & BF \\ 0 & A - K_e C \end{bmatrix} \begin{pmatrix} x \\ e \end{pmatrix} - \begin{bmatrix} BG \\ 0 \end{bmatrix} r + \tilde{v}, \quad \tilde{v} = \begin{pmatrix} v \\ v - K_e w \end{pmatrix} \\ y = \begin{bmatrix} C & 0 \end{bmatrix} \begin{pmatrix} x \\ e \end{pmatrix} + w \end{cases}$$

The closed-loop poles are given by $\lambda(A - BF) \cup \lambda(A - K_e C)$, which are stable.

Example: Consider again the inverse pendulum system characterized by



$$\begin{pmatrix} \dot{\tilde{\theta}} \\ \ddot{\theta} \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ \dfrac{g}{L} & 0 \end{bmatrix} \begin{pmatrix} \tilde{\theta} \\ \dot{\theta} \end{pmatrix} + \begin{bmatrix} 0 \\ \dfrac{1}{ML^2} \end{bmatrix} u, \quad y = [1 \quad 0] \begin{pmatrix} \tilde{\theta} \\ \dot{\theta} \end{pmatrix}$$

For simplicity, we set $ML^2 = 1, \; g/L = 1.$

Also, consider that there are some input noise v and measurement noise w with $Q_e = I$ and $R_e = 1$. We thus have

$$\begin{pmatrix} \dot{\tilde{\theta}} \\ \ddot{\theta} \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{pmatrix} \tilde{\theta} \\ \dot{\theta} \end{pmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + v, \quad y = [1 \quad 0] \begin{pmatrix} \tilde{\theta} \\ \dot{\theta} \end{pmatrix} + w$$

We proceed to design an LQG control law for the given system to keep the output around $\theta = \pi$ and also minimize the cost function

$$J = \lim_{T \to \infty} \frac{1}{T} E\left[\int_0^T \left(x^\top Q x + u^\top R u \right) dt \right], \quad Q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad R = 1.$$

ex2097

Step 1: With the given system data and Q and R, we solve the following ARE

$$PA + A^{\top} P - PBR^{-1} B^{\top} P + Q = 0$$

and obtain

$$P = \begin{bmatrix} 3.1075 & 2.4142 \\ 2.4142 & 2.1974 \end{bmatrix} \quad \Rightarrow \quad F = R^{-1} B^{\top} P = \begin{bmatrix} 2.4142 & 2.1974 \end{bmatrix}$$

which gives the eigenvalues of $A - BF$ at $-1.0987 \pm j0.4551$.

Step 2: Solving the Kalman filter ARE

$$P_e A^{\top} + AP_e - P_e C^{\top} R_e^{-1} C P_e + Q_e = 0$$

we obtain

$$P_e = \begin{bmatrix} 2.4142 & 2.4142 \\ 2.4142 & 3.4142 \end{bmatrix} \quad \Rightarrow \quad K_e = P_e C^{\top} R_e^{-1} = \begin{bmatrix} 2.4142 \\ 2.4142 \end{bmatrix}$$

which places the eigenvalues of $A - K_e C$ at -1 and -1.4142, respectively.

Step 3: The resulting LQG control law is given by

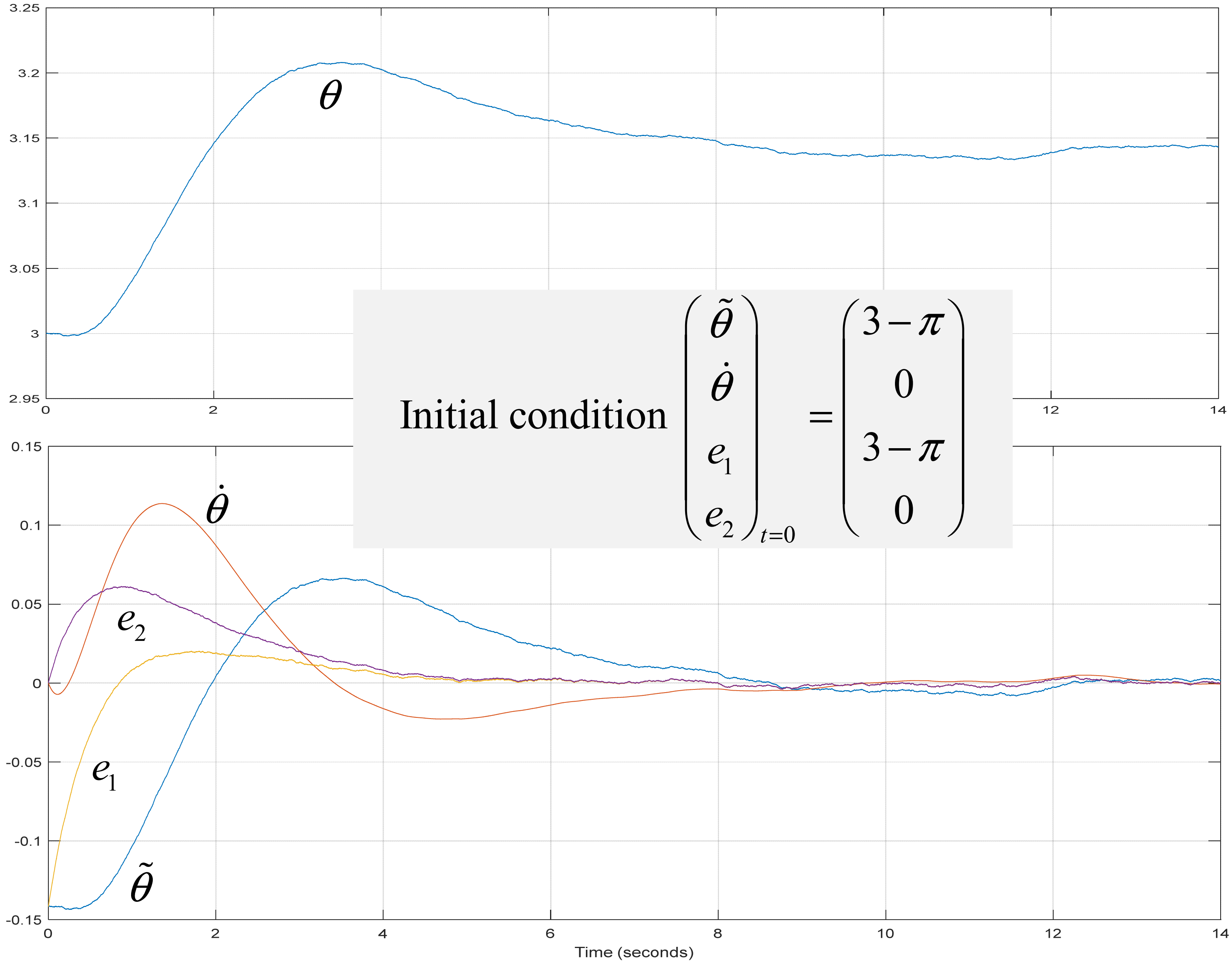
$$\begin{cases} \dot{\hat{x}} = (A - BF - K_e C)\,\hat{x} - BGr + K_e\, y = \begin{bmatrix} -2.4142 & 1 \\ -3.8284 & -2.1974 \end{bmatrix} \hat{x} + \begin{bmatrix} 0 \\ 1.4142 \end{bmatrix} r + \begin{bmatrix} 2.4142 \\ 2.4142 \end{bmatrix} y \\ u = -F\,\hat{x} - Gr \qquad\qquad = \begin{bmatrix} -2.4142 & -2.1974 \end{bmatrix} \hat{x} + 1.4142\, r \end{cases}$$

where $G = [C(A - BF)^{-1} B]^{-1}$.

The closed-loop system is given by

$$\begin{cases} \begin{pmatrix} \dot{x} \\ \dot{e} \end{pmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1.4142 & -2.1974 & 2.4142 & 2.1974 \\ 0 & 0 & -2.4142 & 1 \\ 0 & 0 & -1.4142 & 0 \end{bmatrix} \begin{pmatrix} x \\ e \end{pmatrix} + \begin{bmatrix} 0 \\ 1.4142 \\ 0 \\ 0 \end{bmatrix} r + \begin{pmatrix} v \\ v - K_e w \end{pmatrix} \\ y = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} x \\ e \end{pmatrix} + w \end{cases}$$

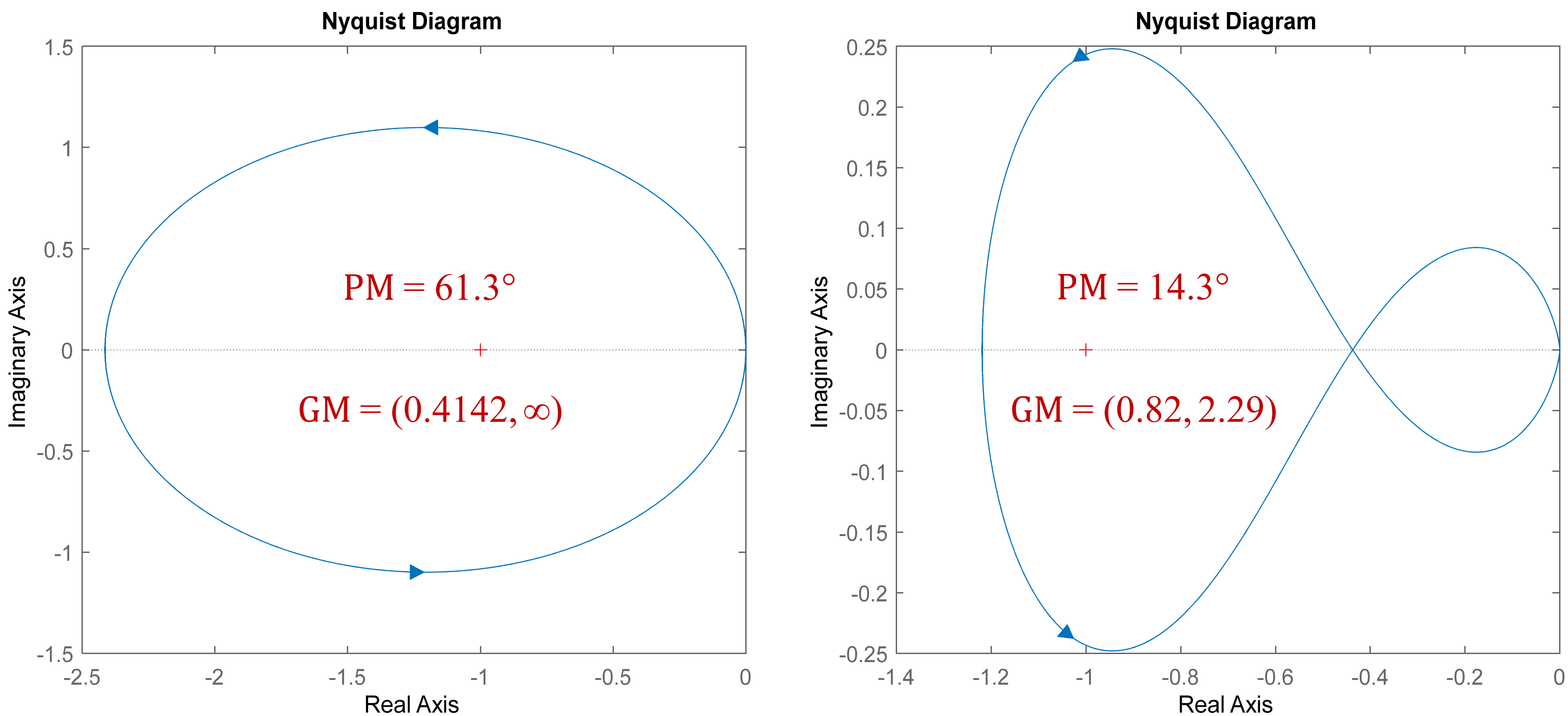
Simulation results given on the next page show that LQG control works, but its performance can be and should be improved before implementing it to the real problem.



$$\text{Initial condition}\quad \begin{pmatrix} \tilde{\theta} \\ \dot{\theta} \\ e_1 \\ e_2 \end{pmatrix}_{t=0} = \begin{pmatrix} 3-\pi \\ 0 \\ 3-\pi \\ 0 \end{pmatrix}$$

What is the shortfall with the LQG design?

- Gain margins and phase margins…



LQR Control

LQG Control

Homework Assignment 5 + Design Problem 1

Reconsider the Double Inverted Pendulum on a Cart (DIPC) in Homework Assignment 4.

1. Assume all state variables of the plant are available for feedback. Find an LQR control law, which minimizes the following performance index:

$$J = \int_0^\infty \left(x^\top Q x + u^\top R u \right) dt, \quad Q = I, \; R = 1$$

   What are the resulting gain and phase margins of your LQR design?

2. Assume that there is an input noise (or disturbance) entering the system as:

$$\dot{x} = A x + B u + B v(t),$$

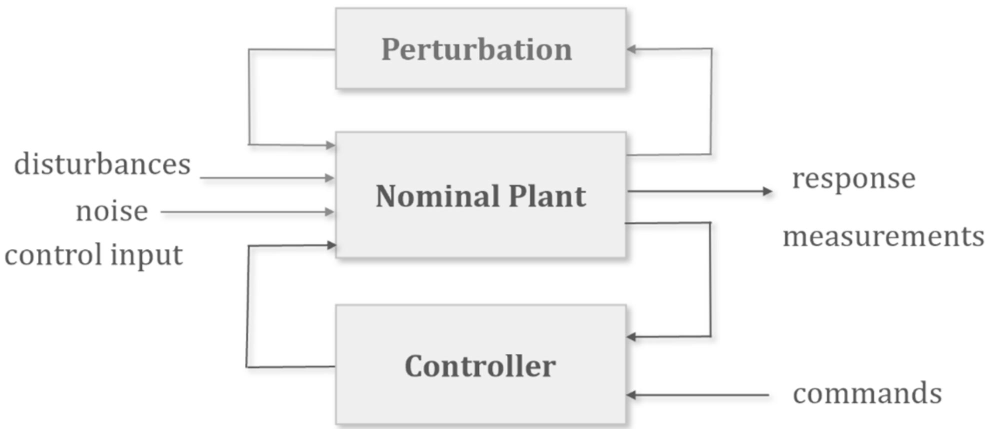
   and the system measurement output is $y = x_c + w(t)$, where $w(t)$ is the measurement noise. Assume both $v(t)$ and $w(t)$ have zero means and

$$E[v(t)v^\top(t)] = Q_e \delta(t-\tau), \; Q_e = 1, \quad E[w(t)w^\top(t)] = R_e \delta(t-\tau), \; R_e = 1.$$

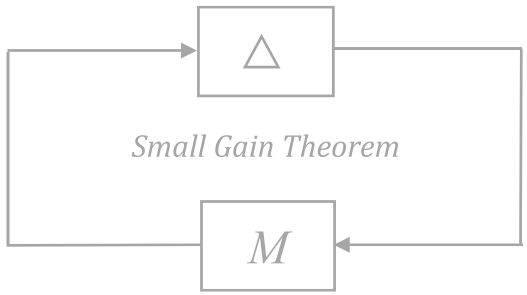
   Design an appropriate Kalman filter.

3. Derive the corresponding LQG control law, i.e., the combination of the LQR law in Step 1 and the Kalman filter in Step 2. What are the closed-loop eigenvalues? What are the resulting gain and phase margins of your LQG control law? Simulate your design with a reference $r = 0$ and with the same initial condition of the state variable as that in Homework Assignment 4 and the initial condition for the Kalman filter being 0.

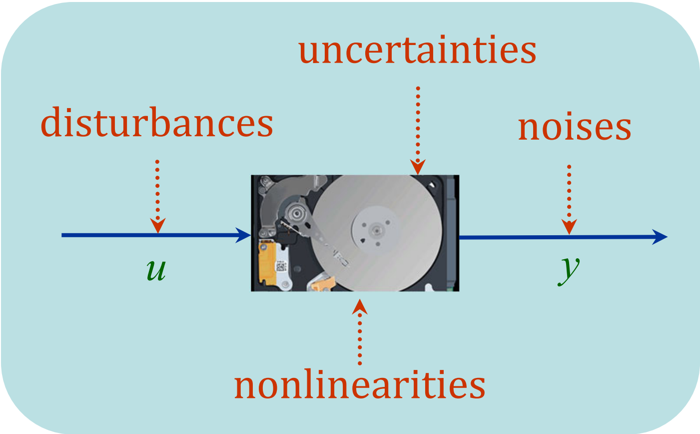
Introduction to Robust Control



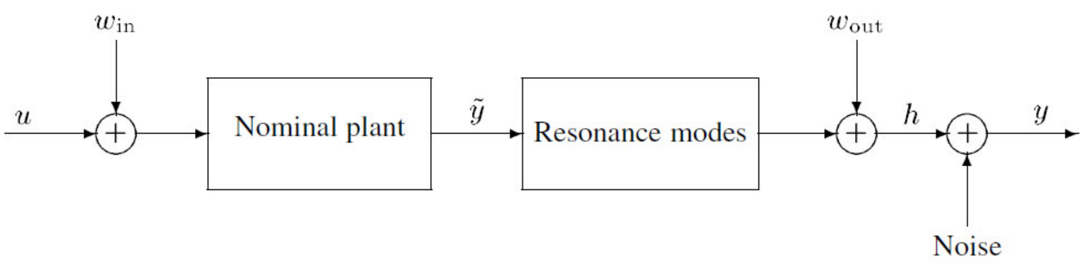
George Zames
Canadian Control Theorist
1934–1997



Representation of uncertain plant dynamics



Example: An HDD servo system…



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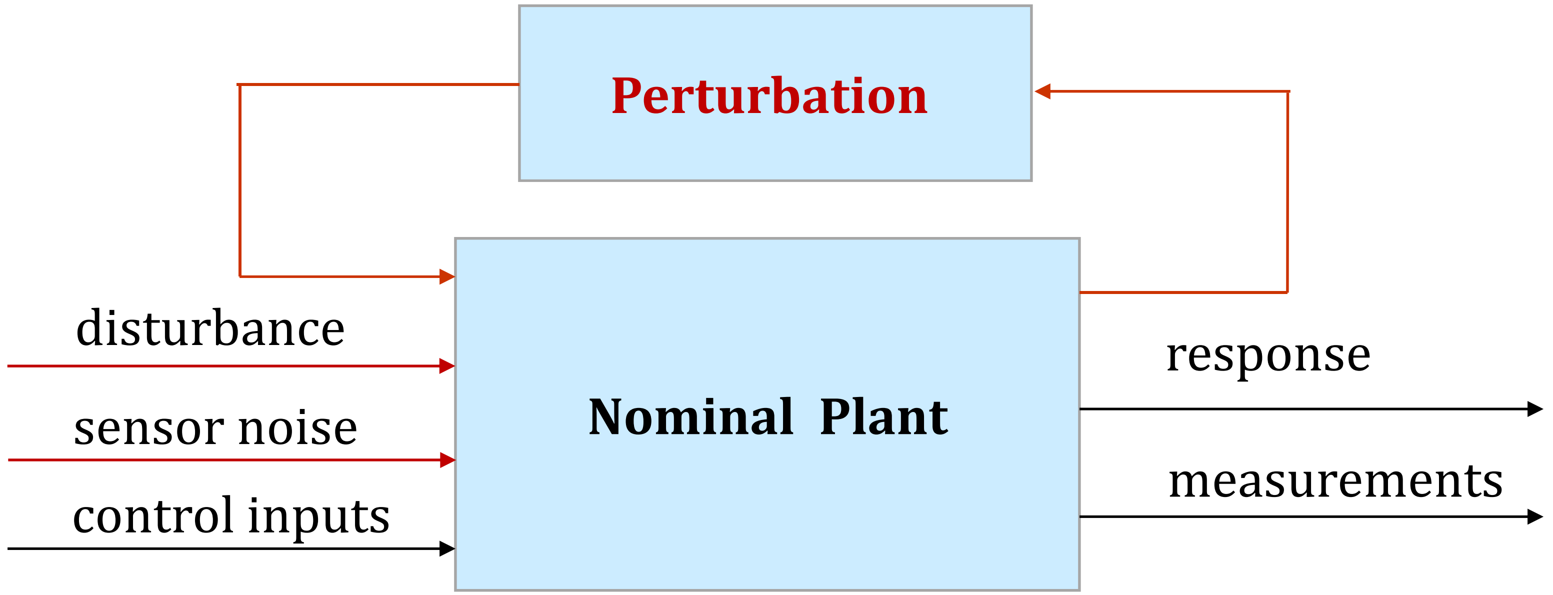
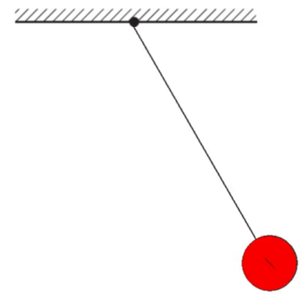
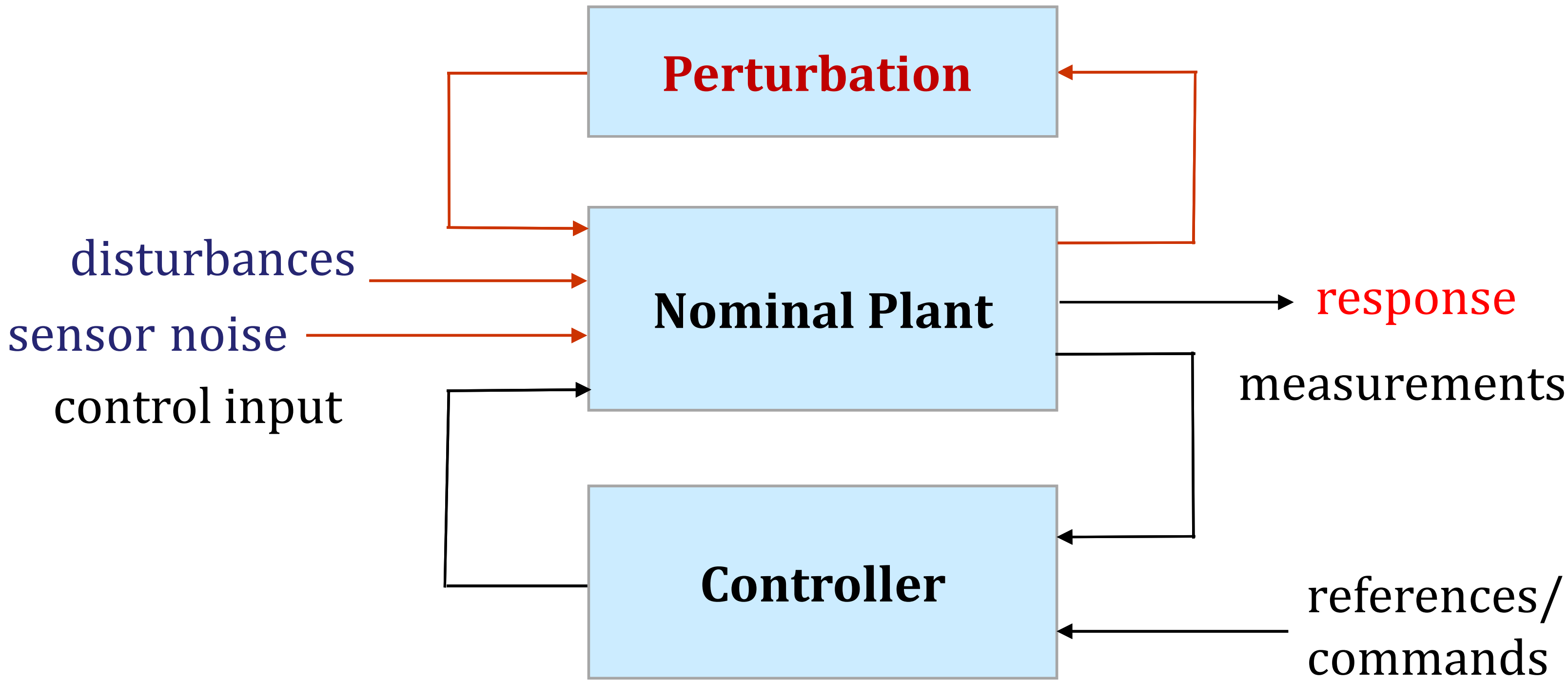


A real control problem

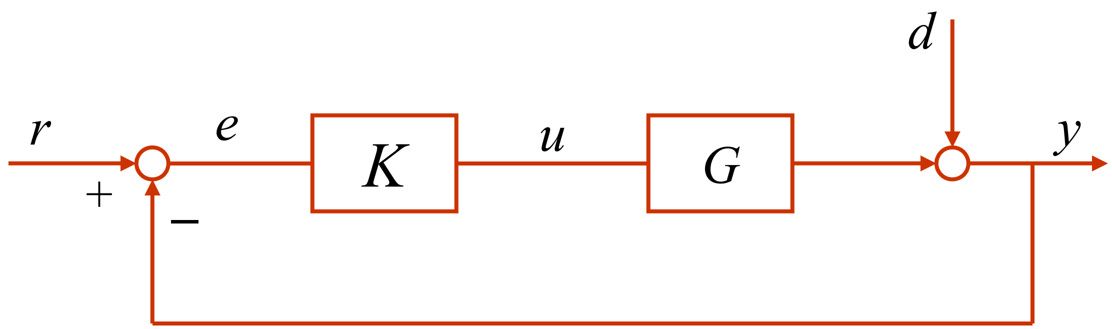




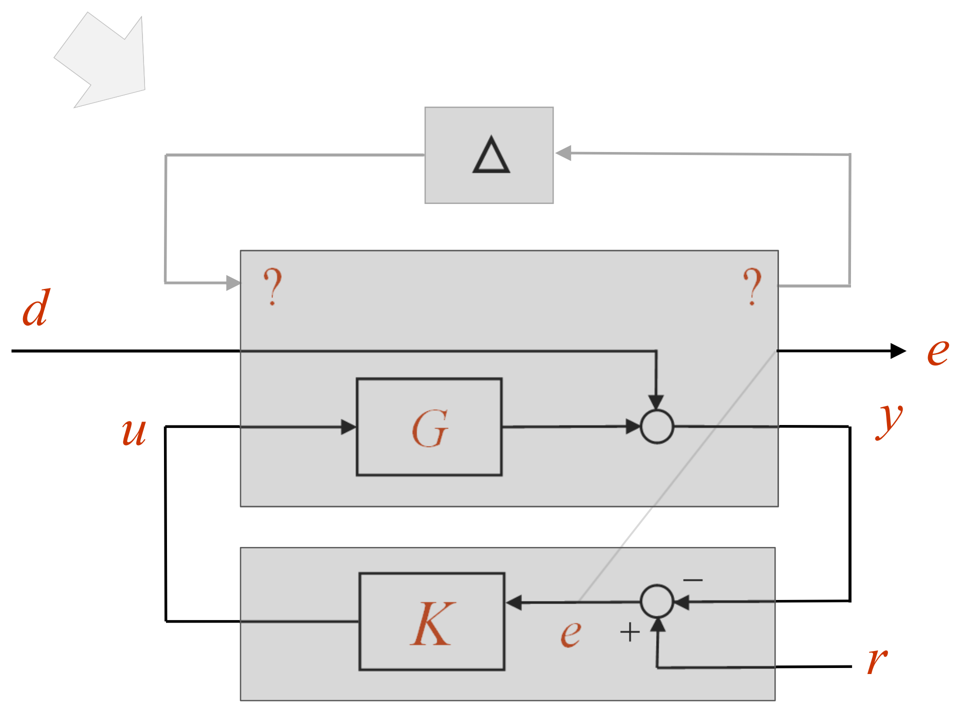
Controller Objective: To provide desired responses in face of

❖ Uncertain plant dynamics + External inputs $\left\{ \begin{array}{l} \text{disturbances} \\ \text{sensor noise} \end{array} \right.$

Standard feedback loops in terms of general interconnection structure



Standard feedback system



Even though it is not directly formulated in the problem formulation, classical control system design deals with system uncertainties through specifications imposed on gain and phase margins…

Analysis objectives

- **Nominal Performance Question (*H_2* Optimal Control):**

  Are closed loop responses acceptable for disturbances? sensor noise?

- **Robust Stability Question (*H_∞* Optimal Control):**

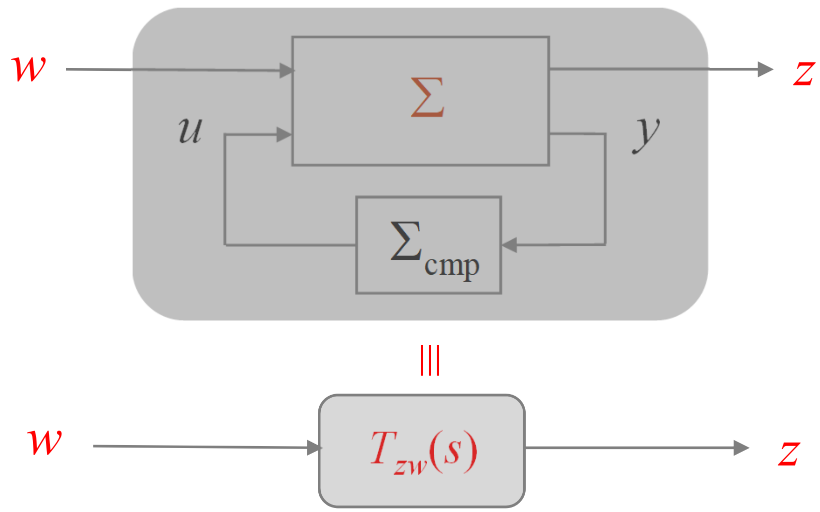
  Is closed loop system stable for nominal plant? for all possible perturbations?

- **Robust Performance Question (Mixed *H_2 / H_∞* Optimal Control):**

  Are closed loop responses acceptable for all possible perturbations and all external inputs? Simultaneously?

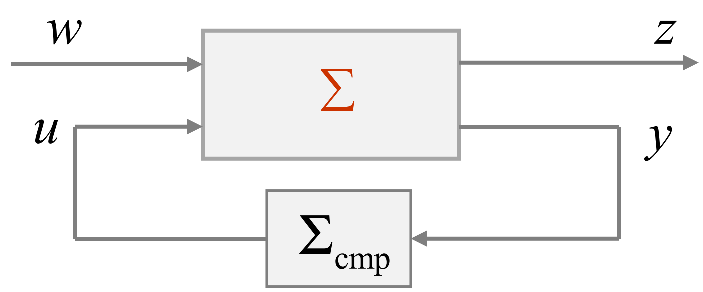
  Many issues related to robust performance problems are still open!...

H_2 and H_∞ Control Techniques



Introduction to the problems

We first ignore uncertainties (perturbations) in the plant. Will bring such an issue back later. Consider a stabilizable and detectable linear time-invariant nominal system Σ with a proper controller Σ_{cmp} as in the configuration below:



$$\Sigma: \begin{cases} \dot{x} = A\,x + B\,u + E\,w \\ y = C_1\,x + 0\,u + D_1\,w \\ z = C_2\,x + D_2\,u + 0\,w \end{cases} \qquad \Sigma_{\mathrm{cmp}}: \begin{cases} \dot{x}_{\mathrm{cmp}} = A_{\mathrm{cmp}}\,x_{\mathrm{cmp}} + B_{\mathrm{cmp}}\,y \\ u = C_{\mathrm{cmp}}\,x_{\mathrm{cmp}} + D_{\mathrm{cmp}}\,y \end{cases}$$

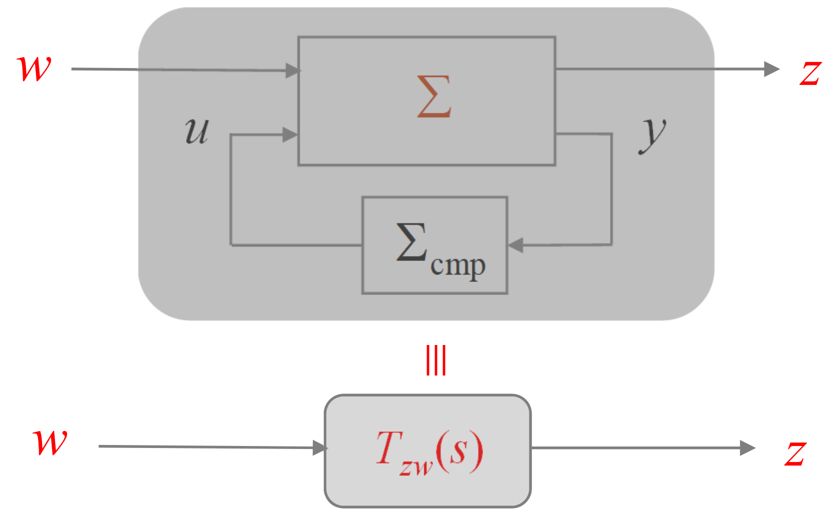
$$\begin{cases} x \in \mathbb{R}^n \Leftrightarrow \text{state variable} \\ y \in \mathbb{R}^p \Leftrightarrow \text{measurement} \\ z \in \mathbb{R}^q \Leftrightarrow \text{controlled output} \end{cases} \quad \& \quad \begin{array}{l} u \in \mathbb{R}^m \Leftrightarrow \text{control input} \\ w \in \mathbb{R}^l \Leftrightarrow \text{disturbance} \\ x_{\mathrm{cmp}} \in \mathbb{R}^k \Leftrightarrow \text{controller state} \end{array}$$

The problems of H_2 and H_∞ optimal control are to design a proper control law Σ_{cmp} such that when it is applied to the given plant with disturbance, i.e., Σ, we have

- The resulting closed loop system is internally stable.

- The resulting closed-loop transfer function matrix from the disturbance input w to the controlled output z, say, $T_{zw}(s)$, is as small as possible, i.e., the effect of the disturbance input on the controlled output is minimized.
  - H_2 optimal control: the H_2-norm of $T_{zw}(s)$ is minimized.
  - H_∞ optimal control: the H_∞-norm of $T_{zw}(s)$ is minimized.

Note: $T_{zw}(s)$ is a function of frequencies. It is meaningless to say if it is large or small. The common practice is to measure its norms instead. H_2-norm and H_∞-norm are two commonly used norms in measuring the sizes of a transfer function matrix.



The closed-loop transfer function matrix: $T_{zw}(s)$

The closed-loop transfer function from disturbance to controlled output can be derived as follows: Recall…

$$\Sigma: \begin{cases} \dot{x} = A\,x + B\,u + E\,w \\ y = C_1\,x + D_1\,w \\ z = C_2\,x + D_2\,u \end{cases} \quad \text{and} \quad \Sigma_{\rm cmp}: \begin{cases} \dot{x}_{\rm cmp} = A_{\rm cmp}\,x_{\rm cmp} + B_{\rm cmp}\,y \\ u = C_{\rm cmp}\,x_{\rm cmp} + D_{\rm cmp}\,y \end{cases}$$

$$\begin{cases} \dot{x} = A\,x + B\,(C_{\rm cmp}x_{\rm cmp} + D_{\rm cmp}y) + E\,w \\ y = C_1\,x + D_1\,w \\ z = C_2\,x + D_2\,(C_{\rm cmp}x_{\rm cmp} + D_{\rm cmp}y) \end{cases}$$

$$\begin{aligned} \dot{x}_{\rm cmp} &= A_{\rm cmp}x_{\rm cmp} + B_{\rm cmp}(C_1x + D_1w) \\ &= A_{\rm cmp}x_{\rm cmp} + B_{\rm cmp}C_1x + B_{\rm cmp}D_1w \end{aligned}$$

$$\begin{cases} \dot{x} = A\,x + BC_{\rm cmp}x_{\rm cmp} + BD_{\rm cmp}\,y + E\,w \\ z = C_2x + D_2C_{\rm cmp}x_{\rm cmp} + D_2D_{\rm cmp}\,y \end{cases}$$

$$\dot{x}_{\rm cmp} = A_{\rm cmp}x_{\rm cmp} + B_{\rm cmp}C_1x + B_{\rm cmp}D_1w$$

$$\begin{cases} \dot{x} = Ax + BC_{\mathrm{cmp}}x_{\mathrm{cmp}} + BD_{\mathrm{cmp}}\left(C_1x + D_1w\right) + Ew \\ z = C_2x + D_2C_{\mathrm{cmp}}x_{\mathrm{cmp}} + D_2D_{\mathrm{cmp}}\left(C_1x + D_1w\right) \end{cases}$$

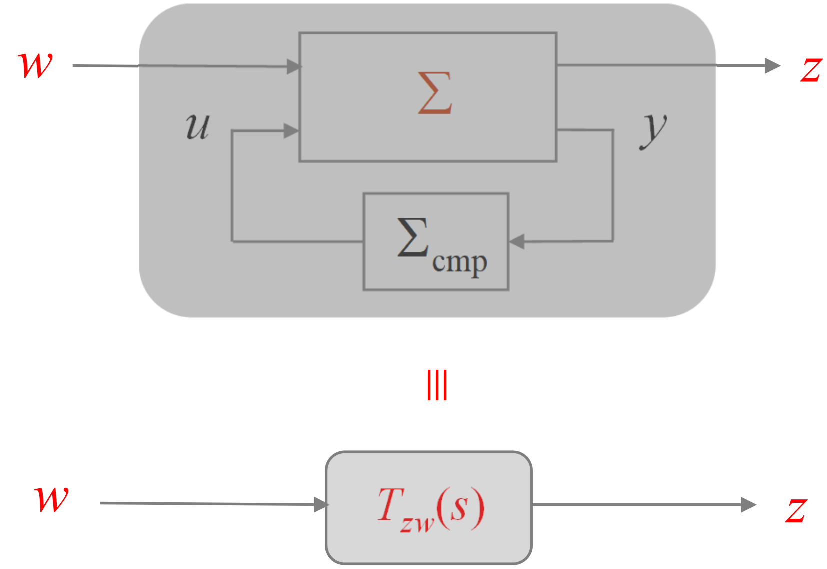
$$\dot{x}_{\mathrm{cmp}} = \begin{bmatrix} B_{\mathrm{cmp}}C_1 & A_{\mathrm{cmp}} \end{bmatrix} \begin{pmatrix} x \\ x_{\mathrm{cmp}} \end{pmatrix} + B_{\mathrm{cmp}}D_1w$$

$$\begin{cases} \dot{x} = \left(A + BD_{\mathrm{cmp}}C_1\right)x + BC_{\mathrm{cmp}}x_{\mathrm{cmp}} + \left(E + BD_{\mathrm{cmp}}D_1\right)w \\ z = \left(C_2 + D_2D_{\mathrm{cmp}}C_1\right)x + D_2C_{\mathrm{cmp}}x_{\mathrm{cmp}} + D_2D_{\mathrm{cmp}}D_1w \end{cases}$$

$$\begin{cases} \begin{pmatrix} \dot{x} \\ \dot{x}_{\mathrm{cmp}} \end{pmatrix} = \overbrace{\begin{bmatrix} A + BD_{\mathrm{cmp}}C_1 & BC_{\mathrm{cmp}} \\ B_{\mathrm{cmp}}C_1 & A_{\mathrm{cmp}} \end{bmatrix}}^{A_{\mathrm{cl}}} \begin{pmatrix} x \\ x_{\mathrm{cmp}} \end{pmatrix} + \overbrace{\begin{bmatrix} E + BD_{\mathrm{cmp}}D_1 \\ B_{\mathrm{cmp}}D_1 \end{bmatrix}}^{B_{\mathrm{cl}}} w \\ z = \underbrace{\begin{bmatrix} C_2 + D_2D_{\mathrm{cmp}}C_1 & D_2C_{\mathrm{cmp}} \end{bmatrix}}_{C_{\mathrm{cl}}} \begin{pmatrix} x \\ x_{\mathrm{cmp}} \end{pmatrix} + \underbrace{D_2D_{\mathrm{cmp}}D_1}_{D_{\mathrm{cl}}} w \end{cases}$$

Thus, $T_{zw}(s)$ is given by

$$T_{zw}(s) = C_{\rm cl}\,(sI - A_{\rm cl})^{-1}\,B_{\rm cl} + D_{\rm cl}$$



The closed-loop system is internally stable if and only if the eigenvalues of

$$A_{\rm cl} = \begin{bmatrix} A + BD_{\rm cmp}C_1 & BC_{\rm cmp} \\ B_{\rm cmp}C_1 & A_{\rm cmp} \end{bmatrix}$$

are all in open left half complex plane.

Remark: For the state feedback case, $C_1 = I$ and $D_1 = 0$, i.e., all the states of the given system can be measured, $\Sigma_{\rm c}$ can then be reduced to $u = Fx$ and the corresponding closed-loop transfer function is reduced to

Tzwo
Tzws

$$T_{zw}(s) = (C_2 + D_2F)\,(sI - A - BF)^{-1}\,E$$

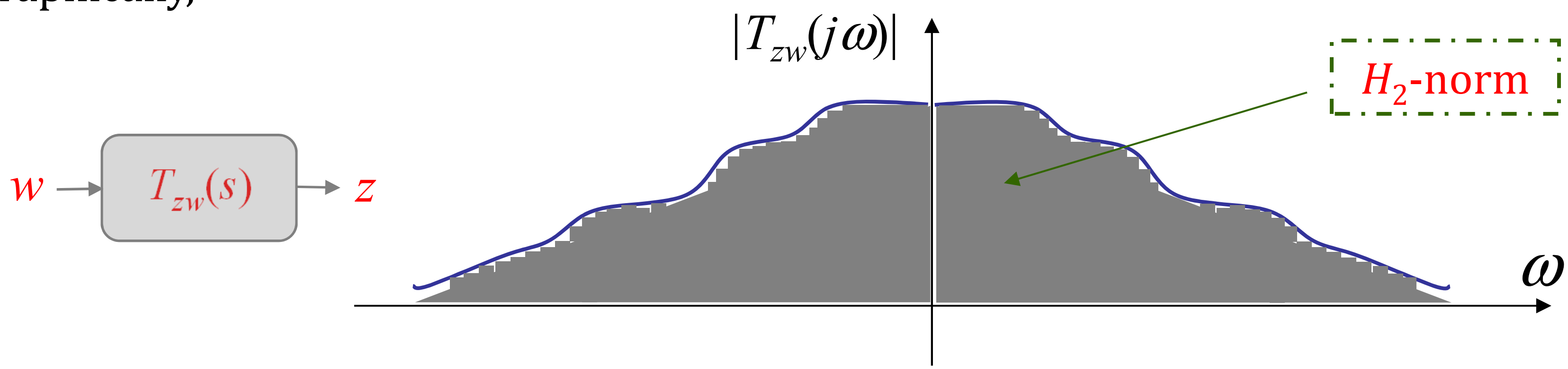
The closed-loop stability implies and is implied that $A + BF$ has stable eigenvalues.

H_2-norm and H_∞-norm of a transfer function matrix

Definition: (H_2-norm) Given a stable and proper transfer function matrix $T_{zw}(s)$, its H_2-norm is defined as

$$\|T_{zw}\|_2 = \left(\frac{1}{2\pi} \operatorname{trace}\left[\int_{-\infty}^{+\infty} T_{zw}(j\omega) T_{zw}(j\omega)^{\mathrm{H}} \, d\omega \right] \right)^{\frac{1}{2}}$$

Graphically,



Note: The H_2-norm is the total energy corresponding to the impulse response of $T_{zw}(s)$. Thus, minimization of the H_2-norm of $T_{zw}(s)$ is equivalent to the minimization of the total energy from the disturbance w to the controlled output z.

♣ Singular value decomposition

Given a matrix $A \in \mathbb{C}^{m \times n}$, its *singular values* are defined as

$$\sigma_i(A) := \sqrt{\lambda_i(A^{\mathrm{H}} A)} = \sqrt{\lambda_i(A A^{\mathrm{H}})}, \quad i = 1, 2, \ldots, k, \tag{2.3.75}$$

where $k := \min\{m, n\}$, assuming that the eigenvalues of $A^{\mathrm{H}} A$ and $A A^{\mathrm{H}}$ are arranged in a descending order. Clearly, we have $\sigma_1(A) \geq \sigma_2(A) \geq \cdots \geq \sigma_k(A) \geq 0$.

Let

$$\Delta_1 = \operatorname{diag}\left\{\sigma_1(A), \sigma_2(A), \ldots, \sigma_k(A)\right\}. \tag{2.3.77}$$

It can be shown that there exist two unitary matrices such that A can be decomposed as:

$$A = U \Delta V^{\mathrm{H}}, \tag{2.3.78}$$

where

$$\Delta = \begin{bmatrix} \Delta_1 \\ 0 \end{bmatrix}, \quad \text{if } m \geq n, \tag{2.3.79}$$

or

$$\Delta = \begin{bmatrix} \Delta_1 & 0 \end{bmatrix}, \quad \text{if } m \leq n. \tag{2.3.80}$$

Decomposition (2.3.78) is called the *singular value decomposition* of A.

♣ **Example:** Given a matrix

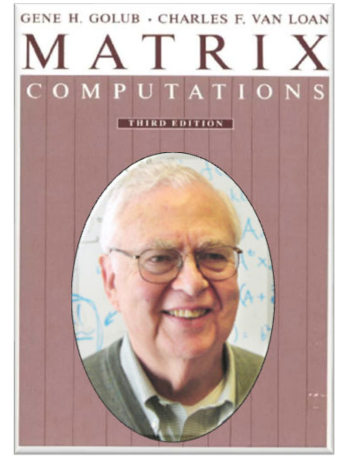
$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \end{bmatrix}$$

its singular value decomposition is given as

$$A = U \Delta V' = U \begin{bmatrix} 12.4472 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} V'$$

where

$$U = \begin{bmatrix} -0.4472 & -0.8944 \\ -0.8944 & 0.4472 \end{bmatrix}, \quad V = \begin{bmatrix} -0.1826 & -0.9832 & 0 & 0 \\ -0.3651 & 0.0678 & -0.5571 & -0.7428 \\ -0.5477 & 0.1017 & 0.7737 & -0.3017 \\ -0.7303 & 0.1356 & -0.3017 & 0.5977 \end{bmatrix}$$



Gene H. Golub
1932–2007
Stanford University

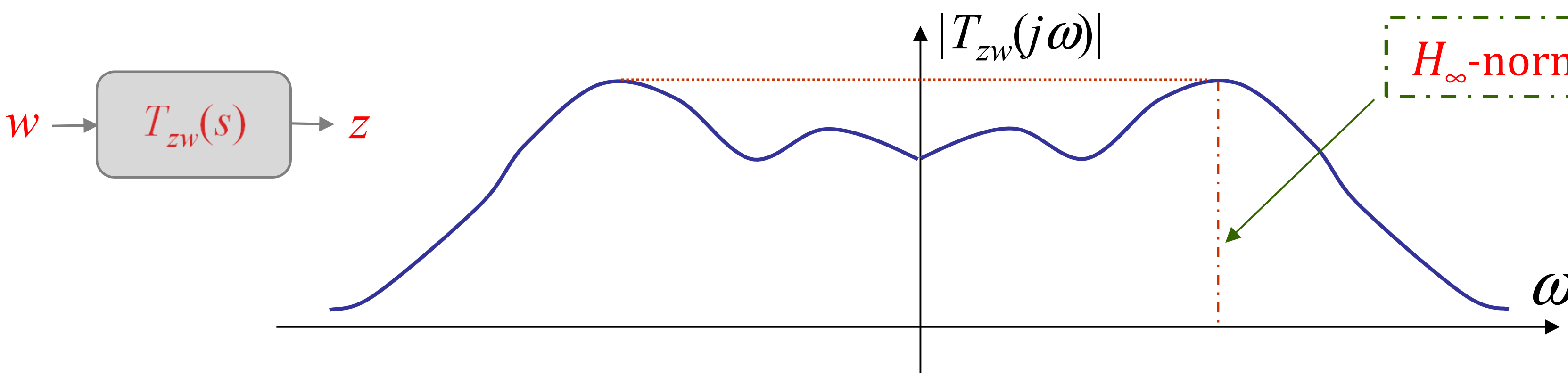
Note: It can be computed using an m-function SVD in MATLAB.

ex1042

Definition: (H_∞-norm) Given a stable and proper transfer function matrix $T_{zw}(s)$, its H_∞-norm is defined as

$$\|T_{zw}\|_\infty = \sup_{0\le\omega<\infty} \sigma_{\max}\left[T_{zw}(j\omega)\right]$$

where $\sigma_{\max}[T_{zw}(j\omega)]$ denotes the maximum singular value of $T_{zw}(j\omega)$. For a single-input-single-output transfer function $T_{zw}(s)$, it is equivalent to the magnitude of $T_{zw}(j\omega)$. Graphically,



Note: The H_∞-norm is the worst-case gain in $T_{zw}(s)$. Thus, minimization of the H_∞-norm of $T_{zw}(s)$ is equivalent to the minimization of the worst-case (gain) situation on the effect from the disturbance w to the controlled output z.

Infima (optimal performance) and optimal controllers

Definition: (The infimum of H_2 optimization) The infimum of the H_2-norm of the closed-loop transfer matrix $T_{zw}(s)$ over all stabilizing proper controllers is denoted by γ_2^*, that is

$$\gamma_2^* := \inf \left\{ \ \|T_{zw}\|_2 \ \ \middle| \ \Sigma_{\rm cmp} \ \text{ internally stabilizes } \Sigma \ \right\}.$$

Definition: (The H_2 optimal controller) A proper controller $\Sigma_{\rm cmp}$ is said to be an H_2 optimal controller if it internally stabilizes Σ and $\| T_{zw} \|_2 = \gamma_2^*$.

Definition: (The infimum of H_∞ optimization) The infimum of the H_∞-norm of the closed-loop transfer matrix $T_{zw}(s)$ over all stabilizing proper controllers is denoted by γ_∞^*, that is

$$\gamma_\infty^* := \inf \left\{ \ \|T_{zw}\|_\infty \ \ \middle| \ \Sigma_{\rm cmp} \ \text{ internally stabilizes } \Sigma \ \right\}.$$

Definition: (The H_∞ γ-suboptimal controller) A proper controller $\Sigma_{\rm cmp}$ is said to be an H_∞ γ- suboptimal controller if it internally stabilizes Σ and $\| T_{zw} \|_\infty < \gamma \left(> \gamma_\infty^* \right)$.

Critical assumptions: regular case *vs* singular case

Most results in H_2 and H_∞ optimal control deal with a so-called a regular problem or regular case because it is simple. An H_2 or H_∞ control problem is said to be **regular** if the following conditions are satisfied,

1. D_2 is of maximal column rank, i.e., D_2 is a tall and full rank matrix
2. The subsystem (A, B, C_2, D_2) has no invariant zeros on the imaginary axis;
3. D_1 is of maximal row rank, i.e., D_1 is a fat and full rank matrix
4. The subsystem (A, E, C_1, D_1) has no invariant zeros on the imaginary axis;

$$\Sigma : \begin{cases} \dot{x} = A\,x + B\,u + E\,w \\ y = C_1\,x + D_1\,w \\ z = C_2\,x + D_2\,u \end{cases}$$

An H_2 or H_∞ control problem is said to be **singular** if it is not regular, i.e., at least one of the above 4 conditions is not satisfied.

Note: For state feedback control, Conditions 1 and 2 are sufficient for the regular case.

Classification of H_2 and H_∞ control problems……

①

Regular problem under static state feedback control laws…

State Feedback

②

Singular problem under static state feedback control laws…

Regular Case

Singular Case

Regular problem under dynamic output feedback control laws…

③

Output Feedback

Singular problem under dynamic output feedback control laws…

④

Solutions to the state feedback problems: the regular case

The state feedback H_2 and H_∞ control problems are referred to the problems in which all the states of the given plant Σ are available for feedback. That is the given system is

$$\Sigma: \begin{cases} \dot{x} = A\,x + B\,u + E\,w \\ y = x \\ z = C_2\,x + D_2\,u \end{cases}$$

where (A, B) is stabilizable, D_2 is of maximal column rank and (A, B, C_2, D_2) has no invariant zeros on the imaginary axis.

In the state feedback case, we look for a static control law, instead of a dynamical control law,

$$u = F\,x$$

which would give us the required H_2 and H_∞ performance.

Solution to the regular H_2 state feedback problem

h2care
h2state
gm2star

Solve the following algebraic Riccati equation (H_2-ARE)

$$A^\top P + PA + C_2^\top C_2 - \left(PB + C_2^\top D_2\right)\left(D_2^\top D_2\right)^{-1}\left(D_2^\top C_2 + B^\top P\right) = 0$$

for a unique positive semi-definite stabilizing solution $P \geq 0$. The H_2 optimal state feedback law is then given by

$$u = F\,x = -(\,D_2^\top D_2\,)^{-1}\left(\,D_2^\top C_2 + B^\top P\,\right)x$$

It can be showed that the resulting closed-loop system $T_{zw}(s)$ has the following property:

$$\|\,T_{zw}\,\|_2 = \gamma_2^* \,.$$

It can also be showed that $\gamma_2^* = [\,\mathrm{trace}\,(E'PE)]^{1/2}$.

Note: the trace of a matrix is defined as the sum of all its diagonal elements.

Example: Consider a system characterized by

$$\Sigma : \begin{cases} \dot{x} = \begin{bmatrix} 5 & 2 \\ 3 & 4 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 1 \\ 2 \end{bmatrix} w \\ y = x \\ z = \begin{bmatrix} 1 & 1 \end{bmatrix} x + 1 \cdot u \end{cases}$$

ex2121

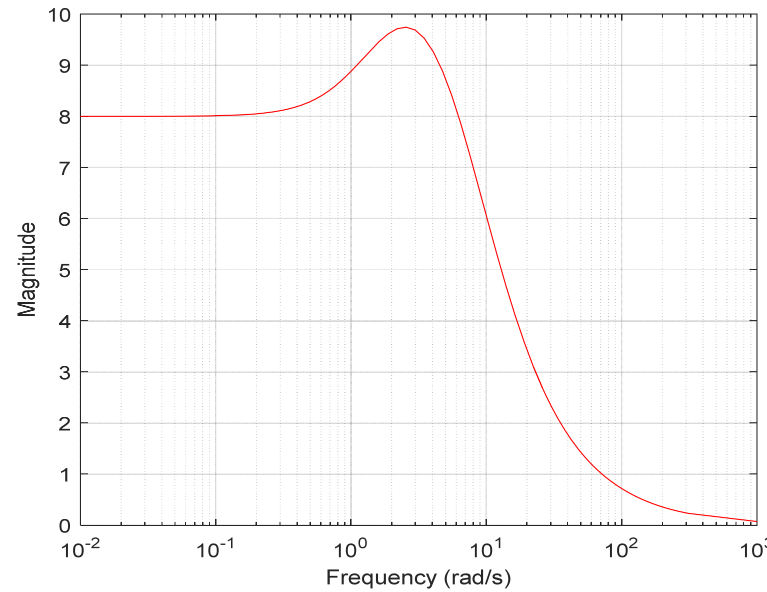
which has two unstable invariant zeros at 1.7639 and 6.2361, respectively. Solving the H_2-ARE using MATLAB, we obtain a positive definite solution

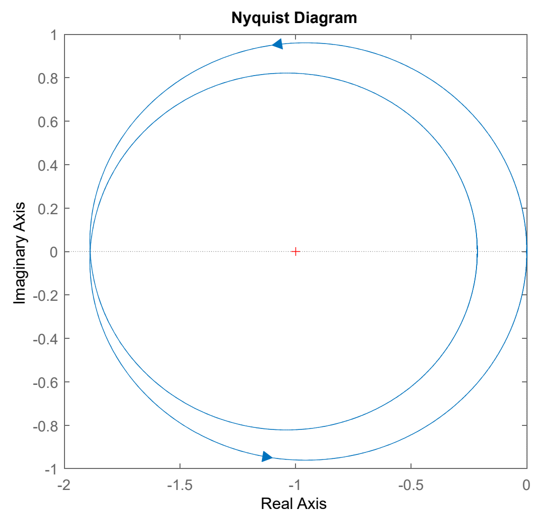
$$P = \begin{bmatrix} 144 & 40 \\ 40 & 16 \end{bmatrix}, \quad F = \begin{bmatrix} -41 & -17 \end{bmatrix}$$

The closed-loop magnitude response from the disturbance to the controlled output is given on the right. The H_2 optimal performance or infimum is given by

$$\gamma_2^* = 19.1833$$

Tzws





Gain Margin = (0.529, 4.673) and Phase Margin = 47.5°

It does **NOT** guarantee an ∞ GM & 60° PM!

Classical LQR problem is a special case of H_2 control

It can be shown that the well-known LQR problem can be re-formulated as an H_2 optimal control problem. Consider a linear system,

$$\dot{x} = A\,x + B\,u, \quad x(0) = X_0$$

The LQR problem is to find a control law $u = F\,x$ such that the following index is minimized:

$$J = \int_0^\infty \left(x^\top Q\,x + u^\top R u \right) dt$$

where $Q \geq 0$ is a positive semi-definite matrix and $R > 0$ is a positive definite matrix. The problem is equivalent to finding a static state feedback H_2 optimal control law $u = F\,x$ for

$$\begin{cases} \dot{x} = A\,x + B\,u + X_0 w \\ y = x \\ z = \begin{bmatrix} 0 \\ Q^{1/2} \end{bmatrix} x + \begin{bmatrix} R^{1/2} \\ 0 \end{bmatrix} u \end{cases}$$

John Doyle
CalTech

Solution to the regular H_∞ state feedback problem

h8care
h8state
gm8star

Given $\gamma > \gamma_\infty^*$, solve the following algebraic Riccati equation (H_∞-ARE)

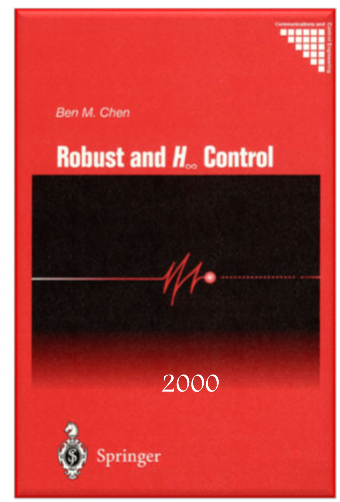
$$A^\top P + PA + C_2^\top C_2 + \gamma^{-2} PEE^\top P - (PB + C_2^\top D_2)\left(D_2^\top D_2\right)^{-1}(D_2^\top C_2 + B^\top P) = 0$$

for a unique positive semi-definite stabilizing solution $P \geq 0$. The H_∞ γ-suboptimal state feedback law is then given by

$$u = F\,x = -(D_2^\top D_2)^{-1}\left(D_2^\top C_2 + B^\top P\right)x$$

The resulting closed-loop system $T_{zw}(s)$ has the following property: $\| T_{zw} \|_\infty < \gamma$.

Remark: The computation of the best achievable H_∞ attenuation level, γ_∞^*, is in general quite complicated. For certain cases, γ_∞^* can be computed exactly. There are cases in which γ_∞^* can only be obtained using some iterative algorithms. One method is to keep solving the H_∞-ARE for different values of γ until it hits γ_∞^* for which and any $\gamma < \gamma_\infty^*$, the H_∞-ARE does not have a solution. Please see the reference by Chen (2000) for details.



Example: Again, consider the following system

$$\Sigma: \begin{cases} \dot{x} = \begin{bmatrix} 5 & 2 \\ 3 & 4 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 1 \\ 2 \end{bmatrix} w \\ y = x \\ z = \begin{bmatrix} 1 & 1 \end{bmatrix} x + 1 \cdot u \end{cases}$$

ex2121

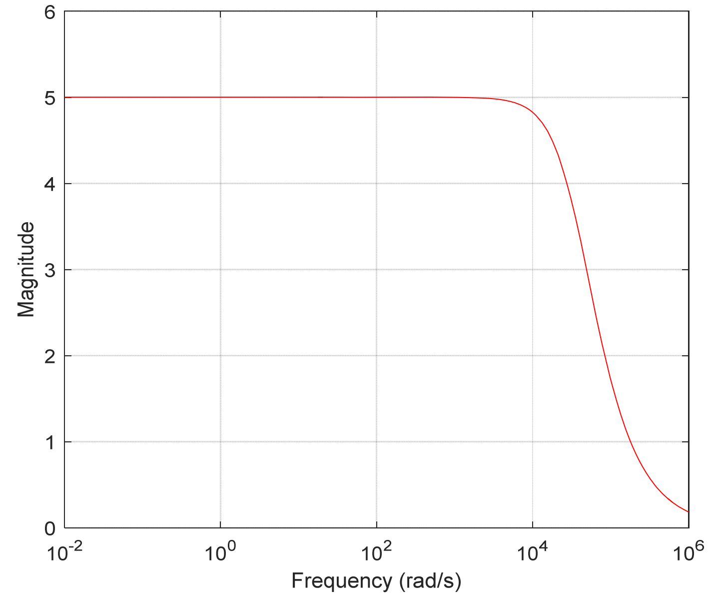
It can be showed that the best achievable H_∞ performance for this system is $\gamma_\infty^* = 5$. Solving the H_∞-ARE using MATLAB with $\gamma = 5.001$, we obtain a solution

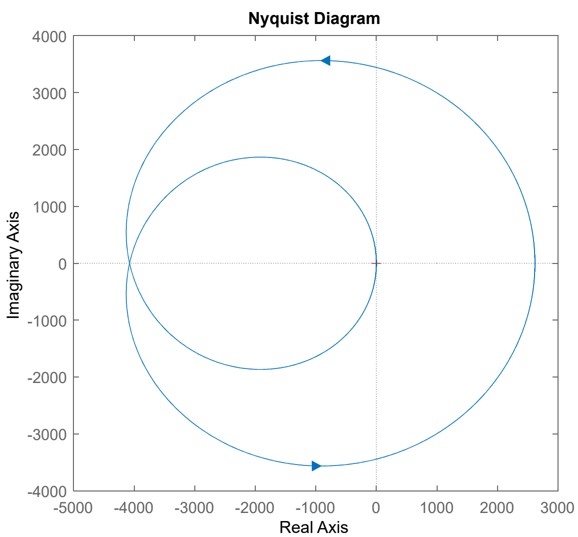
$$P = \begin{bmatrix} 330111.5 & 110028.8 \\ 110028.8 & 26679.1 \end{bmatrix},$$

$$F = \begin{bmatrix} -110029.8 & -36680.1 \end{bmatrix}$$

Tzws

The closed-loop magnitude response from the disturbance to the controlled output is given on the right. The worse case gain, occurred at the low frequency is equal to 5.000999775.





Gain Margin = (0.000245, ∞) and Phase Margin = 90°

About good and bad invariant zeros……

For simplicity, we consider the system

$$\Sigma : \begin{cases} \dot{x} = A\,x + B\,u + E\,w \\ y = x \\ z = C_2\,x + D_2\,u \end{cases}$$

with D_2 being square and full rank, i.e., it is nonsingular. We can then apply a pre-feedback $u = -D_2^{-1}C_2\,x + D_2^{-1}\tilde{u}$ to the given system, which yields

$$\begin{cases} \dot{x} = (A - BD_2^{-1}C_2)\,x + BD_2^{-1}\,\tilde{u} + E\,w \\ y = x \\ z = 0\,x + I\,\tilde{u} \end{cases} \Rightarrow \begin{cases} \dot{x} = \bar{A}\,x + \bar{B}\,\tilde{u} + E\,w \\ y = x \\ z = 0\,x + I\,\tilde{u} \end{cases}$$

and the Rosenbrock system matrix of the subsystem from $\tilde{u}$ to z is given by

$$P_\Sigma(s) = \begin{bmatrix} sI - \bar{A} & -\bar{B} \\ 0 & I \end{bmatrix} \Rightarrow$$

All the eigenvalues of $\bar{A}$ are the invariant zeros of the system!

If (A, B, C_2, D_2) is of minimum phase, i.e., all its invariant zeros are stable, or equivalently $\bar{A}$ is a stable matrix, the its corresponding H_2-ARE, i.e.,

$$\bar{A}^{\top} P + P\bar{A} + \cancel{C_2^{\top} C_2}^{\,0} - \left(P\bar{B} + \cancel{C_2^{\top} D_2}^{\,0}\right)\left(\cancel{D_2^{\top} D_2}^{\,I}\right)^{-1}\left(\cancel{D_2^{\top} C_2}^{\,0} + \bar{B}^{\top} P\right) = 0$$

$$\begin{cases} \dot{x} = \bar{A}\, x + \bar{B}\, \tilde{u} + E\, w \\ y = \quad x \\ z = 0\, x + I\, \tilde{u} \end{cases}$$

can be simplified as

$$\bar{A}^{\top} P + P\bar{A} - P\bar{B}\,\bar{B}^{\top} P = 0$$

Then, it can be seen that $P = 0$ is the required solution!

The optimal solution is given by

$$\tilde{u} = \bar{F}\, x = -(\, I^{\top} I\,)^{-1}\left(\, I^{\top} \cdot 0 + \bar{B}^{\top} \cdot 0\,\right) x = 0$$

and the solution in terms of the original control input is given by

$$\boxed{u = -D_2^{-1} C_2\, x + D_2^{-1} \tilde{u} = -D_2^{-1} C_2\, x}$$

Similarly, the corresponding H_∞-ARE, i.e.,

$$\bar{A}^\top P + P\bar{A} + \overset{0}{C_2^\top C_2} + \gamma^{-2} PEE^\top P - (P\bar{B} + \overset{0}{C_2^\top D_2})\left(\overset{I}{D_2^\top D_2}\right)^{-1}(\overset{0}{D_2^\top C_2} + \bar{B}^\top P) = 0$$

can be simplified as

$$\bar{A}^\top P + P\bar{A} + \gamma^{-2} PEE^\top P - P\bar{B}\bar{B}^\top P = 0$$

Again, $P = 0$ is the required solution. The optimal solution (for this special situation, the H_∞ control has an optimal solution) is given by

$$\tilde{u} = \bar{F}\, x = -(I^\top I)^{-1}\left(I^\top \cdot 0 + \bar{B}^\top \cdot 0\right) x = 0$$

and the solution in terms of the original control input is given by

$$u = -D_2^{-1} C_2 x + D_2^{-1}\tilde{u} = -D_2^{-1} C_2 x$$

In both H_2 and H_∞ cases, the closed-loop transfer function matrix from w to z is

$$T_{zw}(s) = 0 \cdot \left(sI - \bar{A}\right)^{-1} E \equiv 0$$

The disturbance is totally rejected. Also note that the closed-loop system poles are exactly the invariant zeros of (A, B, C_2, D_2). They cancel each other!

If (A, B, C_2, D_2) has all its invariant zeros to be unstable, or equivalently $\bar{A}$ is an anti-stable matrix, the its corresponding H_2-ARE, i.e.,

$$\bar{A}^{\top} P + P\bar{A} - P\bar{B}\,\bar{B}^{\top} P = 0$$

has a solution $P = 0$ too. But it does not give a stabilizing control law (why?)...

However, it can be converted into a Lyapunov equation

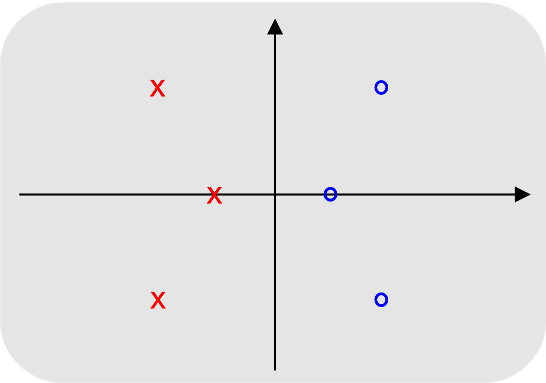
$$P^{-1}\left(-\bar{A}\right)^{\top} + \left(-\bar{A}\right)P^{-1} = -\bar{B}\,\bar{B}^{\top} \qquad \left[\quad \Rightarrow \quad P^{-1}\left(-\bar{A}\right)^{\top} P = \bar{A} - \bar{B}\bar{B}^{\top} P \quad\right]$$

From the Lyapunov stability theorem, it has a unique positive definite solution. The optimal performance is $\gamma_2^* = \sqrt{\operatorname{trace}(E' P E)}$ and the optimal solution is given by

$$\tilde{u} = \bar{F} x = -(I^{\top} I)^{-1}\left(I^{\top} \cdot 0 + \bar{B}^{\top} \cdot P \right) x = -\bar{B}^{\top} P x \quad \Rightarrow \quad u = -\left(D_2^{-1} C_2 + \left(D_2^{\top} D_2 \right)^{-1} B^{\top} P \right) x$$

The resulting closed-loop system matrix

$$\bar{A} + \bar{B}\bar{F} = \bar{A} - \bar{B}\bar{B}^{\top} P = P^{-1}\left(-\bar{A}\right)^{\top} P$$



That is the closed-loop system poles are located right at the mirror images of the unstable invariant zeros of the subsystem (A, B, C_2, D_2).

Similarly, the corresponding H_∞-ARE, i.e.,

$$\bar{A}^\top P + P\bar{A} + \gamma^{-2} PEE^\top P - P\bar{B}\,\bar{B}^\top P = 0$$

can be re-written as

$$P^{-1}\bar{A}^\top + \bar{A}P^{-1} = \bar{B}\,\bar{B}^\top - \gamma^{-2} EE^\top$$

which can be solved by solving two Lyapunov equations:

$$S\bar{A}^\top + \bar{A}S = \bar{B}\,\bar{B}^\top \quad \text{and} \quad T\bar{A}^\top + \bar{A}T = EE^\top$$

Ian Petersen
Australian National University

It can be showed that

$$\gamma_\infty^* = \sqrt{\lambda_{\max}(TS^{-1})} \quad \text{and} \quad P = \left(S - \frac{1}{\gamma^2}T\right)^{-1} > 0, \quad \forall \gamma > \gamma_\infty^*$$

The γ-suboptimal solution is given as

$$u = -\left(D_2^{-1}C_2 + \left(D_2^\top D_2\right)^{-1} B^\top P\right)x$$


Exact Computation of the Infimum in H_∞-Optimization Via Output Feedback

More general results for the singular case can be found in Chen *et al.* (1992).

★ I. R. Petersen, “Disturbance attenuation and H_∞-optimization: A design method based on the algebraic Riccati equation,” *IEEE Transactions on Automatic Control*, Vol. AC-32, pp. 427-429, 1987.

★ B. M. Chen, *et al.*, “Exact computation of the infimum in H_∞-optimization via output feedback,” *IEEE Transactions on Automatic Control*, Vol. 37, pp. 70-78, 1992.

Side note: For the case when $\bar{A}$ has both stable and unstable eigenvalues, there exists a similarity transformation T such that

$$T^{-1}\bar{A}T = \begin{bmatrix} A_- & 0 \\ 0 & A_+ \end{bmatrix}, \quad A_- \text{ stable, } A_+ \text{ anti-stable.}$$

One can then deal with each part separately.

The solution to the ARE corresponding to the stable zero dynamic part is **0** and the solution to the ARE corresponding to the unstable part cannot be set to **0** (it can be calculated by solving Lyapunov equations as on the previous page), which implies…

$$P = \begin{bmatrix} 0 & 0 \\ 0 & P_+ \end{bmatrix}$$

- When the disturbance enters the system through the stable zero dynamic subspace, its effect to the output to be controlled can be totally attenuated.
- When the disturbance enters the system through the unstable zero subspace, the attenuation of its effect to the output to be controlled is limited.

This once again confirms that a nonminimum phase system would result in a bad overall control performance including disturbance attenuation.

Solutions to the state feedback problems – the singular case

Consider the following system again,

$$\Sigma: \begin{cases} \dot{x} = A\,x + B\,u + E\,w \\ y = x \\ z = C_2\,x + D_2\,u \end{cases}$$

Kemin Zhou
Louisiana State University

where (A,B) is stabilizable, D_2 is not necessarily of maximal rank and (A,B,C_2,D_2) might have invariant zeros on the imaginary axis.

Solution to this kind of problems can be done using the following trick (or so-called a **perturbation approach**): Define a new controlled output

$$\tilde{z} = \begin{bmatrix} z \\ \varepsilon x \\ \varepsilon u \end{bmatrix} = \begin{bmatrix} C_2 \\ \varepsilon I \\ 0 \end{bmatrix} x + \begin{bmatrix} D_2 \\ 0 \\ \varepsilon I \end{bmatrix} u$$

small perturbations

P Khargonekar
University of Florida

Clearly, $z \propto \tilde{z}$ if $\varepsilon = 0$.

* K. Zhou and P. Khargonekar, "An algebraic Riccati equation approach to H_∞-optimization," *Systems & Control Letters,* Vol. 11, pp. 85-91, 1988.

The perturbed system is nonunique and can be done in many ways. The whole idea is to make the perturbed system satisfying the regularity assumptions. It can be showed that

$$\tilde{\Sigma} \; : \; \begin{cases} \dot{x} = A\,x + B\,u + E\,w \\ y = \quad x \\ \tilde{z} = \tilde{C}_2 x + \tilde{D}_2 u \end{cases} \qquad \text{with} \qquad \tilde{C}_2 := \begin{bmatrix} C_2 \\ \varepsilon I \\ 0 \end{bmatrix} \quad \text{and} \quad \tilde{D}_2 := \begin{bmatrix} D_2 \\ 0 \\ \varepsilon I \end{bmatrix}$$

always satisfies the regularity assumptions when $\varepsilon \neq 0$. We should note that the perturbation block in $\tilde{D}_2$ might be replaced by any other perturbed term so long as the resulting $\tilde{D}_2$ is of full rank. Similarly, the perturbation block in $\tilde{C}_2$ might be omitted or replaced by another one as long as $(A, B, \tilde{C}_2, \tilde{D}_2)$ has no zeros on the imaginary axis. Here is an example:

$$\begin{cases} \dot{x} = \begin{bmatrix} 5 & 2 \\ 3 & 4 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 1 \\ 2 \end{bmatrix} w \\ y = x \\ z = \begin{bmatrix} 1 & 1 \end{bmatrix} x + 0 \cdot u \end{cases} \Rightarrow \begin{cases} \dot{x} = \begin{bmatrix} 5 & 2 \\ 3 & 4 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 1 \\ 2 \end{bmatrix} w \\ y = x \\ \tilde{z} = \begin{bmatrix} 1 & 1 \\ \varepsilon & 0 \\ 0 & \varepsilon \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \varepsilon \end{bmatrix} u \end{cases} \quad \text{or} \quad \begin{cases} \dot{x} = \begin{bmatrix} 5 & 2 \\ 3 & 4 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 1 \\ 2 \end{bmatrix} w \\ y = x \\ \tilde{z} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ \varepsilon \end{bmatrix} u \end{cases}$$

Note: The blue perturbation blocks in the above example might be omitted as the perturbed system would also meet the regularity assumptions without these blocks.

Solution to the general H_2 state feedback problem

Given a small $\varepsilon > 0$, Solve the following algebraic Riccati equation (H_2-ARE)

$$A^\top \tilde{P} + \tilde{P} A + \tilde{C}_2^\top \tilde{C}_2 - \left(\tilde{P} B + \tilde{C}_2^\top \tilde{D}_2 \right) \left(\tilde{D}_2^\top \tilde{D}_2 \right)^{-1} \left(\tilde{D}_2^\top \tilde{C}_2 + B^\top \tilde{P} \right) = 0$$

for a unique positive definite solution $\tilde{P} > 0$. Obviously, $\tilde{P}$ is a function of ε. The H_2 suboptimal state feedback law is then given by

$$u = \tilde{F}\, x = -(\tilde{D}_2^\top \tilde{D}_2)^{-1} \left(\tilde{D}_2^\top \tilde{C}_2 + B^\top \tilde{P} \right) x$$

It can be showed that the resulting closed-loop system $T_{zw}(s)$ has

$$\| T_{zw} \|_2 \to \gamma_2^* \quad \text{as} \quad \varepsilon \to 0$$

It can also be showed that

$$\left[\operatorname{trace}(E' \tilde{P} E) \right]^{1/2} \to \gamma_2^* \quad \text{as} \quad \varepsilon \to 0.$$

gm2star
gm2sos
h2out

Example: Consider a system characterized by

$$\Sigma: \begin{cases} \dot{x} = \begin{bmatrix} 5 & 2 \\ 3 & 4 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 1 \\ 2 \end{bmatrix} w \\ y = x \\ z = \begin{bmatrix} 1 & 1 \end{bmatrix} x + 0 \cdot u \end{cases}$$

Solving the following H_2-ARE using MATLAB with $\varepsilon = 1$, we obtain

$$\tilde{P} = \begin{bmatrix} 186.1968 & 46.2778 \\ 46.2778 & 18.2517 \end{bmatrix}, \quad F = \begin{bmatrix} -46.2778 & -18.2517 \end{bmatrix}$$

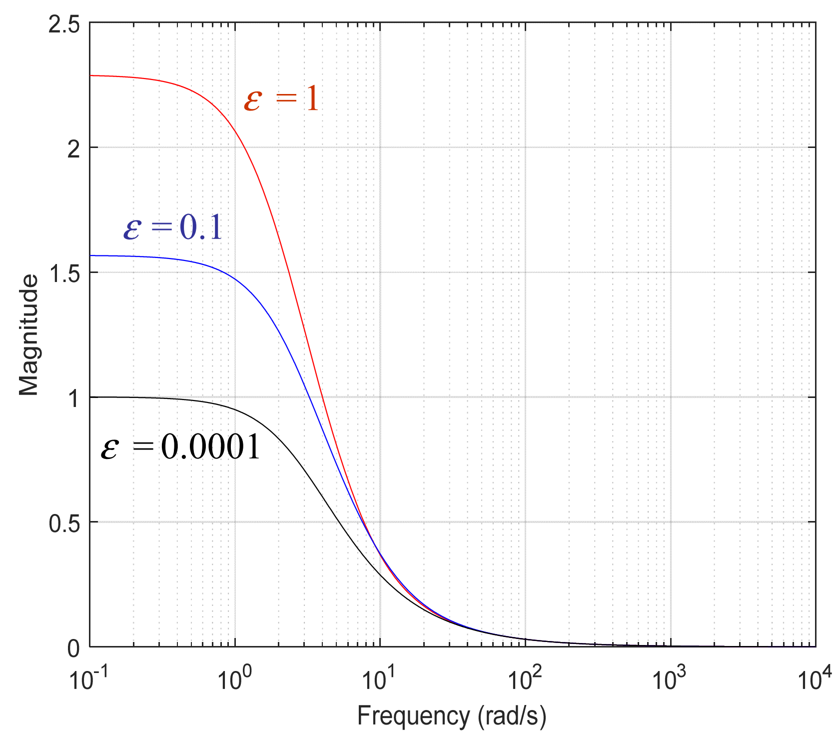
- $\varepsilon = 0.1$

$$\tilde{P} = \begin{bmatrix} 4.3046 & 0.6944 \\ 0.6944 & 0.2387 \end{bmatrix}, \quad F = \begin{bmatrix} -69.4426 & -23.8688 \end{bmatrix}$$

- $\varepsilon = 0.0001$

$$\tilde{P} = \begin{bmatrix} 1.5016 & 0.0004 \\ 0.0004 & 0.0001 \end{bmatrix}, \quad F = \begin{bmatrix} -4.0023 & -1.0012 \end{bmatrix} \times 10^4$$

The closed-loop magnitude response from the disturbance to the controlled output:



The optimal performance or infimum is given by

$$\gamma_2^* = 1.225$$

Solution to the general H_∞ state feedback problem

$$\tilde{C}_2 := \begin{bmatrix} C_2 \\ \varepsilon I \\ 0 \end{bmatrix} \quad \text{and} \quad \tilde{D}_2 := \begin{bmatrix} D_2 \\ 0 \\ \varepsilon I \end{bmatrix}$$

Step 1: Given a $\gamma > \gamma_\infty^*$, choose $\varepsilon = 1$.

Step 2: Define the corresponding $\tilde{C}_2$ and $\tilde{D}_2$

Step 3: Solve the following algebraic Riccati equation (H_∞-ARE) for $\tilde{P}$:

$$A^\top \tilde{P} + \tilde{P} A + \tilde{C}_2^\top \tilde{C}_2 + \gamma^{-2} \tilde{P} E E^\top \tilde{P} - (\tilde{P} B + \tilde{C}_2^\top \tilde{D}_2) \left(\tilde{D}_2^\top \tilde{D}_2 \right)^{-1} (\tilde{D}_2^\top \tilde{C}_2 + B^\top \tilde{P}) = 0$$

Step 4: If $\tilde{P} > 0$, go to Step 5. Otherwise, reduce the value of ε and go to Step 2.

Step 5: Compute the required state feedback control law

$$u = \tilde{F} x = -(\tilde{D}_2^\top \tilde{D}_2)^{-1} \left(\tilde{D}_2^\top \tilde{C}_2 + B^\top \tilde{P} \right) x$$

It can be showed that the resulting closed-loop system $T_{zw}(s)$ has: $\| T_{zw} \|_\infty < \gamma$.

More general results for the singular case can be found in Chen (2000).

gm8star
gm8sos
h8out

Example: Again, consider the following system

$$\Sigma: \begin{cases} \dot{x} = \begin{bmatrix} 5 & 2 \\ 3 & 4 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 1 \\ 2 \end{bmatrix} w \\ y = x \\ z = \begin{bmatrix} 1 & 1 \end{bmatrix} x + 0 \cdot u \end{cases}$$

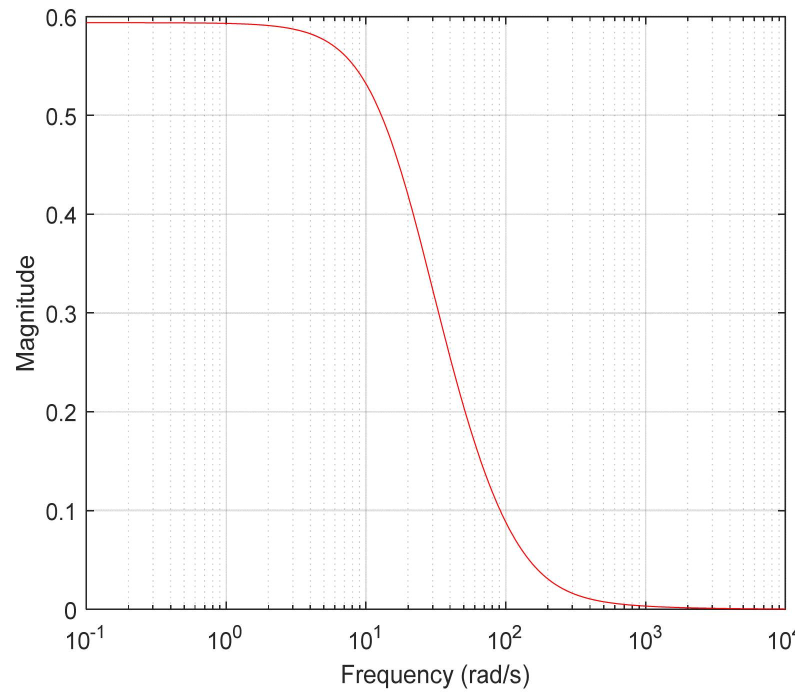
It can be showed that the best achievable H_∞ performance for this system is $\gamma_\infty^* = 0.5$. Solving the following H_∞-ARE using MATLAB with $\gamma = 0.6$ and $\varepsilon = 0.01$, we obtain a positive definite solution

$$P = \begin{bmatrix} 6.3774 & 0.1373 \\ 0.1373 & 0.0131 \end{bmatrix}$$

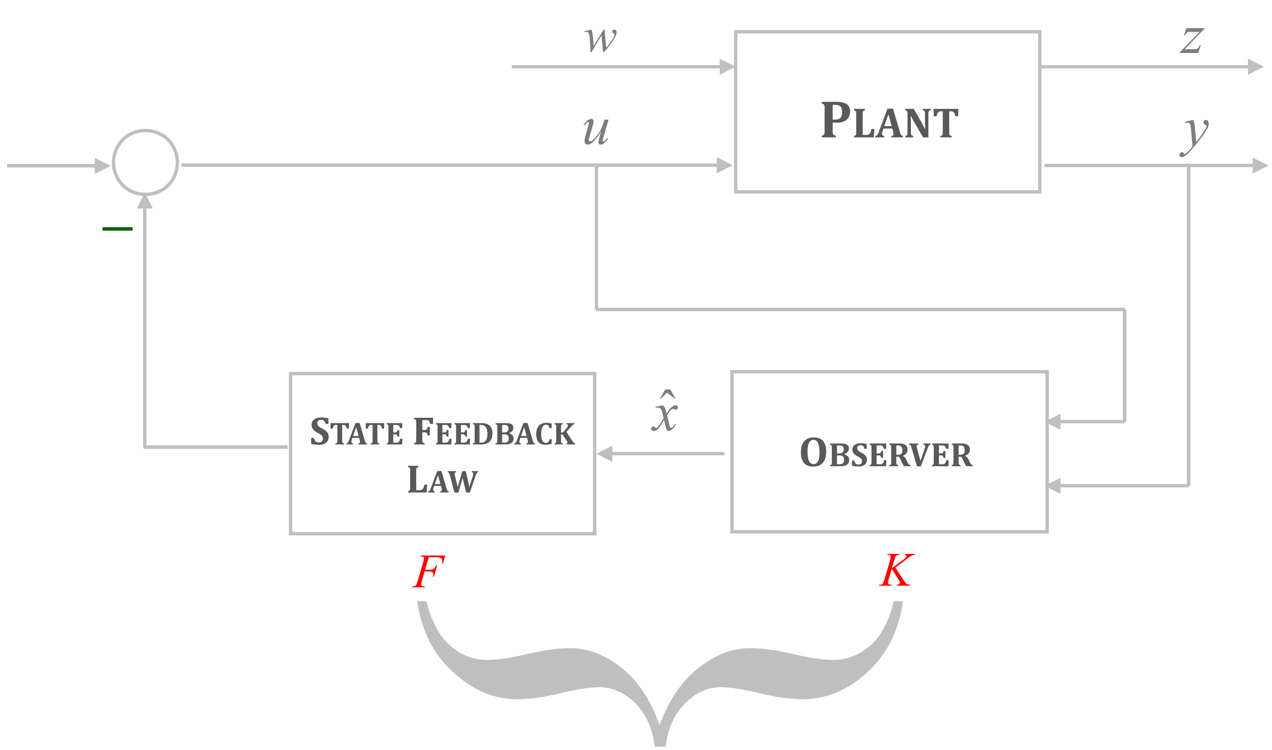
and

$$F = \begin{bmatrix} -1373 & -131.5 \end{bmatrix}$$

The closed-loop magnitude response from the disturbance to the controlled output:



Clearly, the worse case gain, occurred at the low frequency is slightly less than 0.6. The design specification is achieved.



∞ 2 ∞ Output Feedback Problems ∞ 2 ∞

Solutions to output feedback problems – the regular case

Recall the system with measurement feedback, i.e.,

$$\Sigma: \quad \begin{cases} \dot{x} = A\,x + B\,u + E\,w \\ y = C_1\,x + D_1\,w \\ z = C_2\,x + D_2\,u \end{cases}$$

where (A, B) is stabilizable and (A, C_1) is detectable. Also, it satisfies the following regularity assumptions:

1. D_2 is of maximal column rank, i.e., D_2 is a tall and full rank matrix
2. The subsystem (A, B, C_2, D_2) has no invariant zeros on the imaginary axis
3. D_1 is of maximal row rank, i.e., D_1 is a fat and full rank matrix
4. The subsystem (A, E, C_1, D_1) has no invariant zeros on the imaginary axis

Solution to the regular H_2 output feedback problem

Solve the following algebraic Riccati equation (H_2-ARE)

$$A^\top P + PA + C_2^\top C_2 - \left(PB + C_2^\top D_2\right)\left(D_2^\top D_2\right)^{-1}\left(D_2^\top C_2 + B^\top P\right) = 0$$

for a unique positive semi-definite stabilizing solution $P \ge 0$, and the following ARE

$$QA^\top + AQ + EE^\top - \left(QC_1^\top + ED_1^\top\right)\left(D_1 D_1^\top\right)^{-1}\left(D_1 E^\top + C_1 Q\right) = 0$$

for a unique positive semi-definite stabilizing solution $Q \ge 0$. The H_2 optimal output feedback law is then given by

$$\Sigma_{\mathrm{cmp}} : \begin{cases} \dot{x}_{\mathrm{cmp}} = \left(A + BF + KC_1\right) x_{\mathrm{cmp}} - K\, y \\ u = F\, x_{\mathrm{cmp}} \end{cases}$$

where $F = -(D_2^\top D_2)^{-1}(D_2^\top C_2 + B^\top P)$ and $K = -(QC_1^\top + ED_1^\top)(D_1 D_1^\top)^{-1}$.

Furthermore,

$$\gamma_2^* = \left\{ \mathrm{trace}(E^\top P E) + \mathrm{trace}\left[\left(A^\top P + PA + C_2^\top C_2\right) Q \right] \right\}^{1/2}.$$

gm2sos
h2out

Example: Consider a system characterized by

$$\Sigma : \begin{cases} \dot{x} = \begin{bmatrix} 5 & 2 \\ 3 & 4 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 1 \\ 2 \end{bmatrix} w \\ y = \begin{bmatrix} 0 & 1 \end{bmatrix} x \qquad + 1 \cdot w \\ z = \begin{bmatrix} 1 & 1 \end{bmatrix} x + 1 \cdot u \end{cases}$$

Solving the following H_2-AREs using MATLAB, we obtain

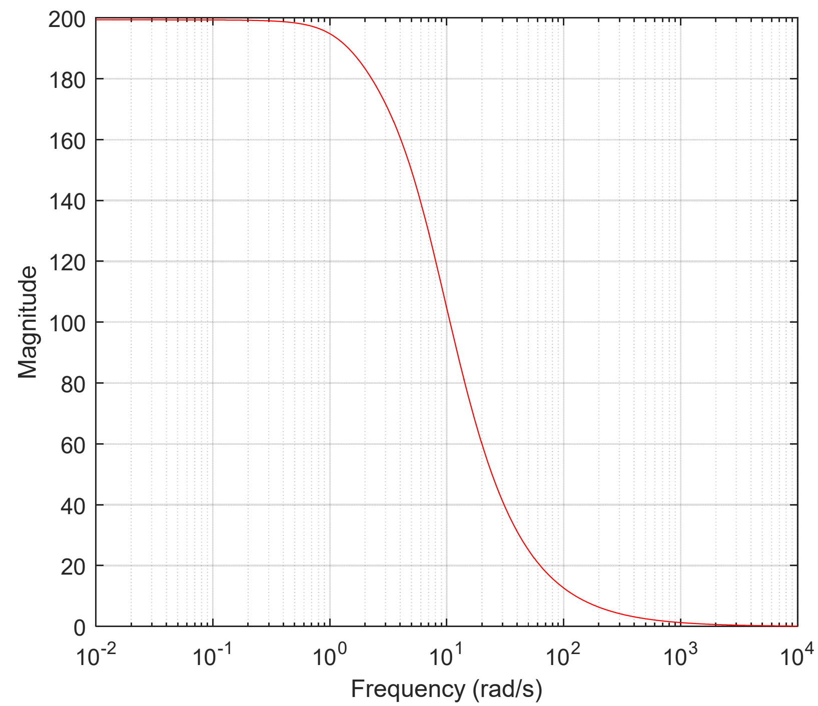
$$P = \begin{bmatrix} 144 & 40 \\ 40 & 16 \end{bmatrix}, \quad F = \begin{bmatrix} -41 & -17 \end{bmatrix}$$

$$Q = \begin{bmatrix} 49.7778 & 23.3333 \\ 23.3333 & 14.0000 \end{bmatrix}, \quad K = \begin{bmatrix} -24.3333 \\ -16.0000 \end{bmatrix}$$

and an output feedback control law,

$$\Sigma_{\text{cmp}} : \begin{cases} \dot{x}_{\text{cmp}} = \begin{bmatrix} 5 & -22.3333 \\ -38 & -29 \end{bmatrix} x_{\text{cmp}} + \begin{bmatrix} 24.3333 \\ 16 \end{bmatrix} y \\ u = \begin{bmatrix} -41 & -17 \end{bmatrix} x_{\text{cmp}} \end{cases}$$

The closed-loop magnitude response from the disturbance to the controlled output:



The optimal performance or infimum is given by

$$\gamma_2^* = 347.3$$

Solution to the regular H_∞ output feedback problem

Given a $\gamma > \gamma_\infty^*$, solve the following algebraic Riccati equation (H_∞-ARE)

$$A^\top P + PA + C_2^\top C_2 + \gamma^{-2} PEE^\top P - (PB + C_2^\top D_2)(D_2^\top D_2)^{-1}(D_2^\top C_2 + B^\top P) = 0$$

for a positive semi-definite stabilizing solution $P \geq 0$, and the ARE

$$QA^\top + AQ + EE^\top + \gamma^{-2} QC_2^\top C_2 Q - (QC_1^\top + ED_1^\top)(D_1 D_1^\top)^{-1}(D_1 E^\top + C_1 Q) = 0$$

for a positive semi-definite stabilizing solution $Q \geq 0$. In fact, these P and Q satisfy the so-called coupling condition: $\rho(PQ) < \gamma^2$.

The H_∞ γ-suboptimal output feedback law is then given by [DGKF]

$$\Sigma_{\text{cmp}} : \begin{cases} \dot{x}_{\text{cmp}} = A_{\text{cmp}}\, x_{\text{cmp}} + B_{\text{cmp}}\, y \\ u = C_{\text{cmp}}\, x_{\text{cmp}} \end{cases}$$

where

$$B_{\text{cmp}} = -(I - \gamma^{-2} QP)^{-1} K, \quad C_{\text{cmp}} = F,$$

$$A_{\text{cmp}} = A + \gamma^{-2} EE^\top P + BF + (I - \gamma^{-2} QP)^{-1} K (C_1 + \gamma^{-2} D_1 E^\top P),$$

and where $F = -(D_2^\top D_2)^{-1}(D_2^\top C_2 + B^\top P)$, $K = -(QC_1^\top + ED_1^\top)(D_1 D_1^\top)^{-1}$.


John Doyle


Keith Glover


P Khargonekar


Gilead Tadmor


Bruce Francis
1947–2018

Example: Consider a system characterized by

$$\Sigma: \begin{cases} \dot{x} = \begin{bmatrix} 5 & 2 \\ 3 & 4 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 1 \\ 2 \end{bmatrix} w \\ y = \begin{bmatrix} 0 & 1 \end{bmatrix} x \qquad + 1 \cdot w \\ z = \begin{bmatrix} 1 & 1 \end{bmatrix} x + 1 \cdot u \end{cases}$$

gm8sos
h8out

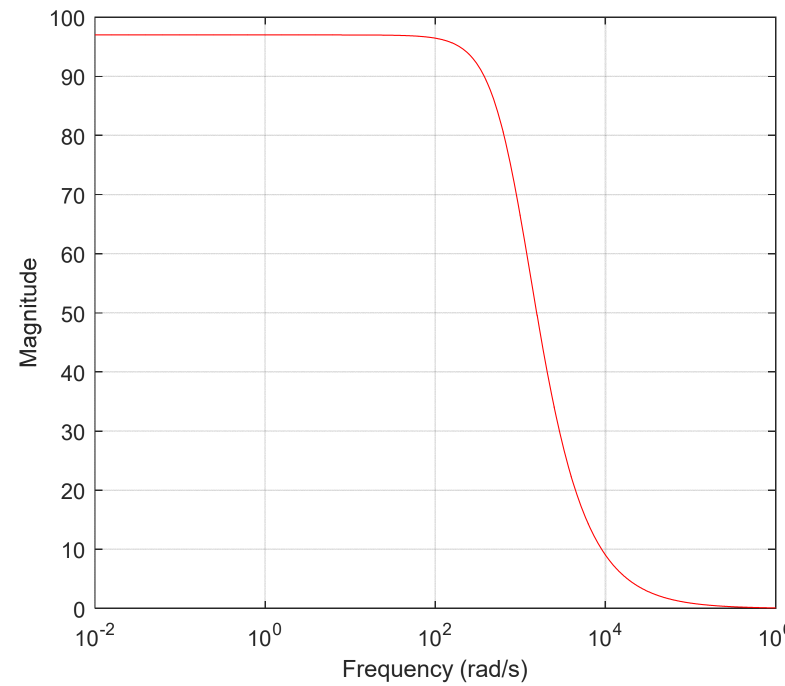
It can be showed that the best achievable H_∞ performance for this system is $\gamma_\infty^* = 96.32864$. Solving the following H_∞-AREs using MATLAB with $\gamma = 97$, we obtain

$$P = \begin{bmatrix} 144.353 & 40.1168 \\ 40.1168 & 16.0392 \end{bmatrix}, \quad Q = \begin{bmatrix} 49.8205 & 23.3556 \\ 23.3556 & 14.0118 \end{bmatrix}$$

and the corresponding controller

$$\begin{cases} \dot{x}_{\text{cmp}} = \begin{bmatrix} -38.808668 & -1848.4365 \\ -59.411030 & -914.00139 \end{bmatrix} x_{\text{cmp}} + \begin{bmatrix} 1836.35389 \\ 894.116965 \end{bmatrix} y \\ u \;= \begin{bmatrix} -41.116796 & -17.039215 \end{bmatrix} x_{\text{cmp}} \end{cases}$$

The closed-loop magnitude response from the disturbance to the controlled output:



Clearly, the worse case gain, occurred at the low frequency, is slightly less than 97 (96.998). The H_∞ performance specification is achieved.

Solutions to the output feedback problems – the singular case

For general systems for which the regularity conditions are not satisfied, it can be solved again using the perturbation approach. We define a new controlled output:

$$\tilde{z} = \begin{bmatrix} z \\ \varepsilon x \\ \varepsilon u \end{bmatrix} = \begin{bmatrix} C_2 \\ \varepsilon I \\ 0 \end{bmatrix} x + \begin{bmatrix} D_2 \\ 0 \\ \varepsilon I \end{bmatrix} u$$

gm2sos
h2out
gm8sos
h8out

and new matrices associated with the disturbance inputs:

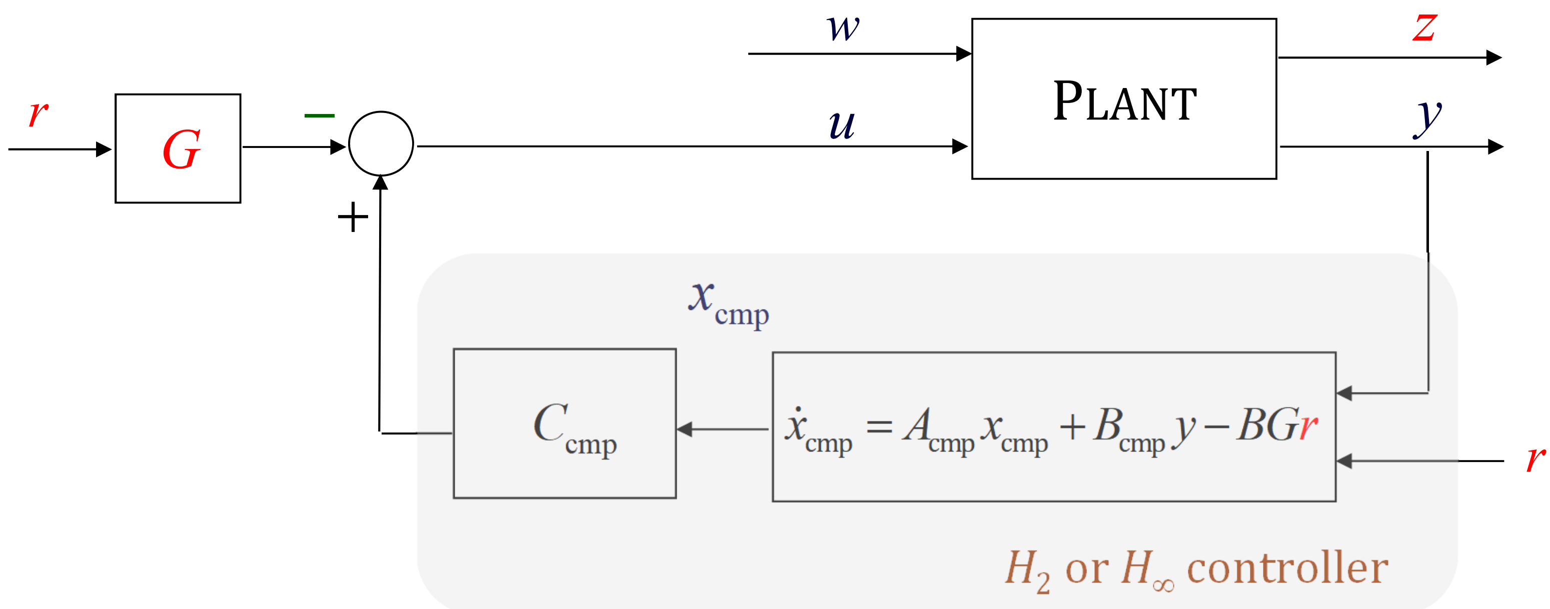
$$\tilde{E} = [\, E \quad \varepsilon I \quad 0 \,] \qquad \text{and} \qquad \tilde{D}_1 = [\, D_1 \quad 0 \quad \varepsilon I \,].$$

The H_2 and H_∞ control problems for singular output feedback case can be obtained by solving the following perturbed **regular** system with sufficiently small ε:

$$\tilde{\Sigma} : \begin{cases} \dot{x} = A\, x + B\, u + \tilde{E}\, \tilde{w} \\ y = C_1\, x + \tilde{D}_1\, \tilde{w} \\ \tilde{z} = \tilde{C}_2\, x + \tilde{D}_2\, u \end{cases}$$

***Remark:** Perturbation approach might have serious numerical problems!*

Block diagram of an H_2 or H_∞ control law with a reference……



$$G = [C_2 (A + BF)^{-1} B]^{-1} \text{ assuming } z = C_2 x \quad \Rightarrow \quad z(t) \to r, \text{ as } t \to \infty.$$

Some applications…

Side notes on the H_∞ singular case

1. ~~D_2 is of maximal column rank, i.e., D_2 is a tall and full rank matrix~~
2. ~~(A,B,C_2,D_2) has no invariant zeros on the imaginary axis~~
3. ~~D_1 is of maximal row rank, i.e., D_1 is a fat and full rank matrix~~
4. ~~(A,E,C_1,D_1) has no invariant zeros on the imaginary axis~~

Construction of closed-form solutions and computation of γ_∞^* etc…

Anton Stoorvogel & coworkers
University of Twente

Carsten Scherer
University of Stuttgart

BMC & coworkers

Side notes on (almost) disturbance decoupling

1. If $\gamma_2^* = 0$, then the corresponding H_2 optimal control problem is also called an H_2 (almost) disturbance decoupling problem. It can be showed that the H_2 almost disturbance decoupling problem is solvable if the following conditions are satisfied (**Good systems!**):

   - (A, B) is stabilizable and (A, C_1) is detectable
   - (A, B, C_2, D_2) is right invertible and has no invariant zeros on open RHP
   - (A, E, C_1, D_1) is left invertible and has no invariant zeros on open RHP

   Necessary and sufficient conditions for the solvability of the almost disturbance decoupling problem is available in the literature. However, they can only be expressed in terms of certain geometric subspaces on the given system…

★ B. M. Chen, Z. Lin and C. C. Hang, "Design for general H_∞ almost disturbance decoupling problem with measurement feedback and internal stability - An eigenstructure assignment approach," *International Journal of Control*, Vol. 71, pp. 653-685, 1998.

2. If $\gamma_\infty^* = 0$, then the corresponding H_∞ optimal control problem is also called an H_∞ almost disturbance decoupling problem. It can be showed that the H_∞ almost disturbance decoupling problem is solvable if the following conditions are satisfied:

- (A, B) is stabilizable and (A, C_1) is detectable
- (A, B, C_2, D_2) is right invertible and of minimum phase
- (A, E, C_1, D_1) is left invertible and of minimum phase

Studies on disturbance decoupling problems led to the development of the geometric theory in linear systems…

W. M. Wonham
Canadian Control Theorist
1934–2023

Jan C. Willems
Belgian Scientist
1939–2013

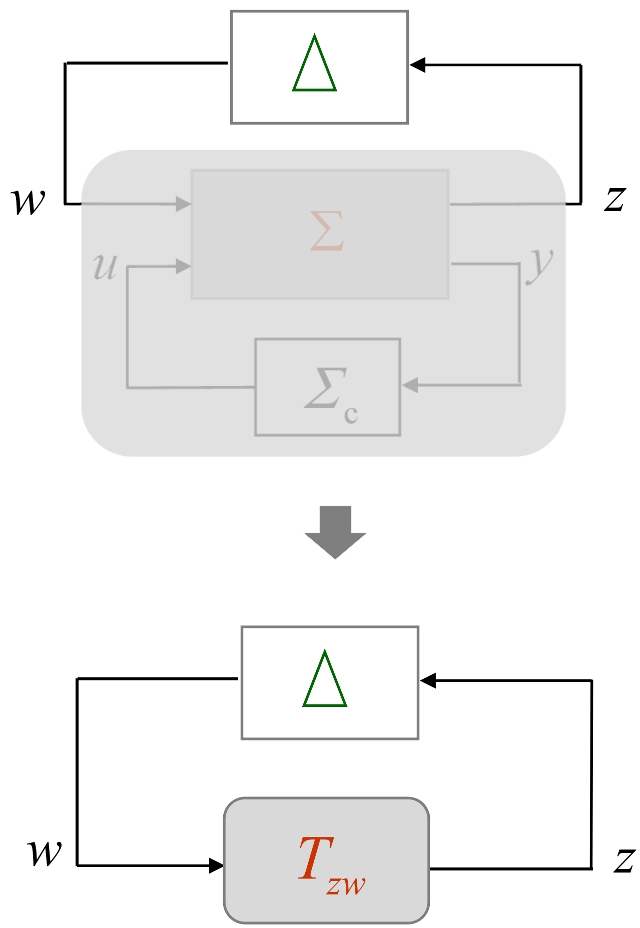
Carsten Scherer
University of Stuttgart

……

BMC & coworkers

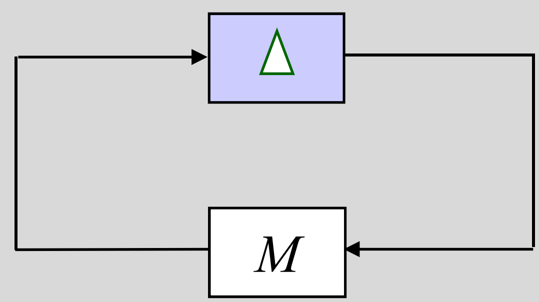
Robust stabilization of systems with unstructured uncertainties

Consider an uncertain plant with an unstructured perturbation,



Small Gain Theory (!)

If $\triangle$ is stable and $\|\Delta\|_\infty \cdot \|M\|_\infty < 1$, then the interconnected system is stable.

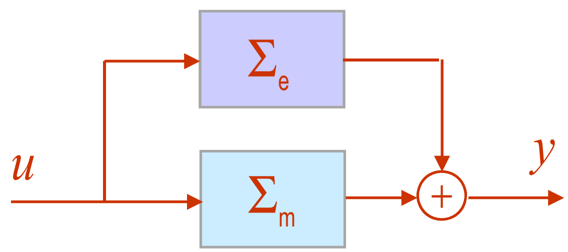


Assume $\| T_{zw} \|_\infty < \gamma$. Then the system with unstructured uncertainty is stable if

$$\| T_{zw} \|_\infty \cdot \| \Delta \|_\infty < \gamma \cdot \| \Delta \|_\infty < 1 \quad \Rightarrow \quad \| \Delta \|_\infty < \frac{1}{\gamma}$$

Robust stabilization with additive perturbation★

Consider an uncertain plant with additive perturbations,



Σ_m has a transfer function $G_m(s) = C_m(sI - A_m)^{-1}B_m + D_m$

Σ_e is an unknown perturbation.

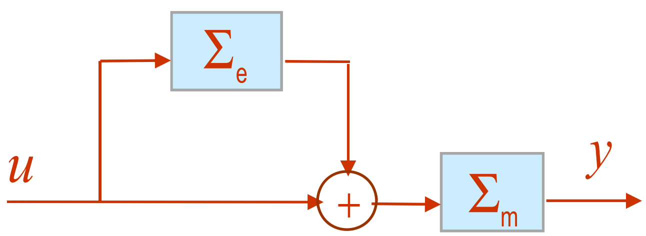
Σ_m and $\Sigma_m + \Sigma_e$ have same number of unstable poles.

Given a $\gamma_a > 0$, the problem of robust stabilization for plants with additive perturbations is to find a proper controller such that when it is applied to the uncertain plant, the resulting closed-loop system is stable for all possible perturbations with their L_∞-norm $\le \gamma_a$. (The definition of L_∞-norm is the same as that of H_∞-norm except for L_∞-norm, the system need not be stable.) Such a problem is equivalent to find an H_∞ γ-suboptimal control law (with $\gamma = 1/\gamma_a$) for

$$\Sigma_{\text{add}} : \begin{cases} \dot{x} = A_m\, x + B_m\, u + 0\, w \\ y = C_m\, x + D_m\, u + I\, w \\ z = \ 0 \ \ x + \ \ I \ \ u \end{cases}$$

Robust stabilization with multiplicative perturbation★

Consider an uncertain plant with multiplicative perturbations,



Σ_m has a transfer function $G_m(s) = C_m(sI - A_m)^{-1}B_m + D_m$

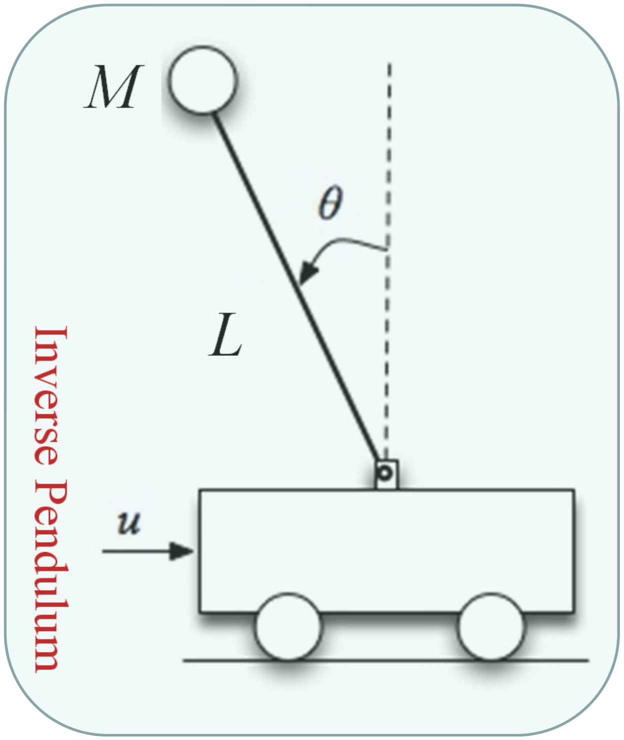
Σ_e is an unknown perturbation.

Σ_m and $\Sigma_m \times \Sigma_e$ have same number of unstable poles.

Given a $\gamma_m > 0$, the problem of robust stabilization for plants with multiplicative perturbations is to find a proper controller such that when it is applied to the uncertain plant, the resulting closed-loop system is stable for all possible perturbations with their L_∞-norm $\le \gamma_m$. Again, such a problem is equivalent to find an H_∞ γ-suboptimal control law (with $\gamma = 1/\gamma_m$) for the following system,

$$\Sigma_{\text{multi}} : \quad \begin{cases} \dot{x} = A_m\, x + B_m\, u + B_m\, w \\ y = C_m\, x + D_m\, u + D_m\, w \\ z = \;0\; x + \;I\; u \end{cases}$$

Example: Consider again the inverse pendulum system characterized by



$$\begin{pmatrix} \dot{\tilde{\theta}} \\ \ddot{\theta} \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{pmatrix} \tilde{\theta} \\ \dot{\theta} \end{pmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \tilde{w}$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{pmatrix} \tilde{\theta} \\ \dot{\theta} \end{pmatrix} + \begin{bmatrix} 0 & 1 \end{bmatrix} \tilde{w}$$

$$z = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{pmatrix} \tilde{\theta} \\ \dot{\theta} \end{pmatrix}$$

ex2149

where $\tilde{w}$ contains both the input disturbance and measurement noise (we treat both of them as system disturbance).

Design H_2 and H_∞ controllers such that the resulting closed-loop system is stable, and the controlled output z is kept around $\theta = \pi$ as it was done in the LQG design.

Clearly, this is a singular problem as $D_2 = 0$. It can be calculated that the optimal performance γ_∞^* and γ_2^* are given as $\gamma_\infty^* = 1$ and $\gamma_2^* = 1.4824$, respectively.

Since it is a singular problem, we adopt the perturbation approach to make it regular, i.e.,

$$\begin{pmatrix} \dot{\tilde{\theta}} \\ \ddot{\theta} \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{pmatrix} \tilde{\theta} \\ \dot{\theta} \end{pmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \tilde{w}$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{pmatrix} \tilde{\theta} \\ \dot{\theta} \end{pmatrix} + \begin{bmatrix} 0 & 1 \end{bmatrix} \tilde{w}$$

$$\tilde{z} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{pmatrix} \tilde{\theta} \\ \dot{\theta} \end{pmatrix} + \begin{bmatrix} \varepsilon \\ 0 \end{bmatrix} \cdot u$$

Note that this perturbation ε is good enough to make the problem regular… No need to perturb E and D_1 as Conditions 3 and 4 are already satisfied…

Thus, we have

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad E = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$C_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 0 & 1 \end{bmatrix}, \quad \tilde{C}_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad \tilde{D}_2 = \begin{bmatrix} \varepsilon \\ 0 \end{bmatrix}$$

where ε is a small perturbation variable. For our design, we select $\varepsilon = 0.01$.

H_2 control law

Solving the following H_2-AREs

$$A^\top P + PA + \tilde{C}_2^\top \tilde{C}_2 - \left(PB + \tilde{C}_2^\top \tilde{D}_2\right)\left(\tilde{D}_2^\top \tilde{D}_2\right)^{-1}\left(\tilde{D}_2^\top \tilde{C}_2 + B^\top P\right) = 0$$

$$QA^\top + AQ + EE^\top - \left(QC_1^\top + ED_1^\top\right)\left(D_1 D_1^\top\right)^{-1}\left(D_1 E^\top + C_1 Q\right) = 0$$

we obtain

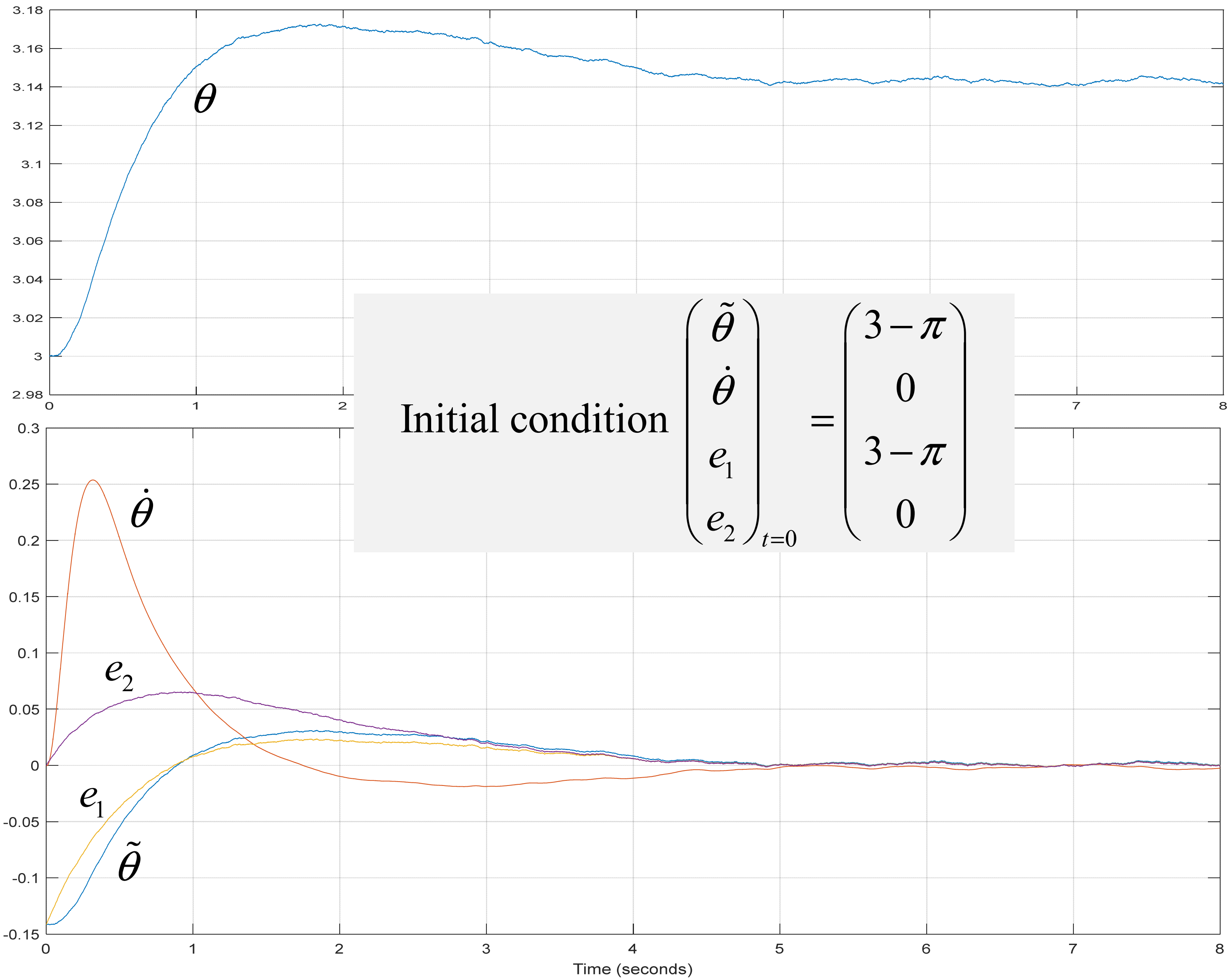
$$P = \begin{bmatrix} 0.1421 & 0.0101 \\ 0.0101 & 0.0014 \end{bmatrix}, \quad Q = \begin{bmatrix} 2.1974 & 2.4142 \\ 2.4142 & 3.1075 \end{bmatrix}$$

and the corresponding state feedback gain and observer gain matrices

$$F = -[101.005 \quad 14.213], \quad K = -\begin{bmatrix} 2.1974 \\ 2.4142 \end{bmatrix}$$

The resulting H_2 control law is given as

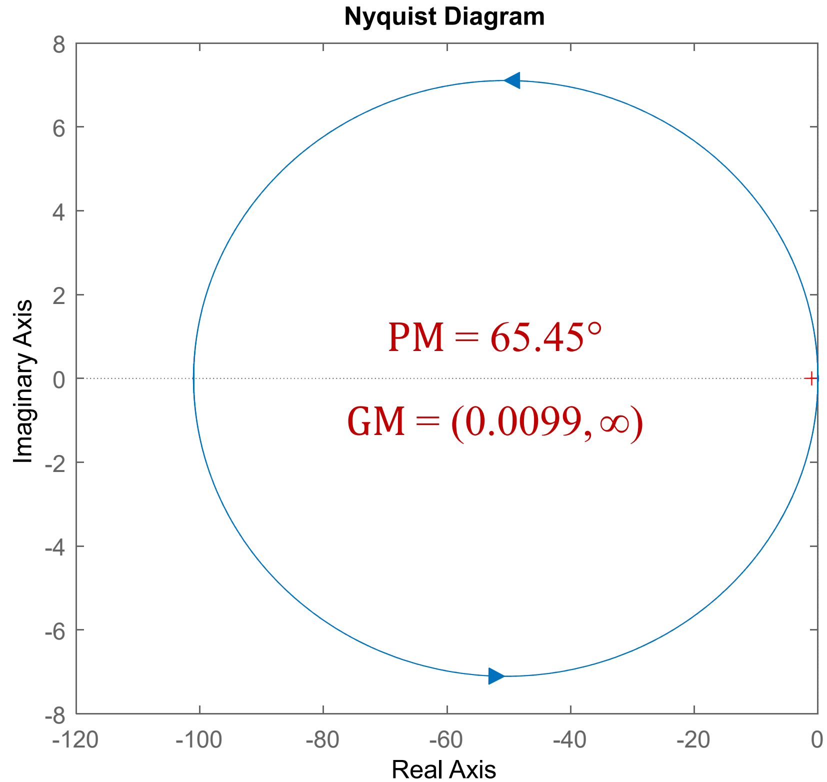
$$\begin{cases} \dot{x}_{\text{cmp}} = \left(A + BF + KC_1\right) x_{\text{cmp}} - K\, y = \begin{bmatrix} -2.1974 & 1 \\ -102.4192 & -14.213 \end{bmatrix} x_{\text{cmp}} + \begin{bmatrix} 2.1974 \\ 2.4142 \end{bmatrix} y \\ u \quad = F\, x_{\text{cmp}} - \left[C_2 (A + BF)^{-1} B\right]^{-1} r = -[101.005 \quad 14.213]\, x_{\text{cmp}} + 100.005\, r \end{cases}$$

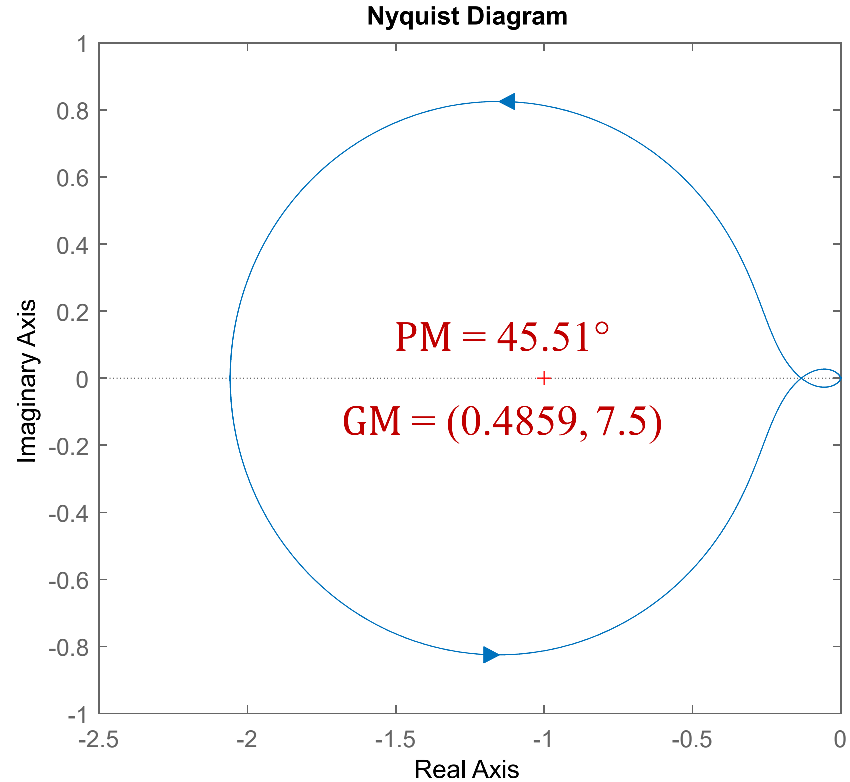


Initial condition $\begin{pmatrix} \tilde{\theta} \\ \dot{\theta} \\ e_1 \\ e_2 \end{pmatrix}_{t=0} = \begin{pmatrix} 3-\pi \\ 0 \\ 3-\pi \\ 0 \end{pmatrix}$

Gain margins and phase margins of the H_2 control laws

State Feedback Case **Output Feedback Case**





★ Here we note that the open-loop transfer matrix for the H_2 output feedback control is

$$L_o(s) = \left[F(sI - A - BF - KC_1)^{-1} K \right] \cdot \left[C_1(sI - A)^{-1} B \right] = -\kappa(s) G(s)$$

which will be studied in detail later in the topic of loop transfer recovery (LTR)…

H_∞ control law

Solving the following H_∞-AREs with $\gamma = 2$

$$A^\top P + PA + \tilde{C}_2^\top \tilde{C}_2 + \gamma^{-2} PEE^\top P - (PB + \tilde{C}_2^\top \tilde{D}_2)\left(\tilde{D}_2^\top \tilde{D}_2\right)^{-1}(\tilde{D}_2^\top \tilde{C}_2 + B^\top P) = 0$$

$$QA^\top + AQ + EE^\top + \gamma^{-2} Q\tilde{C}_2^\top \tilde{C}_2 Q - \left(QC_1^\top + ED_1^\top\right)\left(D_1 D_1^\top\right)^{-1}\left(D_1 E^\top + C_1 Q\right) = 0$$

we obtain

$$P = \begin{bmatrix} 0.1421 & 0.0101 \\ 0.0101 & 0.0014 \end{bmatrix}, \quad Q = \begin{bmatrix} 2.8739 & 3.0972 \\ 3.0972 & 3.8018 \end{bmatrix}$$

and the corresponding state feedback gain and observer gain matrices

$$F = -\begin{bmatrix} 101.0063 & 14.2133 \end{bmatrix}, \quad K = -\begin{bmatrix} 2.8739 \\ 3.0972 \end{bmatrix}$$

The resulting H_∞ control law is given as

$$\Sigma_{\text{cmp}} : \begin{cases} \dot{x}_{\text{cmp}} = A_{\text{cmp}} x_{\text{cmp}} + B_{\text{cmp}} \, y \\ u = C_{\text{cmp}} x_{\text{cmp}} \end{cases}$$

where

$$A_{\text{cmp}} = A + \gamma^{-2} E E^{\mathsf{T}} P + BF + \left(I - \gamma^{-2} Q P\right)^{-1} K \left(C_1 + \gamma^{-2} D_1 E^{\mathsf{T}} P\right)$$
$$= \begin{bmatrix} -3.2619 & 1 \\ -103.5235 & -14.2129 \end{bmatrix}$$

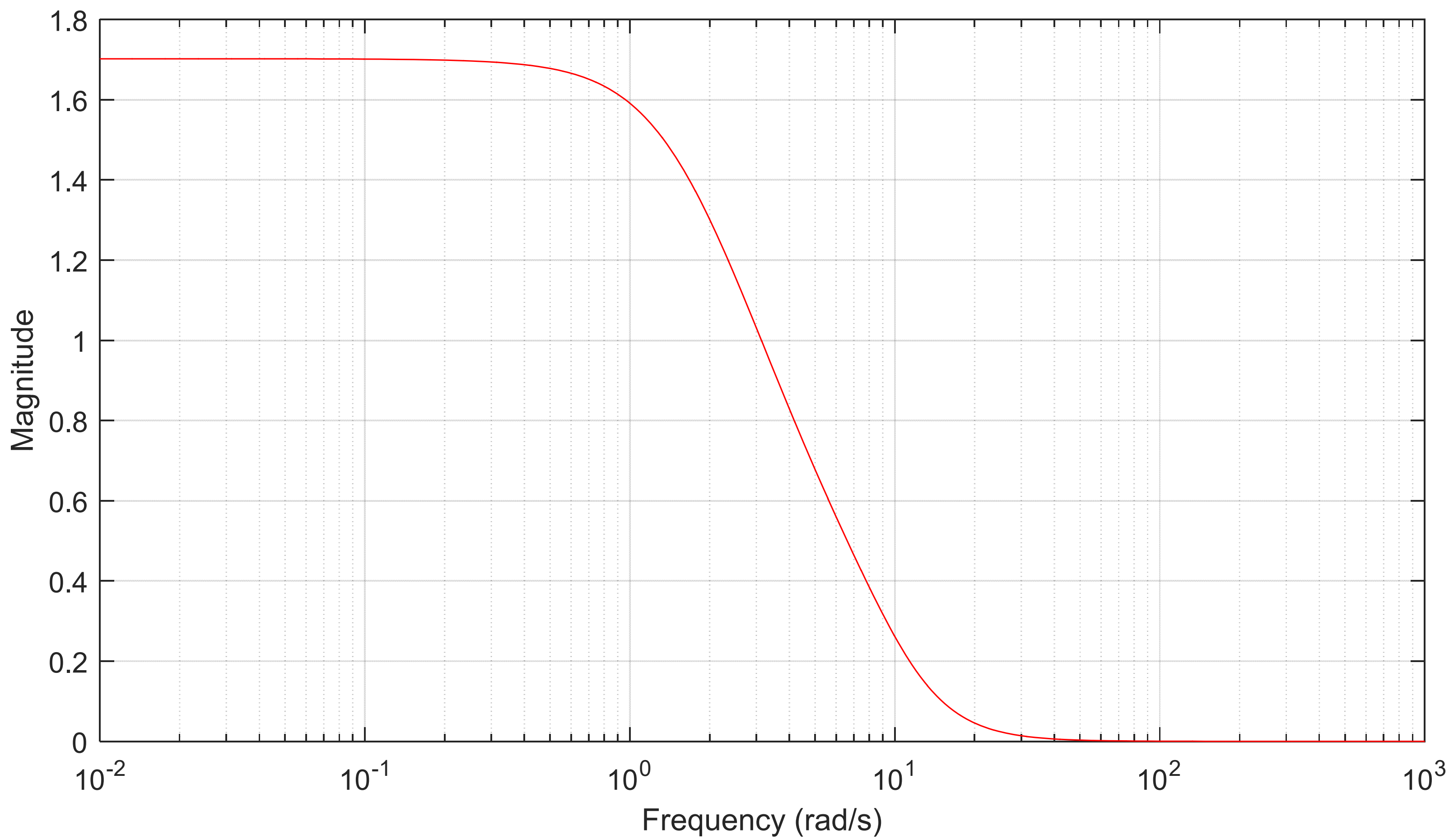
$$B_{\text{cmp}} = -\left(I - \gamma^{-2} Q P\right)^{-1} K = \begin{bmatrix} 3.2619 \\ 3.5198 \end{bmatrix}, \quad C_{\text{cmp}} = -\begin{bmatrix} 101.0063 & 14.2133 \end{bmatrix}$$

The resulting closed-loop system was derived earlier and is given as follows

$$\begin{cases} \begin{pmatrix} \dot{x} \\ \dot{x}_{\text{cmp}} \end{pmatrix} = \begin{bmatrix} A + B D_{\text{cmp}} C_1 & B C_{\text{cmp}} \\ B_{\text{cmp}} C_1 & A_{\text{cmp}} \end{bmatrix} \begin{pmatrix} x \\ x_{\text{cmp}} \end{pmatrix} + \begin{bmatrix} E + B D_{\text{cmp}} D_1 \\ B_{\text{cmp}} D_1 \end{bmatrix} \tilde{w} \\ z \quad = \begin{bmatrix} C_2 + D_2 D_{\text{cmp}} C_1 & D_2 C_{\text{cmp}} \end{bmatrix} \begin{pmatrix} x \\ x_{\text{cmp}} \end{pmatrix} + D_2 D_{\text{cmp}} D_1 \quad \tilde{w} \end{cases}$$

It is shown in the singular value plot on the next page that the resulting closed-loop is indeed has an H_∞ norm of $1.7021 < \gamma = 2$.

Singular values of the closed-loop system…

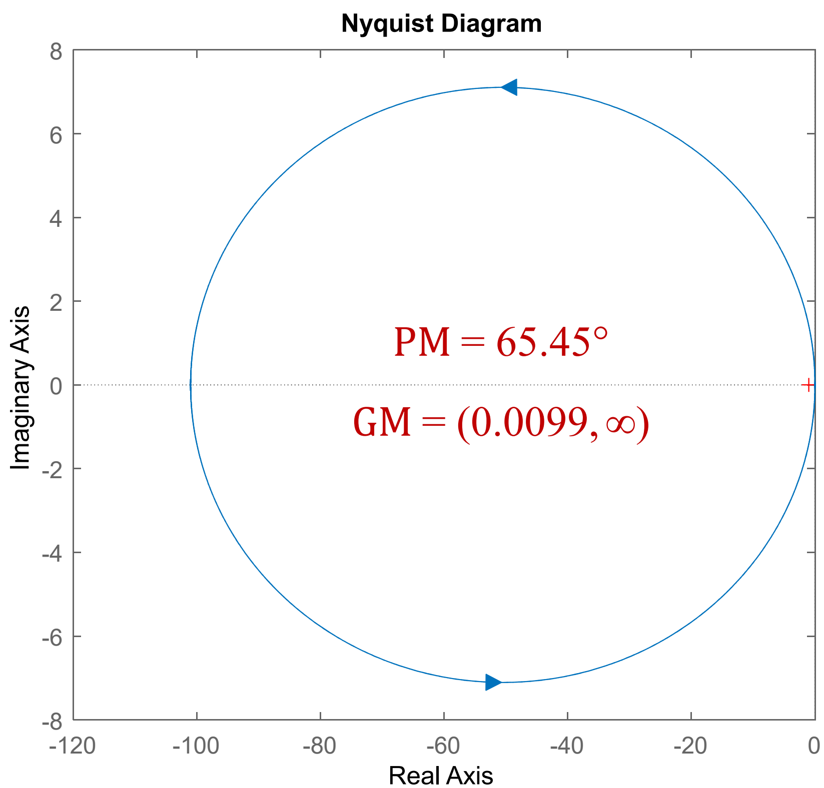


Tzwo
Tzws

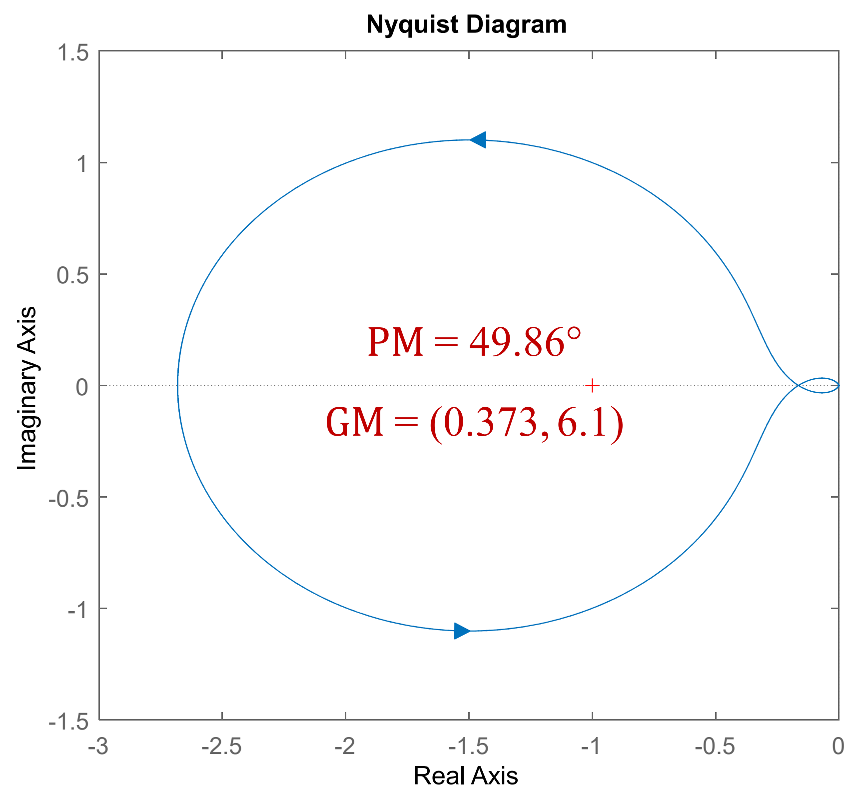
Time-domain simulation can be done similarly as those in the previous cases…

Gain margins and phase margins of the H_∞ control laws

State Feedback Case



Output Feedback Case



★ Here we note that the open-loop transfer matrix for the H_∞ output feedback control is

$$L_o(s) = -\left[C_{\text{cmp}}(sI - A_{\text{cmp}})^{-1} B_{\text{cmp}}\right] \cdot \left[C_1(sI - A)^{-1} B\right] = -\kappa(s) G(s)$$

which will be studied in detail later in the topic of loop transfer recovery (LTR)…

Homework Assignment 6 + Design Problem 2

P.1: Write the system to be controlled in Homework Assignment 5 in the following form

$$\Sigma : \begin{cases} \dot{x} = A\,x + B\,u + E\,\tilde{w} \\ y = C_1\,x + D_1\,\tilde{w} \\ z = C_2\,x + D_2\,u \end{cases}$$

with

$$\tilde{w} = \begin{pmatrix} v(t) \\ w(t) \end{pmatrix} \quad \text{and} \quad z = \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}.$$

1. Determine the best achievable H_∞-norm of the closed-loop system from $\tilde{w}$ to z?
2. Design an H_∞ suboptimal control law such that the H_∞-norm of the resulting closed-loop system is reasonably close to the optimal value.
3. Plot the singular value of the closed-loop system and find its H_∞-norm.
4. Find the resulting gain and phase margins of the system under the control law.
5. Assume that there is an unstructured but stable perturbation, Δ, presented in the given plant. Give the range of $\|\Delta\|_\infty$ so that the closed-loop would remain stable.

P.2: Consider a linear time-invariant system characterized by

$$\Sigma : \begin{cases} \dot{x} = \; A\, x + \; B\, u + \; E\, w \\ z = C_2\, x + D_2\, u, \end{cases}$$

where $C_2 = 0_{m\times n}$, $D_2 = I_m$, and where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $w \in \mathbb{R}^l$ and $z \in \mathbb{R}^m$, are the state, control input, disturbance input and controlled output, respectively. Assume that the state variable x is available for feedback, i.e., the measurement output $y = x$, and assume that (A, B) is stabilizable and (A, B, C_2, D_2) has no invariant zeros on the imaginary axis.

(a) Show that the subsystem (A, B, C_2, D_2) has a total of n invariant zeros and are given by $\lambda(A)$, i.e., the eigenvalues of A.

(b) Show that there exist an $n \times n$ nonsingular transformation T such that

$$\tilde{A} = T^{-1}AT = \begin{bmatrix} A_- & 0 \\ 0 & A_+ \end{bmatrix},$$

where A_- and A_+ are stable and unstable matrices, respectively.

(c) Let us define a state transformation $x = T\tilde{x}$, where T as given in Part (b). It is easy to verify that the given system Σ can be transformed into the following:

$$\begin{cases} \dot{\tilde{x}} = \begin{bmatrix} A_- & 0 \\ 0 & A_+ \end{bmatrix} \tilde{x} + \begin{bmatrix} B_- \\ B_+ \end{bmatrix} u + \begin{bmatrix} E_- \\ E_+ \end{bmatrix} w \\ \\ z = \begin{bmatrix} 0 & 0 \end{bmatrix} \tilde{x} + \quad I \quad u, \end{cases}$$

where B_-, B_+, E_- and E_+ are respectively appropriate constant matrices. Show that (A, B) is stabilizable if and only if (A_+, B_+) is controllable.

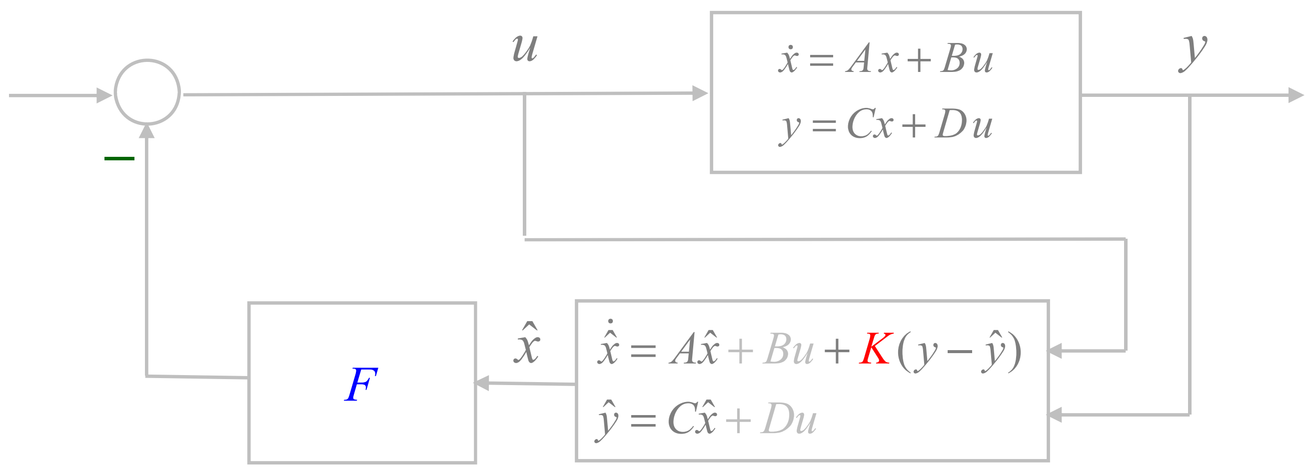
(d) Show that the solution to the corresponding H_2 Riccati equation for the transformed system in Part (c), if existent, can be partitioned as follows:

$$P = \begin{bmatrix} 0 & 0 \\ 0 & P_+ \end{bmatrix}, \quad P_+ > 0.$$

Find the H_2 optimal state feedback control law $u = F\tilde{x}$ for the transformed system in terms of P_+. Show that the resulting closed-loop system has poles at $\lambda(A_-)$ and $\lambda(-A_+)$.

(e) Show that $\gamma_2^* = 0$, i.e., the disturbance can be totally rejected from the controlled output, if and only if $E_+ = 0$, i.e., the disturbance is not allowed to enter the unstable invariant zero subspace.

Loop Transfer Recovery (LTR) Technique



Is LQG controller robust?

LQG H2 H∞

It is now well-known that the linear quadratic regulator (LQR) has very impressive robustness properties, including guaranteed infinite gain margins and at least 60^0 phase margins. The result is only valid, however, for the full state feedback case. If observers or Kalman filters (i.e., LQG regulators) are used in implementation, no guaranteed robustness properties hold. Still worse, the closed-loop system may become unstable if you do not design the observer of Kalman filter properly. The following example given in Doyle (1978) shows the unrobustness of the LQG regulators.

Example: Consider the following system characterized by

$$\dot{x} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 1 \\ 1 \end{bmatrix} v, \quad y = [1 \quad 0] x + w$$

where x, u and y denote the usual states, control input and measured output, and w and v are white noises with intensities 1 and $\sigma > 0$, respectively.

John Doyle
CalTech

* J. Doyle, "Guaranteed margins for LQG regulators," *IEEE Transactions on Automatic Control*, Vol. 23, pp. 756-757, 1978.

The LQG controller consists of an LQR control law and a Kalman filter.

LQR design: Suppose we wish to minimize the performance index

$$J=\int_0^\infty (x^\mathrm{T}Qx+u^\mathrm{T}Ru)dt, \qquad R=1,\ Q=q\begin{pmatrix}1\\1\end{pmatrix}\begin{bmatrix}1 & 1\end{bmatrix}, q>0$$

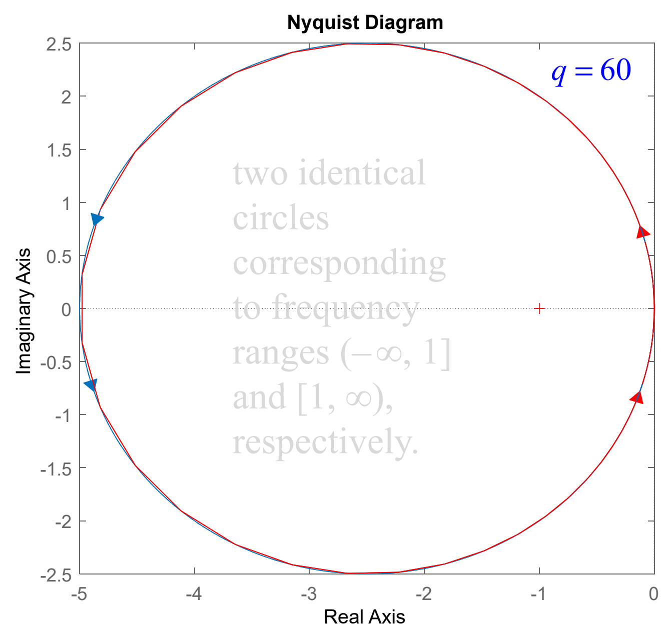
It is known that the state feedback law $u=-F\,x$ which minimize the performance index J is given by $F=R^{-1}B^\mathrm{T}P,$ where

$$PA+A^\mathrm{T}P-PBR^{-1}B^\mathrm{T}P+Q=0, \quad P>0.$$

We can obtain a closed-form solution,

$$F=\left(2+\sqrt{4+q}\right)[1 \quad 1]=f\,[1 \quad 1].$$

It can be verified that the open loop of the LQR design with any $q=60$ has a gain margin of $(0.2,\infty)$ and a phase margin of 101.5 degrees. Thus, it is very robust.



It can also be shown that the Kalman filter gain for this problem can be expressed as

$$K = \left(2+\sqrt{4+\sigma}\right)\begin{bmatrix}1\\1\end{bmatrix} = k\begin{bmatrix}1\\1\end{bmatrix}$$

which together with the LQR law result an LQG controller,

$$\begin{cases} \dot{\hat{x}} = (A-BF-KC)\,\hat{x} + K\,y \\ u = -F\,\hat{x} \end{cases} \quad \text{or} \quad u = -F(sI - A + BF + KC)^{-1}Ky$$

Suppose that the resulting closed-loop controller has a scalar gain $1+\varepsilon$ (nominally unity) associated with the input matrix, i.e.,

$$\text{the actual input matrix} = (1+\varepsilon)B = \begin{bmatrix}0\\1+\varepsilon\end{bmatrix}$$

Tedious manipulations show that the characteristic function of the closed-loop system comprising the given system an the LQG controller is given by

$$\boxed{K(s) = \Pi\cdot s^4 + \Theta\cdot s^3 + \Omega\cdot s^2 + \left(2\varepsilon k f + k + f - 4\right)s + \left(1-\varepsilon k f\right)}$$

A necessary condition for stability is that

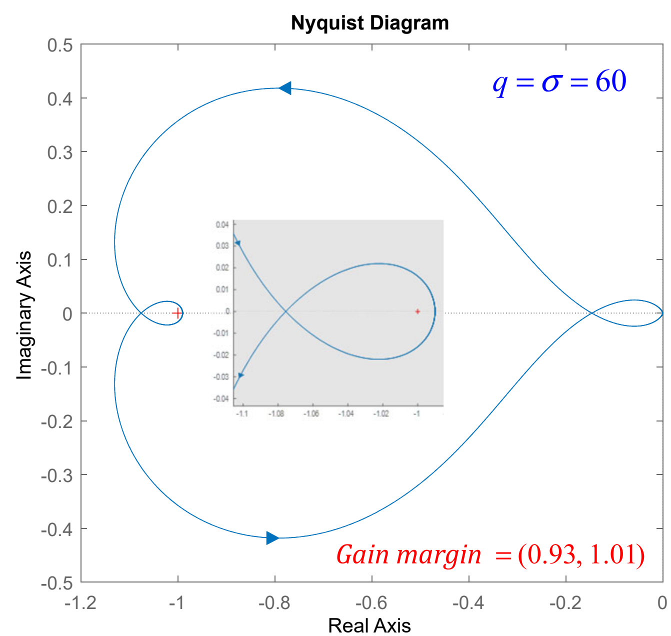
$$2\varepsilon k f + k + f - 4 > 0 \quad \text{and} \quad 1 - \varepsilon k f > 0$$

It is easy to see that for sufficient large q and σ, the closed-loop could be unstable for a small perturbation in B in either direction. For instance, let us choose $q = \sigma = 60$. Then it is simple to verify the closed-loop system remains stable only when $-0.07 < \varepsilon < 0.01$.

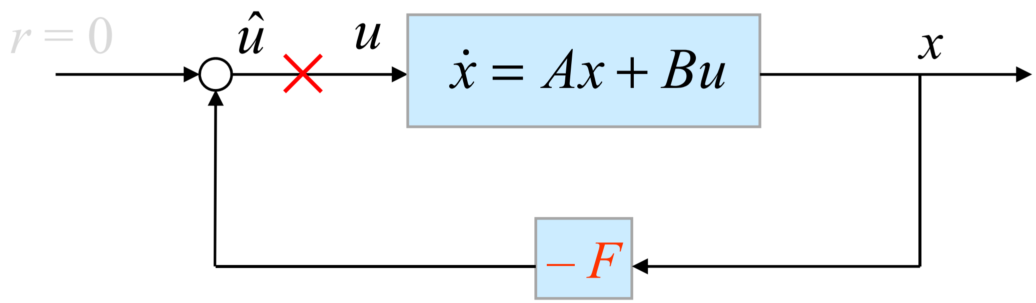
This shows that the LQG controller is not robust at all!

What is wrong?

The answer is that the open-loop transfer function of the LQR design and the open-loop transfer function of the LQG design are totally different and thus, all the nice properties associated with the LQR design vanish in the LQG controller. It can be seen more clearly from the precise mathematical expressions of these two open-loop transfer functions, and this leads to the birth of the loop transfer recovery technique.



Open-loop transfer function of LQR



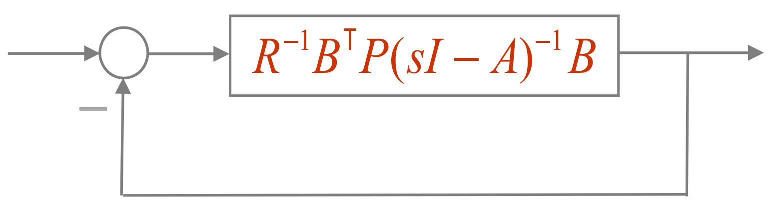
Open-loop transfer function: When the loop is broken at the input point of the plant, i.e., the point marked X, we have

$$\hat{u} = -F(sI - A)^{-1} B u$$

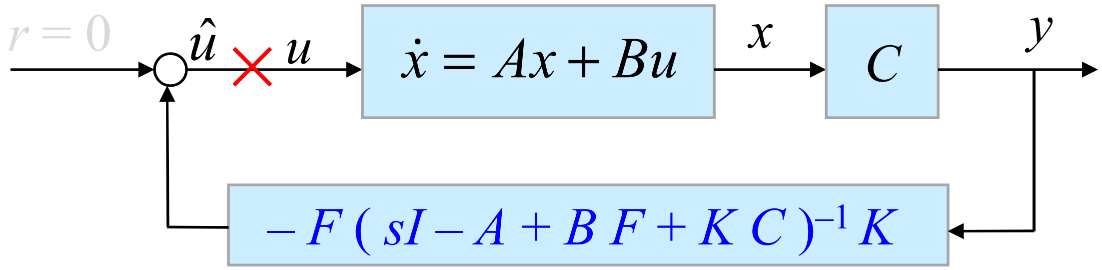
Thus, the loop transfer matrix from u to $-\hat{u}$ is given by

$$L_{\mathrm{t}}(s) = F(sI - A)^{-1} B$$

We have learnt from the previous lectures that the open loop transfer $L_{\mathrm{t}}(s)$ have very impressive properties if the gain matrix F comes from the LQR design, i.e.,



Open-loop transfer function of LQG



Open-loop transfer function: When the loop is broken at the input point of the plant, i.e., the point marked X, we have

$$\hat{u} = \left[-F(sI - A + BF + KC)^{-1} K \right] \cdot \left[C(sI - A)^{-1} B \right] u$$

Thus, the loop transfer matrix from u to $-\hat{u}$ is given by

$$L_{\mathrm{o}}(s) = \left[F(sI - A + BF + KC)^{-1} K \right] \cdot \left[C(sI - A)^{-1} B \right]$$

Clearly, $L_{\mathrm{t}}(s)$ and $L_{\mathrm{o}}(s)$ are very different and that is why LQG in general does not have nice properties as LQR does.

We note that the stability of the closed-loop system with the observer-based output feedback control law is determined by this open-loop transfer function, $L_{\mathrm{o}}(s)$. It is acting like $T(s)$ in the Nyquist stability criterion formulation.

Loop transfer recovery

The above problem can be fixed by choosing an appropriate Kalman filter gain matrix K such that $L_t(s)$ and $L_o(s)$ are either exactly or almost matched over a certain range of frequencies. Such a technique is called **Loop Transfer Recovery (LTR)**.

The idea was first pointed out by Doyle and Stein in 1979. They had given a sufficient condition under which $L_o(s) = L_t(s)$. They had also developed a procedure to design the Kalman filter gain matrix K in terms of a tuning parameter q such that the resulting $L_o(s) \to L_t(s)$ as $q \to \infty$, for invertible and minimum phase systems. The technique is known as **LQG/LTR** in the literature.

Doyle-Stein conditions: It can be shown that $L_o(s)$ and $L_t(s)$ are identical if the observer gain K satisfies

$$K\left(I + C\Phi K\right)^{-1} = B\left(C\Phi B\right)^{-1}, \quad \Phi = \left(sI - A\right)^{-1}$$

which is equivalent to $B = 0$ (prove it!). Thus, it is impractical.

Gunter Stein
Honeywell

[*] J. Doyle and G. Stein, "Robustness with observers," *IEEE Transactions on Automatic Control*, Vol. 24, pp. 607-611, 1979.

Classical LTR design

The following procedure was proposed by Doyle and Stein in 1979 for left invertible and minimum phase systems (good systems): Define

$$Q_q = Q_0 + q^2 B V B^\top, \quad R = R_0$$

where Q_0 and R_0 are noise intensities appropriate for the nominal plant (in fact, Q_0 can be chosen as a zero matrix and $R_0 = I$), and V is any positive definite symmetric matrix (V can be chosen as an identity matrix). Then the observer (or Kalman filter) gain is given by

$$K = P C^\top R^{-1}$$

where P is the positive definite solution of

$$AP + PA^\top + Q_q - PC^\top R^{-1} C P = 0$$

It can be shown that the resulting open-loop transfer function $L_\mathrm{o}(s)$ from the above observer or Kalman filter has

$$L_\mathrm{o}(s) \to L_\mathrm{t}(s), \quad \text{as } q \to \infty.$$

Example: Consider a given plant characterized by

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 35 \\ -61 \end{bmatrix} v, \qquad y = [2 \quad 1]\, x + w$$

with $E[v(t)] = E[w(t)] = 0$ and $E[v(t)v(\tau)] = E[w(t)w(\tau)] = \delta(t-\tau)$.

This system is of minimum phase with one invariant zero at $s = -2$. The LQR control law is given by

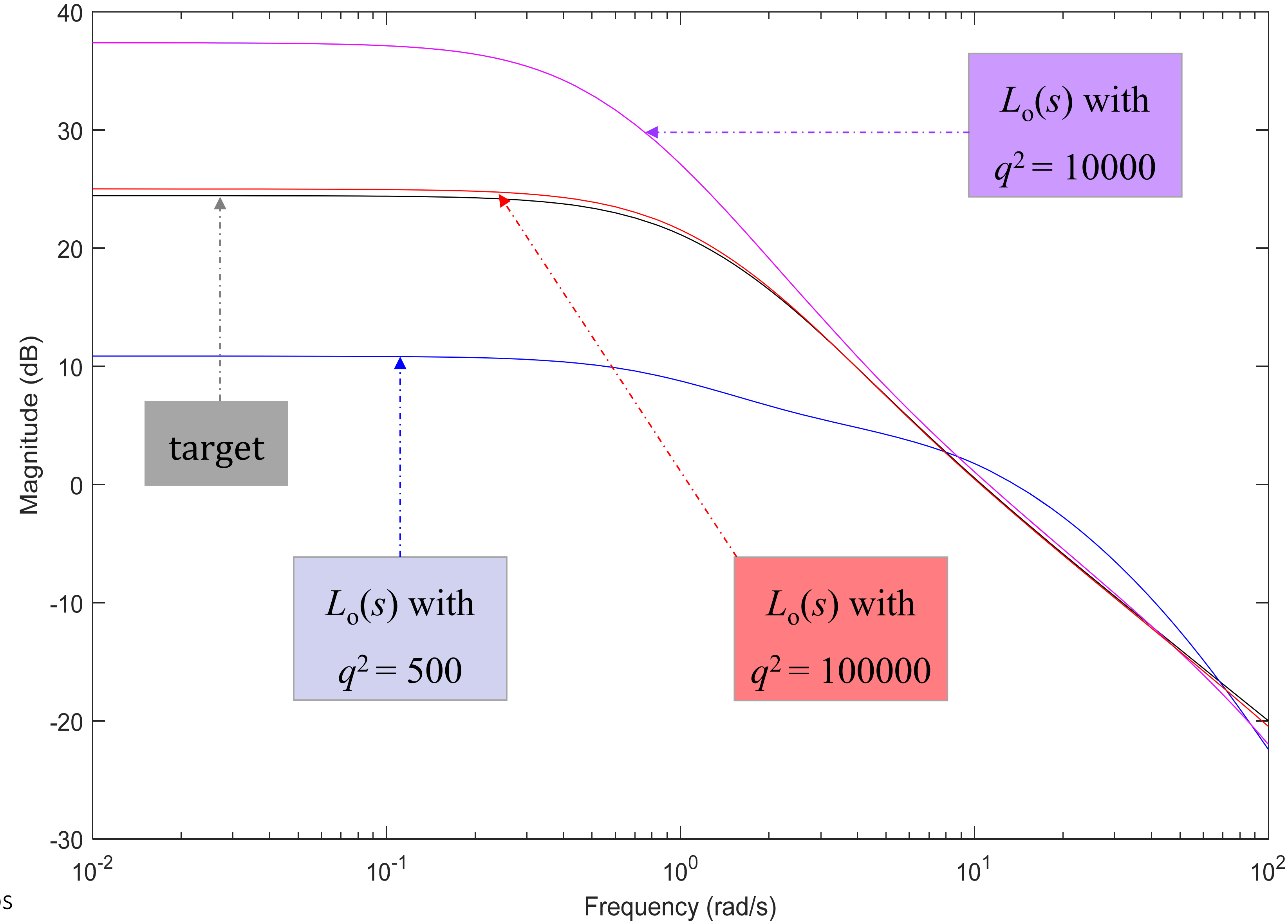
$$u = -F x = -[50 \quad 10]\, x$$

The resulting open-loop transfer function $L_t(s)$ has an infinity gain margin and a phase margin over $85°$. We apply the Doyle-Stein LTR procedure to design an observer-based controller, i.e.,

$$u = -F[\Phi^{-1} + BF + KC]^{-1} K\, y$$

where K is computed as on the previous page with

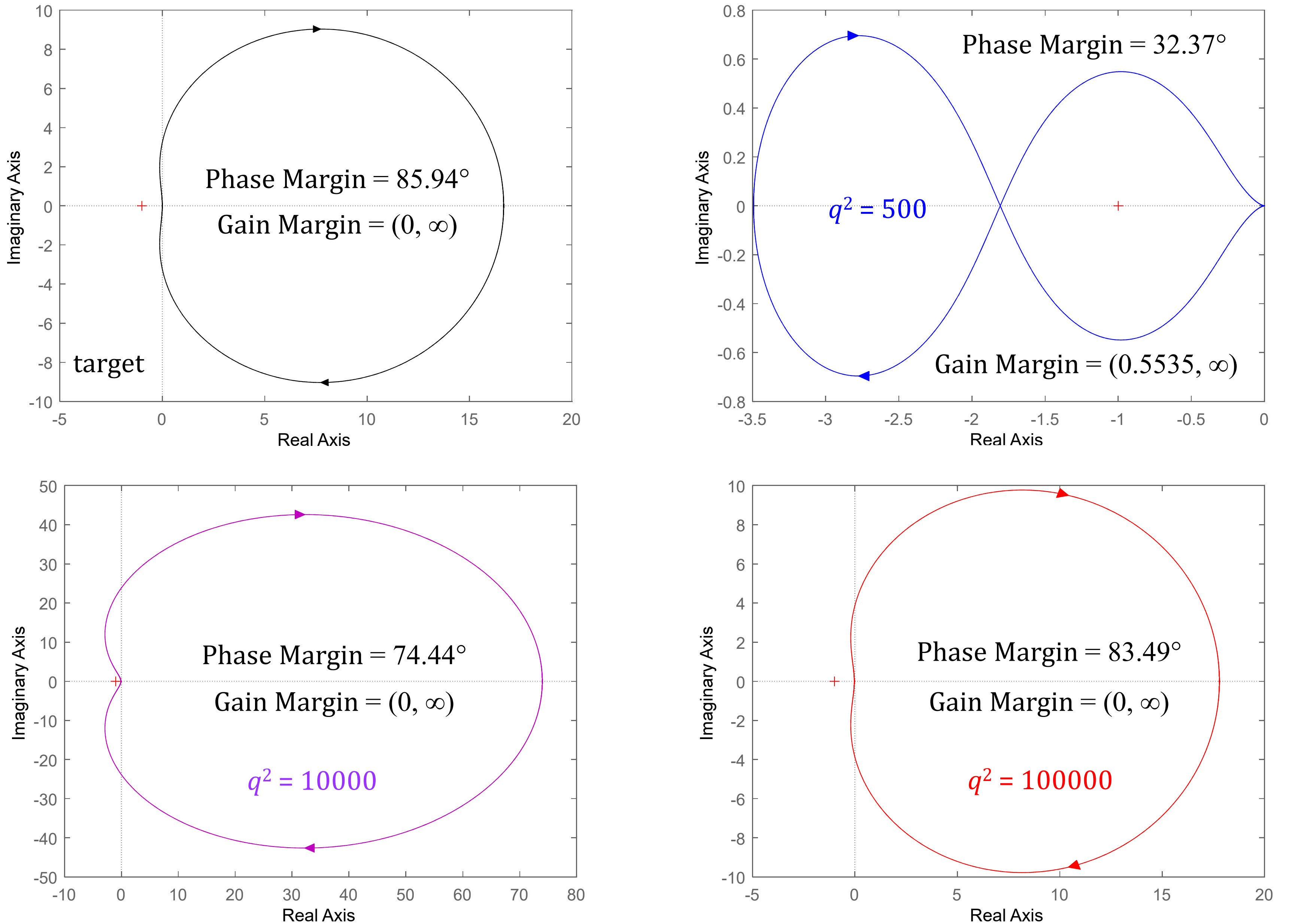
$$Q_q = \begin{bmatrix} 35 \\ -61 \end{bmatrix} [35 \quad -61] + q^2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} [0 \quad 1] = \begin{bmatrix} 1225 & -2135 \\ -2135 & 3721 + q^2 \end{bmatrix}.$$



40
30
20
10
0
-10
-20
-30
Magnitude (dB)
10^{-2}
10^{-1}
10^{0}
10^{1}
10^{2}
Frequency (rad/s)
$L_o(s)$ with $q^2 = 10000$
target
$L_o(s)$ with $q^2 = 500$
$L_o(s)$ with $q^2 = 100000$

ltrloops

Nyquist plots…

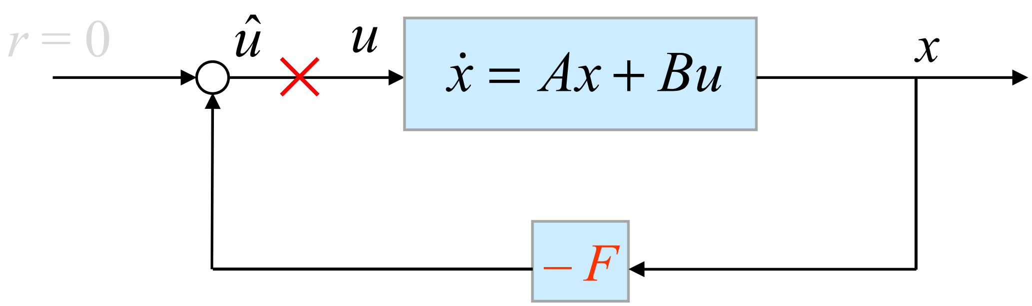


New formulation for loop transfer recovery

Consider a general stabilizable and detectable plant,

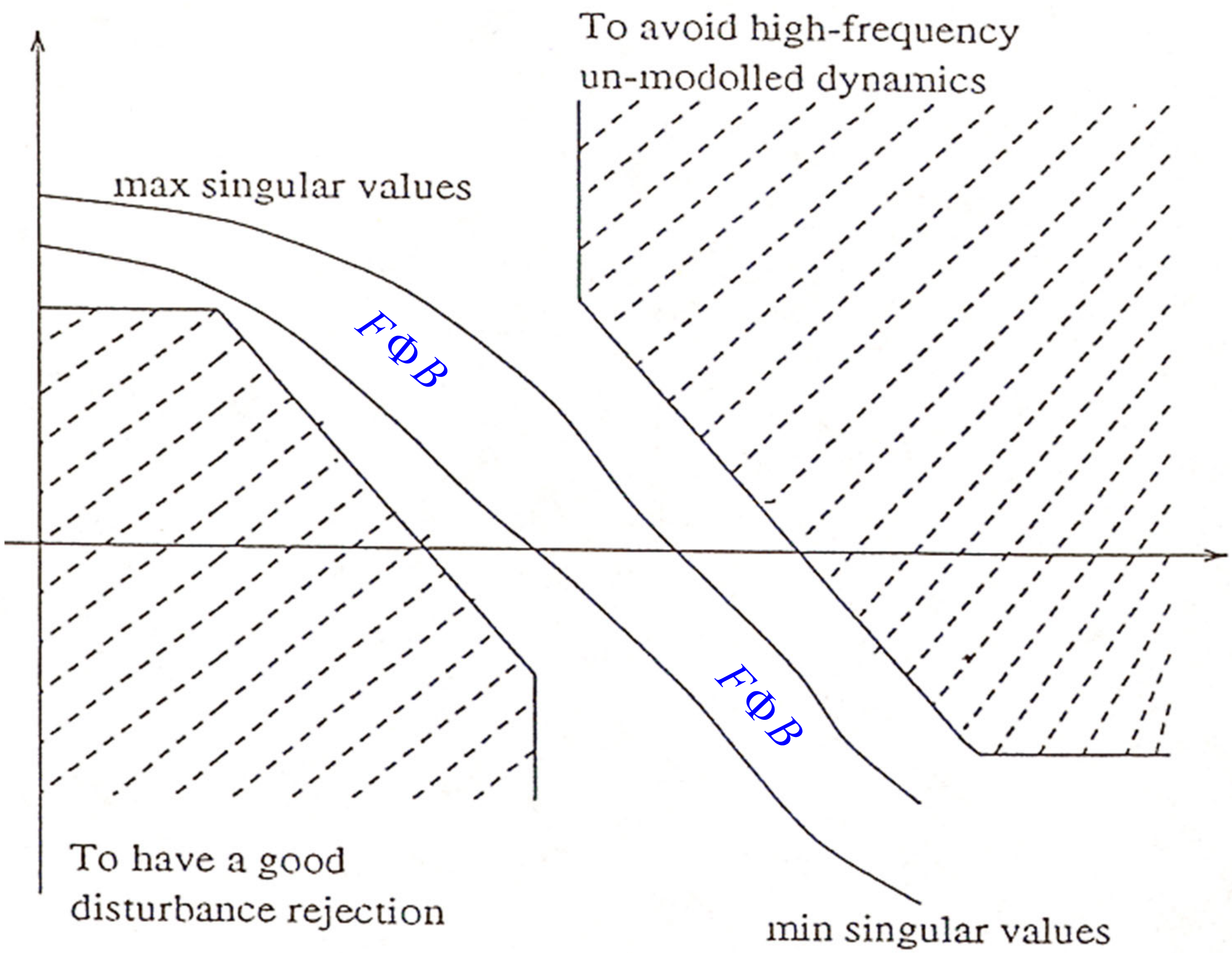
$$\begin{cases} \dot{x} = A\,x + B\,u \\ y = C\,x + D\,u \end{cases}$$

The transfer function is given by $G(s) = C\Phi B + D,\ \ \Phi = (sI - A)^{-1}$. Also, let F be a state feedback gain matrix such that under the state feedback control law $u = -F\,x$ has the following properties: i) the resulting closed-loop system is asymptotically stable; and ii) the resulting **target loop** $L_{\mathrm{t}}(s) = F\Phi B$ meets design specifications.



Such a state feedback can be obtained using LQR design or any other design methods so long as it meets your design specifications. Usually, a desired target loop would have the shape as given in the figure on the next page.

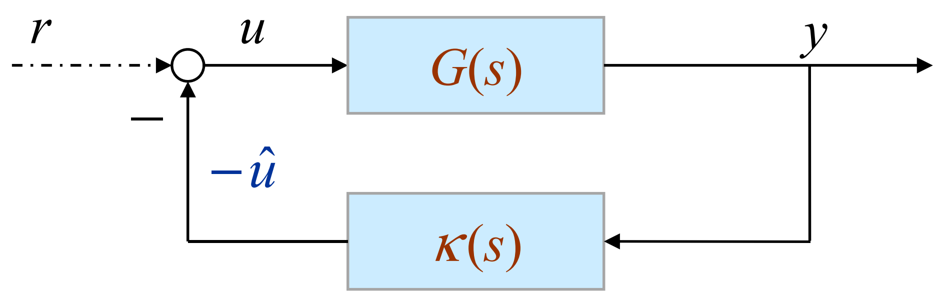
Typical desired open-loop characteristics…



Siva S. Banda
Air Force Research Lab
USA

★ D. B. Ridgely and S. S. Banda, *Introduction to Robust Multivariable Control,* Report No. AFWAL-TR-85-3102, Flight Dynamics Laboratories, Air Force Systems Command, Wright-Patterson Air Force Base, Ohio, 1986.

The problem of LTR is to find a stabilizing controller $u = -\kappa(s)y$



such that the resulting open-loop transfer function from u to $-\hat{u}$, i.e.,

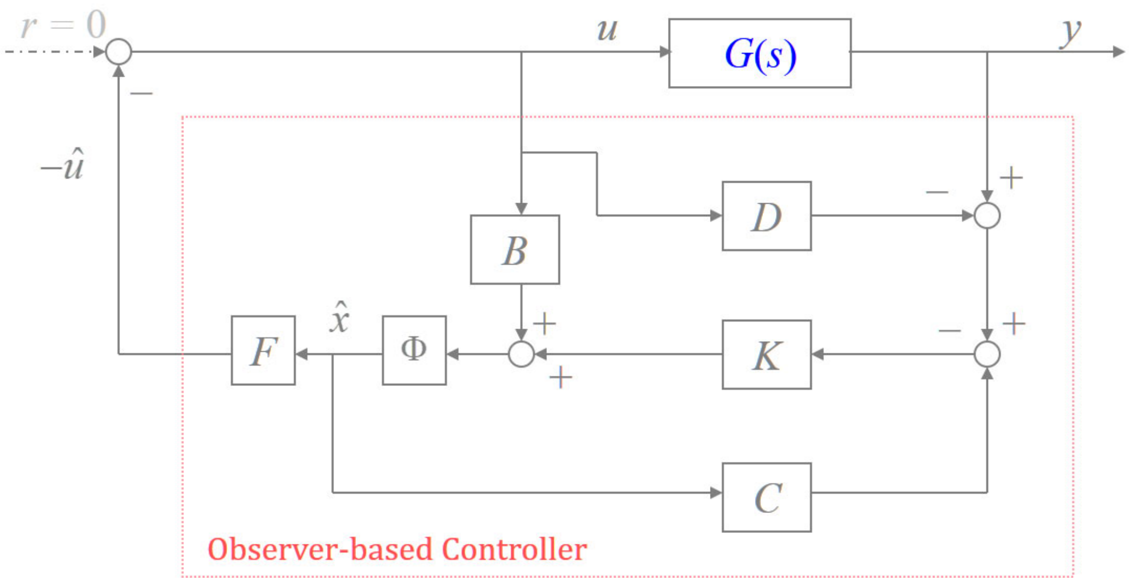
$$L_o(s) = \kappa(s)G(s)$$

is either exactly or approximately equal to the target loop $L_t(s)$. Let us define the **recovery error** as the difference between the target loop and the achieved loop, i.e.,

$$E(s) = L_t(s) - L_o(s) = F\Phi B - \kappa(s)G(s)$$

Then, we say exact LTR is achievable if $E(s)$ can be made identically zero, or almost LTR is achievable if $E(s)$ can be made arbitrarily small.

Observer-based structure for $\kappa(s)$



Dynamic equations of $\kappa(s)$: $\dot{\hat{x}} = A\,\hat{x} + B\,u + K(y - C\hat{x} - D\,u)$

$$\hat{u} = u = -F\,\hat{x}$$

Transfer function of $\kappa(s) = \kappa_o(s) = F(\Phi^{-1} + BF + KC - KDF)^{-1}K$

Achieved open-loop: $L_o(s) = \kappa_o(s)G(s)$

$$= \left[F(\Phi^{-1} + BF + KC - KDF)^{-1}K\right] \cdot \left(C\Phi B + D\right)$$

Lemma: Recovery error, $E_\mathrm{o}(s)$, i.e., the mismatch between the target loop and the resulting open-loop of the observer-based controller is given by

$$E_\mathrm{o}(s) = M(s)\left[I + M(s)\right]^{-1}\left(I + F\Phi B\right), \quad M(s) = F\left(\Phi^{-1} + KC\right)^{-1}\left(B - KD\right)$$

Proof. $L_\mathrm{o}(s) = \kappa_\mathrm{o}(s)G(s) = F\left(\Phi^{-1} + BF + KC - KDF\right)^{-1}K\left(C\Phi B + D\right)$

$$= F\left[I + \left(\Phi^{-1} + KC\right)^{-1}(B - KD)F\right]^{-1}\left(\Phi^{-1} + KC\right)^{-1}K\left(C\Phi B + D\right)$$

$$= \left[I + F(\Phi^{-1} + KC)^{-1}(B - KD)\right]^{-1}F(\Phi^{-1} + KC)^{-1}K\left(C\Phi B + D\right)$$

$A(I + BA)^{-1}$

$= (I + AB)^{-1}A$

$$= (I + M)^{-1}\left[F(\Phi^{-1} + KC)^{-1}KC\Phi B + F\left(\Phi^{-1} + KC\right)^{-1}KD\right]$$

$$= (I + M)^{-1}\left\{F\left[I - (\Phi^{-1} + KC)^{-1}\Phi^{-1}\right]\Phi B + F\left(\Phi^{-1} + KC\right)^{-1}KD\right\}$$

$(\Phi^{-1} + KC)^{-1}KC$

$= I - (\Phi^{-1} + KC)^{-1}\Phi^{-1}$

$$= (I + M)^{-1}\left[F\Phi B - F\left(\Phi^{-1} + KC\right)^{-1}B + F\left(\Phi^{-1} + KC\right)^{-1}KD\right]$$

$$= (I + M)^{-1}\left[F\Phi B - F(\Phi^{-1} + KC)^{-1}(B - KD)\right]$$

$$= (I + M)^{-1}\left(F\Phi B - M\right)$$

Credit to G C Goodman in a master thesis conducted at MIT in 1984

Michael Athans
MIT
1937–2020

$$E_\mathrm{o}(s) = F\Phi B - [I + M(s)]^{-1}[F\Phi B - M(s)] = M(s)[I + M(s)]^{-1}\left(I + F\Phi B\right)$$

Loop transfer recovery design

H2 H∞

It is simple to observe from the above lemma that the loop transfer recovery is achievable if and only if we can design a gain matrix K such that $M(s)$ can be made either identically zero or arbitrarily small, where $M(s) = F(\Phi^{-1} + KC)^{-1}(B - KD)$.

Let us define an auxiliary system

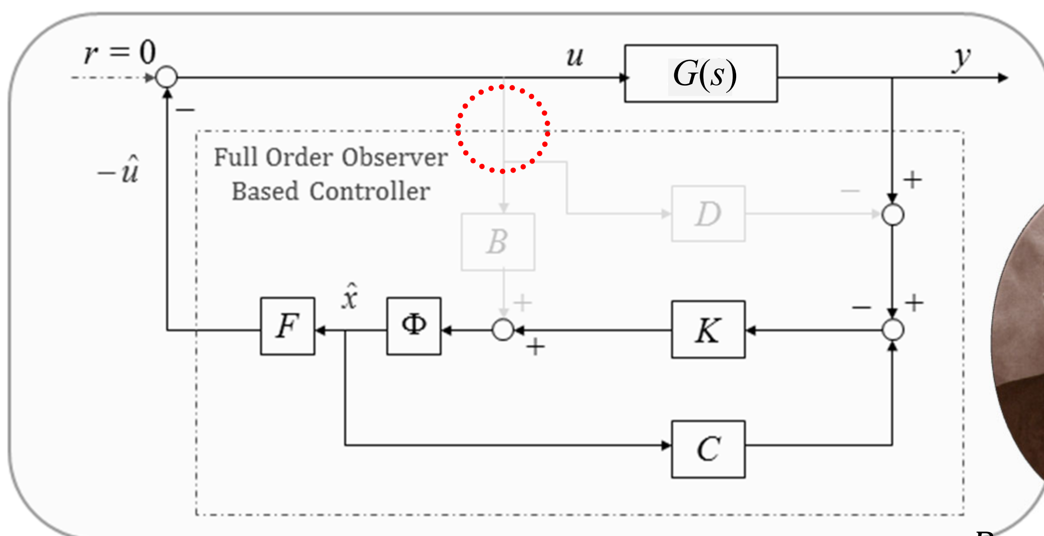
$$\Sigma_{\mathrm{aux}} : \begin{cases} \dot{x} = A^{\mathsf{T}} x + C^{\mathsf{T}} u + F^{\mathsf{T}} w \\ y = x \\ z = B^{\mathsf{T}} x + D^{\mathsf{T}} u \end{cases} \quad + \quad u = -K^{\mathsf{T}} x$$

- Closed-loop transfer function from w to z is $(B^{\mathsf{T}} - D^{\mathsf{T}} K^{\mathsf{T}})(sI - A^{\mathsf{T}} + C^{\mathsf{T}} K^{\mathsf{T}})^{-1} F^{\mathsf{T}} = M^{\mathsf{T}}(s)$.

Thus, LTR design is equivalent to design a state feedback law for the above auxiliary system such that certain norm of the resulting closed-loop transfer function is made either zero or arbitrarily small. As such, the H_2 and H_∞ optimization techniques (with $\gamma_2^* = 0$ and $\gamma_\infty^* = 0$, i.e., the corresponding almost disturbance decoupling problems) can be used to solve the LTR problem.

What really got me interested in control was my first unintentional discovery. I was asked to simulate some examples on loop transfer recovery (LTR) in the book *Control System Design*, by Bernard Friedland. It was mentioned in the text that under the Doyle-Stein condition for LTR, the link feeding the control input signal to an observer-based control law might be omitted. When I simulated examples without satisfying the Doyle-Stein condition (which can never be met in any physical system, by the way) by removing the link to the observer, to my surprise, the recovery performance turned out to be unbelievably good. When I showed the result to my advisor, I got kicked out of his office, as apparently I had violated the common belief in control systems design—the separation principle. Nevertheless, the discovery eventually led to a new controller structure for the LTR design

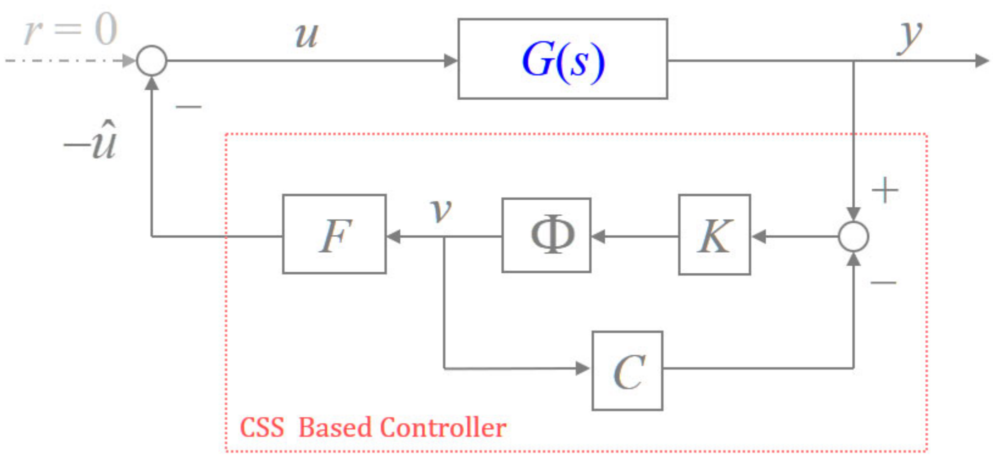
A story behind a new controller structure for LTR…



Bernard Friedland
New Jersey Institute of Technology

LTR design via CSS architecture-based controller

Proposed by Chen, Saberi & Sannuti in 1991, the CSS based controller has the following characteristics:



Ali Saberi
Washington State University

Dynamic equations of $\kappa(s)$: $\dot{x}_{\rm cmp} = (A - KC)\, x_{\rm cmp} + K y,$

$$\hat{u} = u = -F x_{\rm cmp}$$

Transfer function of $\kappa(s) = \kappa_{\rm c}(s) = F(\Phi^{-1} + KC)^{-1} K$

Pedda Sannuti
Rutgers University

Achieved open-loop: $L_{\rm c}(s) = \kappa_{\rm c}(s) G(s)$

$$= \left[F(\Phi^{-1} + KC)^{-1} K \right] \cdot \left(C\Phi B + D \right)$$

Lemma: Recovery error, $E_c(s)$, i.e., the mismatch between the target loop and the resulting open-loop of the CSS architecture-based controller is given by

$$E_c(s) = M(s) = F\left(\Phi^{-1} + KC\right)^{-1}(B - KD)$$

Proof.

$$\begin{aligned} E_c(s) &= L_t(s) - L_c(s) \\ &= F\Phi B - F\left(\Phi^{-1} + KC\right)^{-1} K\left(C\Phi B + D\right) \\ &= F\left(\Phi^{-1} + KC\right)^{-1}\left[\left(\Phi^{-1} + KC\right)\Phi B - KC\Phi B - KD\right] \\ &= F\left(\Phi^{-1} + KC\right)^{-1}\left(B + KC\Phi B - KC\Phi B - KD\right) \\ &= M(s) \end{aligned}$$


A New Stable Compensator Design for Exact and Approximate Loop Transfer Recovery*

B. M. CHEN,† A. SABERI‡ and P. SANNUTI§

It is clear that LTR via the CSS architecture-based controller is achievable iff one can design a gain matrix K such that the resulting $M(s)$ can be made either zero or arbitrarily small. This is the same as the LTR design via the observer-based controller.

★ B. M. Chen, A. Saberi and P. Sannuti, “A new stable compensator design for exact and approximate loop transfer recovery,” *Automatica,* Vol. 27, pp. 257–280, 1991.

Collected in Bibliography on Robust Control by P. Dorato, R. Tempo, G. Muscato in Automatica, Vol. 29, 1993.

What is the advantage of CSS structure?

Theorem. Consider a stabilizable and detectable system Σ characterized by (A, B, C, D) and target loop transfer function $L_t(s) = F\Phi B$.

- Assume that Σ is left invertible and of minimum phase (the so-called good systems), which implies that the target loop $L_t(s)$ is recoverable by both observer-based and CSS architecture- based controllers.

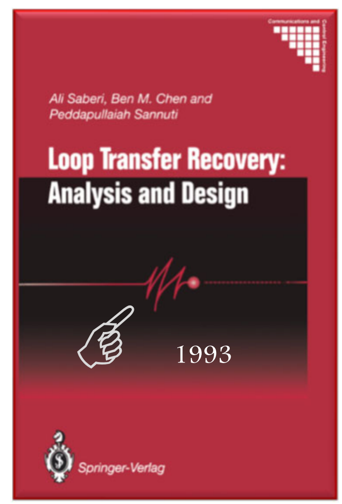
- Also, assume that the same gain K is used for both observer-based controller and CSS architecture-based controller and is such that for all $\omega \in \Omega$, where Ω is some frequency region of interest,

$$\sigma_{\min}\left[F\Phi B\right] \gg 1$$

Then, for all $\omega \in \Omega$,

$$\sigma_{\max}\left[E_c(j\omega)\right] \ll \sigma_{\max}\left[E_o(j\omega)\right]$$

Proof of this result can be found in Chen *et al.*, *Automatica*, vol. 27, 1991; and a monograph by Saberi *et al*. (1993).



Remark: In order to have good command following and desired disturbance rejection properties, the target loop transfer function $L_t(j\omega)$ has to be large and consequently, the minimum singular value $\sigma_{\min}\left[L_t(j\omega)\right]$ should be relatively large in the appropriate frequency region. Thus, the assumption in the above theorem is very practical.

Example: Consider a given plant characterized by

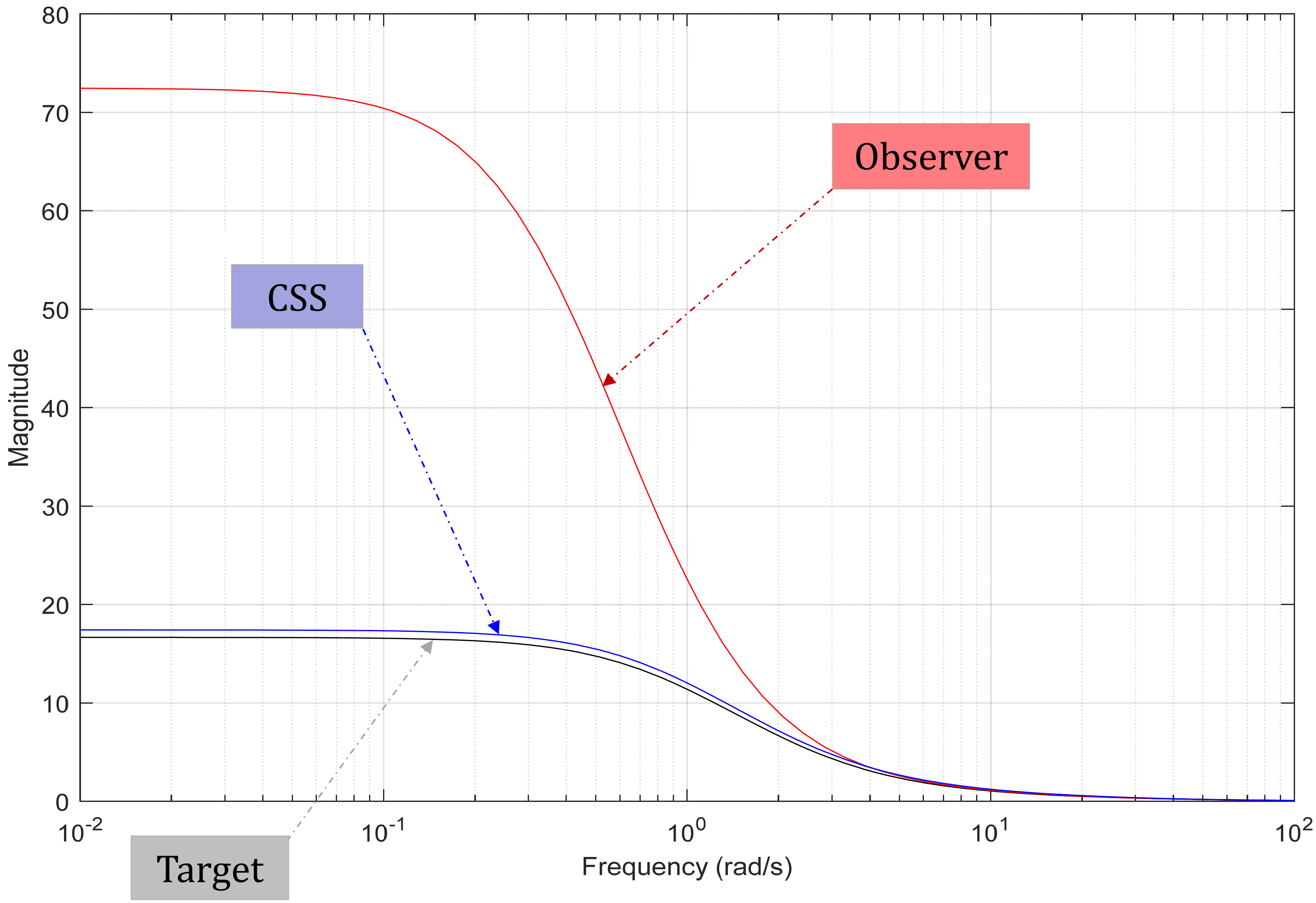
$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \quad y = \begin{bmatrix} 2 & 1 \end{bmatrix} x + 0 \cdot u$$

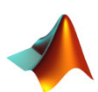
Let the target loop $L_t(s) = F\,\Phi B$ be characterized by a state feedback gain $F = \begin{bmatrix} 50 & 10 \end{bmatrix}$.

Using MATLAB, we know that the above system has an invariant zero at $s = -2$. Hence it is of minimum phase. Also, it is invertible. Thus, the target loop $L_t(s)$ is recoverable by both the observer based and CSS architecture-based controllers.

Using the H_2 optimization method, we obtain matrix $K = \begin{bmatrix} 6.9 \\ 84.6 \end{bmatrix}$.

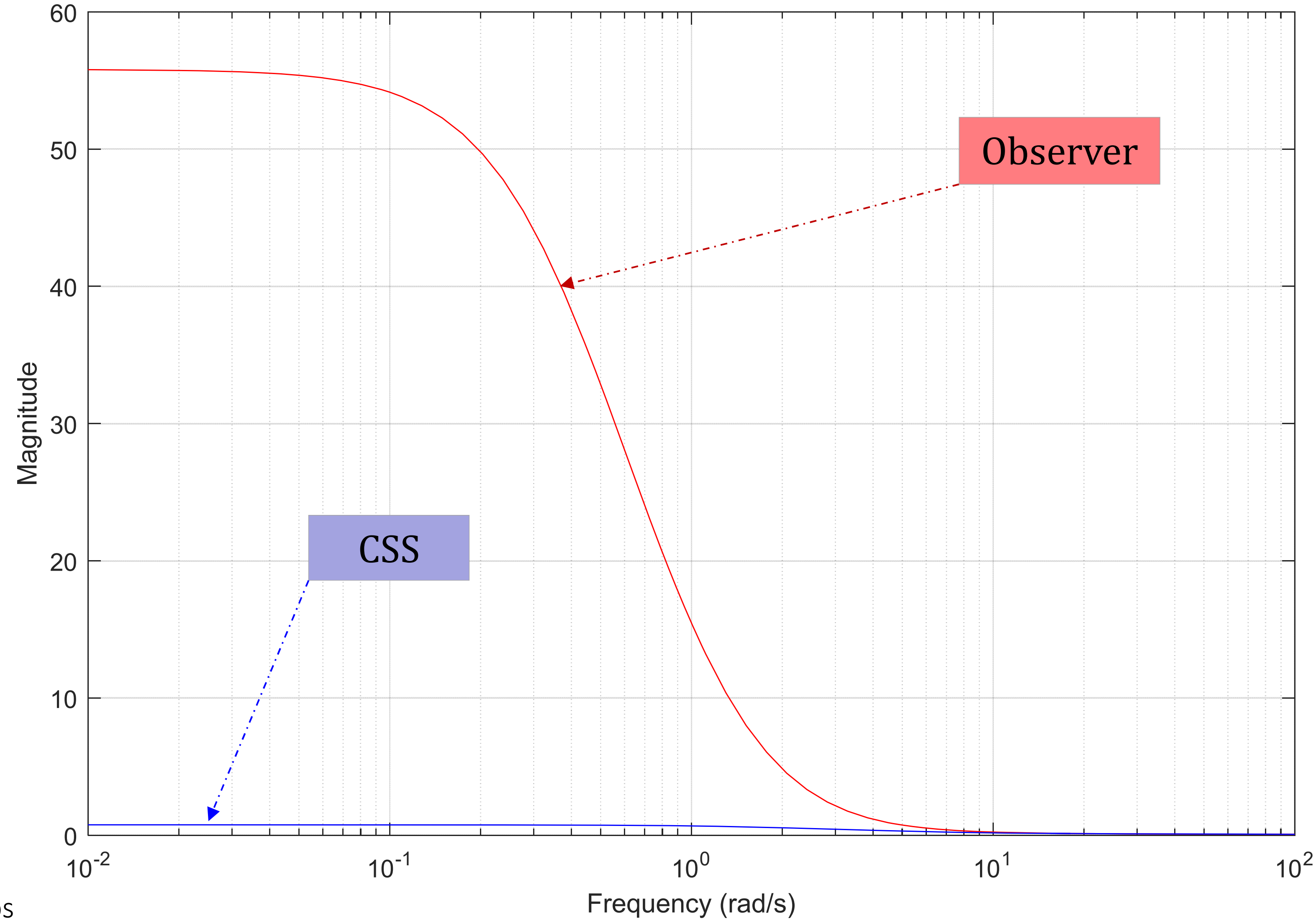
Target loops and achieved loops…

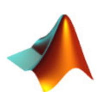




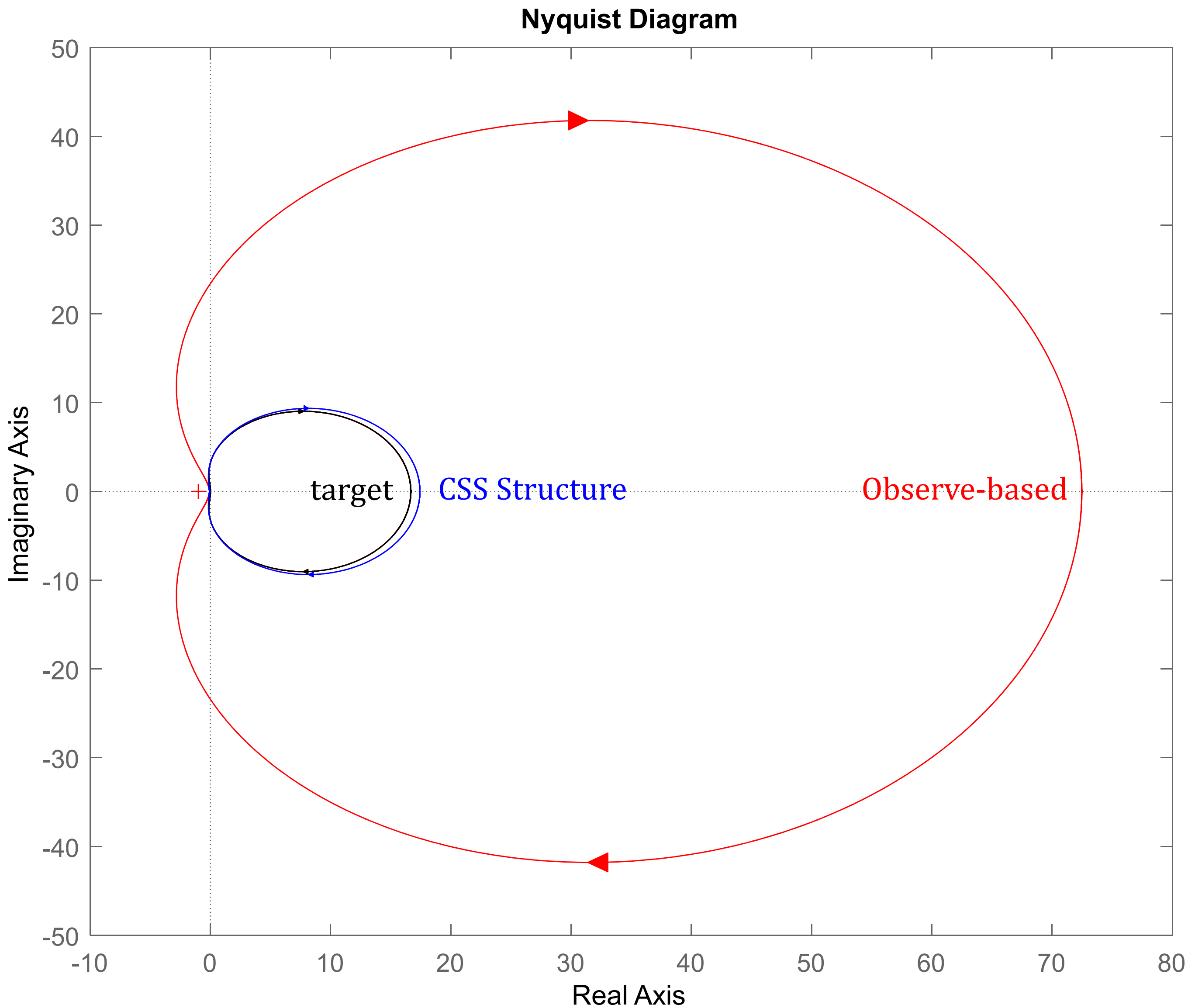
ltrloops

Recovery errors…





ltrloops



Gain Margin

=

(0, ∞)

for all 3 loops

Phase Margin

=

85.94°

(target)

=

81.47°

(CSS)

=

74.47°

(Observer)

Advantages and drawbacks of multivariable control techniques

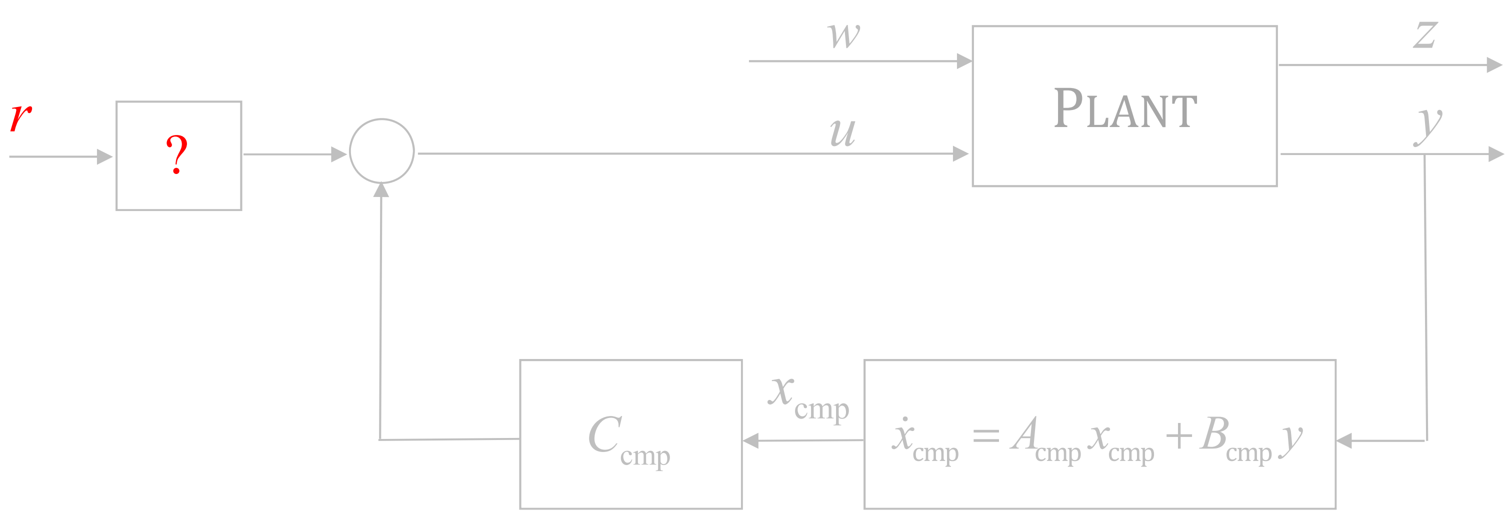
The advantages of the multivariable control techniques covered:

- It is relatively easy to formulate the control system design process into some optimization problems, which can effectively be solved using MATLAB.
- The problem formulations are mathematically elegant and are applicable to general MIMO systems.
- Some techniques, such as LQR, can automatically guarantee remarkable robust properties (such as impressive gain and phase margins). Some, such as H_∞ control, could yield a design that is robust to perturbation and uncertainties.

The drawbacks are also very obvious:

- It is tedious to tune the parameters (e.g., Q and R in LQR, weighting functions in H_2 and H_∞ control) used in optimization associated with the design process.
- It is hard, if not impossible, to formulate the design process directly linked to the **time-domain specifications** (such as overshoot, settling time and/or rise time), as it is done in classical control.

Robust & Perfect Tracking (RPT) Control*

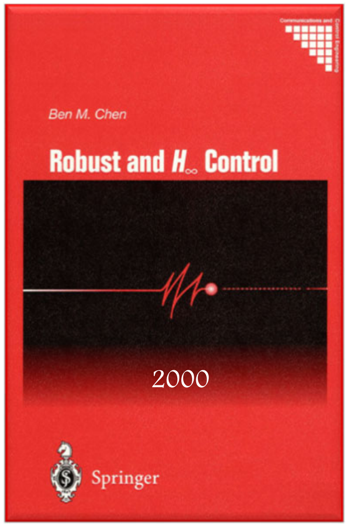


We have spent too much time so far on frequency-domain methods……

What about the time-domain performance?

Robust and perfect tracking control

The robust and perfect tracking (RPT) control technique developed by Chen and his co-workers is to design a controller such that the resulting closed-loop system is stable, and the controlled output almost perfectly tracks a given reference signal in the presence of any initial conditions and external disturbances.



One of the most interesting features in the RPT control method is its capability of utilizing all possible information available in its controller structure. Such a feature is highly desirable for UAV flight missions or other unmanned vehicles involving complicated maneuvers, in which not only the position reference is useful, but also its velocity and even acceleration information are important or even necessary to be used in order to achieve a good overall performance.

BMC

Problem formulation

Consider the following continuous-time system:

$$\Sigma : \begin{cases} \dot{\boldsymbol{x}} = A\,\boldsymbol{x} + B\,\boldsymbol{u} + E\,\boldsymbol{w}, & \boldsymbol{x}(0) = \boldsymbol{x}_0, \\ \boldsymbol{y} = C_1\,\boldsymbol{x} + D_1\,\boldsymbol{w} \\ \boldsymbol{z} = C_2\,\boldsymbol{x} + D_2\,\boldsymbol{u} + D_{22}\,\boldsymbol{w} \end{cases} \tag{8.1}$$

Robust to disturbance & initial condition

where $\boldsymbol{x} \in \mathbb{R}^n$ is the state, $\boldsymbol{u} \in \mathbb{R}^m$ is the control input, $\boldsymbol{w} \in \mathbb{R}^q$ is the external disturbance, $\boldsymbol{y} \in \mathbb{R}^p$ is the measurement output, and $\boldsymbol{z} \in \mathbb{R}^\ell$ is the output to be controlled. Given the external disturbance $\boldsymbol{w} \in L_{\boldsymbol{p}}$, $\boldsymbol{p} \in [1,\infty)$, and any reference signal vector $\boldsymbol{r} \in \mathbb{R}^\ell$ with $\boldsymbol{r}$, $\dot{\boldsymbol{r}}$, $\cdots$, $\boldsymbol{r}^{(\kappa-1)}$, $\kappa \geq 1$, being available, and $\boldsymbol{r}^{(\kappa)}$ being either a vector of delta functions or in $L_{\boldsymbol{p}}$, the RPT problem for the system in (8.1) is to find a parameterized dynamic measurement control law of the following form:

$$\begin{cases} \dot{\boldsymbol{v}} = A_{\mathrm{cmp}}(\varepsilon)\boldsymbol{v} + B_{\mathrm{cmp}}(\varepsilon)\boldsymbol{y} + G_0(\varepsilon)\boldsymbol{r} + \cdots + G_{\kappa-1}(\varepsilon)\boldsymbol{r}^{(\kappa-1)} \\ \boldsymbol{u} = C_{\mathrm{cmp}}(\varepsilon)\boldsymbol{v} + D_{\mathrm{cmp}}(\varepsilon)\boldsymbol{y} + H_0(\varepsilon)\boldsymbol{r} + \cdots + H_{\kappa-1}(\varepsilon)\boldsymbol{r}^{(\kappa-1)} \end{cases} \tag{8.2}$$

RPT CONTROL

such that when the controller of (8.2) is applied to the system of (8.1), we have the following

1. There exists an $\varepsilon^* > 0$ such that the resulting closed-loop system with $\boldsymbol{r} = 0$ and $\boldsymbol{w} = 0$ is asymptotically stable for all $\varepsilon \in (0, \varepsilon^*]$.
2. Let $\boldsymbol{z}(t,\varepsilon)$ be the closed-loop controlled output response and let $\boldsymbol{e}(t,\varepsilon)$ be the resulting tracking error, i.e., $\boldsymbol{e}(t,\varepsilon) := \boldsymbol{z}(t,\varepsilon) - \boldsymbol{r}(t)$. Then, for any initial condition of the state, $\boldsymbol{x}_0 \in \mathbb{R}^n$,

Perfect in Tracking

$$\|\boldsymbol{e}\|_{\boldsymbol{p}} = \left(\int_0^\infty |\boldsymbol{e}(t)|^{\boldsymbol{p}}\, \mathrm{d}t \right)^{1/\boldsymbol{p}} \to 0 \text{ as } \varepsilon \to 0. \tag{8.3}$$

Solvability conditions:

Corollary 8.0.1. Consider the given system (8.1) with its external disturbance $w \in L_p$, $p \in [1, \infty)$, its initial condition $x(0) = x_0$. Assume that all its states are measured for feedback, i.e., $C_1 = I$ and $D_1 = 0$. Then, for any reference signal $r(t)$, which has all its i-th order derivatives, $i = 1, 2, \cdots, \kappa - 1$, $\kappa \geq 1$, being available and $r^{(\kappa)}(t)$ being either a vector of delta functions or in L_p, the proposed robust and perfect tracking (RPT) problem is solvable by the control law of (8.2) if and only if the following conditions are satisfied:

1. (A, B) is stabilizable;
2. $D_{22} = 0$;
3. Σ_{P}, i.e., (A, B, C_2, D_2), is right invertible and of minimum phase. $\boxed{\text{C}}$

Good system!

The solvability condition for the general measurement feedback case is rather complicated. Please refer to the reference text for details (Theorem 9.2.1).

Solution:

Remark 8.0.1. Note that the required gain matrices for the state feedback RPT problem might be computed by solving the following Riccati equation,

$$\boldsymbol{P}\tilde{\boldsymbol{A}}+\tilde{\boldsymbol{A}}'\boldsymbol{P}+\tilde{\boldsymbol{C}}_2'\tilde{\boldsymbol{C}}_2-\left(\boldsymbol{P}\boldsymbol{B}+\tilde{\boldsymbol{C}}_2'\tilde{\boldsymbol{D}}_2\right)\left(\tilde{\boldsymbol{D}}_2'\tilde{\boldsymbol{D}}_2\right)^{-1}\left(\boldsymbol{P}\boldsymbol{B}+\tilde{\boldsymbol{C}}_2'\tilde{\boldsymbol{D}}_2\right)'=0, \qquad (8.4)$$

for a positive definite solution $\boldsymbol{P} > 0$, where

$$\tilde{\boldsymbol{C}}_2 = \begin{bmatrix} \boldsymbol{C}_2 \\ \varepsilon I_{\kappa\ell+n} \\ 0 \end{bmatrix}, \quad \tilde{\boldsymbol{D}}_2 = \begin{bmatrix} \boldsymbol{D}_2 \\ 0 \\ \varepsilon I_m \end{bmatrix},$$

$$\tilde{\boldsymbol{A}} = \begin{bmatrix} \tilde{A}_0 & 0 \\ 0 & A \end{bmatrix}, \quad \tilde{A}_0 = -\varepsilon I_{\kappa\ell} + \begin{bmatrix} 0 & I_\ell & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I_\ell \\ 0 & 0 & \cdots & 0 \end{bmatrix},$$

$$\boldsymbol{B} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ B \end{bmatrix}, \quad \boldsymbol{C}_2 = [\,-I_\ell \quad 0 \quad 0 \quad \cdots \quad 0 \quad C_2\,], \quad \boldsymbol{D}_2 = D_2.$$

The required gain matrix is then given by

$$\tilde{\boldsymbol{F}}(\varepsilon) = -\left(\tilde{\boldsymbol{D}}_2'\tilde{\boldsymbol{D}}_2\right)^{-1}\left(\boldsymbol{P}\boldsymbol{B} + \tilde{\boldsymbol{C}}_2'\tilde{\boldsymbol{D}}_2\right)' = [\,H_0(\varepsilon) \quad \cdots \quad H_{\kappa-1}(\varepsilon) \quad F(\varepsilon)\,],$$

where $H_i(\varepsilon) \in \mathbb{R}^{m\times\ell}$ and $F(\varepsilon) \in \mathbb{R}^{m\times n}$. Finally, we note that solutions to the Riccati equation (8.4) might have severe numerical problems as ε tends smaller and smaller. $\boxed{\mathbb{R}}$

The RPT state feedback control law:

$$u = F(\varepsilon)x + H_0(\varepsilon)r + H_1(\varepsilon)\dot{r} + \cdots + H_{\kappa-1}(\varepsilon)r^{(\kappa-1)}$$

which feeds in all the possible reference signals.

Such a controller structure is a perfect choice for flight control systems, in which not only the position reference is relevant, but also the velocity and acceleration references are crucial in many applications.

Special case…

For the special case when the given plant is of a double integrator, i.e.,

$$\begin{pmatrix} \dot{p} \\ \dot{v} \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} p \\ v \end{pmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + E\,w, \quad y = \begin{pmatrix} p \\ v \end{pmatrix}, \quad z = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{pmatrix} p \\ v \end{pmatrix}$$

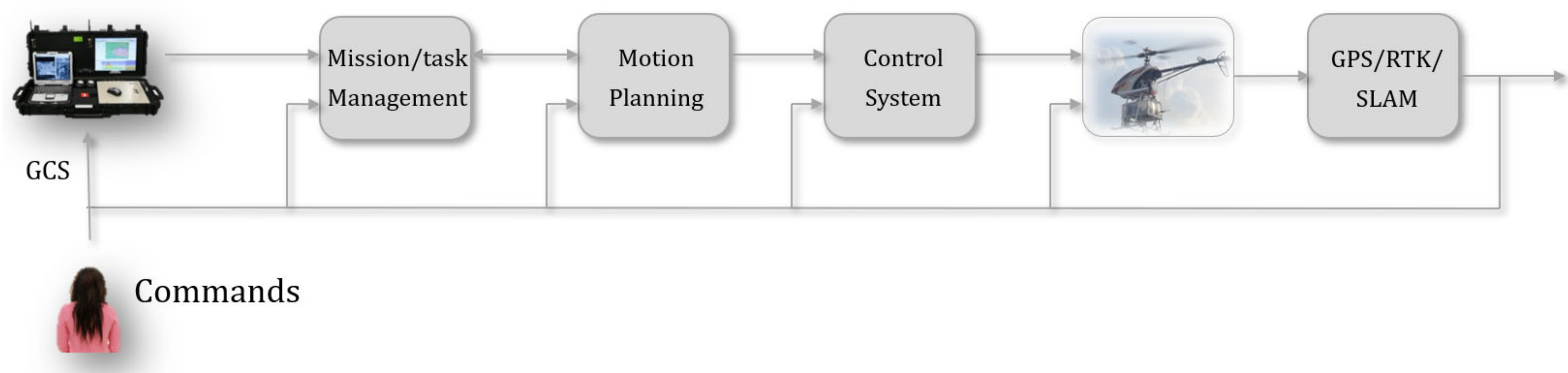
where p is the position and v is the acceleration, assuming the reference position (p_r), velocity (v_r) and acceleration (a_r) are all available, it can be shown that the RPT control law can be calculated in the following closed-form

$$u = -\begin{bmatrix} \dfrac{\omega_\mathrm{n}^2}{\varepsilon^2} & \dfrac{2\varsigma\omega_\mathrm{n}}{\varepsilon} \end{bmatrix} \begin{pmatrix} p \\ v \end{pmatrix} + \left(\dfrac{\omega_\mathrm{n}^2}{\varepsilon^2} \right) p_\mathrm{r} + \left(\dfrac{2\varsigma\omega_\mathrm{n}}{\varepsilon} \right) v_\mathrm{r} + a_\mathrm{r}$$

where ζ is the damping ratio and ω_n is the natural frequency of the closed-loop system, and ε is the tuning parameter.

We note such a plant is very common in real applications including the outer loop flight control systems. In fact, the RPT control is very effective in improving flight performance for UAVs.

Case study: Unmanned helicopter systems



The unmanned helicopter flight control system consists of two loop: An inner loop and an outer loop…

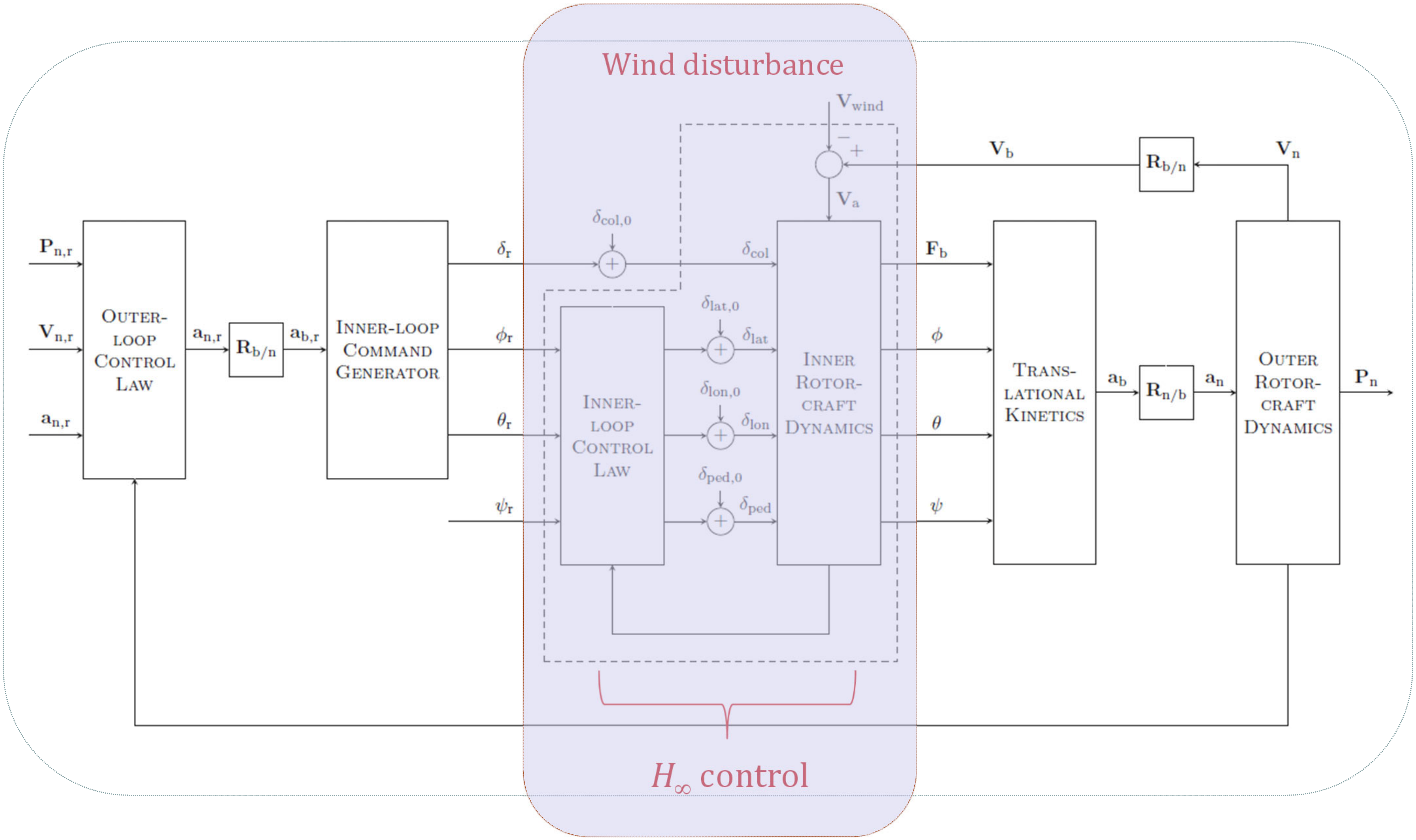
✓ Inner Loop to stabilize UAV attitude

- PID Control (commonly used)
- Optimal Control
- Robust Control
- Nonlinear Control
- ……

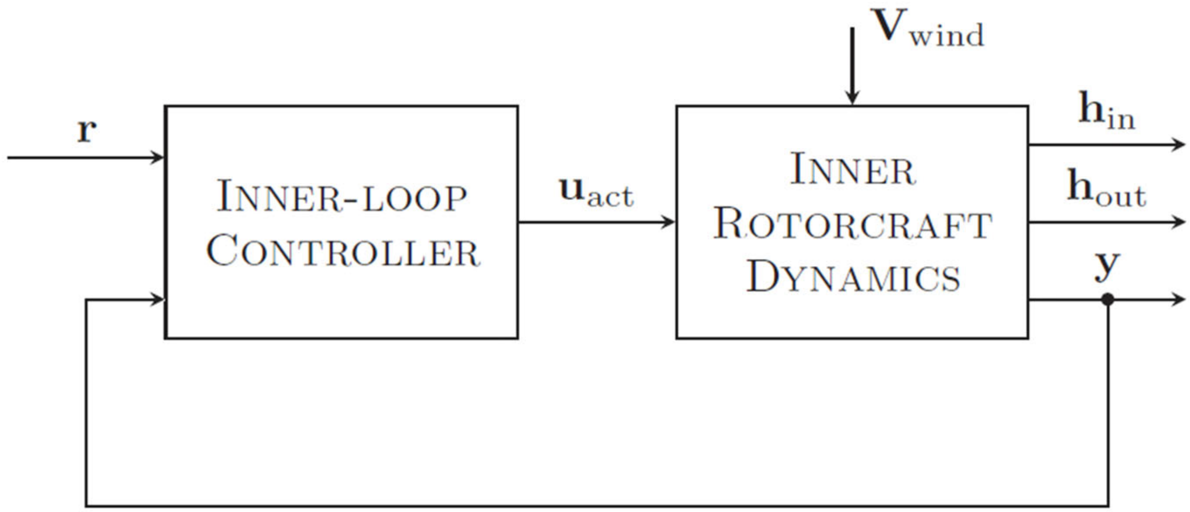
✓ Outer Loop to control position/velocity

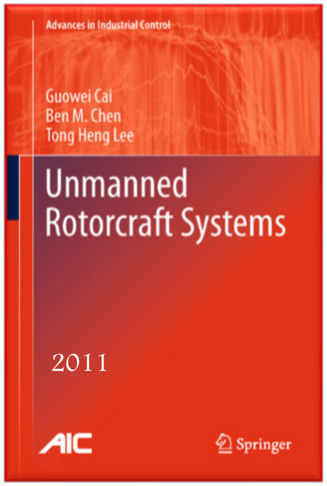
- PID Control (commonly used)
- Pole placement
- RPT Control
- Robust Control
- ……

Detailed control structure



Inner-loop control system design setup





$$\mathbf{x} = [\phi \quad \theta \quad p \quad q \quad a_s \quad b_s \quad r \quad \delta_{\text{ped,int}} \quad \psi]^{\mathrm{T}}$$

$$\mathbf{u}_{\text{act}} = [\delta_{\text{lat}} \quad \delta_{\text{lon}} \quad \delta_{\text{ped}}]^{\mathrm{T}}$$

$$\mathbf{y} = [\phi \quad \theta \quad p \quad q \quad r \quad \psi]^{\mathrm{T}}$$

$$\mathbf{h}_{\text{out}} := [\phi \quad \theta \quad \psi]^{\mathrm{T}}$$

This part is actually what are solving in design problems for this course.

Inner-loop linearized model at hover

$$\begin{cases} \dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u} + E\mathbf{w} \\ \mathbf{y} = C_1\mathbf{x} \\ \mathbf{h}_{\text{out}} = C_{\text{out}}\mathbf{x} \end{cases}$$

$$\mathbf{y} = \begin{pmatrix} \phi \\ \theta \\ p \\ q \\ r \\ \psi \end{pmatrix}$$

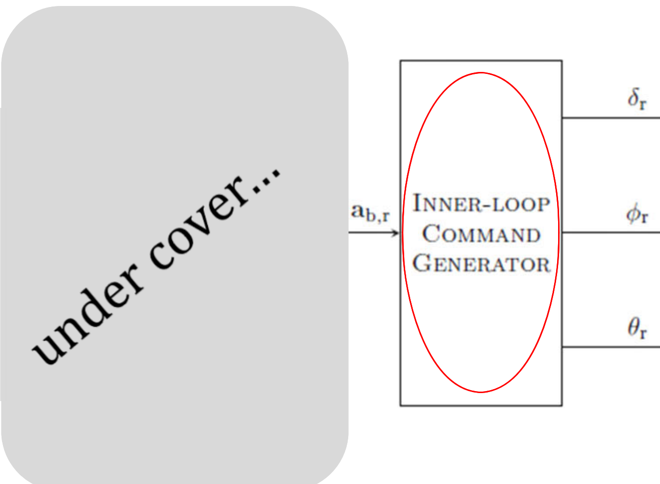
$$\mathbf{h}_{\text{out}} = \begin{pmatrix} \phi \\ \theta \\ \psi \end{pmatrix}$$

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0.0009 & 0 & 0 \\ 0 & 0 & 0 & 0.9992 & 0 & 0 & -0.0389 & 0 & 0 \\ 0 & 0 & -0.0302 & -0.0056 & -0.0003 & 585.1165 & 11.4448 & -59.529 & 0 \\ 0 & 0 & 0 & -0.0707 & 267.7499 & -0.0003 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1.0000 & -3.3607 & 2.2223 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 2.4483 & -3.3607 & 0 & 0 & 0 \\ 0 & 0 & 0.0579 & 0.0108 & 0.0049 & 0.0037 & -21.9557 & 114.2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0.0389 & 0 & 0 & 0.9992 & 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 43.3635 \\ 0 & 0 & 0 \\ 0.2026 & 2.5878 & 0 \\ 2.5878 & -0.0663 & 0 \\ 0 & 0 & -83.1883 \\ 0 & 0 & -3.8500 \\ 0 & 0 & 0 \end{bmatrix} \qquad E = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -0.0001 & 0.1756 & -0.0395 \\ 0.0000 & 0.0003 & 0.0338 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ -0.0002 & -0.3396 & 0.6424 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

One can use the techniques covered earlier, i.e., H_2 control, H_∞ control, or LQG to design an appropriate inner-loop controller for the above system.

Inner-loop command generator

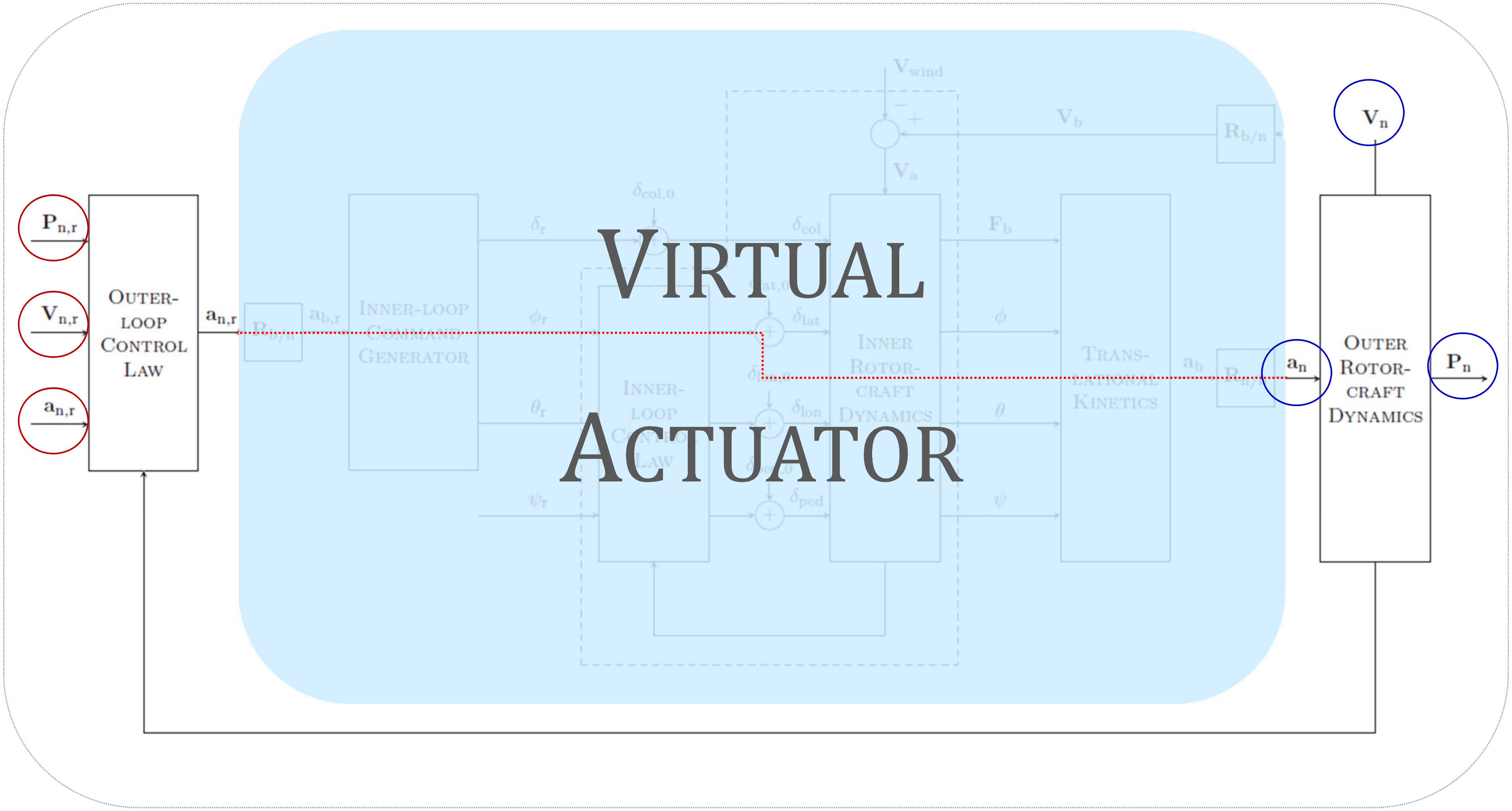


The inner-loop command generator is given as

$$\begin{pmatrix} \delta_r \\ \phi_r \\ \theta_r \end{pmatrix} = \begin{bmatrix} 0 & 0.0019 & 0.0477 \\ 0 & 0.1022 & -0.0037 \\ -0.1022 & 0 & 0.0001 \end{bmatrix} \mathbf{a}_{b,r}$$

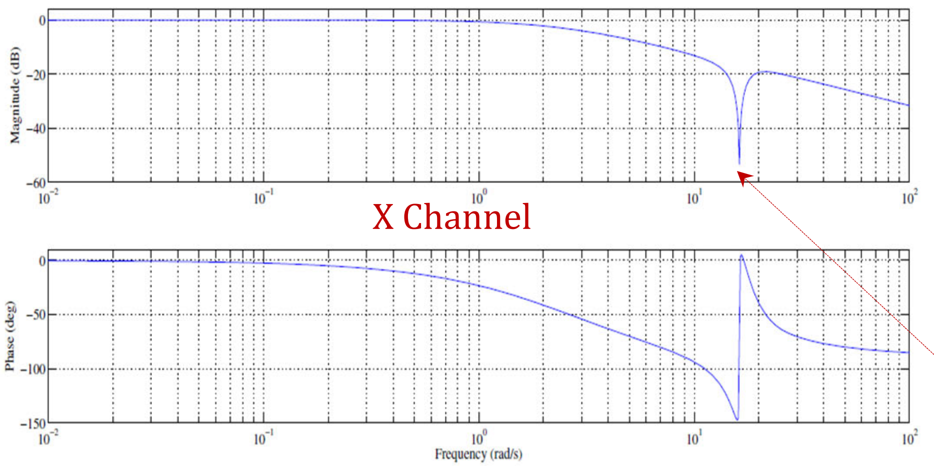
Here we note that the purpose of adding the inner-loop command generator is to yield a unity DC gain…

Outer-loop control system design setup

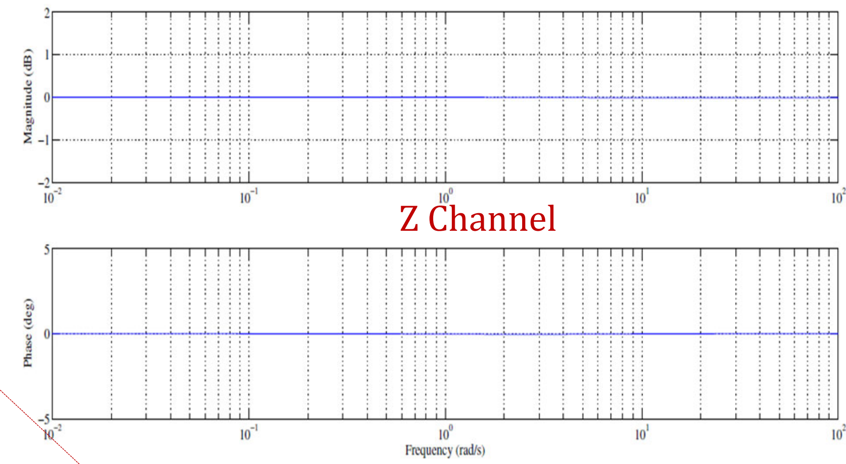


Properties of the virtual actuator

X Channel

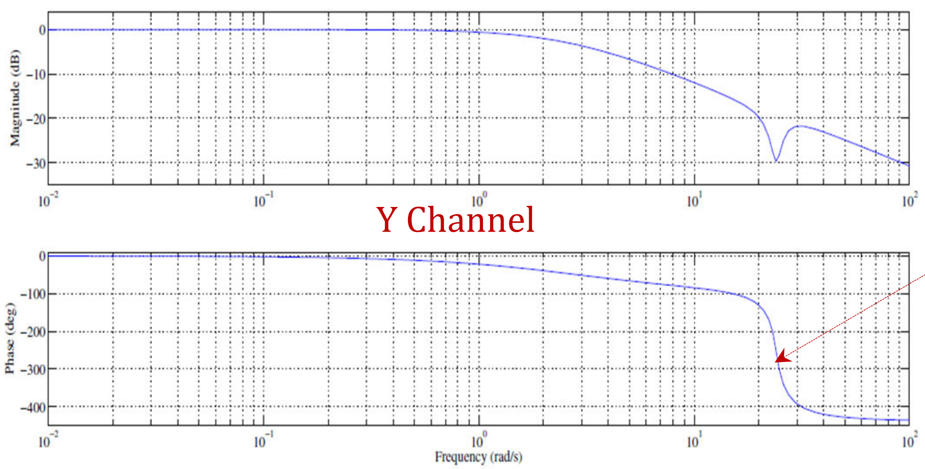


Z Channel



Frequency response of the virtual actuator…

Y Channel



Unstable Zeros!

From practical point of view, it is safe to ignore them so long as the outer-loop bandwidth is within 1 rad/sec…

Properties of the outer-loop dynamics

It can also be verified that coupling among each channel of the outer loop dynamics is very weak and thus can be ignored. As a result, all the x, y and z channels of the rotorcraft dynamics can be treated as decoupled, and each channel can be characterized by

$$\begin{pmatrix} \dot{p}_* \\ \dot{v}_* \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} p_* \\ v_* \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} a_*$$

where p_* is the position, v_* is the velocity and a_* is the acceleration, which is treated a control input in our formulation.

For such a simple system, it can be controlled by almost all the control techniques available in the literature, which include the most popular and the simplest one such as PID control…

Outer-loop RPT control law


$$a_{\mathrm{x,n}} = -\begin{bmatrix} \dfrac{\omega_{\mathrm{n,x}}^2}{\varepsilon_{\mathrm{x}}^2} & \dfrac{2\zeta_{\mathrm{x}}\omega_{\mathrm{n,x}}}{\varepsilon_{\mathrm{x}}} \end{bmatrix} \begin{pmatrix} x_{\mathrm{n}} \\ u_{\mathrm{n}} \end{pmatrix} + \left(\dfrac{\omega_{\mathrm{n,x}}^2}{\varepsilon_{\mathrm{x}}^2}\right) x_{\mathrm{n,r}} + \left(\dfrac{2\zeta_{\mathrm{x}}\omega_{\mathrm{n,x}}}{\varepsilon_{\mathrm{x}}}\right) u_{\mathrm{n,r}} + a_{\mathrm{x,n,r}}$$

$$a_{\mathrm{y,n}} = -\begin{bmatrix} \dfrac{\omega_{\mathrm{n,y}}^2}{\varepsilon^2} & \dfrac{2\zeta_{\mathrm{y}}\omega_{\mathrm{n,y}}}{\varepsilon} \end{bmatrix} \begin{pmatrix} y_{\mathrm{n}} \\ v_{\mathrm{n}} \end{pmatrix} + \left(\dfrac{\omega_{\mathrm{n,y}}^2}{\varepsilon^2}\right) y_{\mathrm{n,r}} + \left(\dfrac{2\zeta_{\mathrm{y}}\omega_{\mathrm{n,y}}}{\varepsilon}\right) v_{\mathrm{n,r}} + a_{\mathrm{y,n,r}}$$

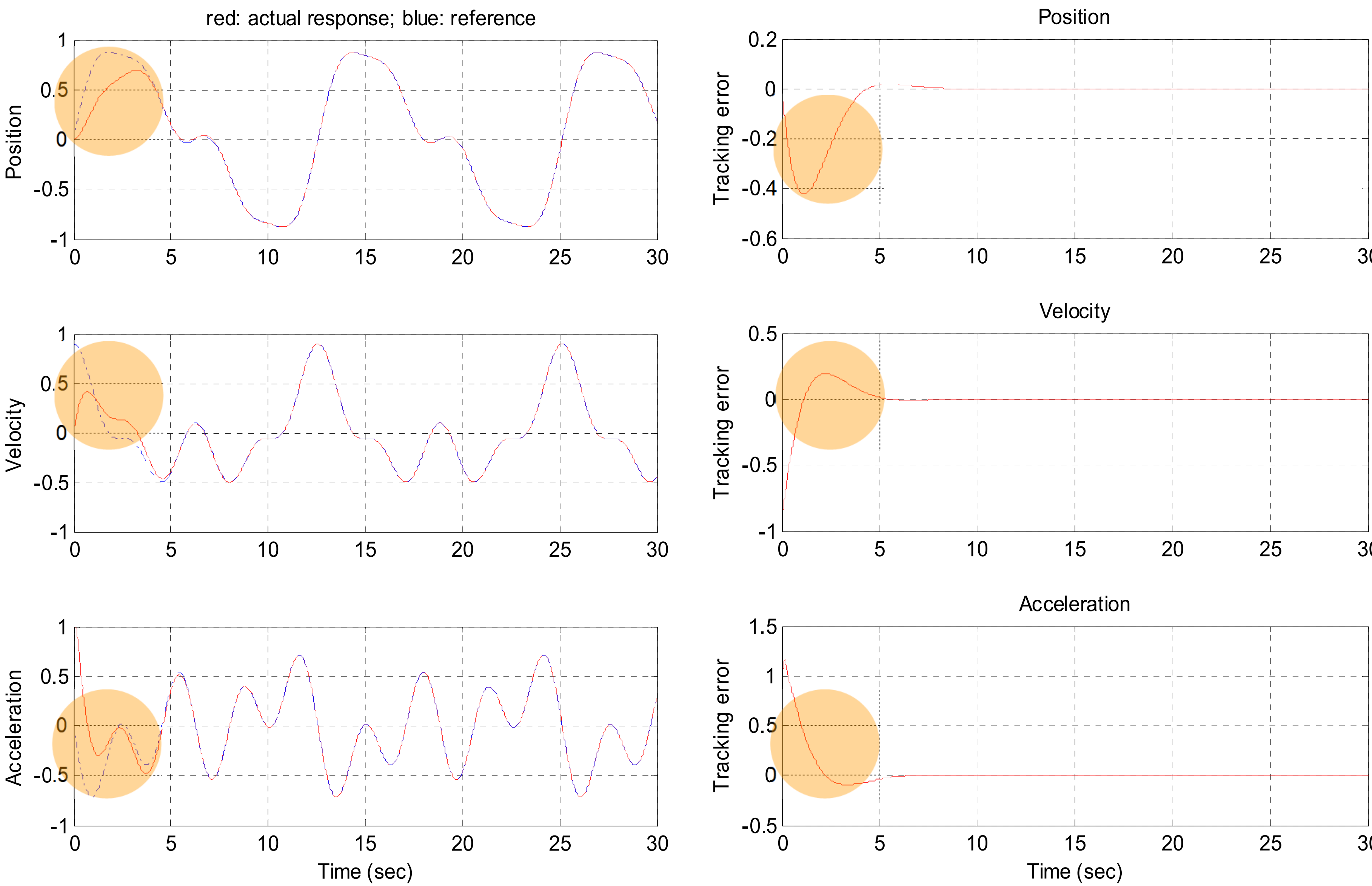
$$a_{\mathrm{z,n}} = -\begin{bmatrix} \dfrac{\omega_{\mathrm{n,z}}^2}{\varepsilon^2} & \dfrac{2\zeta_{\mathrm{z}}\omega_{\mathrm{n,z}}}{\varepsilon} \end{bmatrix} \begin{pmatrix} z_{\mathrm{n}} \\ w_{\mathrm{n}} \end{pmatrix} + \left(\dfrac{\omega_{\mathrm{n,z}}^2}{\varepsilon^2}\right) z_{\mathrm{n,r}} + \left(\dfrac{2\zeta_{\mathrm{z}}\omega_{\mathrm{n,z}}}{\varepsilon}\right) w_{\mathrm{n,r}} + a_{\mathrm{z,n,r}}$$

$$\varepsilon_{\mathrm{x}} = \varepsilon_{\mathrm{y}} = \varepsilon_{\mathrm{z}} = 1$$

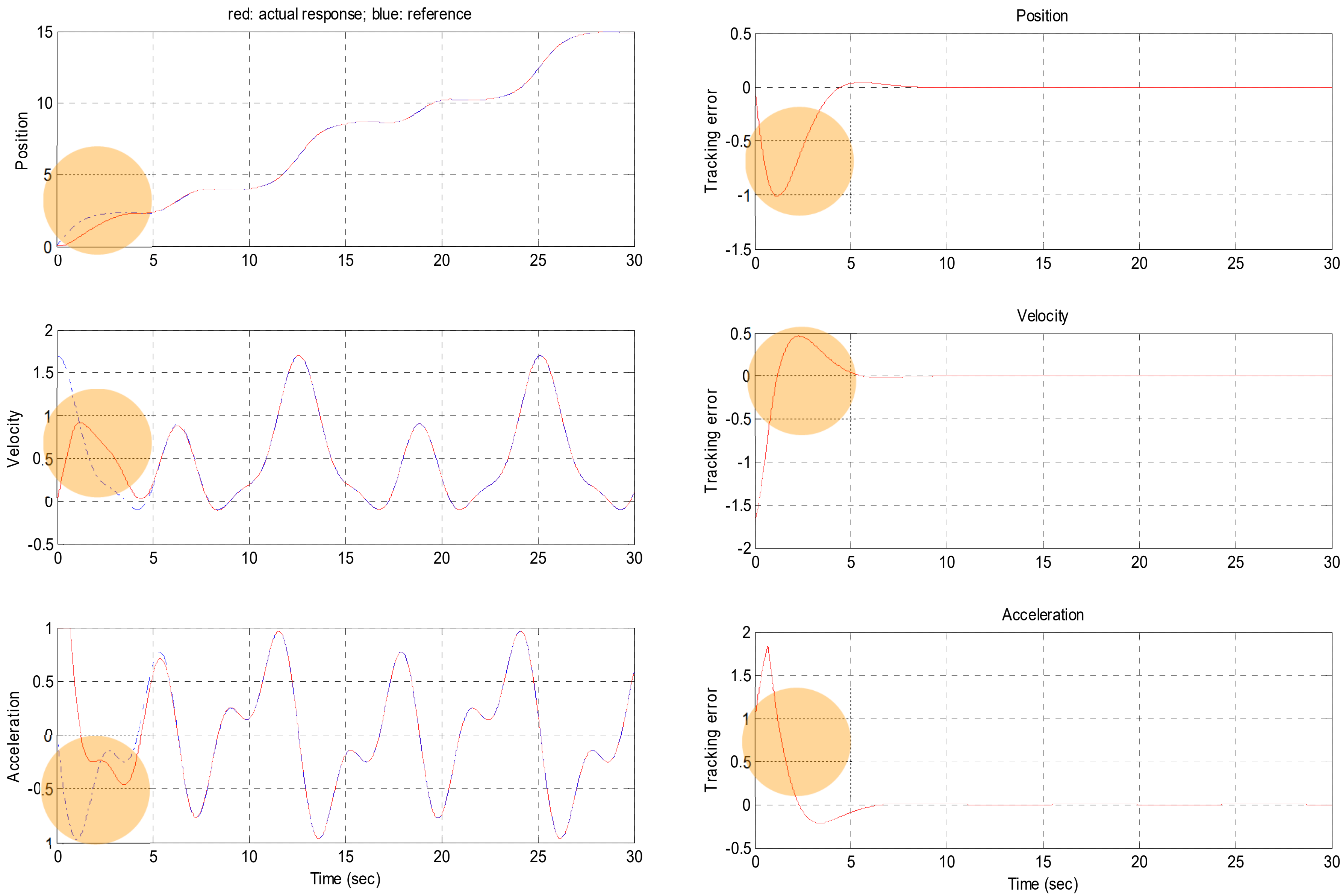
$$\zeta_{\mathrm{x}} = 1, \quad \zeta_{\mathrm{y}} = 1, \quad \zeta_{\mathrm{z}} = 1.1$$

$$\omega_{\mathrm{n,x}} = 0.54, \quad \omega_{\mathrm{n,y}} = 0.62, \quad \omega_{\mathrm{n,z}} = 0.78$$

Simulation of RPT control with $\zeta = 0.7$ & $\omega_n = 1$…



Simulation of RPT control with $\zeta = 0.7$ & $\omega_n = 1$ (cont.)



Concluding Remarks

Improvement of transient performance.

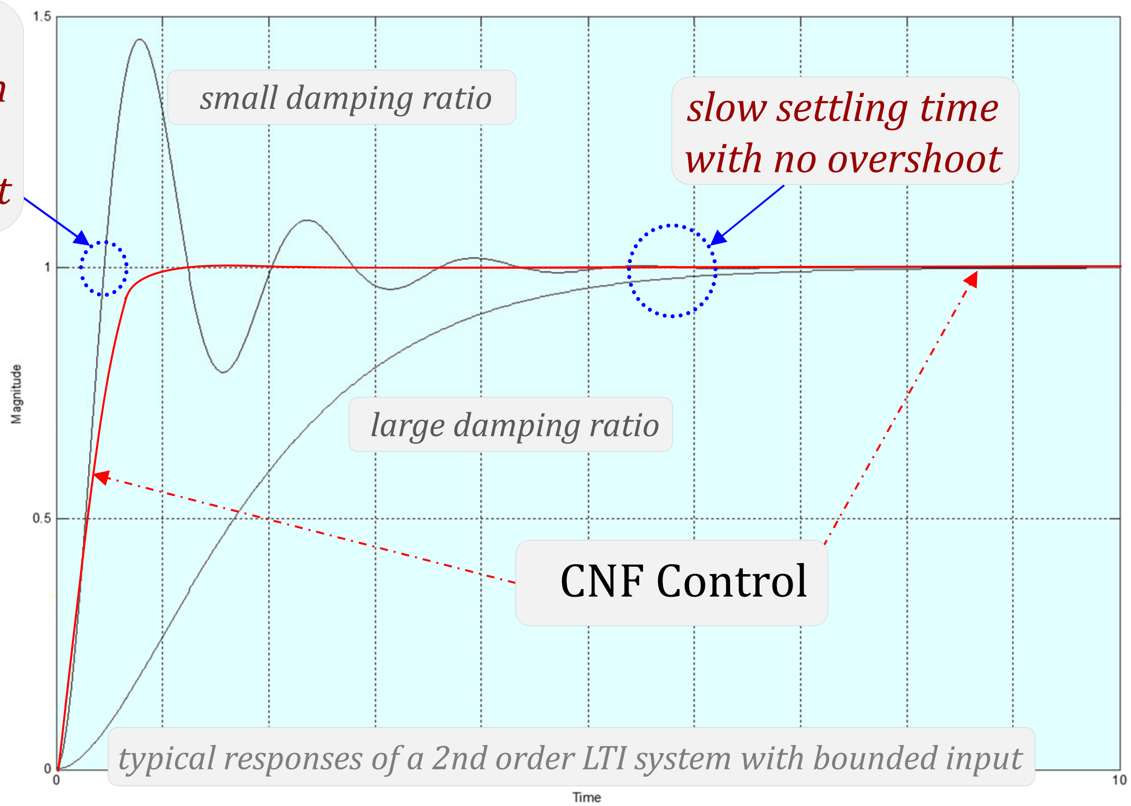
Nonlinear control techniques.

Issues on implementation of controllers on physical systems…

Composite Nonlinear Feedback (CNF) Control…

Nonlinear switching for transient improvement

Key idea… $u = Fx + Gr \underbrace{- \beta e^{-\alpha|e|} \cdot BP' \left[x + (A+BF)^{-1} BGr \right]}$



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Siva S. Banda
US Air Force Research Lab

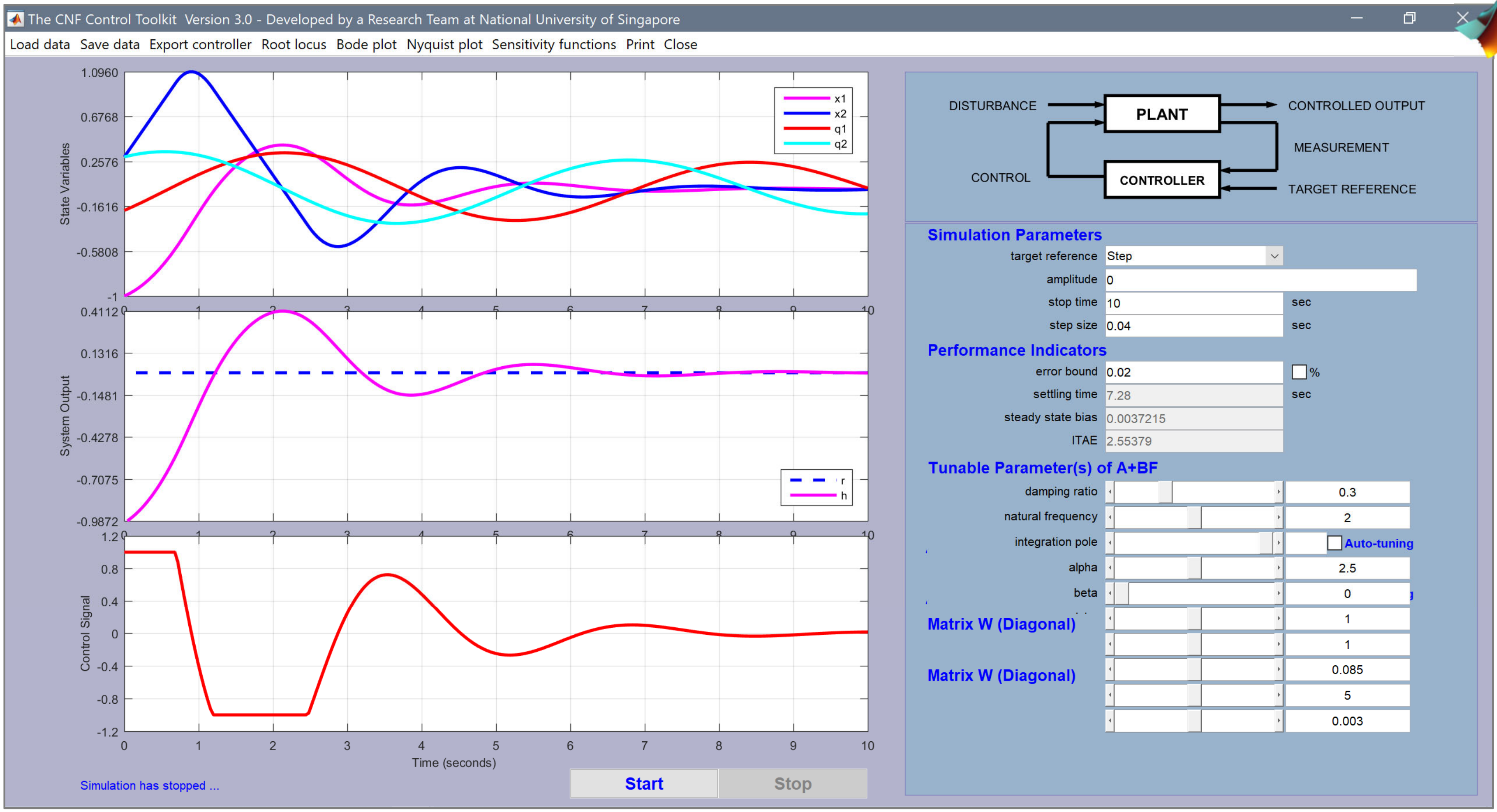
★ Z. Lin, M. Pachter, and S. Banda, "Toward improvement of tracking performance — nonlinear feedback for linear systems," *International Journal of Control,* vol. 70, pp. 1–11, 1998.

★ B. M. Chen, T. H. Lee, K. Peng and V. Venkataramanan, "Composite nonlinear feedback control for linear systems with input saturation: Theory and an application," *IEEE Transactions on Automatic Control,* Vol. 48, pp. 427-439, 2003.

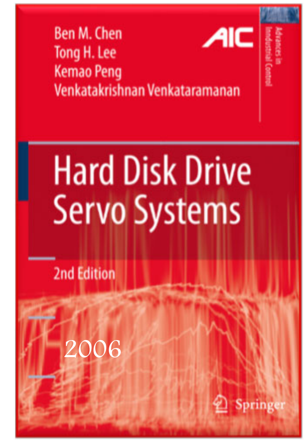
CNF Control Toolkit…

The toolkit, programmed by Guoyang Cheng, provides a user-friendly interface to tune nonlinear parameters…

Guoyang Cheng
Fuzhou University



A Benchmark Problem



Before ending this book, we post in this chapter a typical HDD servo control design problem. The problem has been tackled in the previous chapters using several design methods, such as PID, RPT, CNF, PTOS and MSC control. We feel that it can serve as an interesting and excellent benchmark example for testing other linear and nonlinear control techniques.

We recall that the complete dynamics model of a Maxtor (Model 51536U3) hard drive VCM actuator can be depicted as in Figure 11.1:

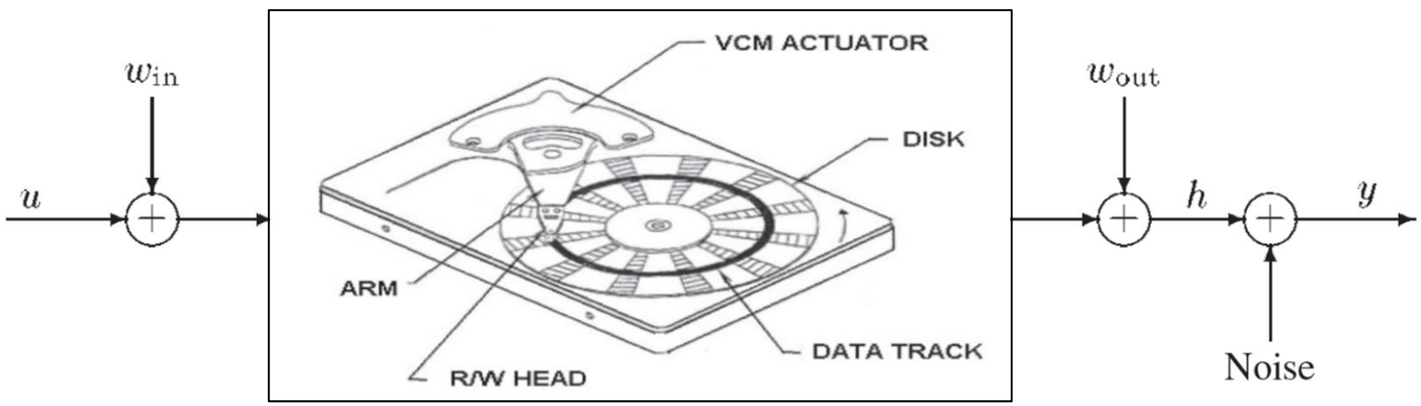
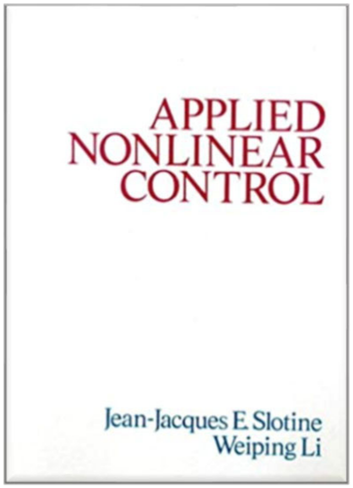
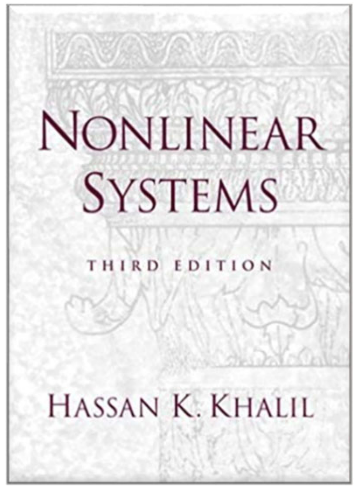
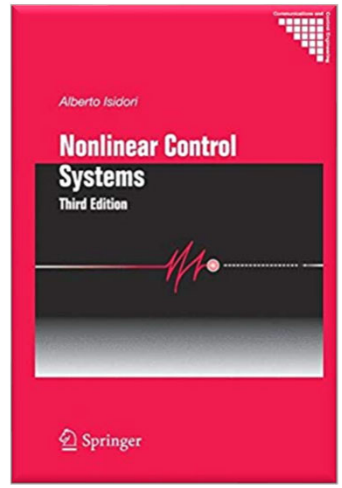


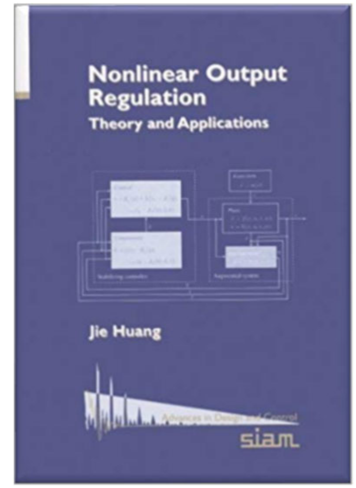
Figure 11.1. Block diagram of the dynamical model of the hard drive VCM actuator

Some good references in nonlinear systems and control…









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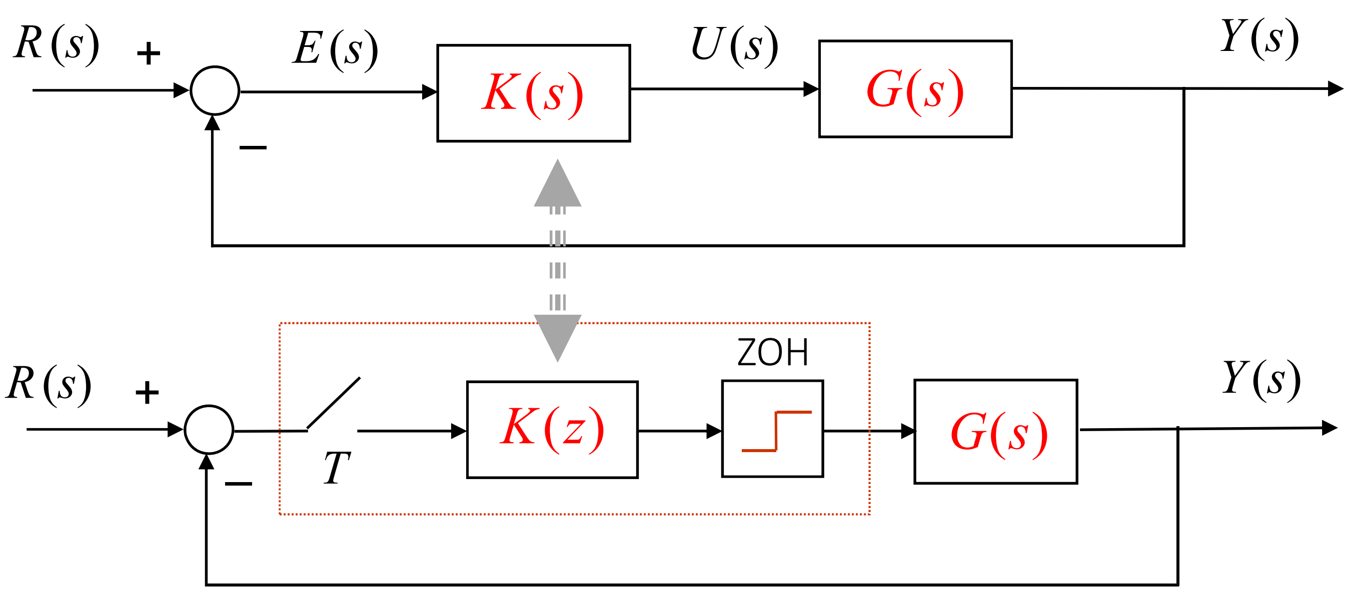
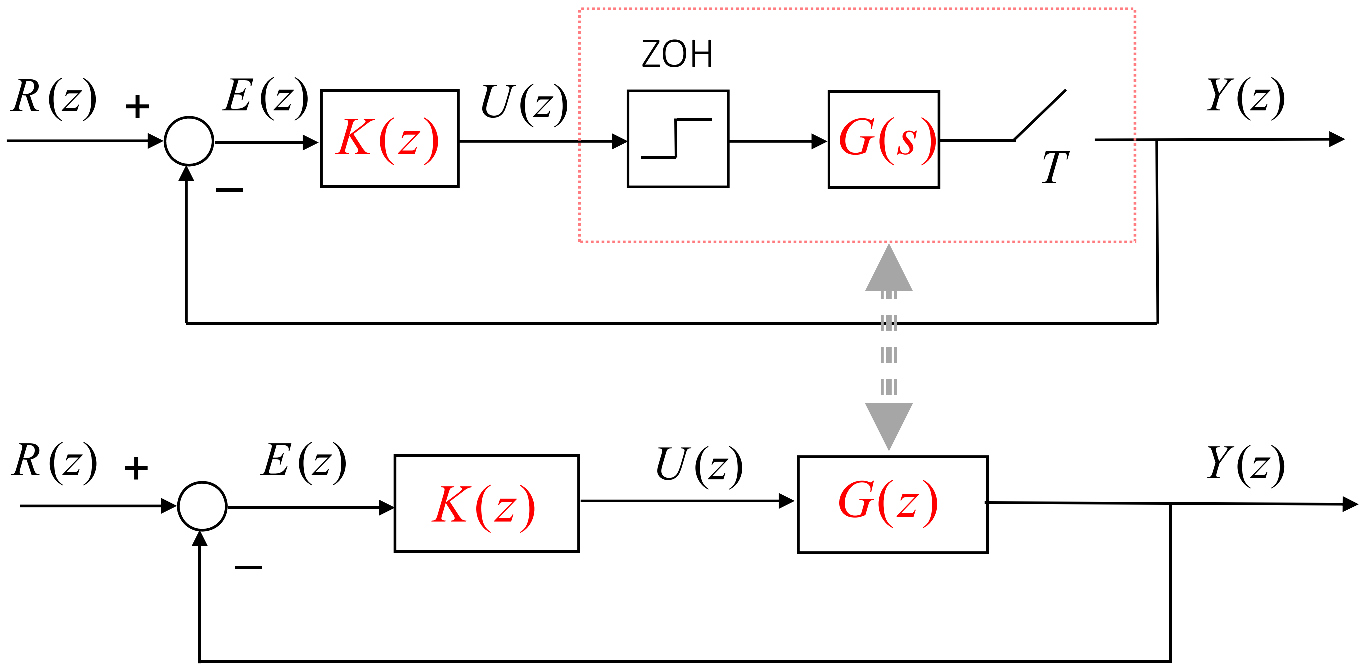
Implementation of controllers……

Implementation of control laws obtained in this part in the real systems can be done using analog devices. It is, however, much more convenient and efficient to realize a controller using a computer or digital signal processor (DSP) instead. There are two ways to design an implementable controller:

i. Design a continuous-time controller like we have done so far in this class and then discretize it using some discretization techniques such as ZOH or bilinear transformation to obtain an equivalent digital controller.



ii. Alternatively, we can discretize the system to be controlled first to obtain a **sampled-data system** or **discrete-time system** and design a controller in the discrete-time setting. The discrete-time controller obtained can then be implemented directly using a computer or digital signal processor (DSP).



Such an approach is to be covered in a course on **computer control systems** or **digital control systems**.

Final remarks……

- We have learned in this course the most fundamental linear systems theory. It is sufficient to understand multivariable control design methods and to carry out some basic multivariable control systems design.

- None of the multivariable control techniques covered in this course can be directly applied to solve real-life problems unless one fully understands the nature of the problem to be solved. However, the design methods presented in this course can be used as the first attempt (and guideline) in solving the actual problem.

- A good system to be controlled is always superior to a good control technique to achieve good performance. A good system is referred to a plant with good system structural properties. As mentioned in Part 1, one can find such topics in the text by Chen, Lin and Shamash (2004).

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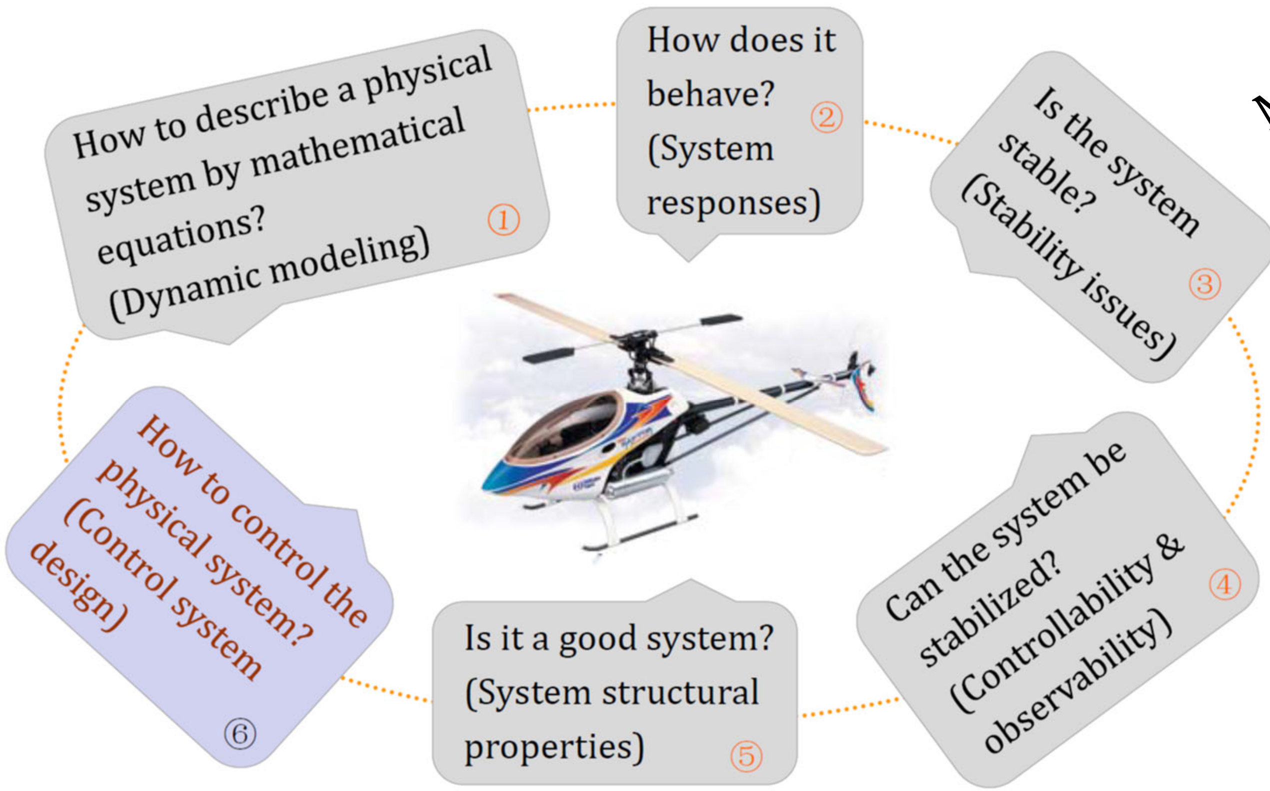
Stony Brook University

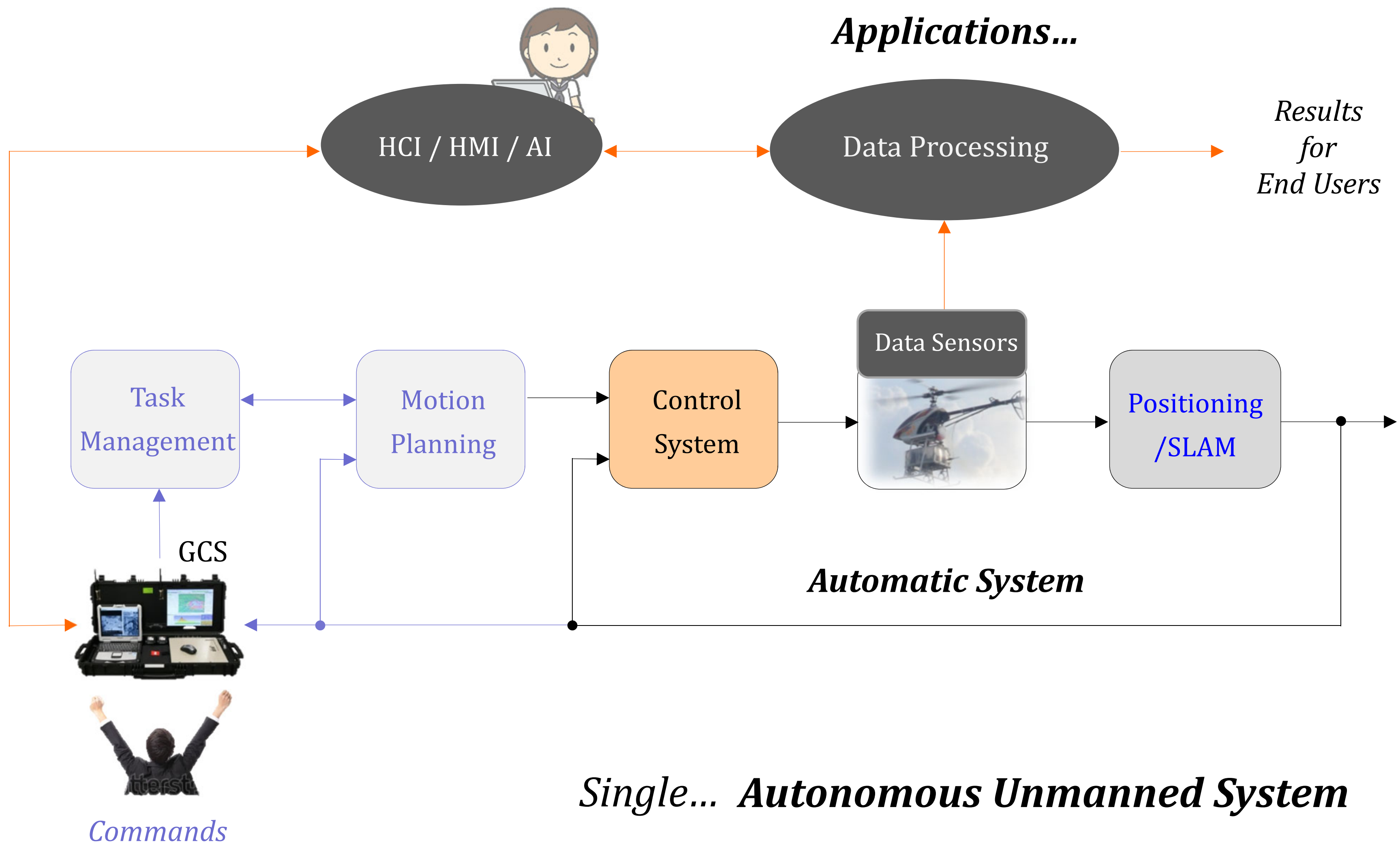
BMC

Mission completed?



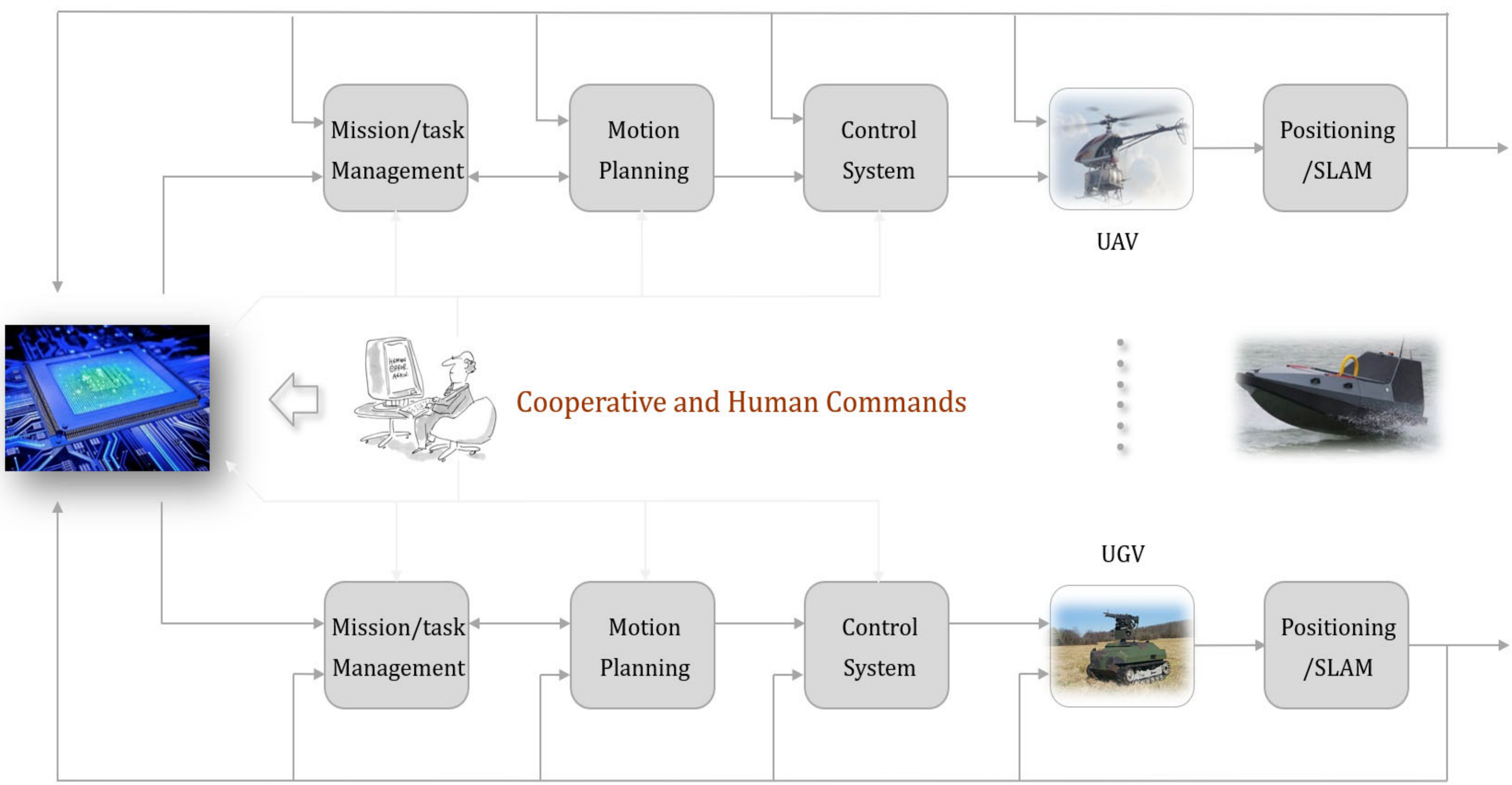
Do not forget: Engineering is to solve real-life problems!……

Beyond automatic control…



*B. M. Chen, On the trends of autonomous unmanned systems research, *Engineering*, 2022. https://doi.org/10.1016/j.eng.2021.10.014

A Cooperative Multi-Agent Systems Framework……



A real-world application:

Industrial building inspection with multiple unmanned aerial systems……

香 港 中 文 大 學
The Chinese University of Hong Kong

香港物流機械人研究中心
HONG KONG CENTRE FOR
LOGISTICS ROBOTICS

基礎設施檢測及管理中的無人系統和人工智能技術

無人系統研究團隊
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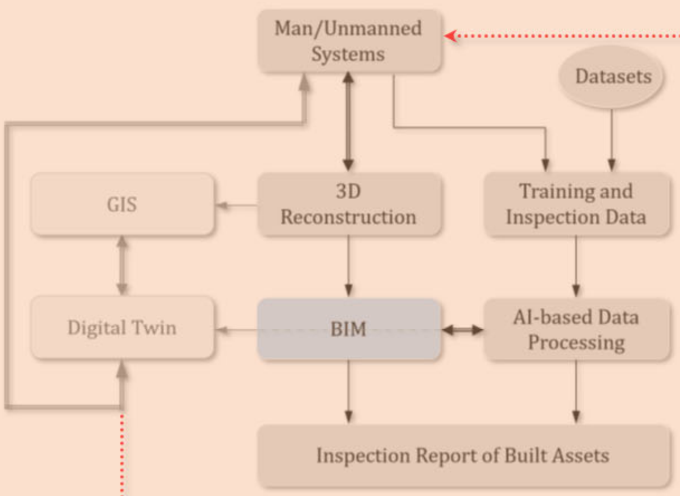
UAS and AI Technologies for Infrastructure Inspection and Information Management

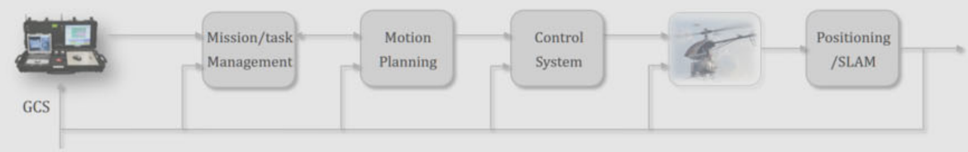
Unmanned Systems Research Group
*Department of Mechanical and Automation Engineering, Chinese University of Hong Kong
**Hong Kong Centre for Logistics Robotics

What's the next?... Just a personal view

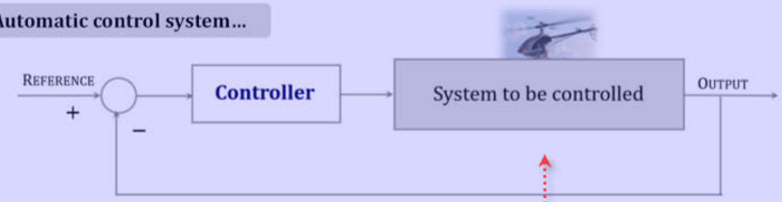
Autonomous inspection system…

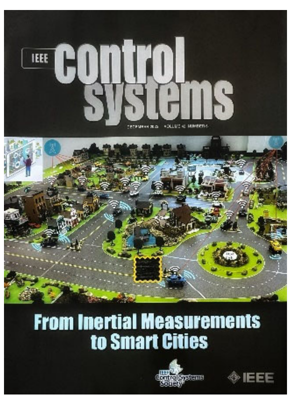


Autonomous unmanned system…



Automatic control system…





Dynamic model… $\dot{x} = f(x,u), \;\; y = h(x,u)$

smart cities……

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NUS UAV GROUP

for MATLAB toolkits and for various flight video clips used throughout this course

Dr Li Qiu 

for sharing the source files of the examples adopted from his textbook

WIKIPEDIA The Free Encyclopedia YouTube

CUHK USR GROUP

for video clips on geographical informational systems (GIS) and 3D reconstruction…

謝謝

That's all, folks! Thank you!