ENGG 2720/ESTR 2014

Complex Variables for Engineers

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Course Outlines

- **Complex Numbers**

  Basic algebraic properties; Algebraic and geometric representations of complex numbers

- **Complex Functions, Complex Differentiation**

  Elementary complex functions – Exponential and trigonometric functions, logarithm and power functions; Limit, continuity, and derivative; Analytic functions, Cauchy–Riemann equations; Harmonic functions, Laplace's equation

- **Complex Integration**

  Line integral on the complex plane; Cauchy's integral theorem and formulae

- **Series**

  Sequences and series, Convergence tests; Power series and basic manipulations; Taylor and Maclaurin series; Laurent series

- **Residue Integration**

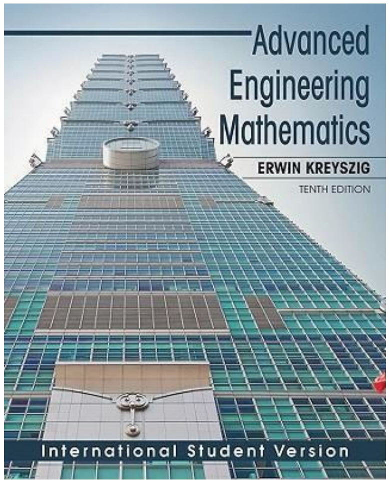
  Singularities, residues; Cauchy's residue theorem; Applications

textbook & reference

E. Kreyszig

Advanced Engineering Mathematics

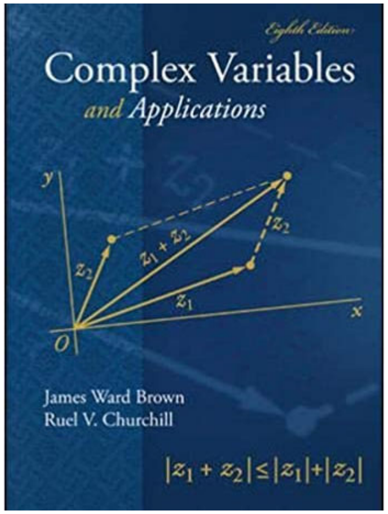
10th Edition, John Wiley & Sons, 2011



J. W. Brown & R. V. Churchill

Complex Variables and Applications

8th Edition, McGraw-Hill, 2009



General Announcements

1. Assessments and Tests
   - Homework Assignments 15%
   - Quizzes (to be randomly announced in the class) 10%
   - Mid-term Exam (common) 25%
   - Final Exam (common) 50%

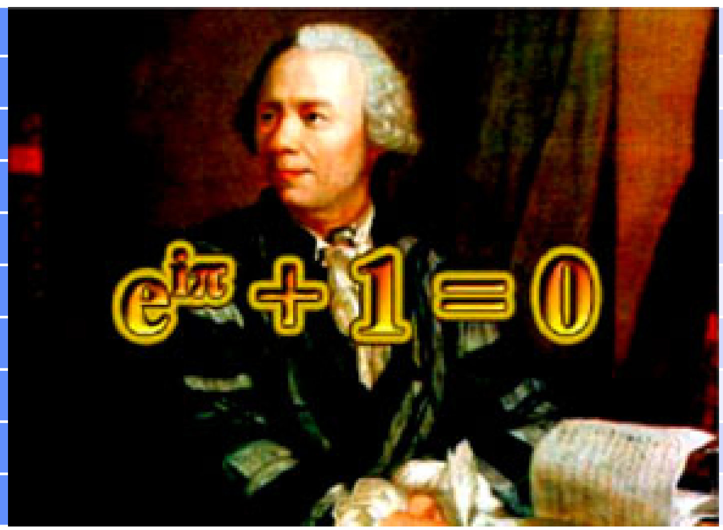
2. The mid-term exam is scheduled to be held from 10:30–12:15, Thursday in the last week of October, in the tutorial session.

3. Both the mid-term and final exam are closed-book. One double-sided A4 handwritten cheat sheet and calculators are allowed.

4. Students are not allowed to switch classes.

General Announcements

5. Tutorial classes start in Week 2. The following are the assignments of tutors for this class…



Tutors	**email: …@mae.cuhk.edu.hk**	**Tutorial Sessions in Charge**	**Homework Assignments**
		Week 01	**Lecture**
		Weeks 02 & 03	To mark Quiz 1
		Weeks 04 & 05	To mark HW 1
		Weeks 06 & 07	To mark HW 2
		Week 08	**Midterm**
		Week 09	To mark HW 3
		Week 10	**Congregation**
		Weeks 11 & 12	To mark HW 4
		Week 13	To mark HW 5
		Coordinator	To mark Quiz 2

Feel free to approach them if you have any problem and/or question about the materials covered in this course. They can be reached by email as in the link above.

Flow Chart of Materials to be Covered

$i=\sqrt{-1}$

$z = x + iy$

Complex number manipulations
Euler's formula

$z(t) = x(t) + iy(t)$

Complex function of a real variable
~ define a ***curve*** on a complex plane

$f(z) = u + iv = u(x,y) + iv(x,y)$

Complex function of a complex variable
~ define a complex mapping from one complex plane to another

$f'(z)$

Complex Differentiation
Cauchy-Riemann equations, analytic functions, singularities

$\int_C f(z)\,dz = \int_\alpha^\beta f[z(t)]\,z'(t)\,dt$

Complex integral
~ integration of a complex function (of a complex variable) on a complex curve
~ *upper bound of a complex integration*

$\oint_C f(z)\,dz = 0$

Cauchy's Integral theorem
~ valid when $f(z)$ is ***analytic*** for every point encircled by a closed curve C

Flow Chart of Materials to be Covered in (cont.)

$$\oint_C \frac{f(z)}{z-z_0}\,dz = 2\pi i\, f(z_0)$$

Cauchy's integral formula
~ C is a closed curve encloses z_0 and $f(z)$ is analytic for every point inside C

$$\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz = 2\pi i\,\frac{f^{(n)}(z_0)}{n!}$$

Generalized Cauchy's integral formula
~ $n \ge 0$, C is a closed curve encloses z_0 and $f(z)$ is analytic inside C

$$\sum_{n=0}^{\infty} z_n = z_0 + z_1 + z_2 + \cdots$$

Complex series
~ *Convergence, divergence, absolute convergence, ratio test*

$$\sum_{n=0}^{\infty} a_n (z-z_0)^n$$

Power series
~ *Region of convergence, ratio test Taylor series*

Flow Chart of Materials to be Covered (cont.)

$$\sum_{n=0}^{\infty} a_n (z-z_0)^n \quad \& \quad \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n$$

$$\oint_C f(z)\,dz = 2\pi i\,\mathbf{Res}\,(f, z_0) = 2\pi i \cdot a_{-1}$$

Taylor series, Laurent series, Order of singularities, Residues

Complex integral in terms of residues

Applications of complex integral

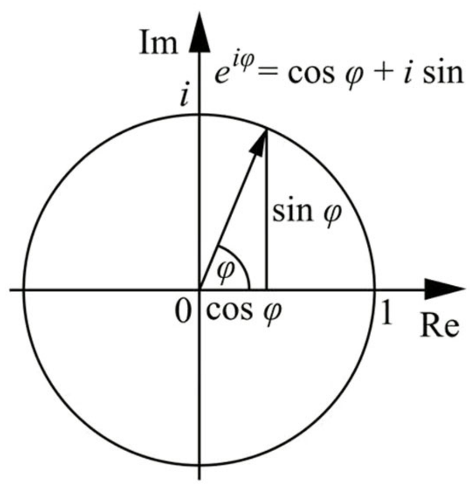
$$\int_0^{2\pi} f(\cos\theta, \sin\theta)\,d\theta$$

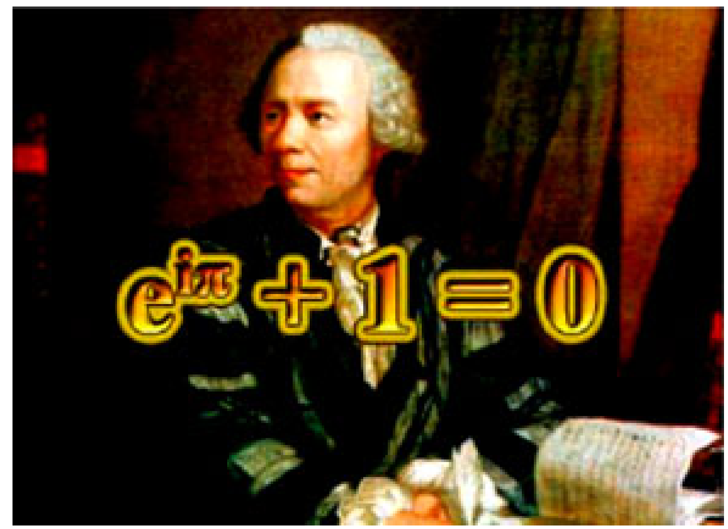
$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}\,dx$$

$$\int_{-\infty}^{\infty} f(x)\cos mx\,dx$$

$$\int_{-\infty}^{\infty} f(x)\sin mx\,dx$$

Euler's formula





Leonhard Euler
(1707–1783)
Swiss Mathematician

$$e^{ix} = \cos x + i \sin x$$

Complex Analysis – 1…

1 **Complex Numbers**

2 Functions of One Complex Variable

3 Complex Differentiation

4 Complex Integration and Cauchy’s Theorem

5 Cauchy Integral Formula

6 Complex Series, Power Series, Taylor Series, Laurent Series

7 Residue Integration

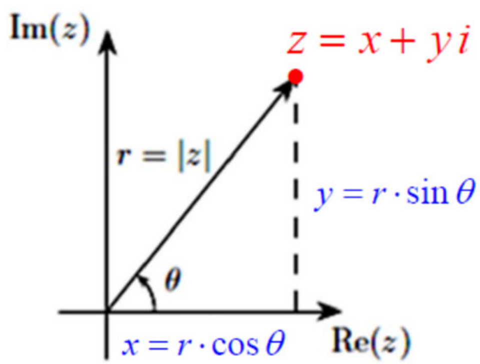
Material flow...

$i = \sqrt{-1}$ ⇨ An imaginary number

$z = x + i\,y = r\,e^{i\theta}$ ⇨ Complex number ~ *Complex conjugate, modulus, argument, principal argument, Euler's formula, de Moivre's formula, n-th root of z...*

$$\bar{z} = x - i\,y = r\,e^{-i\theta}$$

$$r = |z| = \sqrt{x^2 + y^2}, \quad \theta = \arg(z) = \tan^{-1}\frac{y}{x}$$



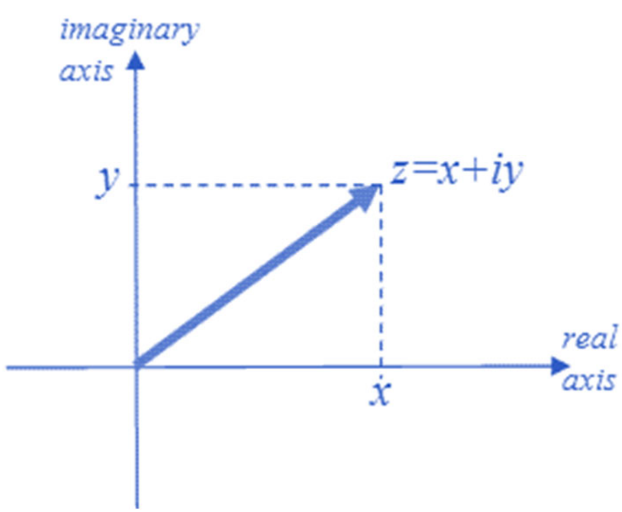
$$-\pi < \mathrm{Arg}(z) \le \pi, \quad \arg(z) = \mathrm{Arg}(z) + 2k\pi, \ \ k = 0, \pm 1, \pm 2, \cdots$$

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$z^{\frac{1}{n}} = r^{\frac{1}{n}}\,e^{\frac{i(\theta+2k\pi)}{n}}, \quad k = 0, 1, \cdots, n-1$$

What is the solution of $x^2 + 1 = 0$? $\Rightarrow$ $x = \pm\sqrt{-1} = \pm i$

It was believed that this equation has no solution before the introduction of imaginary numbers. The use of imaginary numbers was not widely accepted until the work of Leonhard Euler and Carl Friedrich Gauss. The geometric significance of complex numbers as points in a plane was first described by Caspar Wessel.



Leonhard Euler
(1707–1783)
Swiss
Mathematician

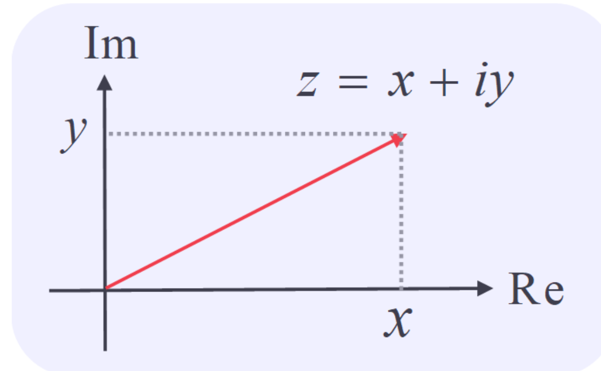
Carl F. Gauss
(1777–1855)
German
Mathematician

Caspar Wessel
(1745–1818)
Danish–Norwegian
Mathematician

Complex Numbers

A complex number is defined (in the Cartesian form) as

$$z = x + iy$$



where

$$i = \sqrt{-1} \qquad \text{or} \qquad i^2 = -1$$

x is called the **Re**al part and y the **Im**aginary part of z, written

$$x = \text{Re } z, \qquad y = \text{Im } z$$

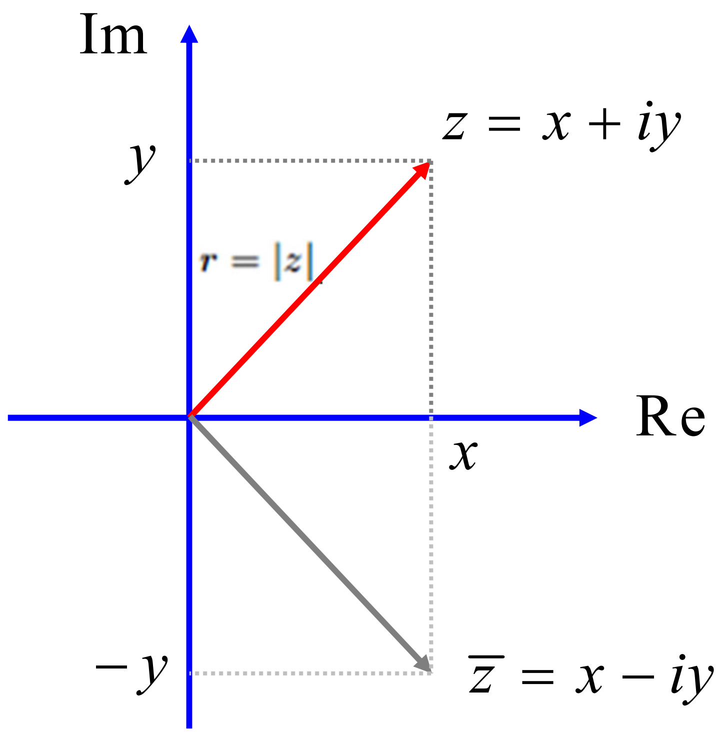
both being a real number. For example,

$$z = -4 + 2i, \quad z = -3 - 5i, \quad z = 5 - 8i$$

all are complex numbers. Occasionally, we might treat a complex number as an ordered pair (x, y) of real numbers x and y, written

$$z = (x, y)$$

Since a complex number has two parts, we can depict it on a 2D-plane, which is called a complex plane.



$\overline{z} = x - iy$ is called the conjugate of z

$$r^2 = |z|^2 = x^2 + y^2$$
$$\Downarrow$$
$$r = |z| = \sqrt{x^2 + y^2}$$

Additions: It is easy to do additions (subtractions) in Cartesian coordinate, i.e.,

$$(a + i\,b) + (v + i\,w) = (a + v) + i\,(b + w)$$

Multiplication:

$$z_1 z_2 = (x_1 + y_1 i)(x_2 + y_2 i) = x_1 x_2 + x_1 y_2 i + x_2 y_1 i + y_1 y_2 i^2$$
$$= (x_1 x_2 - y_1 y_2) + (x_1 y_2 + x_2 y_1) i.$$

Division: The Quotient $z = \frac{z_1}{z_2} (z_2 \neq 0)$

$$\frac{z_1}{z_2} = \frac{z_1}{z_2} \cdot \frac{\overline{z}_2}{\overline{z}_2} = \left(\frac{x_1 + y_1 i}{x_2 + y_2 i} \right) \cdot \left(\frac{x_2 - y_2 i}{x_2 - y_2 i} \right) = \left(\frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2} \right) + \left(\frac{x_2 y_1 - x_1 y_2}{x_2^2 + y_2^2} \right) \cdot i$$

$$z \cdot \overline{z} = (x + iy) \cdot (x - iy) = x^2 + ixy - ixy - i^2 y^2 = x^2 + y^2$$

Example (1a)

- $$\frac{2+i}{3-4i} = \left(\frac{2+i}{3-4i}\right)\cdot\left(\frac{3+4i}{3+4i}\right)$$
$$= \left(\frac{2\cdot 3-1\cdot 4}{3^2+4^2}\right) + \left(\frac{3\cdot 1+2\cdot 4}{3^2+4^2}\right)\cdot i = \frac{2}{25}+\frac{11}{25}\cdot i$$

  which result can be checked by showing that $3-4i$ times $\frac{2+11i}{25}$ gives $2+i$.
- The real part and the imaginary part are $\frac{2}{25}$ and $\frac{11}{25}$, respectively.
- Also, the modulus of this complex number is

$$\left|\frac{2+i}{3-4i}\right| = \left|\frac{2}{25}+\frac{11}{25}\cdot i\right| = \sqrt{\left(\frac{2}{25}\right)^2+\left(\frac{11}{25}\right)^2} = \frac{1}{\sqrt{5}}$$

$$\left|\frac{2+i}{3-4i}\right| = \frac{|2+i|}{|3-4i|} = \frac{\sqrt{2^2+1^2}}{\sqrt{3^2+4^2}} = \frac{\sqrt{5}}{5} = \frac{1}{\sqrt{5}} \quad \Rightarrow \quad \left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$$

- The **complex conjugate** of $z = x + yi$ is defined as

$$\boxed{\bar{z} = x - yi.} \tag{1.7}$$

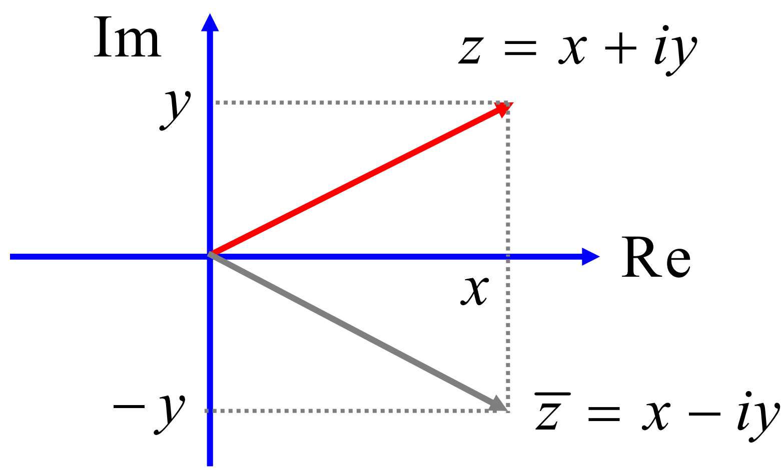
It can be shown that

$$\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2, \quad \overline{z_1 z_2} = \bar{z}_1 \bar{z}_2,$$

and

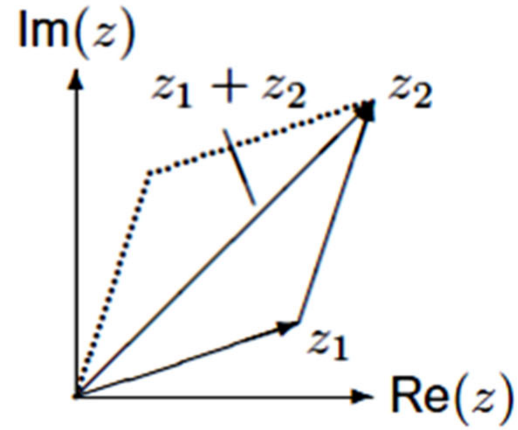
$$\boxed{|z|^2 = z\bar{z}.} \tag{1.8}$$

Geometrically, $\bar{z}$ is the reflection of z along the real axis.



- We have the **triangle inequality**

$$|z_1 + z_2| \leq |z_1| + |z_2|,$$



and the **reverse triangle inequality**

$$|z_1 - z_2| \geq \big|\, |z_1| - |z_2| \,\big|$$

Proof… $|z_1| = |(z_1 - z_2) + z_2| = |\alpha_1 + z_2| \leq |\alpha_1| + |z_2| = |z_1 - z_2| + |z_2|$

$$\Rightarrow \quad |z_1 - z_2| \geq |z_1| - |z_2|$$

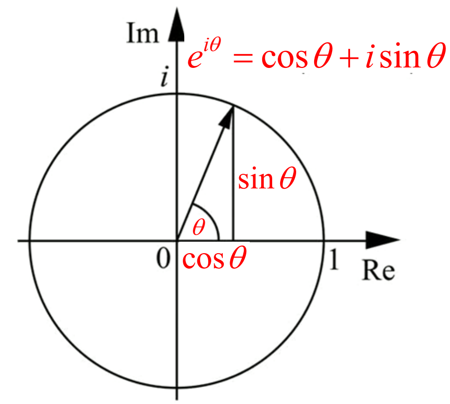
$$|z_2| = |(z_2 - z_1) + z_1| \leq |z_2 - z_1| + |z_1| \quad \Rightarrow \quad |z_2 - z_1| \geq |z_2| - |z_1|$$

$$|z_1 - z_2| = |z_2 - z_1| \geq \big|\, |z_1| - |z_2| \,\big| = \big|\, |z_2| - |z_1| \,\big|$$

Polar Representation of Complex Numbers

There is another way (polar representation) to represent a complex scalar

$$z = re^{i\theta}$$



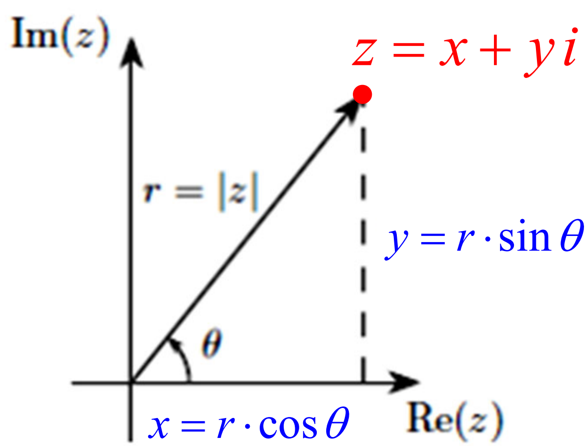
Using the Euler's formula

$$e^{i\theta} = \cos\theta + i\sin\theta$$

we obtain (Polar to Cartesian representations)

$$z = re^{i\theta} = r(\cos\theta + i\sin\theta) = (r\cos\theta) + (r\sin\theta)i = x + yi$$

Conversely,



$$r^2 = |z|^2 = x^2 + y^2$$

$$\tan\theta = \frac{y}{x}$$

$$\Rightarrow \quad \theta = \tan^{-1}\frac{y}{x}$$

Cartesian to Polar Representation

$$x + yi = re^{i\theta}$$

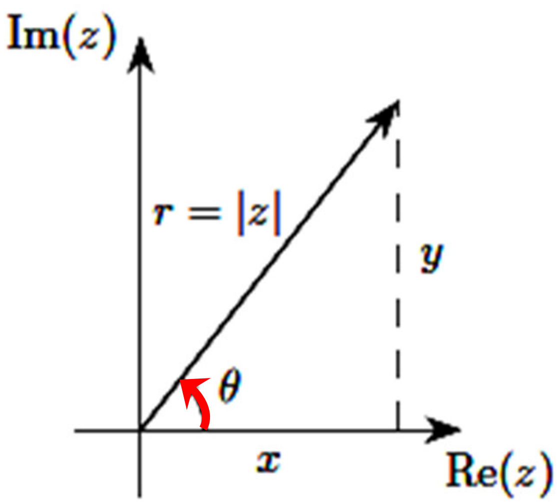
- Note that r is the **absolute value** or **modulus** of z, i.e.,

$$\boxed{r = |z| = \sqrt{x^2 + y^2}.} \tag{1.11}$$

The angle θ, called the **argument** of z, is denoted by $\theta = \arg(z)$, which can be determined from the formula

$$\boxed{\theta = \arg(z) = \tan^{-1}\left(\frac{y}{x}\right)} \tag{1.12}$$

for $z \neq 0$; for $z = 0$ the angle θ is undefined.



All angles (arguments) are measured in radians and **positive** in counter-clockwise sense.

Example: MATLAB DEMO…

The argument of a complex number…

$$z = 4.33 + 2.5\,i$$

⇩

$$r = \sqrt{4.33^2 + 2.5^2} = 5$$

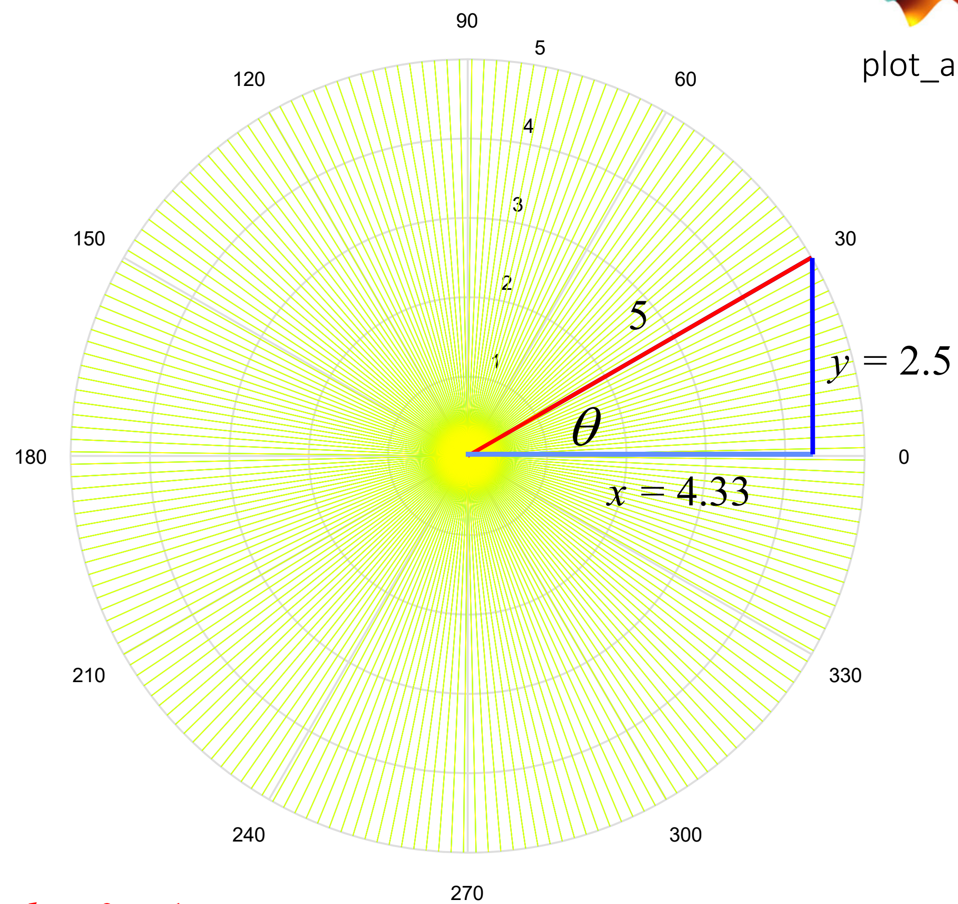
$$\theta = \tan^{-1}\left(\frac{2.5}{4.33}\right) = 0.5236 = \frac{\pi}{6} = 30^\circ$$

⇩

$$z = 5\,e^{i\frac{\pi}{6}} = 5\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$$

$$= 5\left[\cos\left(\frac{\pi}{6} + 2k\pi\right) + i\sin\left(\frac{\pi}{6} + 2k\pi\right)\right], \quad k = 0, \pm 1, \cdots$$

$$= 5\,e^{i\left(\frac{\pi}{6} + 2k\pi\right)}, \quad k = 0, \pm 1, \pm 2, \cdots$$



plot_arg

Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$

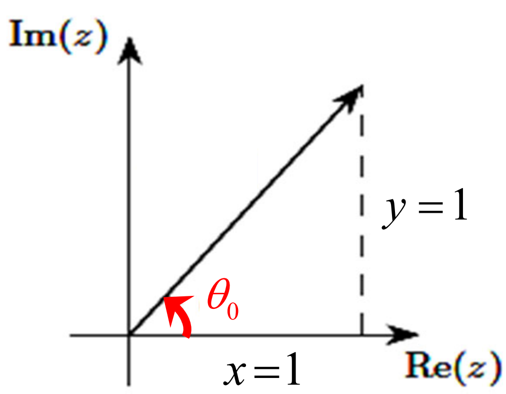
Given any point $z \neq 0$, the angle θ can be determined only to within an arbitrary integer multiple of 2π.

It is **conventional** to define the value of θ in the range $-\pi < \theta \leq \pi$ to be the so-called **principal argument** of z, denoted by $\theta_0 = \text{Arg}\,(z)$. Then, we have

$$\theta = \arg(z) = \text{Arg}(z) + 2k\pi = \theta_0 + 2k\pi$$

for $k = 0, \pm 1, \pm 2, \cdots$, as is evident in the demonstration on the previous page.

For example, $z = x + iy = 1 + i$, its principal argument is given by

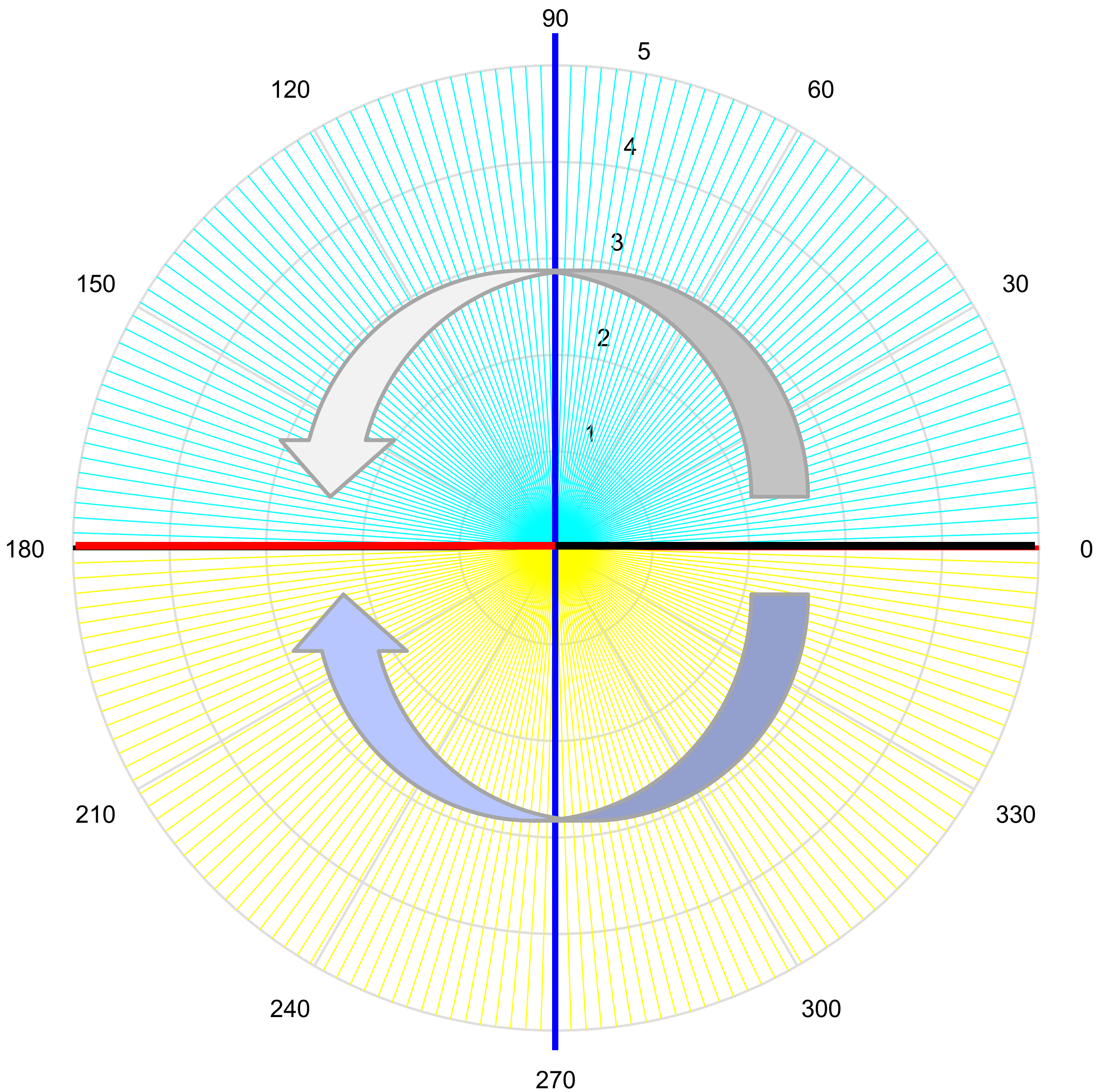


$$\theta_0 = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}(1) = \frac{\pi}{4}$$

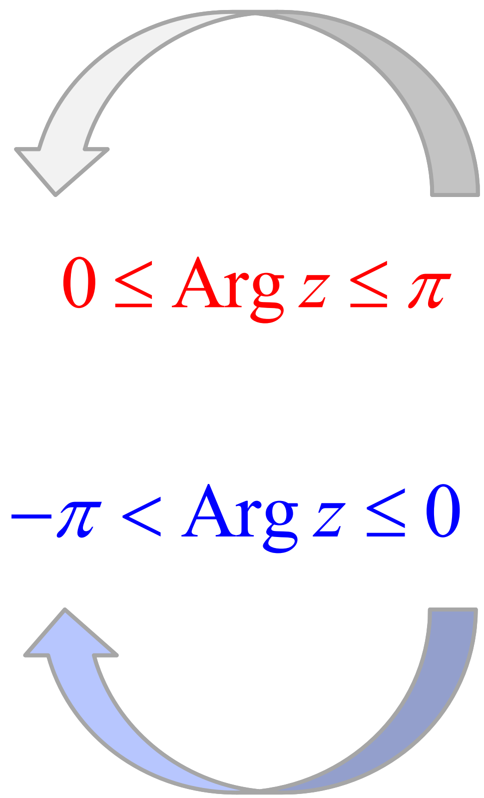
$$\Downarrow$$

$$\theta = \theta_0 + 2k\pi = \frac{\pi}{4} + 2k\pi, \quad k = 0, \pm 1, \pm 2, \cdots$$

Principle Argument…



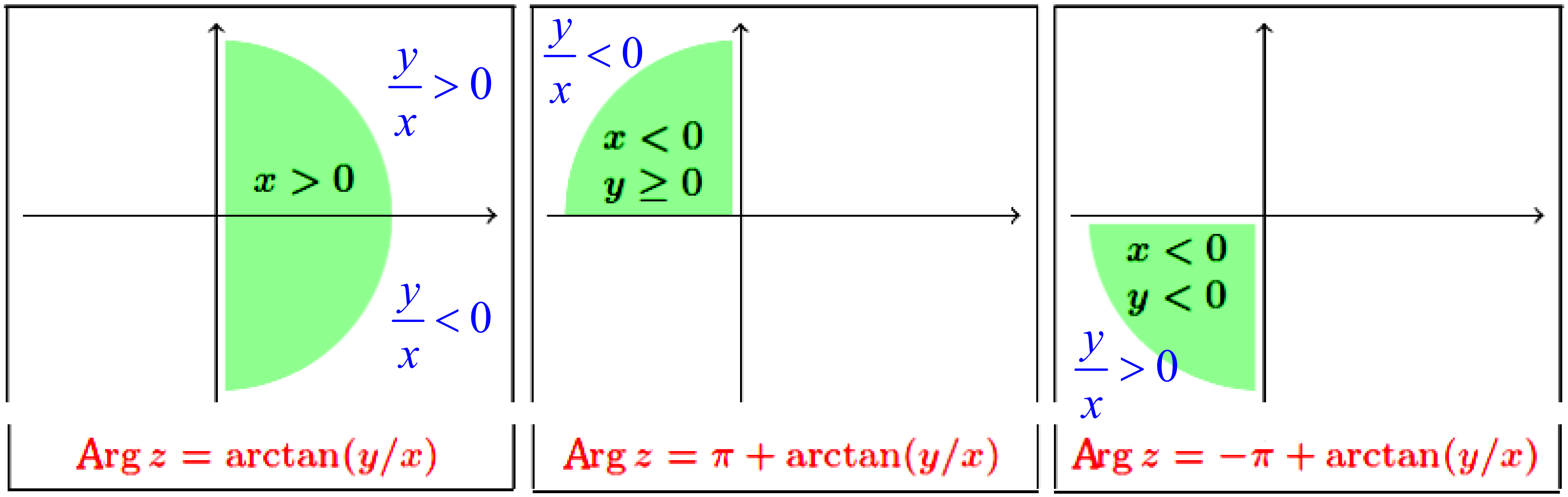
$$-\pi < \operatorname{Arg} z \leq \pi$$



$$0 \leq \operatorname{Arg} z \leq \pi$$

$$-\pi < \operatorname{Arg} z \leq 0$$

- Explicit expression of Arg(z): depends on the location of $z = x + iy$.



$$z = 1 + i \;\Rightarrow\; \mathrm{Arg}\, z = \tan^{-1}\left(\frac{1}{1}\right) = \frac{\pi}{4}$$

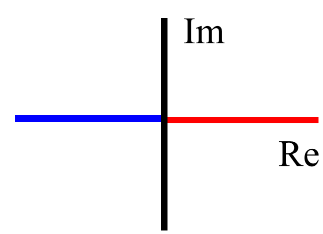
$$z = 1 - i \;\Rightarrow\; \mathrm{Arg}\, z = \tan^{-1}\left(\frac{1}{-1}\right) = -\frac{\pi}{4}$$

$$z = -1 + i \;\Rightarrow\; \mathrm{Arg}\, z = \tan^{-1}\left(\frac{-1}{1}\right) = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$$

$$z = -1 - i \;\Rightarrow\; \mathrm{Arg}\, z = \tan^{-1}\left(\frac{-1}{-1}\right) = -\pi + \frac{\pi}{4} = -\frac{3\pi}{4}$$

For the case when $y = 0$,

$$\mathrm{Arg}\, z = \begin{cases} 0, & \text{if } x > 0 \text{ and } y = 0 \\ \pi, & \text{if } x < 0 \text{ and } y = 0 \\ \text{undefined}, & \text{if } x = 0 \text{ and } y = 0 \end{cases}$$



The polar form of complex numbers is especially convenient for their multiplication and division. For example, let $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$. Then

$$z = z_1 z_2 = (r_1 e^{i\theta_1})(r_2 e^{i\theta_2}) = r_1 r_2 e^{i(\theta_1+\theta_2)} = re^{i\theta}$$

or

$$z_1 z_2 = r_1 r_2[\cos(\theta_1 + \theta_2) + i(\sin(\theta_1 + \theta_2))]$$

and

$$\frac{z_1}{z_2} = \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)}$$

or

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i(\sin(\theta_1 - \theta_2))].$$

Euler's formula $\boxed{e^{i\theta} = \cos\theta + i\sin\theta,}$ $\Rightarrow \left|e^{i\theta}\right| = 1, \quad |z_1 z_2| = r_1\, r_2, \quad \left|\frac{z_1}{z_2}\right| = \frac{r_1}{r_2}$

- In particular, the integer power of z can be computed easily by

$$z^n = \left(re^{i\theta}\right)^n = r^n e^{in\theta} = r^n\left(\cos n\theta + i\sin n\theta\right) \qquad (1.13)$$

where n is any integer. This is known as the **de Moivre's formula**.

$$z^2 = \left(re^{i\theta}\right)^2 = r^2 e^{i2\theta} = r^2\left(\cos 2\theta + i\sin 2\theta\right)$$

$$z^2 = \left(re^{i\theta}\right)^2 = \left(re^{i\theta}\right)\left(re^{i\theta}\right) = r^2\left(\cos\theta + i\sin\theta\right)\left(\cos\theta + i\sin\theta\right) = r^2\left[\left(\cos^2\theta - \sin^2\theta\right) + i\left(2\sin\theta\cos\theta\right)\right]$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$

$$\sin 2\theta = 2\sin\theta\cos\theta$$

Exercise: Prove that

$$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta, \quad \sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$$

- The de Moivre's formula (1.13) gives a way to compute the fractional power $z^{\frac{1}{n}}$. We call $z^{\frac{1}{n}}$ the **n-th root** of z, then

$$z^{\frac{1}{n}} = r^{\frac{1}{n}} \cdot e^{\frac{i(\theta_0 + 2k\pi)}{n}} \qquad (1.14)$$

for $k = 0, 1, \ldots, n-1$. Note that $z^{\frac{1}{n}}$ is a **multi-valued** function.

Generally, we would focus on our attention to solutions whose arguments are within $(-\pi, \pi]$, i.e., we are to choose appropriate values of k such that the corresponding roots with a principal argument.

We will illustrate it through the following examples…

$z = r\, e^{i\theta}$

$e^{i\theta} = \cos\theta + i \sin\theta$

Abraham de Moivre
(1667–1754)
French Mathematician

Example (1b)

Evaluate the values of $(1+i)^{\frac{1}{3}}$.

principal argument

$$r = \sqrt{1^2 + 1^2} = \sqrt{2} \qquad \theta_\circ = \tan^{-1} 1 = \frac{\pi}{4}$$

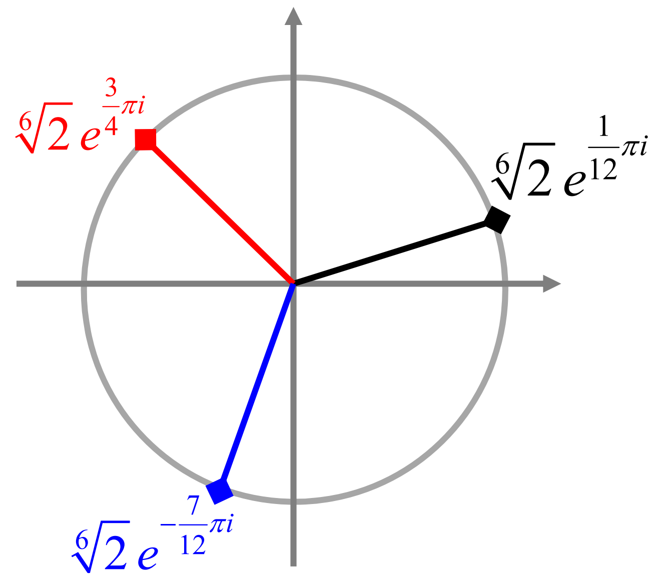
$$1 + i = \sqrt{2}e^{\frac{i\pi}{4}}$$

$$z^{\frac{1}{n}} = r^{\frac{1}{n}} \cdot e^{\frac{i(\theta_0 + 2k\pi)}{n}}$$

$$(1+i)^{\frac{1}{3}} = 2^{\frac{1}{6}} e^{i\left(\frac{\pi}{12} + \frac{2k\pi}{3}\right)}$$

$$= \sqrt[6]{2}\, e^{\frac{1}{12}\pi i}, \quad \sqrt[6]{2}\, e^{\frac{3}{4}\pi i}, \quad \sqrt[6]{2}\, e^{-\frac{7}{12}\pi i}$$

$k = 0$, $k = 1$, $k = -1$



We are selecting appropriate k such that the arguments of the solutions are within $(-\pi, \pi]$…

Example (1c)

Evaluate the values of $z^4 = 1$

To ensure the arguments of all solutions being in $(-\pi, \pi]$

$$z = 1^{\frac{1}{4}} = (1 + 0 \cdot i)^{\frac{1}{4}} = e^{\frac{2k\pi i}{4}} = e^{\frac{k\pi i}{2}}, \quad k = -1, 0, 1, 2$$

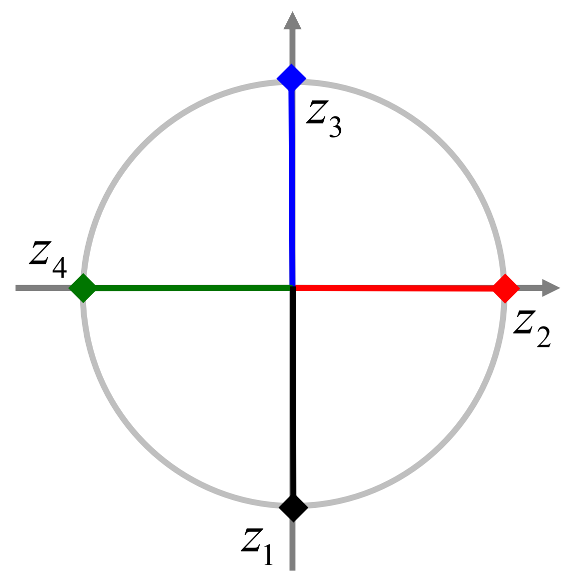
$$z_1 = e^{-\frac{\pi i}{2}} = -i \quad (k = -1)$$

$$z_2 = e^{0 \cdot \pi i} = 1 \quad (k = 0)$$

$$z_3 = e^{\frac{\pi i}{2}} = i \quad (k = 1)$$

$$z_4 = e^{\pi i} = -1 \quad (k = 2)$$

$$z^{\frac{1}{n}} = r^{\frac{1}{n}} \cdot e^{\frac{i(\theta_0 + 2k\pi)}{n}}$$



Homework Assignment No: 1 (Due in two weeks)

Question 1.1: Write the following complex numbers in the form of $a + bi$

(a) $e^{-\pi i/4}$; (b) $e^{1+\pi i}$; (c) e^{3+i}

Question 1.2: Write the following complex numbers in the form of polar representation

(a) $-1+i$; (b) $(2+2i)^{12}$; (c) $\left(\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}i\right)^{19}$

Question 1.3: Solve the equation: $z^2+\sqrt{32}\,zi-6i=0$.

Question 1.4: Let $z_0=2+2i$, answer the following questions

(a) Draw this complex number on the complex plane

(b) Calculate $|z_0|$ and $\arg(z_0)$

(c) Calculate z_0^4

(d) Calculate ALL the fourth roots of z_0 and draw the on the complex plane

Question 1.5: Find all complex numbers z such that $z^4 = -1$ (You can leave your answer in the polar form). How many different solutions are there?

Question 1.6: Let $z_1 = 2 - i$ and $z_2 = 3 + 2i$.

(a) Find $z_1 \cdot (-i) + {}^{z_2}\!/_{i}$.

(b) Find $\mathrm{Re}(z_1 \cdot z_2)$

(c) Find $\overline{z_1 - \overline{z}_2}$

(d) Find ${}^{\overline{z}_1 z_2}\!/_{z_1}$

Question 1.7: Given any two complex scalars z_1 and z_2, show that

(a) The reverse triangle inequality $|z_1 - z_2| \geq \big||z_1| - |z_2|\big|$ holds

(b) $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2\left(|z_1|^2 + |z_2|^2\right)$

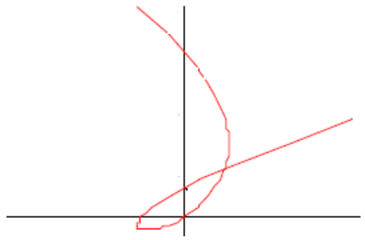
Complex Analysis – 2…

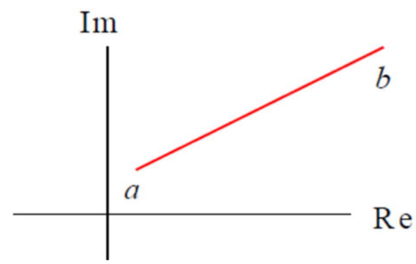
1. Complex Numbers
2. **Functions of One Complex Variable**
3. Complex Differentiation
4. Complex Integration and Cauchy's Theorem
5. Cauchy Integral Formula
6. Complex Series, Power Series, Taylor Series, and Laurent Series
7. Residue Integration

Material flow...

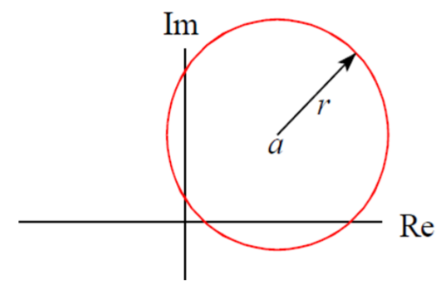
$$z(t) = x(t) + i\,y(t), \quad t \in [a, b]$$

Complex functions of a real variable ~
Curves, circle and straight line





$$z(t) = a + (b-a)t, \quad t \in [0, 1]$$



$$z(t) = a + re^{it}, \quad t \in [a, b]$$

$$f(z) = u + iv = u(x, y) + iv(x, y)$$

Complex functions of a complex variable

$$e^z = 1 + z + \frac{1}{2!}z^2 + \frac{1}{3!}z^3 + \cdots$$

$$\operatorname{Ln} z = \ln|z| + i\,\operatorname{Arg}(z)$$

$$\ln z = \operatorname{Ln} z + 2k\pi i, \quad k = 0, \pm 1, \pm 2, \cdots$$

• • •

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}$$

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

$$z^c = e^{\ln z^c} = e^{c \ln z}$$

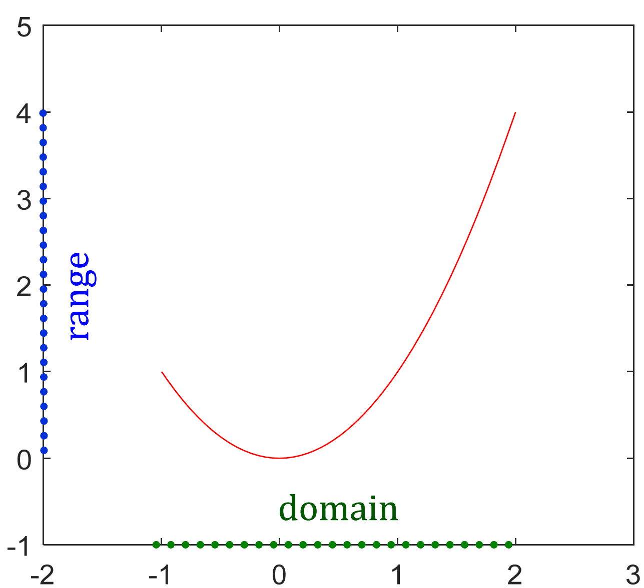
Revisit: Real functions of a real variable

$$y = f(x)$$

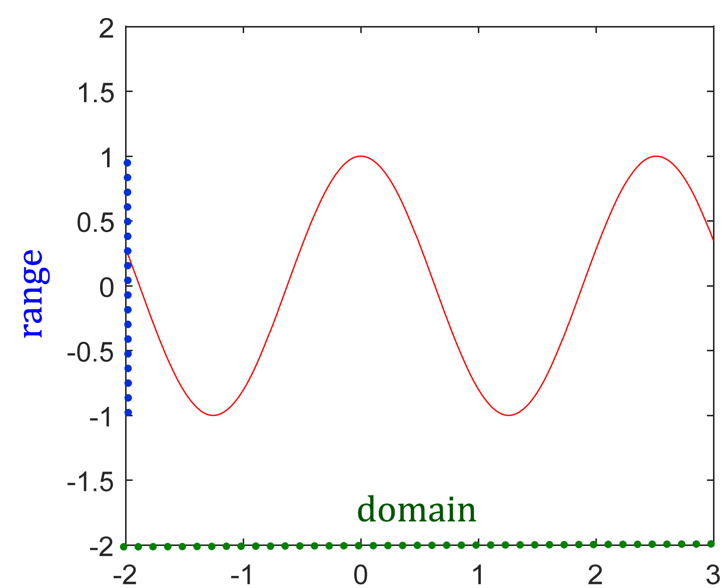
which is a mapping from a set of real scalar x to another set of real scalar y.

Examples:

$$y = x^2, \quad -1 \le x \le 2$$



$$y = \cos 2.5x, \quad -2 \le x \le 3$$



Complex Functions of a Real Variable

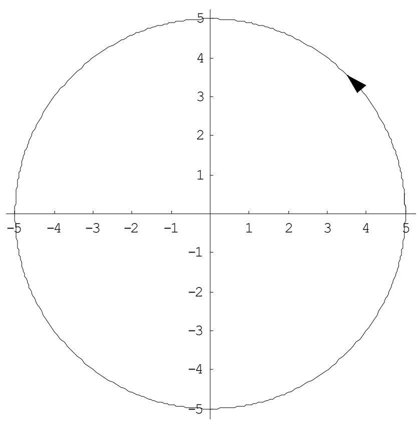
Complex functions of a real variable are needed to represent paths or contours in the complex plane.

$$z(t) = x(t) + i\,y(t), \qquad t \in [a, b]$$

plot_zt
1, 2

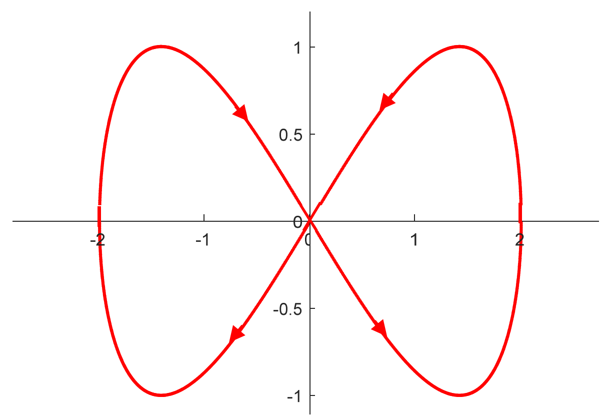
Example 1:

$$z(t) = 5e^{it}, \quad t \in [0, 2\pi]$$
$$= 5\cos t + i\,5\sin t = x(t) + i\,y(t)$$



Example 2:

$$z(t) = 2\cos t + i\sin 2t, \quad t \in [0, 2\pi]$$
$$= x(t) + i\,y(t)$$



Properties of Complex Function of Real Variable

- $\lim_{t\to a} z(t) = \lim_{t\to a} x(t) + i \lim_{t\to a} y(t)$

- z is continuous if x and y are continuous, i.e., $\lim_{t\to a} x(t) = x(a)$, $\lim_{t\to a} y(t) = y(a)$

- $z'(t) = x'(t) + i\, y'(t)$

- $z(t)$ is smooth if $z'(t)$ is continuous, i.e., if $x'(t)$ and $y'(t)$ are continuous.

- $z(t)$ is piecewise smooth if $z'(t)$ is continuous everywhere except for a finite number of discontinuities.

Properties of Complex Function of Real Variable (cont.)

- Normal differentiation and integration rules are applicable:

$$(c_1 z_1 + c_2 z_2)' = c_1 z_1' + c_2 z_2'$$

$$\int_a^b (c_1 z_1 + c_2 z_2)\,dt = c_1 \int_a^b z_1\,dt + c_2 \int_a^b z_2\,dt$$

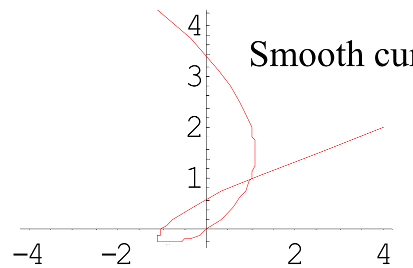
$$\int_a^b z'\,dt = z(b) - z(a)$$

Curves

- The set of images

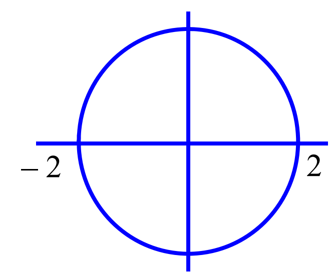
$$C = \{z(t) \mid t \in [a,b]\}$$

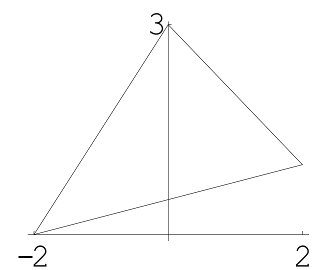
is called a **curve** in the complex plane



Smooth curve

Smooth closed curve





Piecewise smooth (pws) curve

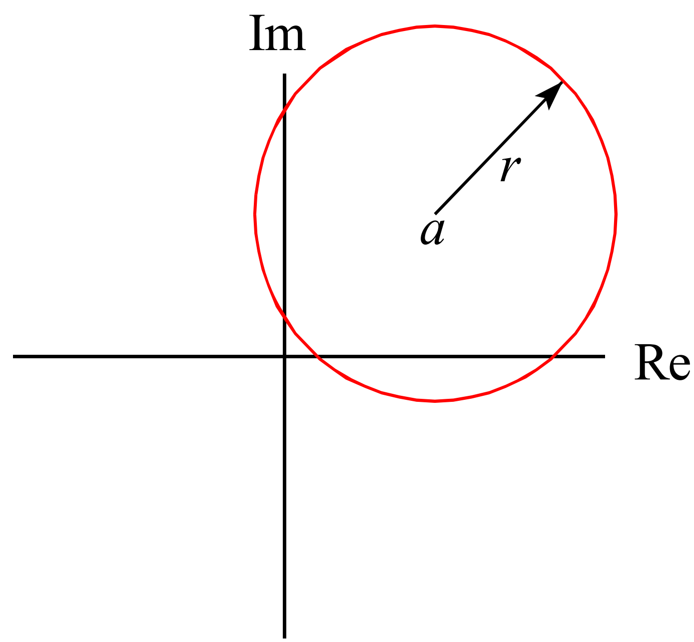
- The curve is smooth if $z'(t)$ is continuous

Two Special Curves

- **Circle**

  The parametric description for a **circle** centred at complex point *a* and with a radius r is

$$z(t) = a + re^{it}, \quad t \in [0, 2\pi]$$

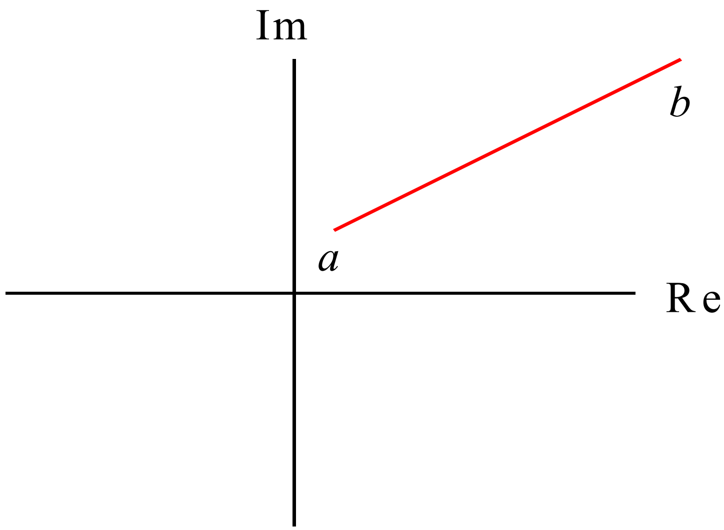


Two Special Curves

- **<u>Straight Line</u>**

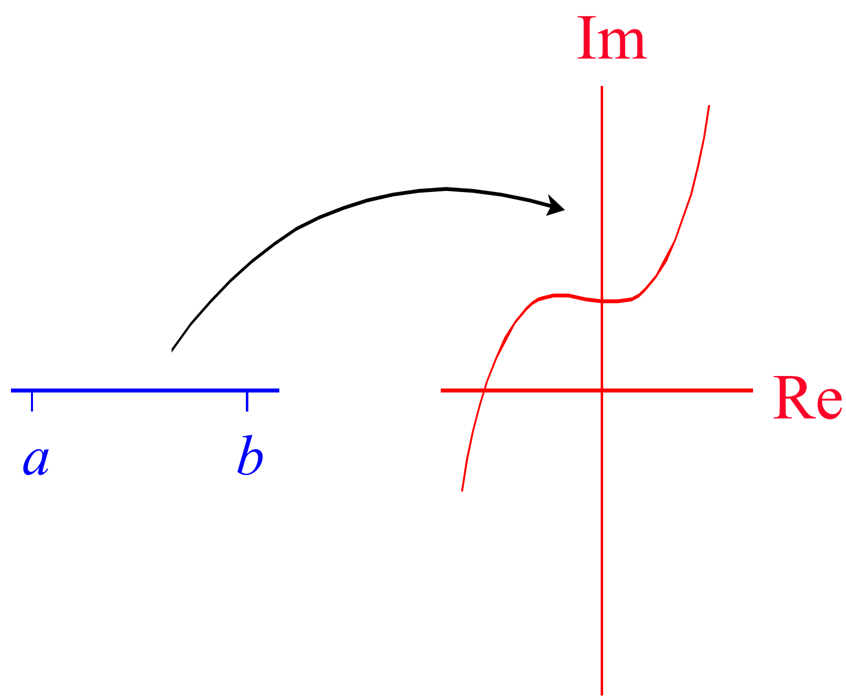
  The parametric description of a **straight-line** segment with starting point a and endpoint b is

$$z(t) = (b - a)\,t + a\,, \quad t \in [0,1]$$



The length of a curve

- A curve is thus a mapping of the real number line onto the complex plane



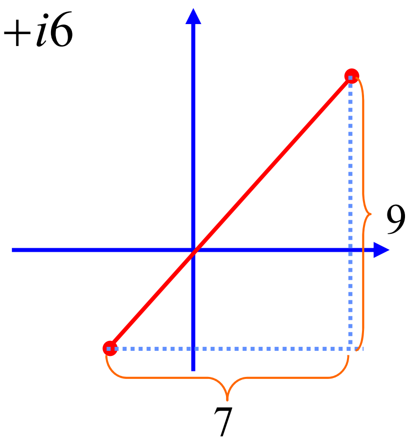
- The **length of a curve** is given by

$$L = \int_a^b |z'(t)|\, dt$$

Ben M. Chen, CUHK MAE

Examples: Parametric Representation and Length of Curves

a) The line segment that connects the points $-2-i3$ and $5+i6$



$$z(t)=(7+i9)\,t+(-2-i3),\quad t\in[0,1]$$

$$L=\int_a^b |z'(t)|dt=\int_0^1 |7+i9|dt=\int_0^1 \sqrt{7^2+9^2}\,dt=\sqrt{130}$$

b) The circle with radius 2 and centre $1-i$

$$z(t)=(1-i)+2e^{it},\quad t\in[0,2\pi]$$

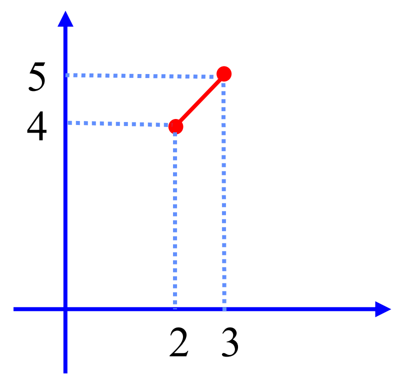
$$L=\int_a^b |z'(t)|\,dt=\int_0^{2\pi} \left|2i\,e^{it}\right|dt=\int_0^{2\pi} |2i|\times\left|e^{it}\right|dt=\int_0^{2\pi} 2\times 1\,dt=4\pi$$

Example (cont.)

c) $y = x + 2, \quad 2 \le x \le 3$

Let $x = t, \quad 2 \le t \le 3 \quad \Rightarrow \quad y = t + 2$

$$\Rightarrow \quad z(t) = t + i(t+2), \quad 2 \le t \le 3$$



$$L = \int_a^b |z'(t)|\,dt = \int_2^3 |1+i|\,dt = \int_2^3 \sqrt{2}\,dt = \sqrt{2}$$

Alternatively, from the figure above, we have $a = 2 + 4i$ and $b = 3 + 5i$. Thus,

$$z(t) = (b-a)t + a = (1+i)t + 2 + 4i = (2+t) + i(4+t), \quad t \in [0,1]$$

$$L = \int_a^b |z'(t)|\,dt = \int_0^1 |1+i|\,dt = \int_0^1 \sqrt{2}\,dt = \sqrt{2}$$

Complex Functions of a Complex Variable

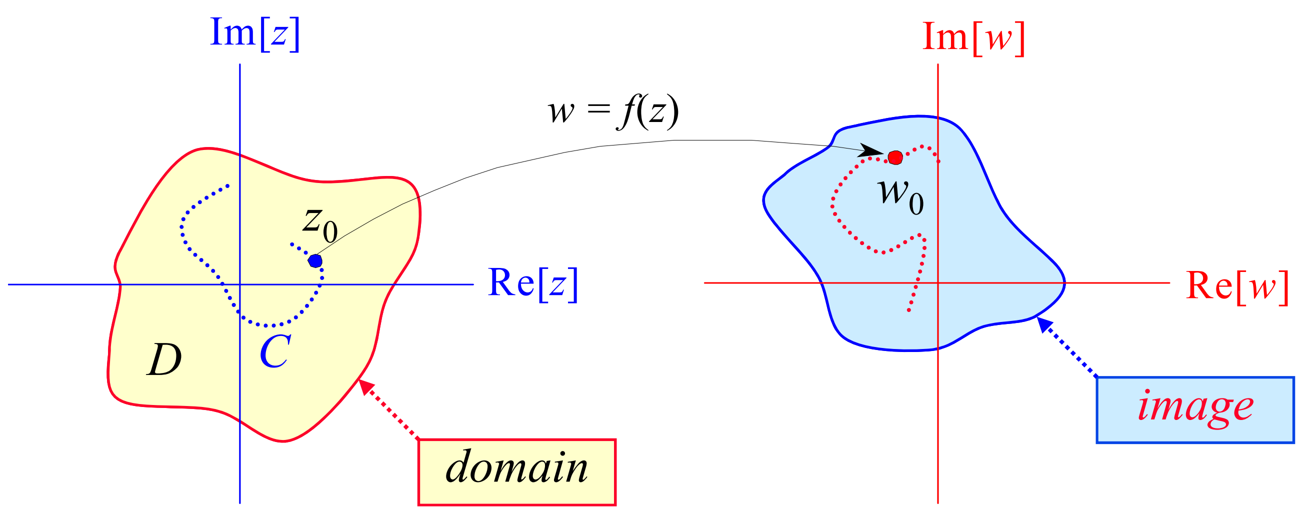
plot_fz

Example: $w = f(z) = z^2 = (x+iy)^2$

$$= \left(x^2 - y^2\right) + i\left(2xy\right) = u(x,y) + i\,v(x,y)$$

$$\Rightarrow \quad u(x,y) = x^2 - y^2, \quad v(x,y) = 2xy$$

A complex function of a complex variable maps one plane to another plane.



These functions are of the form $f(z) = w \Rightarrow f(x+iy) = u + iv = u(x,y) + iv(x,y)$.

The complex function

$$\boxed{w = f(z) = f(x + yi)} \tag{2.1}$$

can be expressed as follows…

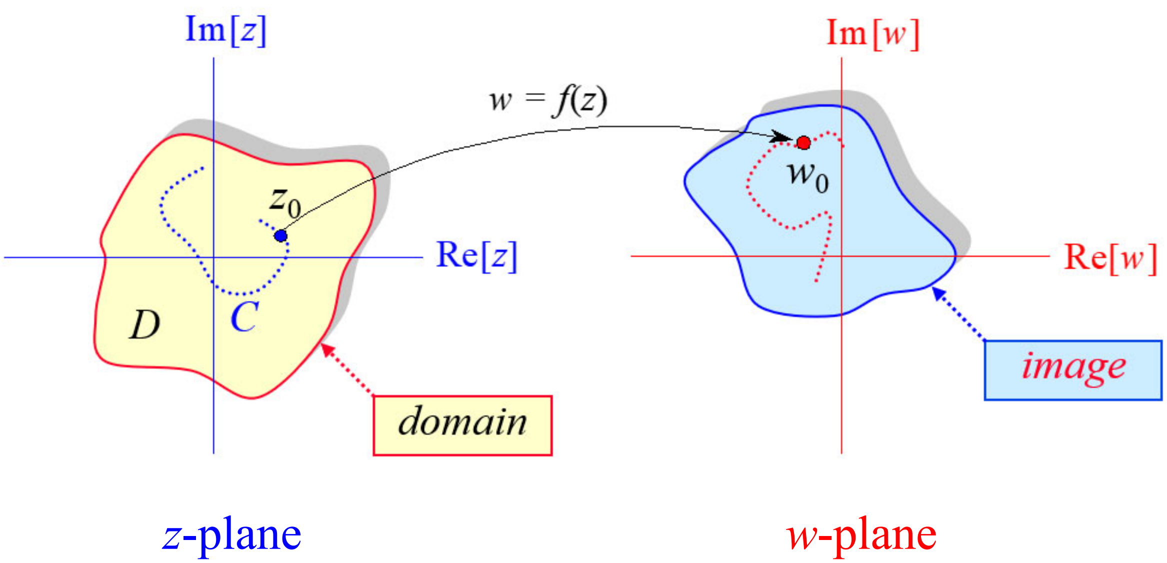
$$\boxed{f(z) = u(x, y) + i\,v(x, y),} \tag{2.2}$$

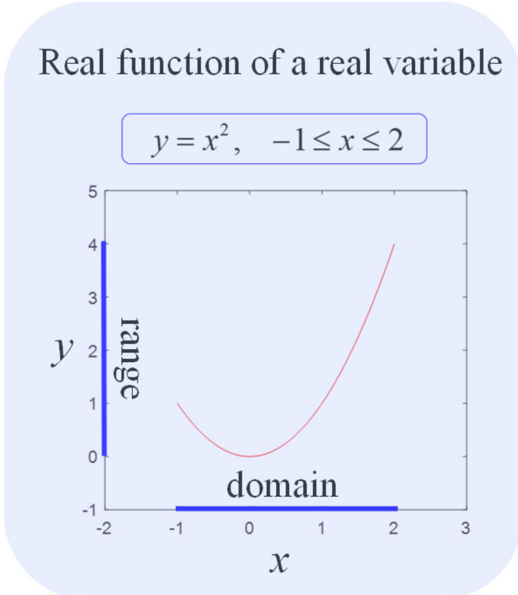
a function of one complex variable z can be regarded as a function of two real variables x and y.

$u(x, y)$ is **real part** of f

$v(x, y)$ is **imaginary part** of f.

As usual, we define the set in the so-called z-plane (see below) on which $w = f(z)$ is defined as the domain of f and the set of all values of $f(z)$ in the so-called w-plane (with z being in the domain of f) as the range of f.





The domain of f can be the whole z-plane or just part of it. Similarly, the range of f can also be the whole w-plane or just part of it.

Example (2a)

- Consider the complex function $w = f(z) = z^2$ defined on the first quadrant of the z-plane: $0 < x < \infty, 0 < y < \infty$. Then

$$f(z) = f(x + yi) = (x + yi)^2 = (x^2 - y^2) + 2xyi.$$

plot_fz

2

Thus, $u(x, y) = x^2 - y^2$ and $v(x, y) = 2xy$.

- Now, $-\infty < u(x, y) < \infty$ and $0 < v(x, y) < \infty$. Therefore, the range of f is the entire upper half plane, i.e., Im$(w) > 0$.

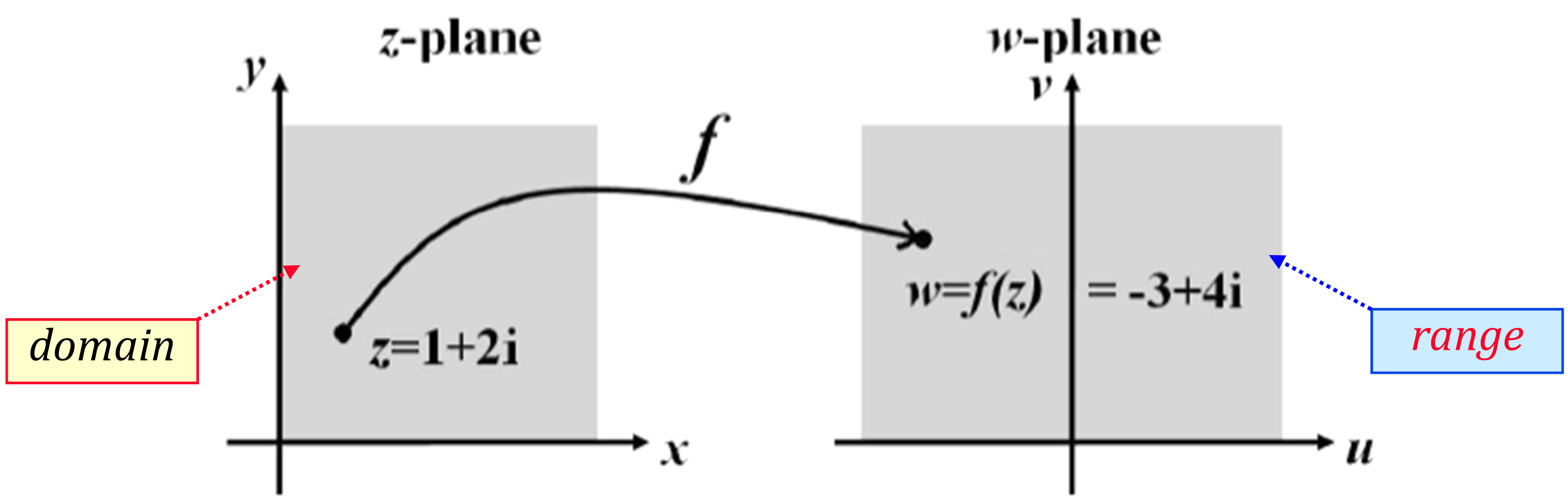


Fig. 2.1. Mapping defined by $w = f(z) = z^2$.

Example: z^2 (cont.)

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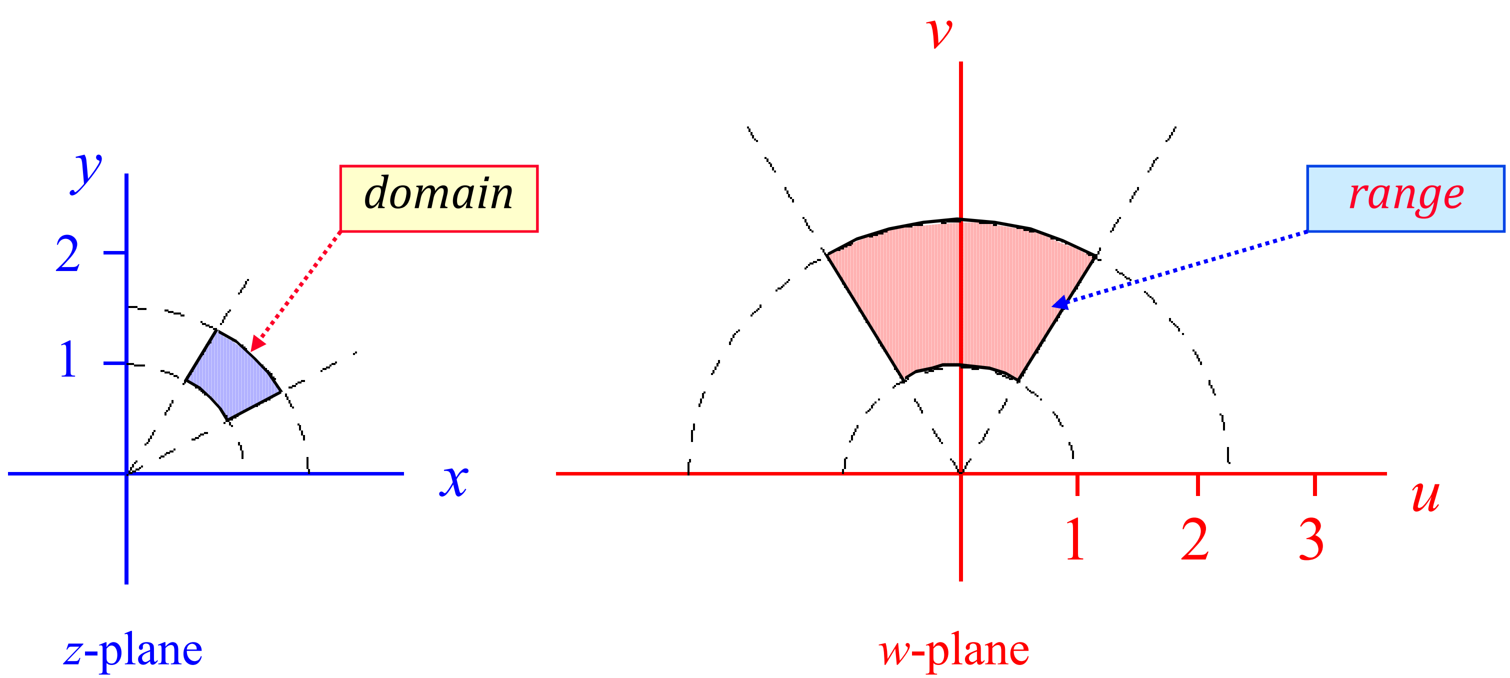
In polar coordinate: $w = f(z) = R\,e^{i\theta} = z^2 = (r\,e^{i\phi})^2 = r^2\,e^{i2\phi}$

For example, the set of the region $1 \le r \le 3/2,\ \ \pi/6 \le \phi \le \pi/3$ under the mapping $w = z^2$ is $1 \le R \le 9/4,\ \ \pi/3 \le \theta \le 2\pi/3$



Elementary functions…

Recall the Taylor series expansion of the real exponential function e^x…

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

The complex exponential function e^z is defined as

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \cdots$$

For the ease of references, we also recall the Taylor series expansions of real sin and cos functions…

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots, \quad \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

⋆The Euler's Formula…

We now prove the formula that we have used many times this far, i.e., the Euler's formula,

$$e^{i\theta} = \cos\theta + i\sin\theta$$

Proof. By the definition of

$$e^{z} = 1 + z + \frac{z^{2}}{2!} + \frac{z^{3}}{3!} + \frac{z^{4}}{4!} + \frac{z^{5}}{5!} + \cdots$$

and by letting $z = i\theta$, we have

$$\begin{aligned} e^{i\theta} &= 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \cdots \\ &= 1 + i\theta + \frac{i^2\theta^2}{2!} + \frac{i^3\theta^3}{3!} + \frac{i^4\theta^4}{4!} + \frac{i^5\theta^5}{5!} + \cdots \\ &= 1 + i\theta + \frac{-\theta^2}{2!} + \frac{-i\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{i\theta^5}{5!} + \cdots \\ &= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right) = \cos\theta + i\sin\theta \end{aligned}$$

From the Euler's formula, we can deduce

$$e^{0i} = 1$$

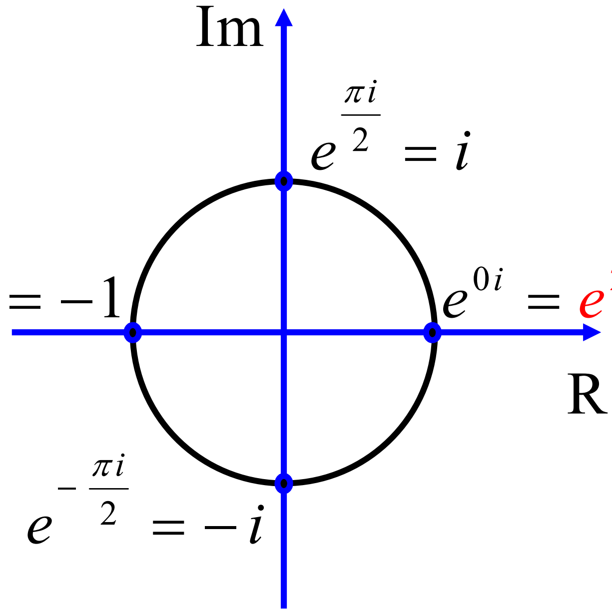
$$e^{2\pi i} = 1$$

$$e^{\frac{1}{2}\pi i} = i$$

$$e^{\pi i} = -1$$

$$e^{-\frac{1}{2}\pi i} = -i$$

$$e^{-\pi i} = -1$$



$$e^{\frac{\pi i}{2}} = i$$

$$e^{\pi i} = e^{-\pi i} = -1$$

$$e^{0i} = e^{2\pi i} = 1$$

$$e^{-\frac{\pi i}{2}} = -i$$

∗ Periodicity of e^z with a period $2\pi i$...

$$e^{z+2\pi i} = e^z\, e^{2\pi i} = e^z \cdot 1 = e^z$$

Writing $z = x + yi$ and using Euler's formula, we have

$$e^z = e^{x+iy} = e^x e^{iy} = e^x \left(\cos y + i \sin y\right)$$

$$= e^x \cos y + i\, e^x \sin y = u(x,y) + i\, v(x,y)$$

- Note that

$$|e^{iy}| = |\cos y + i \sin y| = \sqrt{\cos^2 y + \sin^2 y} = 1 \quad \text{for all } y$$

⇨ $|e^z| = |e^x(\cos y + i \sin y)| = |e^x||\cos y + i \sin y| = e^x$ for all z.

Therefore, $e^z \neq 0$ in the entire z-plane (entire: see section 3).

Example: e^z (cont.)

$$w = e^z = e^{x+iy} = e^x e^{iy} = Re^{i\theta}$$

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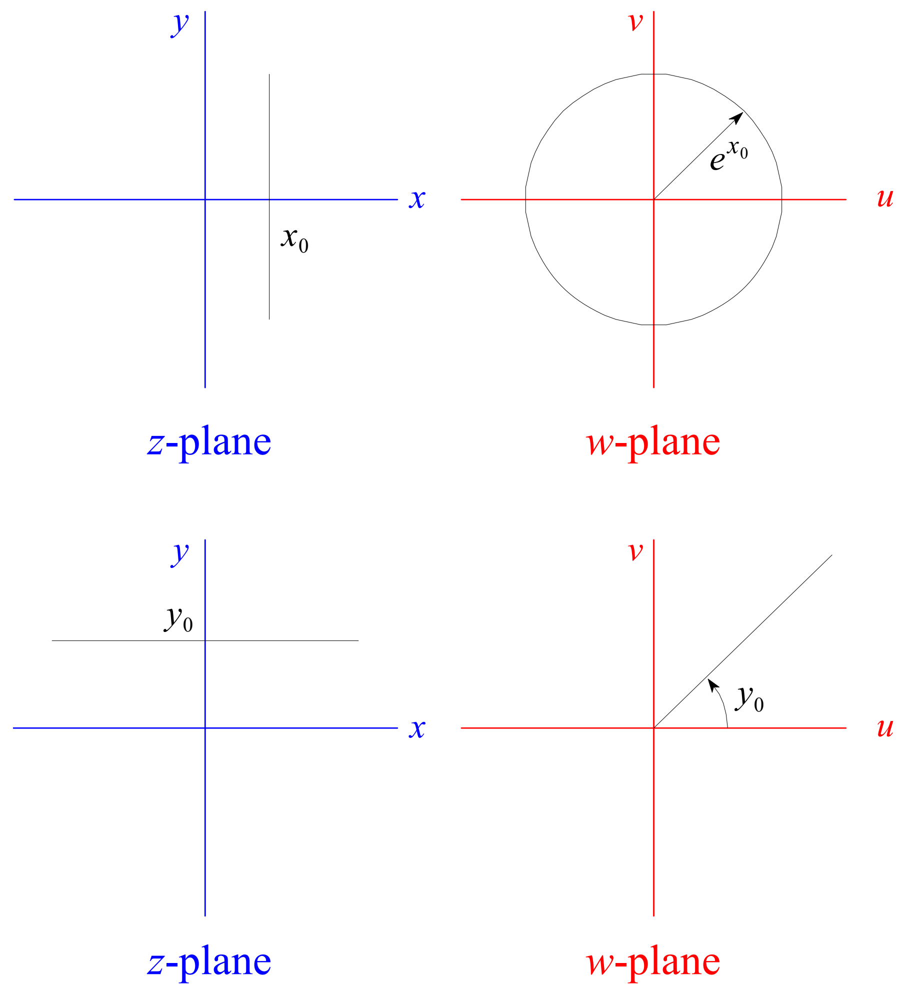
For $w = e^z$, consider the images of:

1. Straight lines $x = x_0 = \text{const}$ and $y = y_0 = \text{const}$

   From $R = e^x, \quad \theta = y$, we see that $x = x_0$ is mapped onto the circle $|w| = e^{x_0}$ and $y = y_0$ is mapped onto the ray

   $\arg(w) = y_0$

$$R = e^x, \quad \theta = y$$



Example: e^z (cont.)

$$R = e^x, \quad \theta = y$$

2. Rectangle $D = \{ z = x + iy \,|\, a \le x \le b, \; c \le y \le d \}$:

From 1, we can conclude that any rectangle with side parallel to the coordinate axes is mapped onto a region bounded by portions of rays and circles. Therefore the range of *D* is

$$D' = \{ \; w = R e^{i\theta} \,\Big|\, e^a \le R \le e^b, c \le \theta \le d \; \}$$

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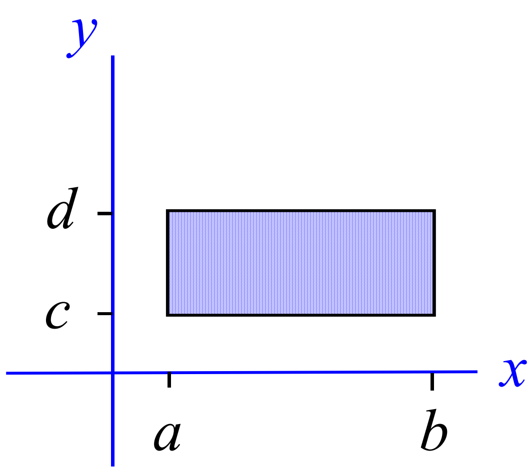
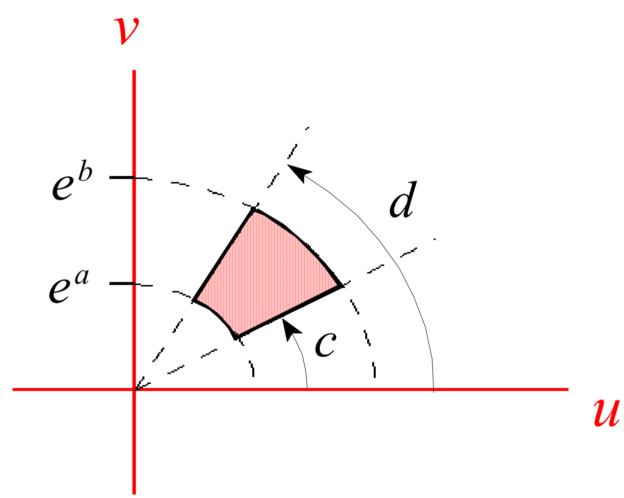


z-plane



w-plane

Example: e^z (cont.)

$$R = e^x\,, \quad \theta = y$$

3. The fundamental region $-\pi \le y \le \pi$:

The fundamental region is mapped onto the entire *w*-plane, excluding the origin. The strip $0 \le y \le \pi$ is mapped onto the upper half-plane

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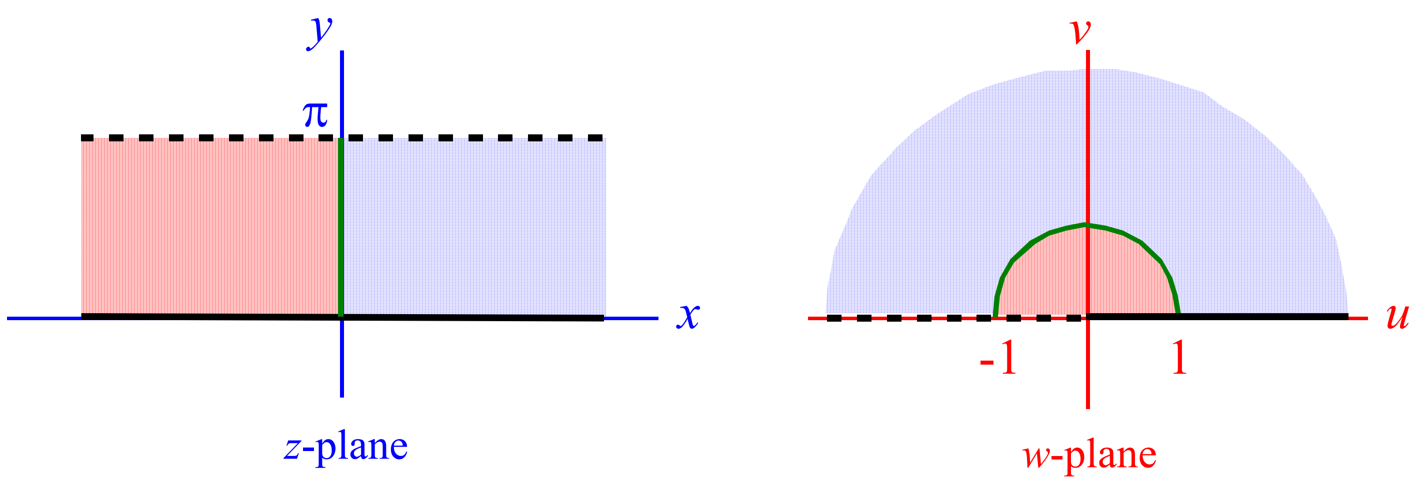


More generally, every horizontal strip $c \le y \le c + 2\pi$ is mapped onto the full *w*-plane excluding the origin.

Trigonometric and Hyperbolic Functions

- Changing θ to y and $-y$ in Euler's formula (1.9), then we get

$$e^{iy} = \cos y + i \sin y, \quad e^{-iy} = \cos y - i \sin y. \tag{2.5}$$

- **Note**: Euler's formula is valid in complex.

Solving (2.5) for $\cos y$ and $\sin y$, we obtain

plot_fz

$$\cos y = \frac{e^{iy} + e^{-iy}}{2}, \quad \sin y = \frac{e^{iy} - e^{-iy}}{2i}$$

$$\cos z = \frac{e^{iz} + e^{-iz}}{2},$$

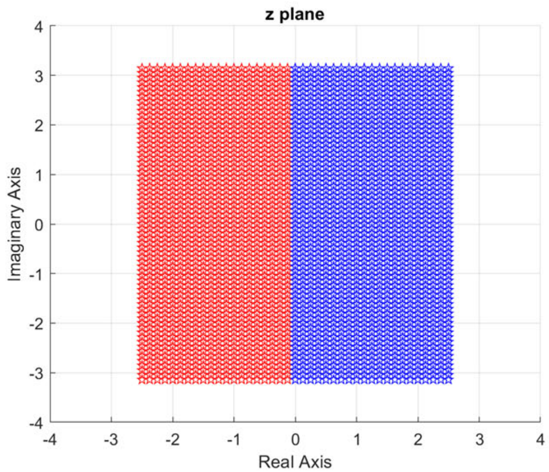
$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}.$$

$$\tan z = \frac{\sin z}{\cos z}, \qquad \sec z = \frac{1}{\cos z}$$

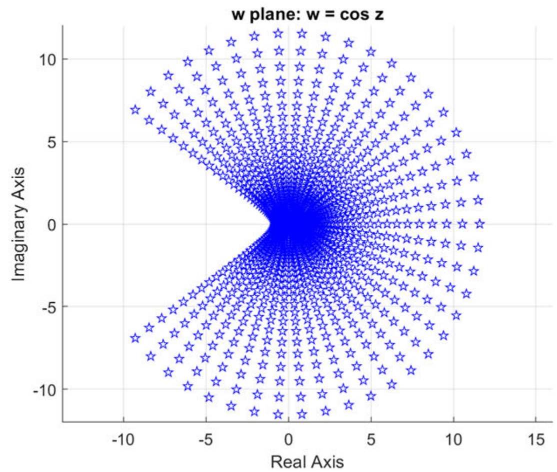
$$\cot z = \frac{\cos z}{\sin z}, \qquad \csc z = \frac{1}{\sin z}$$

Graphical illustration of complex functions:

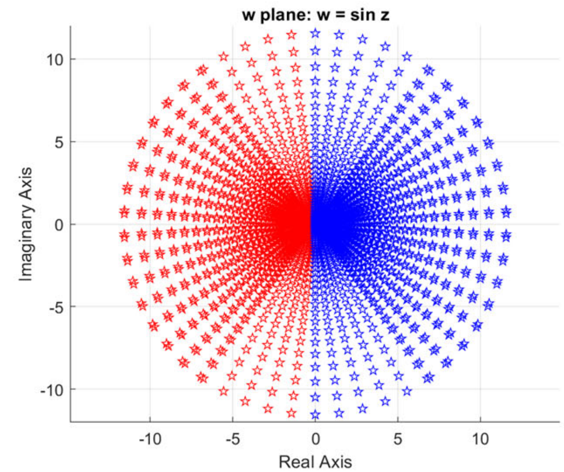
Domain



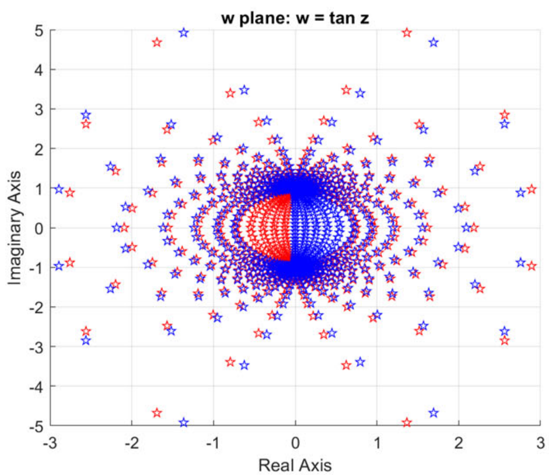
cos *z*



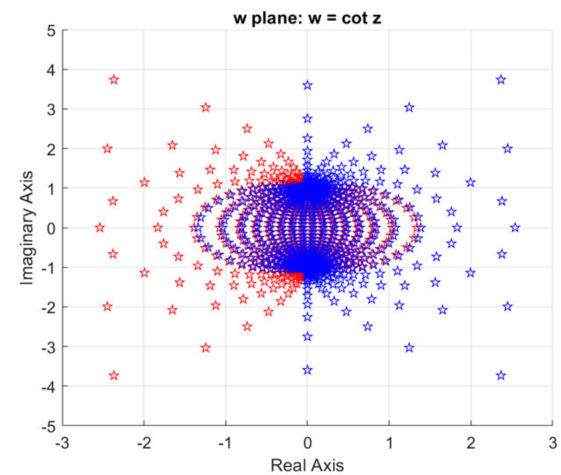
sin *z*



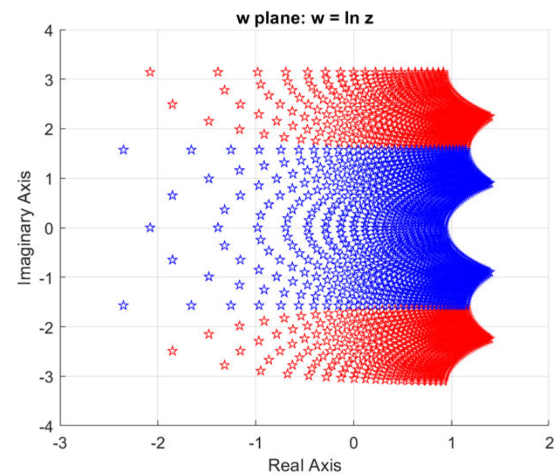
tan *z*



cot *z*



ln *z*



We can also define the hyperbolic cosine and hyperbolic sine functions

$$\cosh z = \frac{e^z + e^{-z}}{2}, \qquad \sinh z = \frac{e^z - e^{-z}}{2}.$$

$$\cosh x = \frac{e^x + e^{-x}}{2}, \qquad \sinh x = \frac{e^x - e^{-x}}{2}$$

It can be shown that $(\cosh z)' = \sinh z, \quad (\sinh z)' = \cosh z.$

Similarly, other complex hyperbolic functions can be also defined…

$$\tanh z = \frac{\sinh z}{\cosh z}, \quad \coth z = \frac{\cosh z}{\sinh z}$$

$$\operatorname{sech} z = \frac{1}{\cosh z}, \quad \operatorname{csch} z = \frac{1}{\sinh z}$$

$$\tanh x = \frac{\sinh x}{\cosh x}, \quad \coth x = \frac{\cosh x}{\sinh x}$$

$$\operatorname{sech} x = \frac{1}{\cosh x}, \quad \operatorname{csch} x = \frac{1}{\sinh x}$$

- Complex trigonometric and hyperbolic functions are related:

the connections between trigonometric and hyperbolic functions are

$$\cos(iz) = \cosh z, \quad \sin(iz) = i \sinh z,$$
$$\cosh(iz) = \cos z, \quad \sinh(iz) = i \sin z.$$

CHECK IT OUT!

- Based on definitions of e^z, $\cos z$, $\sin z$, $\cosh z$ and $\sinh z$ most familiar formulas for real exponentials, trigonometric and hyperbolic functions still apply. For example, we still have

$$\sin^2 z + \cos^2 z = -\frac{1}{4}(e^{iz} - e^{-iz})^2 + \frac{1}{4}(e^{iz} + e^{-iz})^2$$
$$= \frac{1}{4}(\underline{-e^{2iz}} + 2 - \underline{e^{-2iz}} + \underline{e^{2iz}} + 2 + \underline{e^{-2iz}}) = 1$$

*However, it is easy to show that there are z_0 in the z-plane, for which $|\sin z_0| > 1$, $|\cos z_0| > 1$.

CHECK IT OUT!

Logarithmic Functions

- define **logarithmic function** $\ln z$ (sometimes also by $\log z$).

  for $z \neq 0$, express z in polar form and write

$$\ln z = \ln(re^{i\theta}) = \ln r + \ln(e^{i\theta})$$
$$= \ln r + i\theta, \qquad (r = |z| > 0, \theta = \text{arg} z)$$

or

$$\ln z = \ln r + i(\theta_0 + 2k\pi)$$

for $\theta = \text{arg} z = \theta_0 + 2k\pi, -\pi < \theta_0 \leq \pi, k = 0, \pm 1, \pm 2, \ldots.$

- Since the argument of z is determined only up to integer multiples of 2π, the complex natural logarithm $\ln z (z \neq 0)$ is **infinitely many-valued.**

$$\theta = \text{arg} z = \theta_0 + 2k\pi, -\pi < \theta_0 \leq \pi, k = 0, \pm 1, \pm 2, \ldots .$$

$$\Rightarrow \quad \ln z = \ln r + i\theta$$

Example: $\ln 1 = 0 + 2k\pi i = 0, \pm 2\pi i, \pm 4\pi i, \cdots\cdots$

- The value of $\ln z$ corresponding to the principal value Arg$z$ is denoted by Lnz (Ln with capital L) and is called the **Principal value** of $\ln z (z \neq 0)$.

$$\boxed{\text{Ln} z = \ln |z| + i \text{Arg} z} \qquad (2.9)$$

- The uniqueness of Arg$z$ for given $z (\neq 0)$ implies that Lnz is single-valued.

- Since the other values of arg$z$ differ by integer multiples of 2π, the other values of ln$z$ are given by

$$\ln z = \text{Ln}z + 2k\pi i \qquad (k = \pm 1, \pm 2, \ldots),$$

- **Note**: All have the same real part, and imaginary parts differ by integer multiples of 2π.
- If z is positive real, then Arg$z = 0$, and Lnz becomes identical with the real natural logarithm ; If z is negative real, then Arg$z = \pi$ and

$$\text{Ln}z = \ln|z| + \pi i, \qquad \text{(z negative real)}$$

Examples:

$$\text{Ln}(-1) = \ln|-1| + \pi i = \pi i$$

$$\text{Ln}(i) = \ln|i| + \frac{\pi}{2} i = \frac{\pi}{2} i$$

$$\text{Ln}(-i) = \ln|-i| - \frac{\pi}{2} i = -\frac{\pi}{2} i$$

Example (2b)

- Since $1+i=\sqrt{2}e^{\frac{i\pi}{4}}$,

$$\ln(1+i)=\frac{\ln 2}{2}+i\left(\frac{\pi}{4}+2k\pi\right),\quad k=0,\pm1,\pm2,\ldots.$$

$$\mathrm{Ln}(1+i)=\ln\sqrt{2}+\frac{\pi}{4}i$$

- Given $(1-i)^2$, compute its $\ln$ and Ln values…

$$1-i=\sqrt{2}\,e^{\frac{-i\pi}{4}},\quad (1-i)^2=2e^{\frac{-i\pi}{2}}$$

$$\ln(1-i)^2=\ln\left(2e^{\frac{-i\pi}{2}}\right)=\ln 2+i\left(-\frac{\pi}{2}+2k\pi\right),\quad k=0,\pm1,\pm2,\cdots$$

$$\mathrm{Ln}(1-i)^2=\ln 2-\frac{\pi}{2}i$$

General Powers of z

- Suppose z and c are both complex numbers, we have

$$z^c = e^{\ln z^c} = e^{c \ln z}$$

- Since $\ln z$ is infinitely many-valued, z^c will be multivalued, and

$$z^c = e^{c\left[\ln r + i(\theta_0 + 2k\pi)\right]} \qquad \text{for } k = 0, \pm 1, \pm 2, \ldots.$$

c-power of z

The particular value $(k = 0)$

$$z^c = e^{c \operatorname{Ln} z}$$

is called the **principal value** of z^c.

- c-**power** of z as

$$z^c = e^{c\left[\ln r + i(\theta_0 + 2k\pi)\right]}.$$

In particular, when c is real, then

$$z^c = (re^{i\theta})^c = r^c e^{i(\theta_0 + 2k\pi)c}. \qquad (2.11)$$

Example (2c)

- Since $\ln i = \ln 1 + i(\frac{\pi}{2} + 2k\pi) = i(\frac{\pi}{2} + 2k\pi)$,

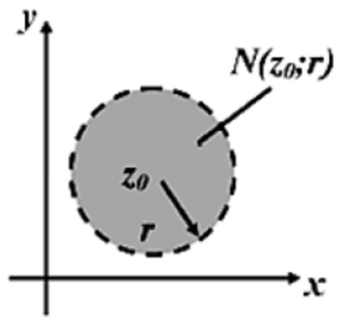
$$i^i = e^{i \ln i} = e^{-(\frac{\pi}{2} + 2k\pi)}, \qquad k = 0, \pm 1, \pm 2, \ldots.$$

$$\ln z = \ln r + i(\theta_0 + 2k\pi)$$

$$z^c = e^{c \ln z}$$

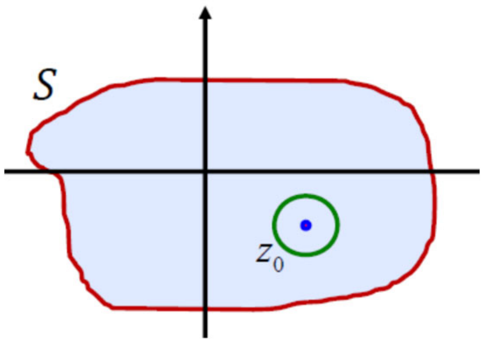
Complex Analysis – 3…

1 Complex Numbers
2 Functions of One Complex Variable
3 **Complex Differentiation**
4 Complex Integration and Cauchy's Theorem
5 Cauchy Integral Formula
6 Complex Series, Power Series, Taylor Series, and Laurent Series
7 Residue Integration

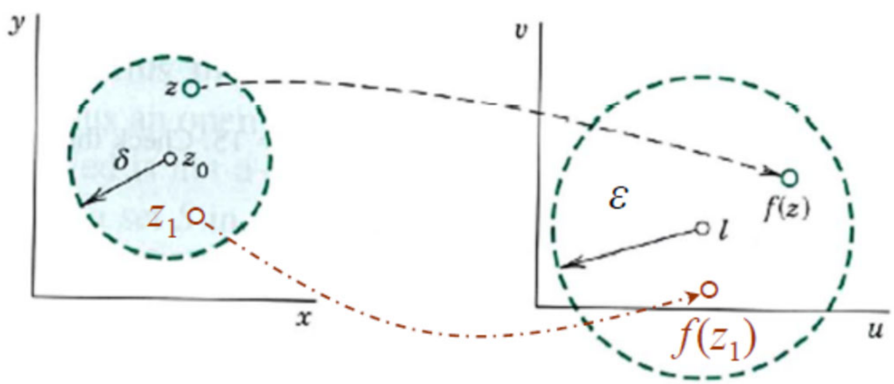


Neighborhood

$$N(z_0;r) = \{z \in \mathbb{C} : |z - z_0| < r\}$$



Connected set, interior points, boundary points, interior, boundary, region, open region, domain, simply connected domain, multiply connected domain



Limit

$$\lim_{z \to z_0} f(z) = l$$

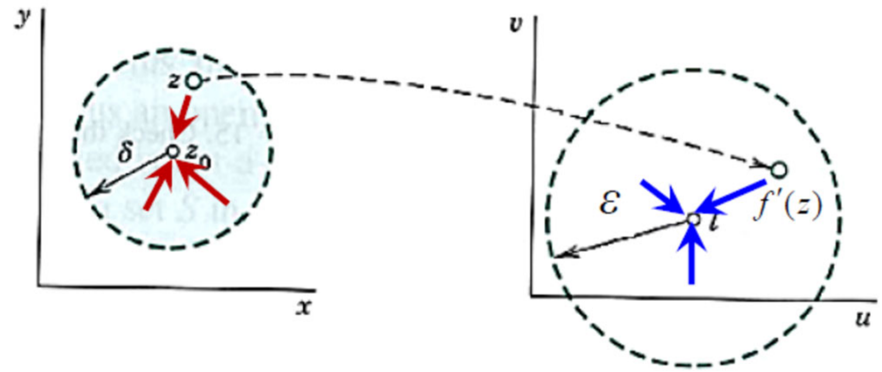
Continuous

$$\lim_{z \to z_0} f(z) = f(z_0)$$

Material flow (cont.)...

Differentiation

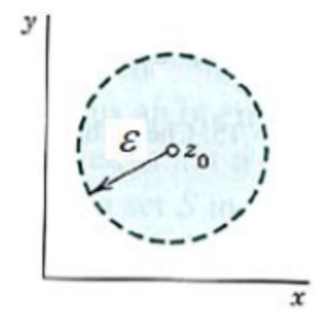
$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$



Analytic functions

A complex function $f(z)$ is **analytic** at $z = z_0$ if there exists a neighborhood $N(z_0; \epsilon)$ of z_0 such that f is differentiable at every point in $N(z_0; \epsilon)$.

Singular point ~ a point on which $f(z)$ is not analytic is called a singular point of the function.



Material flow (cont.)…

Cauchy-Riemann Equations

Let $f(z) = u(x,y) + iv(x,y)$ *be defined and continuous in some neighborhood of a point* $z = x + iy$ *and differentiable at* z *itself. Then, at that point, the first-order partial derivatives of* u *and* v *exist and satisfy the Cauchy-Riemann equations*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$f' = u_x + iv_x = v_y - iu_y = u_x - iu_y = v_y + iv_x$$

Laplace's Equations

If $f(z) = u(x,y) + iv(x,y)$ *is analytic in a domain* D, *then both* u *and* v *satisfy Laplace's equations*

$$\nabla^2 u = u_{xx} + u_{yy} = 0$$
$$\nabla^2 v = v_{xx} + v_{yy} = 0$$

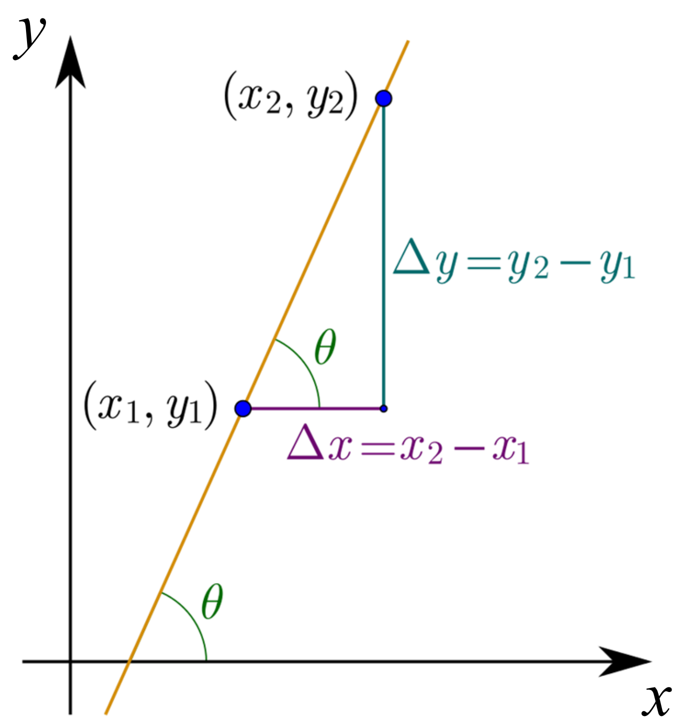
in D and have continuous second partial derivatives in D.

Revisit: The derivative of a real function

$$y = f(x)$$

is a measure of the rate at which the value y of the function changes with respect to the change of x.

Example: Consider the function (a straight line) plotted in the figure on the right. The derivative of the function (or the rate of changes) of the function is its slope.



Note that the derivative cannot be defined on a single point. We need an interval of x, i.e., Δx, to define a derivative.

Similarly, we need a 2D region in a complex function domain to define a complex derivative as a complex function is actually a mapping from a 2D plane to another 2D plane.

Terminologies

- neighborhood of a point z_0 with radius r

$$N(z_0; r) = \{z \in \mathbb{C} : |z - z_0| < r\}.$$

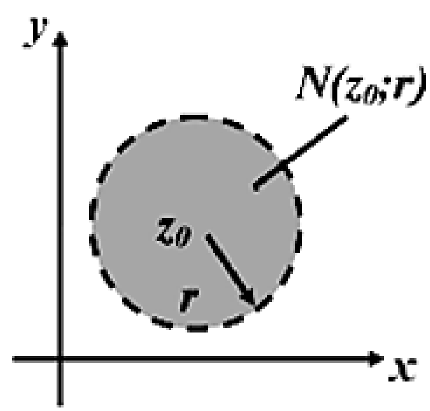
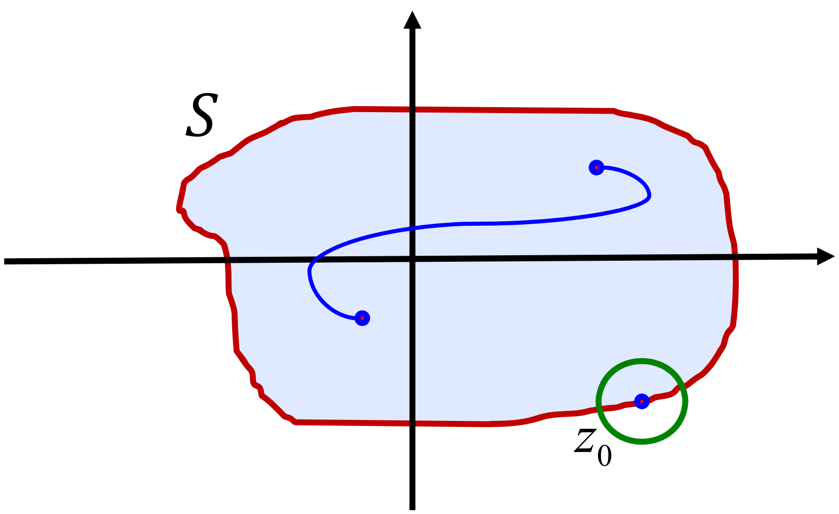
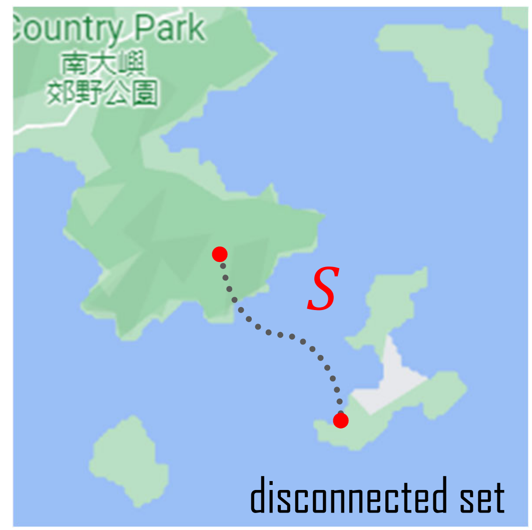


Fig. 1.4. Neighborhood of a point z_0 with radius r.

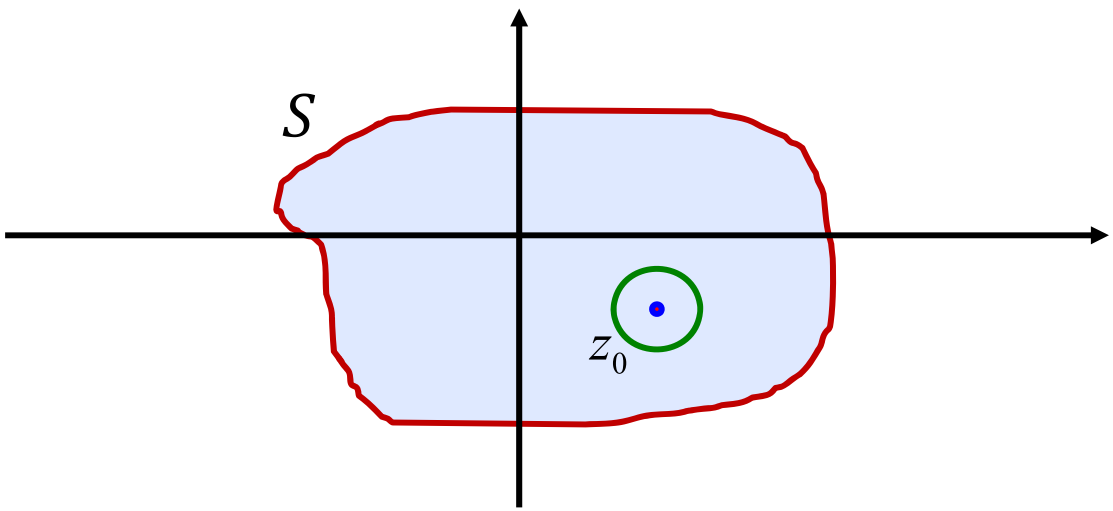
- A set S is called **connected** if every point in S can be joined by an unbroken line entirely within S.





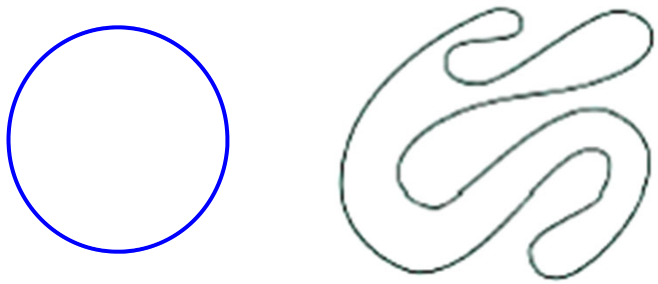
- A point z_0 is called a **boundary point** of S if every neighborhood of z_0 contains a point in S and a point not in S. The set of all boundary points of S is called the **boundary** of S.

- A point z_0 is called an **interior point** of S if there exists a neighborhood $N(z_0; \epsilon)$ of z_0 lying entirely within S. The set of all interior points of S is called the **interior** of S.



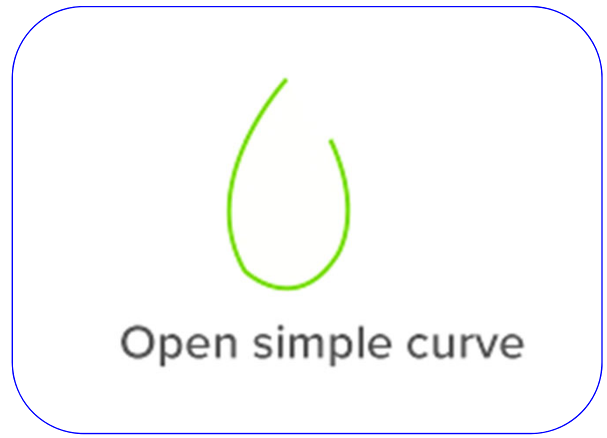
- A connected set is called a **region**.
- S is called an **open region** or a **domain** if it contains none of its boundary points. S is called a **closed region** if it contains all of its boundary points.

- A **simple closed path** is a closed path that does not intersect or touch itself as shown in Fig. 1.5

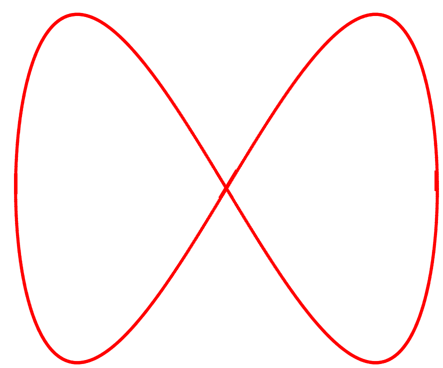


Simple Simple

Fig. 1.5. Closed paths



On the other hand, a **non-simple closed path** is a closed path that intersect or touch itself as shown in the examples on the right.



Non-simple closed path

- A **simply connected domain** D in the complex plane is a domain such that every simple closed path in D encloses only points of D. A domain that is not simply connected is called **multiply connected**

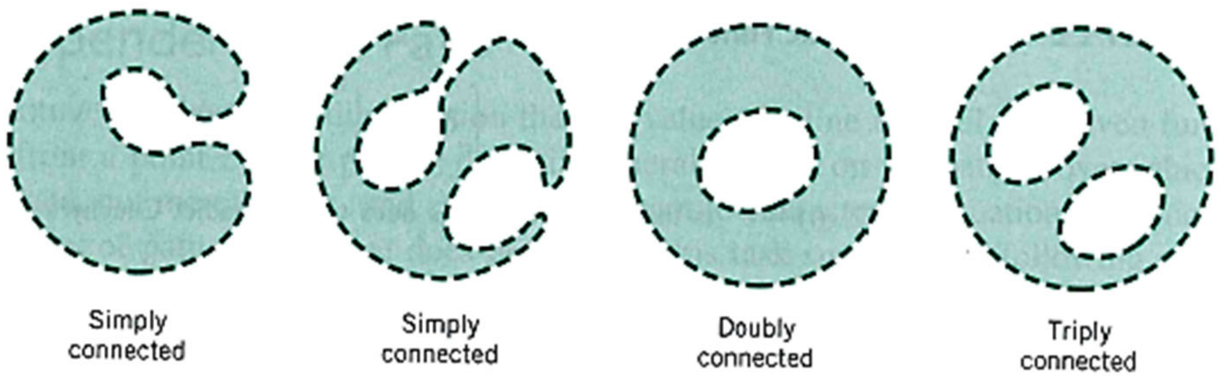
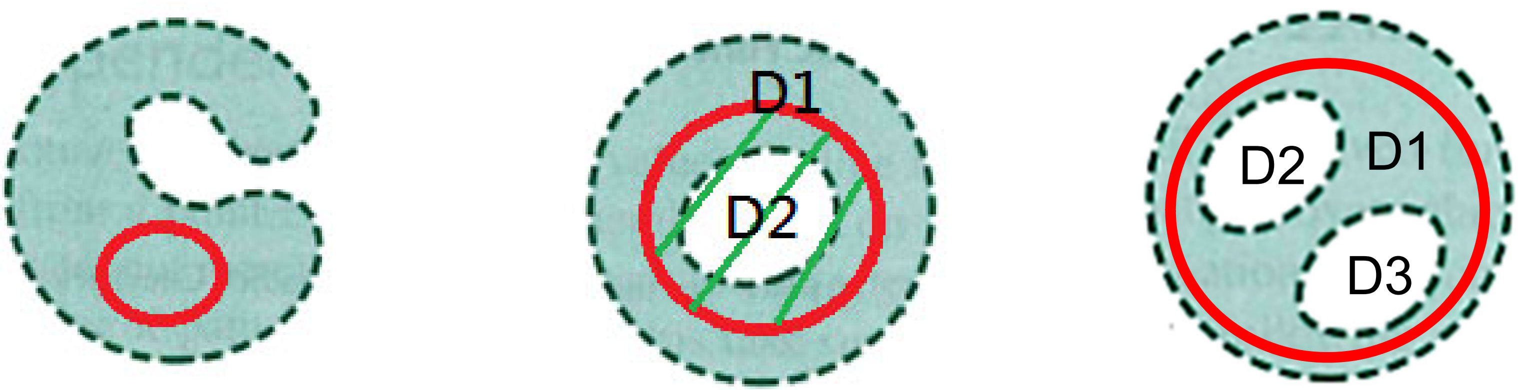


Fig. 1.6. Simply and multiply connected domains



Example

- Consider the set $S' = \{z \in \mathbb{C} : |z - (1+i)| \leq 1\}$.

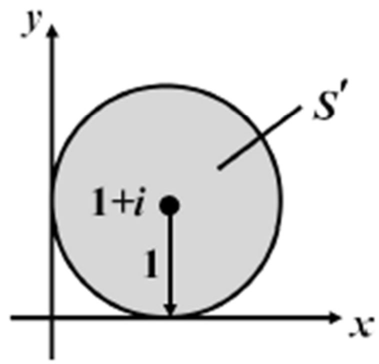


Fig. 1.7. The sketch of S'.

- Then,
  (a) S' is connected, and also simply connected.
  (b) The boundary of S' is $|z-(1+i)| = 1$. The interior of S' is $|z-(1+i)| < 1$.
  (c) The union of boundary of S' and interior of S' is S'.
  (d) The intersection of boundary of S' and interior of S' is $\varnothing$, i.e., the empty set.

Limit and Continuity

- Let z_0 be an interior point in the domain of $f(z)$. We say that the **limit** of $f(z)$, as z approaches a point z_0, is l, i.e.,

$$\boxed{\lim_{z \to z_0} f(z) = l,} \tag{3.1}$$

if for each $\epsilon > 0$ there exists a $\delta > 0$ such that

$$|f(z) - l| < \epsilon \text{ whenever } 0 < |z - z_0| < \delta.$$

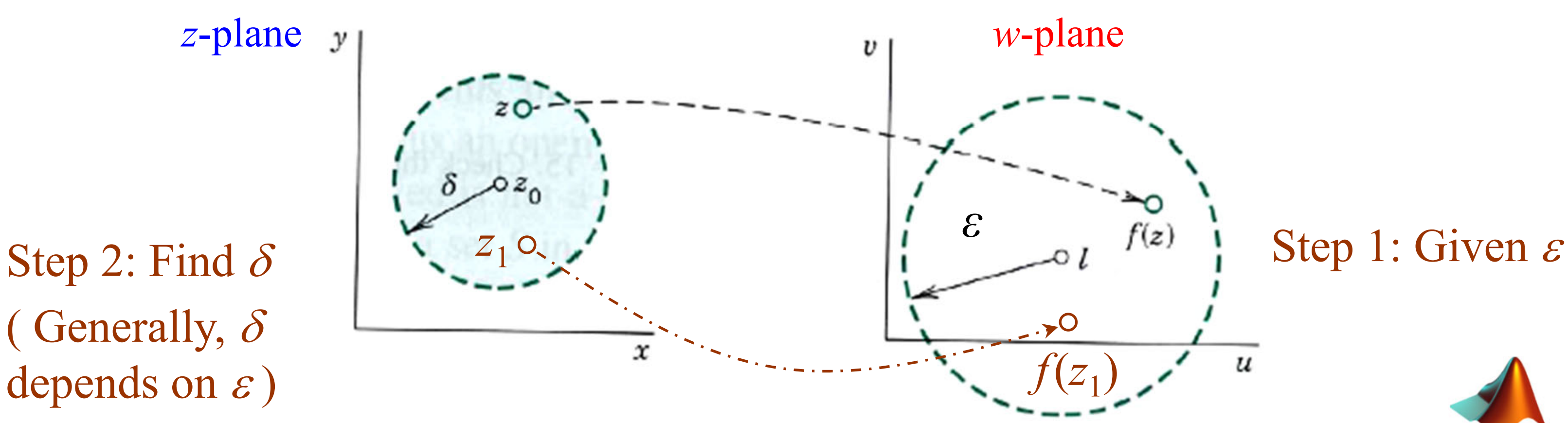


Fig. 3.1. Limit.

plot_fz

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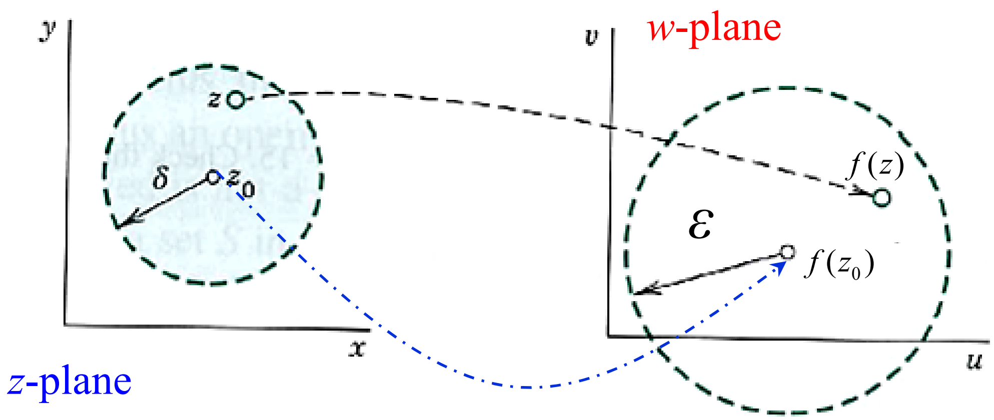
- $f(z)$ is **continuous** at $z = z_0$ if

$$\boxed{\lim_{z \to z_0} f(z) = f(z_0).} \qquad (3.2)$$



- In many cases, we can manipulate complex limits like real limits. example,

$$\lim_{z \to i}(z^2 + iz) = i^2 + i^2 = -2.$$

Complex Differentiation and Analytic Functions

- Let z_0 be an interior point in the domain of $f(z)$. We define the **derivative** of f at $z = z_0$ as

$$f'(z_0) = \lim_{\triangle z \to 0} \frac{f(z_0 + \triangle z) - f(z_0)}{\triangle z} \qquad (3.3)$$

provided that the limit exists.

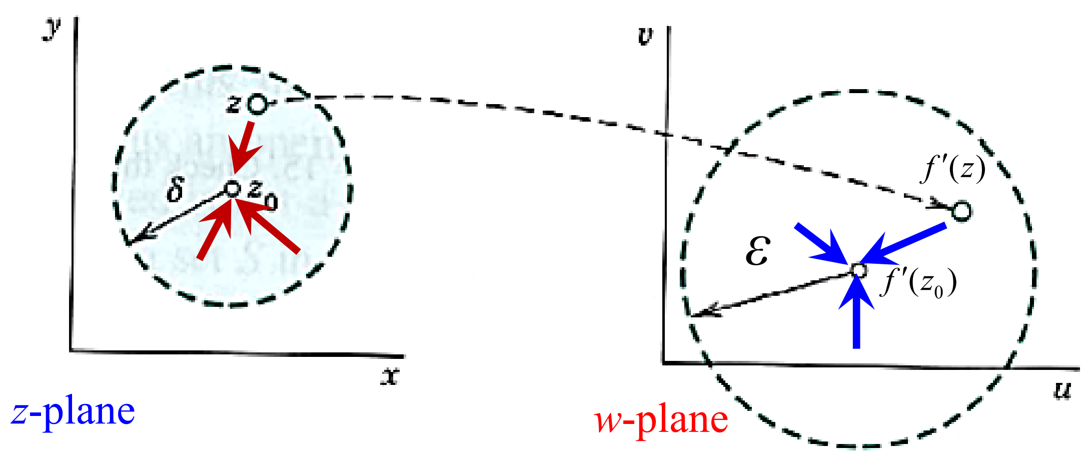
(3.3) can be rewritten as

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \qquad (3.4)$$

If $f'(z_0)$ exists, we say that f is **differentiable** at $z = z_0$.

- Note: By the definition of limit, $f(z)$ is defined in a neighborhood of z_0 and z in (3.4) may approach z_0 from any direction in the complex plane. Hence, differentiability at z_0 means that, along whatever path z approaches z_0, the quotient in (3.4) always approaches a certain value and all these values are equal.

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$



Examples (3a)

- The function $f(z) = z^2$ is differentiable for all z and has the derivative $f'(z) = 2z$ because

$$f'(z) = \lim_{\triangle z \to 0} \frac{(z + \triangle z)^2 - z^2}{\triangle z} = \lim_{\triangle z \to 0} \frac{z^2 + 2z\triangle z + (\triangle z)^2 - z^2}{\triangle z}$$

$$= \lim_{\triangle z \to 0} (2z + \triangle z) = 2z$$

- The function $f(z) = e^z$ is differentiable for all z and has derivative $f'(z) = e^z$.

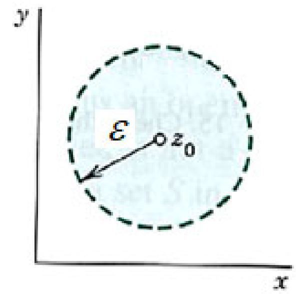
$$f'(z) = \left(e^z\right)' = \lim_{\Delta z \to 0} \frac{e^{z+\Delta z} - e^z}{\Delta z} = \lim_{\Delta z \to 0} \frac{e^{\Delta z} e^z - e^z}{\Delta z} = \lim_{\Delta z \to 0} \frac{(e^{\Delta z} - 1)e^z}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{1}{\Delta z}\left(1 + \Delta z + \frac{(\Delta z)^2}{2!} + \frac{(\Delta z)^3}{3!} + \cdots - 1\right)e^z$$

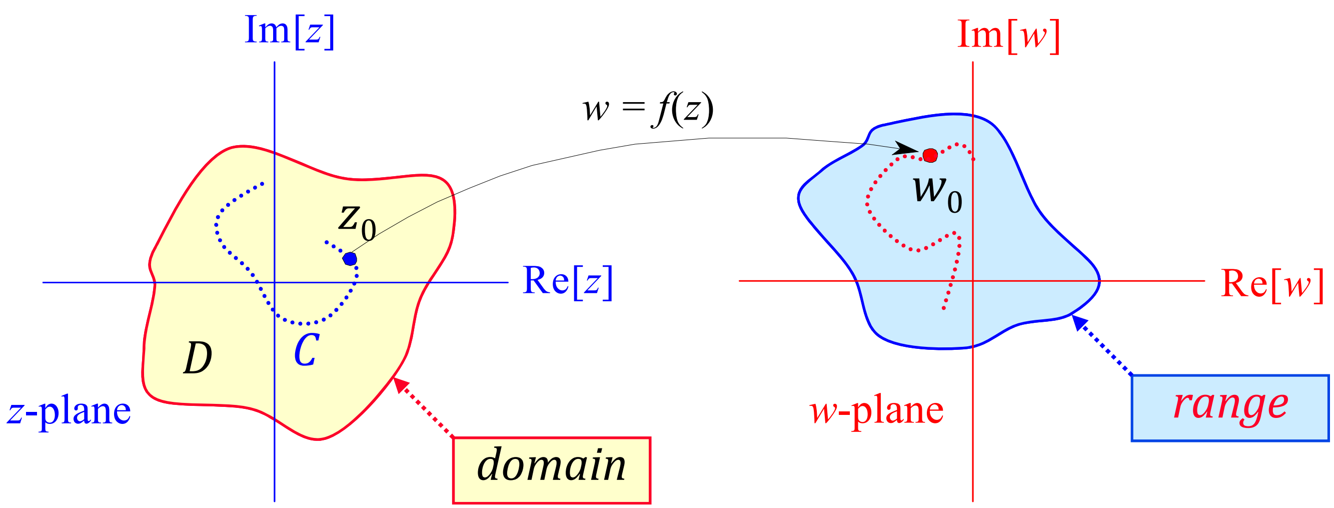
$$= \lim_{\Delta z \to 0}\left(1 + \frac{\Delta z}{2!} + \frac{(\Delta z)^2}{3!} + \cdots\right)e^z = e^z$$

Analytic Functions

- A complex function $f(z)$ is **analytic** at $z = z_0$ if there exists a neighborhood $N(z_0; \epsilon)$ of z_0 such that f is differentiable at every point in $N(z_0; \epsilon)$. If it is not analytic at z_0, it is **singular** there.



- It is called an **analytic function** at a domain $D \subset \mathbb{C}$ if it is analytic at every point in D. Functions that are analytic everywhere in the z-plane are called **entire functions**.

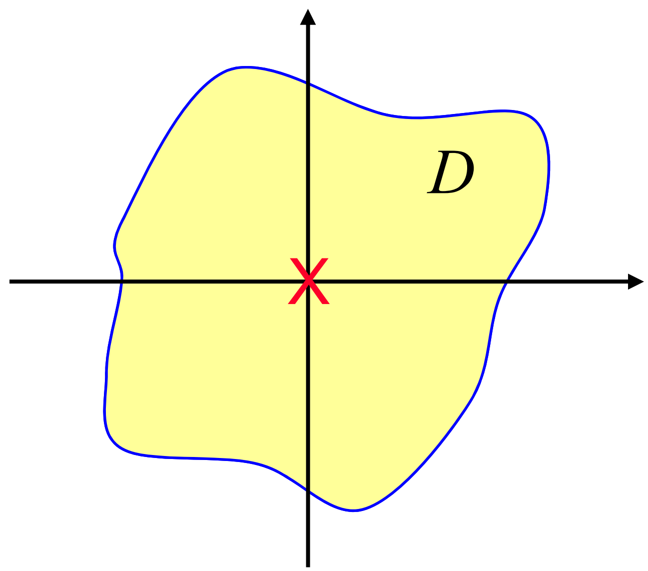


Singularities

Points where a function is not analytic are called **singular points** or **singularities** or **poles** sometimes.

Example:

$f(z) = \dfrac{1}{z}$ is analytic everywhere in D except $z = 0$, which is thus the singular point or pole of the function.



Note that a function is either analytic or singular at any given point…

Rules of Complex Differentiation

- The familiar rules of real differentiation carry over to the complex case. example,

  the product rule

$$[f(z)g(z)]' = f'(z)g(z) + f(z)g'(z)$$

  and the chain rule

$$\frac{d}{dz} f(g(z)) = f'(g(z))g'(z).$$

the complex differentiability of f at a point $z = z_0$ implies the continuity of f at that point.

- useful results of complex differentiation:

$$(z^n)' = nz^{n-1}, \quad (e^z)' = e^z, \quad (\sin z)' = \cos z,$$

$$(\cos z)' = -\sin z, \quad (\sinh z)' = \cosh z, \quad (\cosh z)' = \sinh z$$

$$\left(\frac{1}{z}\right)' = -\frac{1}{z^2}, \quad \frac{d}{dz}\mathrm{Ln}(z) = \frac{1}{z}.$$

- Remark: Exponential, trigonometric and hyperbolic functions are entire functions, while z^{-1} is analytic on $\mathbb{C}\backslash\{0\}$.

Cauchy-Riemann Equations

- A necessary condition for the differentiability of a complex function $f(z) = u(x,y) + iv(x,y)$ is to satisfy the relation

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \tag{3.5}$$

For simplicity, we write

$$u_x = \frac{\partial u}{\partial x}, \quad u_y = \frac{\partial u}{\partial y}, \quad v_x = \frac{\partial v}{\partial x}, \quad v_y = \frac{\partial v}{\partial y}$$

Or simply written as

$$u_x = v_y, \quad u_y = -v_x. \tag{3.6}$$

They are known as the **Cauchy-Riemann equations** (C-R equations).

Augustin-Louis Cauchy
(1789–1857)
French Mathematician

Bernhard Riemann
(1826–1866)
German Mathematician

Theorem (3.1 Cauchy-Riemann Equations)

Let $f(z) = u(x, y) + iv(x, y)$ be defined and continuous in some neighborhood of a point $z = x + iy$ and differentiable at z itself. Then, at that point, the first-order partial derivatives of u and v exist and satisfy the Cauchy-Riemann equations (3.6).

Hence, if $f(z)$ is analytic in a domain D, those partial derivatives exist and satisfy (3.6) at all points of D.

- If f is differentiable, then f' is given by any of these four equivalent expressions:

$$f' = u_x + iv_x = v_y - iu_y = u_x - iu_y = v_y + iv_x. \qquad (3.7)$$

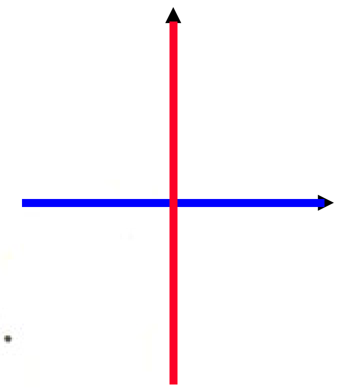
- **Remark:** The four equivalent expressions are obtained by simply applying the Cauchy-Riemann equations (3.6).

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \boxed{u_x = v_y, \quad u_y = -v_x.} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Proof (optional)...

In spite of these similarities, there is a fundamental difference between differentiation for functions of real variables and differentiation for functions of a complex variable. Let $z = (x, y)$ and suppose that h is real. Then

$$f'(z) = \lim_{h\to 0} \frac{f(x+h, y) - f(x, y)}{h} = \frac{\partial f}{\partial x}(z) = f_x(z).$$



But if $h = ik$ is purely imaginary, then

$$f'(z) = \lim_{k\to 0} \frac{f(x, y+k) - f(x, y)}{ik} = \frac{1}{i}\frac{\partial f}{\partial y}(z) = -if_y(z).$$

Thus, the existence of a complex derivative forces the function to satisfy the partial differential equation

$$f'(z) = f_x = -if_y .$$

Writing $f(z) = u(z) + iv(z)$, where u and v are real-valued functions of a complex variable, and equating the real parts and imaginary parts of

$$u_x + iv_x = f_x = -if_y = -i\left(u_y + iv_y\right) = v_y - iu_y \quad \Rightarrow \quad f'(z) = u_x + iv_x$$

we obtain the **Cauchy-Riemann** differential equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad u_x = v_y , \qquad v_x = -u_y . \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Example (3b)

$$f(z) = \bar{z} = x - iy = u + iv \quad \Rightarrow \quad u(x, y) = x, \quad v(x, y) = -y$$

$$\frac{\partial u}{\partial x} = 1 \neq \frac{\partial v}{\partial y} = -1 \quad , \quad \frac{\partial u}{\partial y} = 0 = \frac{\partial v}{\partial x}$$

$$\Rightarrow \quad f(z) \text{ is not analytic anywhere}$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\boxed{u_x = v_y, \quad u_y = -v_x.}$$

Theorem (3.2 Cauchy-Riemann Equations)

If two real-valued continuous functions $u(x,y)$ and $v(x,y)$ of two real variables x and y have continuous first partial derivatives that satisfy the Cauchy-Riemann equations in some domain D, then the complex function $f(z) = u(x,y) + iv(x,y)$ is analytic in D and $f'(z) = u_x + iv_x = v_y - iu_y$.

Example (3c)

Is $f(z) = u(x,y) + iv(x,y) = e^x(\cos y + i \sin y)$ analytic?

e^z

Solution:

we have $u = e^x \cos y, v = e^x \sin y$ and by differentiation

$$u_x = e^x \cos y, \quad v_y = e^x \cos y, \quad u_y = -e^x \sin y, \quad v_x = e^x \sin y$$

The Cauchy-Riemann equations are satisfied and conclude that $f(z)$ is analytic for all z.

$$f'(z) = u_x + iv_x = e^x \cos y + i e^x \sin y = e^x(\cos y + i \sin y) = f(z)$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\boxed{u_x = v_y, \quad u_y = -v_x.}$$

Example:

$$f(z) = z^2 = x^2 - y^2 + i\,2xy = u(x,y) + i\,v(x,y)$$

$$\Downarrow$$

$$u(x,y) = x^2 - y^2, \quad v(x,y) = 2xy$$

$$\frac{\partial u}{\partial x} = 2x = \frac{\partial v}{\partial y} \quad , \quad \frac{\partial u}{\partial y} = -2y = -\frac{\partial v}{\partial x}$$

and the partial derivatives are continuous $\forall\, z$.

Consequently, $f(z)$ is analytic $\forall\, z$.

$$f'(z) = \left(z^2\right)' = u_x + i\,v_x = 2x + i2y = 2(x + i\,y) = 2z$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\boxed{u_x = v_y, \quad u_y = -v_x.}$$

Example:

$$f(z) = \frac{\bar{z}}{|z|^2} = \left(\frac{x}{x^2+y^2}\right) - i\left(\frac{y}{x^2+y^2}\right) = u(x,y) + i\,v(x,y)$$

$$\frac{\partial u}{\partial x} = \frac{y^2 - x^2}{(x^2+y^2)^2} = \frac{\partial v}{\partial y} \quad , \quad \frac{\partial u}{\partial y} = \frac{-2xy}{(x^2+y^2)^2} = -\frac{\partial v}{\partial x}$$

$\Rightarrow$ $f(z)$ is analytic everywhere, except when $x^2 + y^2 = 0$

i.e., at the origin and

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{y^2 - x^2}{(x^2+y^2)^2} + i\frac{2xy}{(x^2+y^2)^2}$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

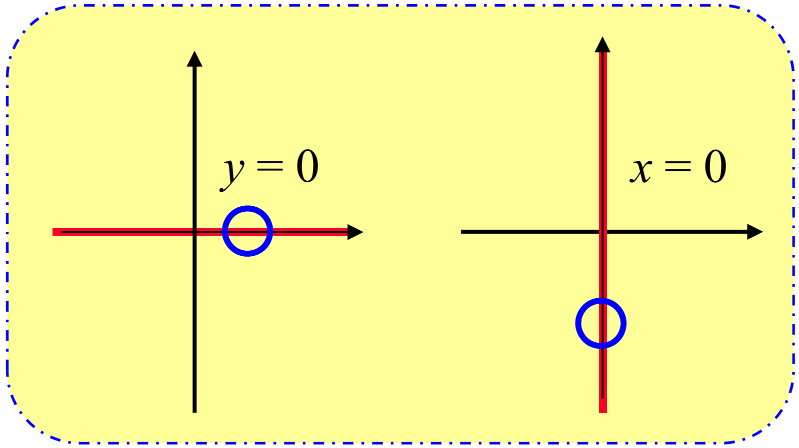
$$\boxed{u_x = v_y, \quad u_y = -v_x.}$$

Example:

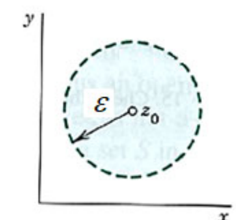
$$f(z) = x^2 y^2 + i\,2x^2 y^2 = u(x,y) + i v(x,y)$$

$$\frac{\partial u}{\partial x} = 2xy^2 \quad , \quad \frac{\partial v}{\partial y} = 4x^2 y \quad , \quad \frac{\partial u}{\partial y} = 2x^2 y \quad , \quad \frac{\partial v}{\partial x} = 4xy^2$$

The Cauchy-Riemann equations only hold for $x = 0$ and/or $y = 0$. Since the function is not analytic in a neighbourhood of $x = 0$ or $y = 0$, $f(z)$ is not analytic anywhere.



- A complex function $f(z)$ is **analytic** at $z = z_0$ if there exists a neighborhood $N(z_0; \epsilon)$ of z_0 such that f is differentiable at every point in $N(z_0; \epsilon)$. If it is not analytic at z_0, it is **singular** there.



Analyticity of the Logarithm…

Recall the logarithm function

$$\ln z = \operatorname{Ln} z \pm 2n\pi i = \ln|z| + i\operatorname{Arg}(z) \pm 2n\pi i, \quad n = 0, 1, 2, \cdots \qquad (3)$$

where $\operatorname{Ln} z$ is the principal value of $\ln z$. Then,

For every $n = 0, \pm 1, \pm 2, \cdots$ formula (3) defines a function, which is analytic, except at 0 and on the negative real axis, and has the derivative

$$(\ln z)' = \frac{1}{z} \qquad (z \text{ not } 0 \text{ or negative real}).$$

The proof of this result can be done by checking its associated CR equations. We rewrite

$$\ln z = \ln r + i(\theta + c) = \frac{1}{2}\ln(x^2 + y^2) + i\left(\arctan\frac{y}{x} + c\right)$$

where c is a multiple of 2π.

$$\frac{d}{dx}\arctan x = \frac{1}{1+x^2}$$

By differentiation,

$$u_x = \frac{x}{x^2+y^2} = v_y = \frac{1}{1+(y/x)^2}\cdot\frac{1}{x}$$

$$u_y = \frac{y}{x^2+y^2} = -v_x = -\frac{1}{1+(y/x)^2}\left(-\frac{y}{x^2}\right).$$

Hence the Cauchy–Riemann equations hold and

$$(\ln z)' = u_x + iv_x = \frac{x}{x^2+y^2} + i\,\frac{1}{1+(y/x)^2}\left(-\frac{y}{x^2}\right) = \frac{x-iy}{x^2+y^2} = \frac{1}{z}.$$

Each of the infinitely many functions in (3) is called a **branch** of the logarithm. The negative real axis is known as a **branch cut** and is usually graphed as shown in Fig. 338. The branch for $n = 0$ is called the **principal branch** of $\ln z$.

$$\operatorname{Ln} z = \ln|z| + i\,\operatorname{Arg}(z)$$

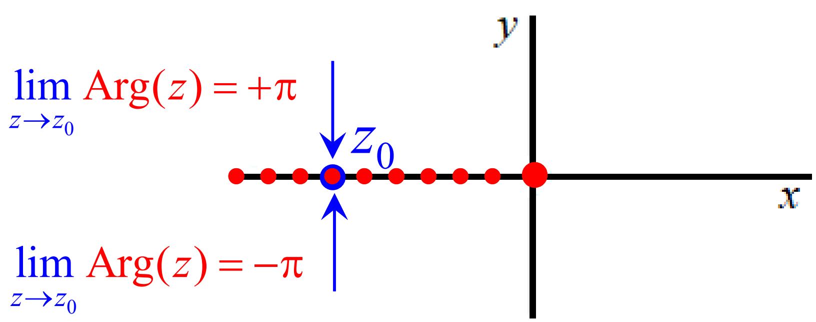


Fig. 338. Branch cut for ln z

Observations

1. The sum or product of analytic functions is analytic.
2. All polynomials are analytic.
3. A rational function (the quotient of two polynomials) is analytic, except at zeroes of the denominator.
4. An analytic function of an analytic function is analytic.
5. Functions e^z, $\sin z$, $\cos z$, $\sinh z$, $\cosh z$ are analytic everywhere.

✦ About Homework Assignment No. 2……

You can start working on Problems 2.1 to 2.4, but hand in the complete set when it is due (to be announced later).

Theorem (3.3 Laplace's Equations)

If $f(z) = u(x,y) + iv(x,y)$ is analytic in a domain D, then both u and v satisfy Laplace's equations

$$\boxed{\begin{aligned} \nabla^2 u = u_{xx} + u_{yy} = 0, \\ \nabla^2 v = v_{xx} + v_{yy} = 0 \end{aligned}} \qquad (3.11)$$

in D and have continuous second partial derivatives in D.

- **Remark:** The above theorem follows from Cauchy-Riemann equations (3.6):

$$u_{xx} + u_{yy} = (u_x)_x + (u_y)_y = (v_y)_x + (-v_x)_y = 0,$$

$$v_{xx} + v_{yy} = (v_x)_x + (v_y)_y = (-u_y)_x + (u_x)_y = 0,$$

assuming that u and v are C^2.

$$\boxed{u_x = v_y, \quad u_y = -v_x.}$$

$$u_{xx} = \frac{\partial^2 u}{\partial x^2}, \quad u_{xy} = \frac{\partial^2 u}{\partial x \partial y}, \quad v_{yy} = \frac{\partial^2 v}{\partial y^2}, \quad \cdots$$

Harmonic Functions

A function $h(x, y)$ is harmonic if it is a twice continuously differentiable that satisfies Laplace's equation: $h_{xx} + h_{yy} = 0$.

Note that if $f(z) = u(x, y) + i\,v(x, y)$ is analytic, then the pair u and v are both harmonic functions. We say that v is a *harmonic conjugate function* of u in the domain D.

If u is harmonic and v is a harmonic conjugate of u, then it can be showed that u is a harmonic conjugate of $-v$ by noting that $g(z) = i\,f(z)$ is analytic

$$g(x,y) = i\,f(x,y) = i\left[u(x,y) + iv(x,y)\right] = -v(x,y) + iu(x,y)$$

$$\nabla^2 u = u_{xx} + u_{yy} = 0$$
$$\nabla^2 v = v_{xx} + v_{yy} = 0$$

Pierre-Simon Laplace
(1749–1827)
French Mathematician

Example*** ...

How to Find a Harmonic Conjugate Function by the Cauchy–Riemann Equations

Verify that $u = x^2 - y^2 - y$ is harmonic in the whole complex plane and find a harmonic conjugate function v of u.

Solution. $\nabla^2 u = 0$ by direct calculation ⋯

$$u_x = 2x, \quad u_{xx} = 2, \quad u_y = -2y - 1, \quad u_{yy} = -2 \quad \Rightarrow \quad \nabla^2 u = u_{xx} + u_{yy} = 0.$$

Hence because of the Cauchy–Riemann equations a conjugate v of u must satisfy

$$v_y = u_x = 2x, \qquad v_x = -u_y = 2y + 1.$$

Integrating the first equation with respect to y and differentiating the result with respect to x, we obtain

$$v(x, y) = \int v_y dy = \int 2x dy = 2xy + h(x), \quad v_x = 2y + \frac{d}{dx} h(x)$$

A comparison with the second equation shows that $dh/dx = 1$. This gives $h(x) = x + c$. Hence $v = 2xy + x + c$ (c any real constant) is the most general harmonic conjugate of the given u. The corresponding analytic function is

$$f(z) = u + iv = x^2 - y^2 - y + i(2xy + x + c)$$

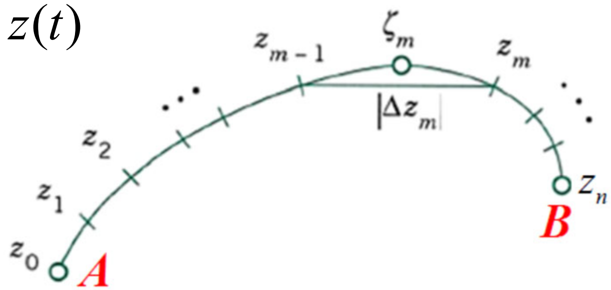
$$= \left(x^2 + i2xy + (iy)^2\right) + i(x + iy) + ic = z^2 + iz + ic.$$

Complex Analysis – 4…

1 Complex Numbers
2 Functions of One Complex Variable
3 Complex Differentiation
4 **Complex Integration and Cauchy's Theorem**
5 Cauchy Integral Formula
6 Complex Series, Power Series, Taylor Series, and Laurent Series
7 Residue Integration

Complex line integral of $f(z)$

Curve: $z(t)$



$$S_n = \sum_{m=1}^{n} f(\zeta_m)(z_m - z_{m-1})$$

$$\int_C f(z)\, dz = \lim_{n \to \infty} S_n$$

Estimation of a complex integral

Let $f(z)$ be continuous on $C: t \to z(t)$, $t \in [\alpha, \beta]$. If $|f(z)| \le M$ on C, then

$$\left| \int_C f(z)\, dz \right| \le M L$$

where L is the length of the curve C, i.e.

$$L = \int_\alpha^\beta |z'(t)|\, dt = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Material flow…

Theorem (4.1 Indefinite Integration of Analytical Functions)

Let $f(z)$ be analytic in a simply connected domian D. Then there exists an indefinite integral of $f(z)$ in the domain D, that is, an analytic function $F(z)$ such that $F'(z) = f(z)$ in D, and for all paths in D joining two points z_0 and z_1 in D we have

$$\int_{z_0}^{z_1} f(z)\,dz = F(z_1) - F(z_0)\ , \quad [F'(z) = f(z)] \qquad (4.5)$$

(Note that we can write z_0 and z_1 instead of C, since we get the same value for all those C from z_0 and z_1.)

Theorem (4.2 Integration by the Use of the Path)

Let C be a piecewise smooth path, represented by $z = z(t)$, where $t \in [a, b]$. Let $f(z)$ be a continuous function on C. Then

$$\int_C f(z)\,dz = \int_a^b f(z(t))z'(t)\,dt. \qquad (4.6)$$

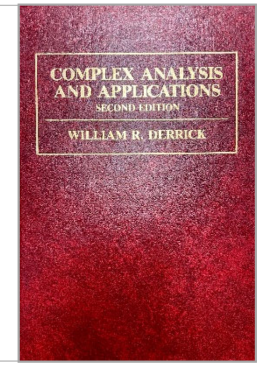
Theorem (4.3 Cauchy's Integral Theorem)

If $f(z)$ is analytic in a simply connected domain D, then

$$\oint_C f(z)\,dz = 0 \qquad (4.9)$$

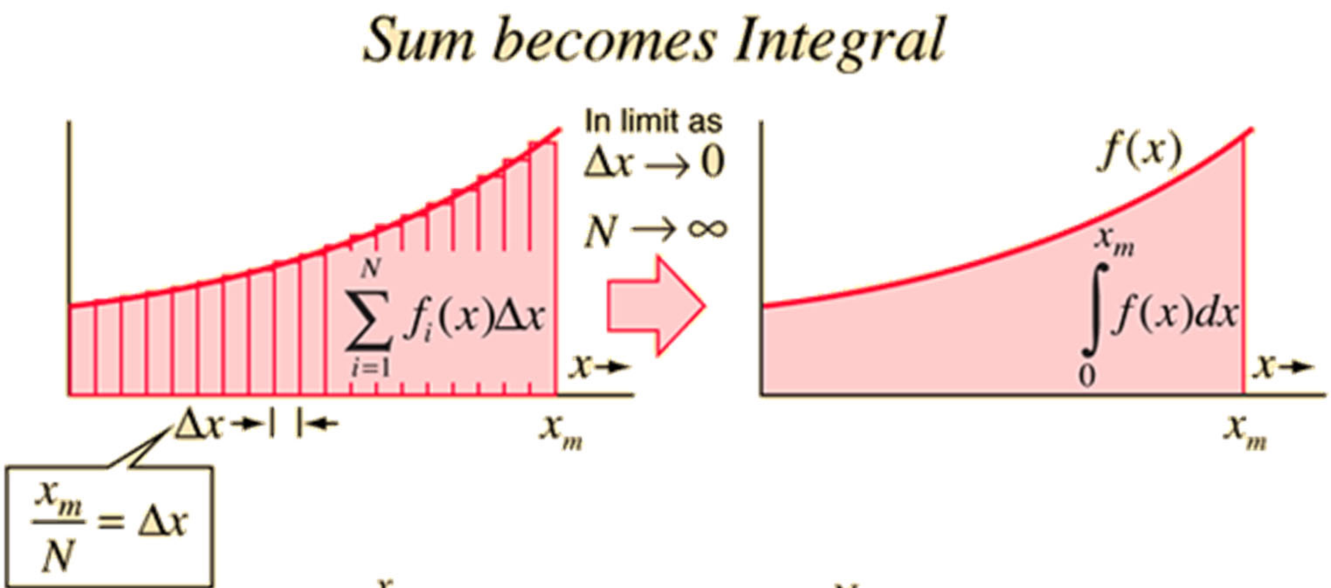
for every simple closed path C in D.

2 COMPLEX INTEGRATION



Integration is an important and useful concept in elementary calculus. The two-dimensional nature of the complex plane suggests the consideration of integrals along arbitrary curves in $\mathbb{C}$ instead of only on segments of the real axis. These "line integrals" have interesting and unusual properties when the function being integrated is analytic. Complex integration is one of the most beautiful and elegant theories in mathematics.

Revisit: Real Integration…



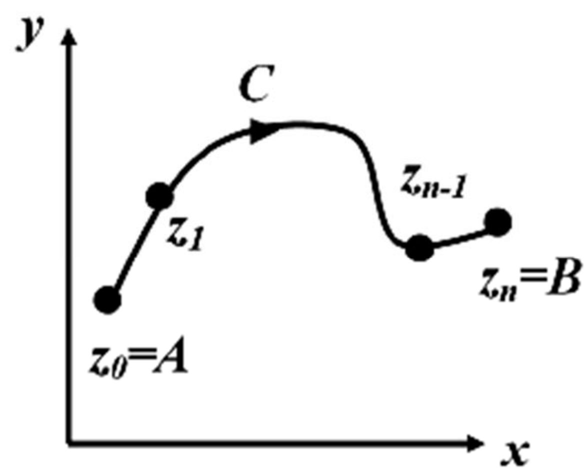
$$Area = \int_0^{x_m} f(x)dx = \lim_{\Delta x \to 0} \sum_{i=1}^{N} f_i(x)\Delta x$$

Line Integral in the Complex Plane

- Complex definite integrals are called (complex) **line integrals**. They are written

$$\boxed{I = \int_C f(z)\, dz} \qquad (4.1)$$

Here the integrand $f(z)$ is integrated over a given curve C in the complex z-plane, called the **path of integration**.



- Such a curve C can be represented by a parametric representation

$$\boxed{z(t) = x(t) + iy(t)} \tag{4.2}$$

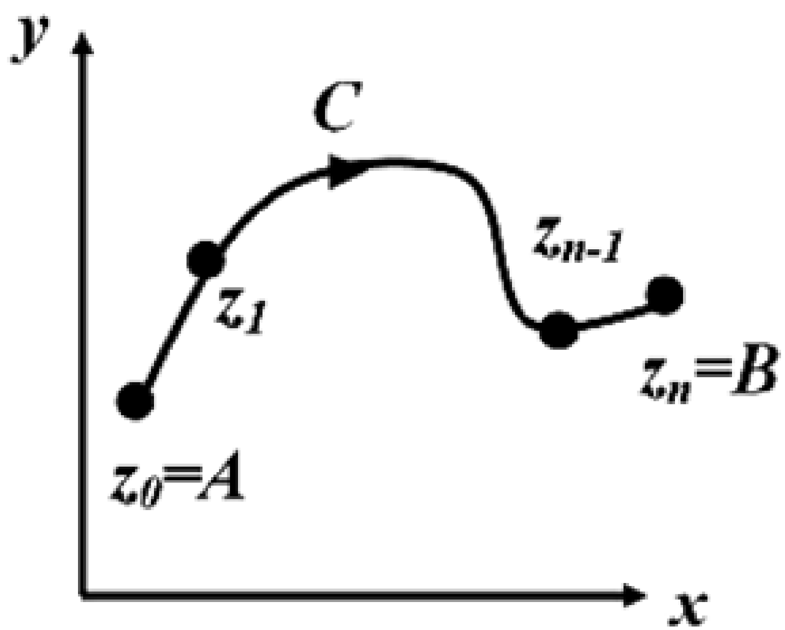


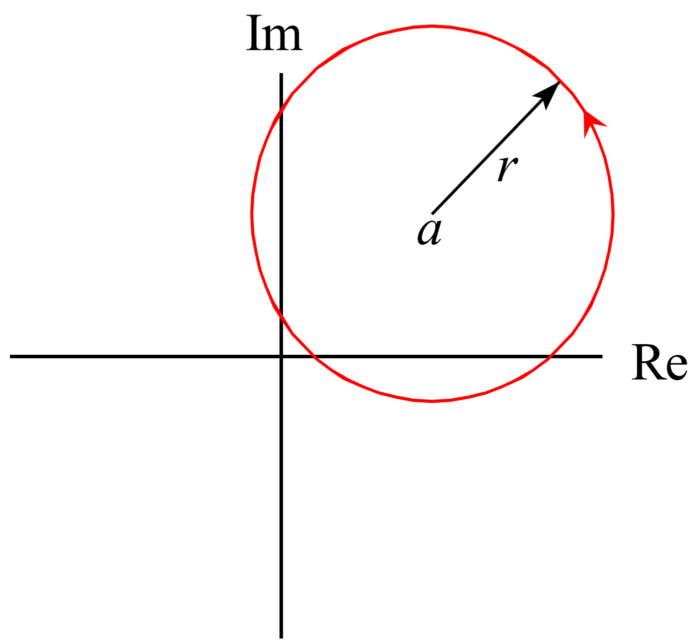
Fig. 4.1

Special Curve: Circle

- **Circle**

  The parametric description for a **circle** centred at complex point a and with a radius r is

  $$z(t) = a + re^{it}, \quad t \in [0, 2\pi]$$

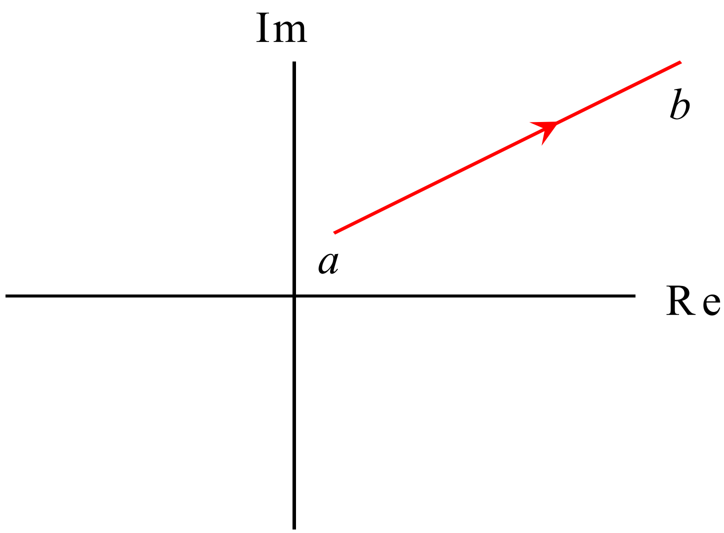


Special Curve: Straight Line

- **Straight Line**

  The parametric description of a **straight-line** segment with starting point a and endpoint b is

  $$z(t) = (b-a)t + a, \quad t \in [0,1]$$



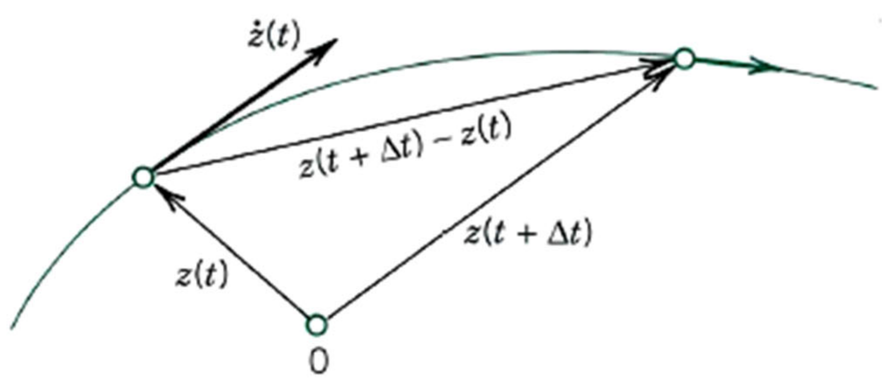
- We assume C to be a **smooth curve**, i.e., C has a continuous and nonzero derivative

$$\dot{z}(t) = \frac{dz}{dt} = \dot{x}(t) + i\dot{y}(t)$$

at each point.

- Geometrically, this means that C has a continuously turning tangent everywhere.

$$\dot{z}(t) = \lim_{\Delta t \to 0} \frac{z(t + \Delta t) - z(t)}{\Delta t}$$



Definition of Complex Line Integral

- Consider a smooth curve C in the complex plane given by

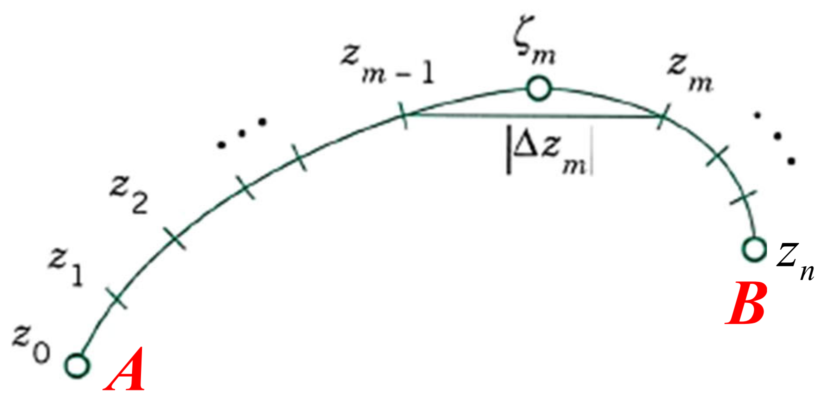
$$z(t) = x(t) + iy(t), \quad a \le t \le b$$

Subdivide the interval $a \le t \le b$ by points

$$a = t_0,\ t_1,\ t_2,\ \ldots,\ t_{n-1},\ t_n = b$$

- Suppose that C has initial point and end points at $z = A$ and $z = B$, respectively, the Corresponding to points on C will be,

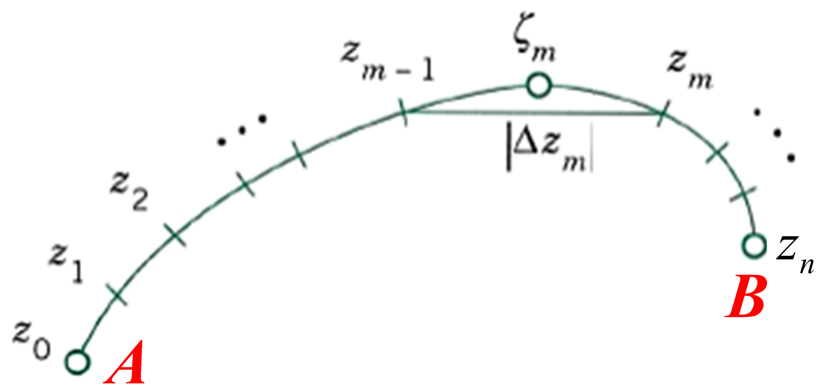
$$A = z_0,\ z_1,\ z_2,\ \ldots,\ z_{n-1},\ z_n = B \qquad z_i = z(t_i)$$



- Form the sum

$$S_n = \sum_{m=1}^{n} f(\zeta_m)(z_m - z_{m-1}) \qquad (4.3)$$

where ζ_m is some point between the arc from z_{m-1} to z_m. The choice of the z_m's and ζ_m's defines a **partition** of C, and we call the largest $|\Delta z_m| = |z_m - z_{m-1}|$ the **norm** of the partition.

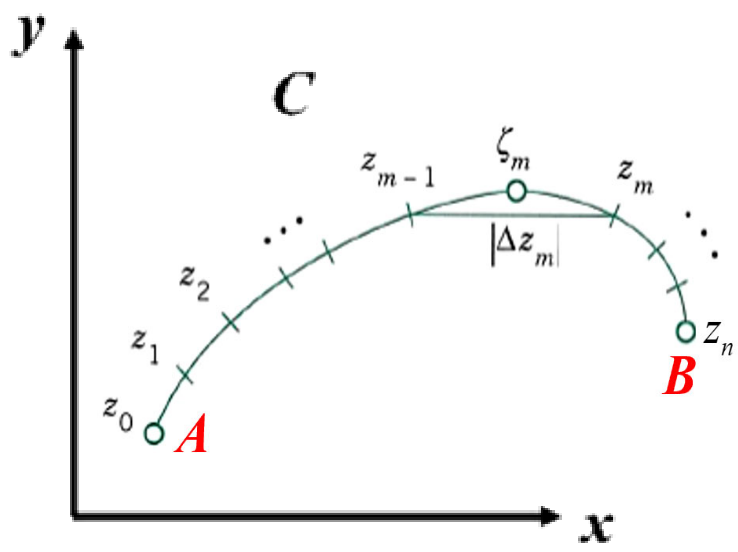


- The partition is chosen such that the norm of the n-th partition tends to zero as $n \to \infty$. If the corresponding sequence of the sums $S_1, S_2, \ldots$ converges to a limit, we call that limit the complex integral $\int_C f(z)\,dz$ and say that the integral exists, i.e.,

$$\int_C f(z)\,dz = \lim_{n\to\infty} S_n \qquad (4.4)$$

- Recall a curve C is simple if it does not intersect itself and it is called a **closed path** if $A = B$ in Fig.4.1. In such case,

$$\boxed{I = \int_C f(z)\, dz} \quad \Rightarrow \quad I = \oint_C f(z)\, dz.$$



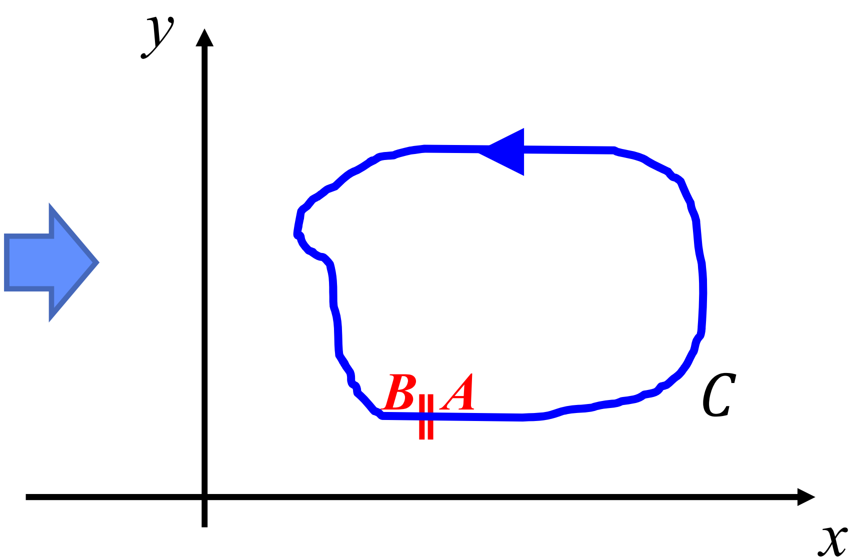


Fig. 4.1

Properties of Complex Integrals

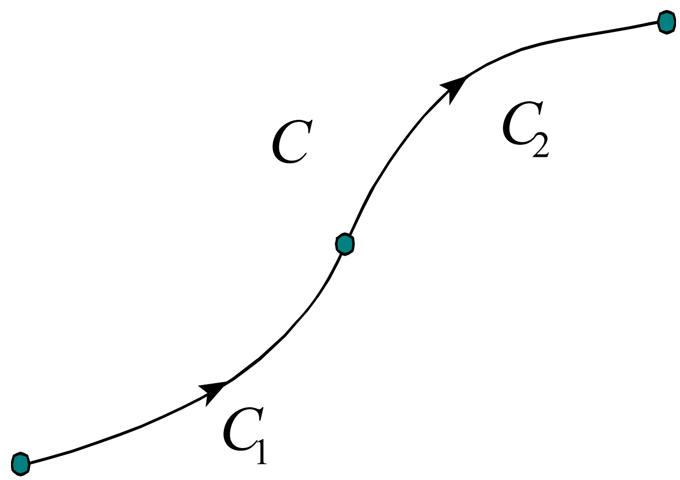
As for real integrals, the following rules apply:

1. $$\int_C [f(z)+g(z)]\,dz = \int_C f(z)\,dz + \int_C g(z)\,dz$$

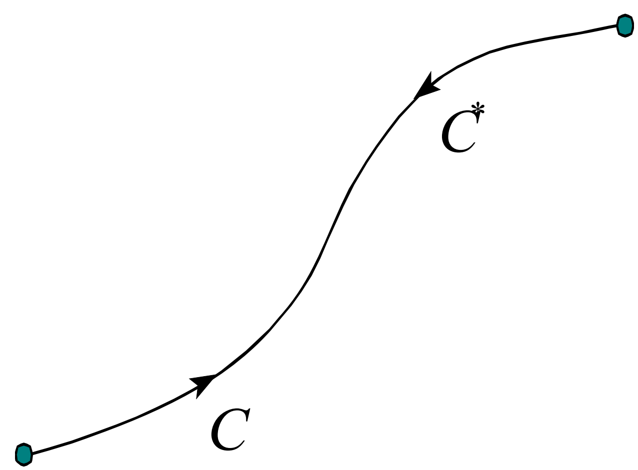
2. $$\int_C k\,f(z)\,dz = k\int_C f(z)\,dz\,,\ \ k \text{ complex}$$

Properties of Complex Integrals

3. $\int_{C} f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz$



4. $\int_{C} f(z)\,dz = -\int_{C^*} f(z)\,dz$



Estimation of a Complex Integral (*ML*-Inequality)

- Let $f(z)$ be continuous on $C: t \to z(t)$, $t \in [\alpha, \beta]$. If $|f(z)| \le M$ on C, then

$$\left|\int_C f(z)\,dz\right| = \left|\int_C f(z)\,z'(t)\,dt\right| \le \int_C |f(z)z'(t)|\,dt = \int_C |f(z)|\cdot|z'(t)|\,dt \le M\int_C |z'(t)|\,dt = ML$$

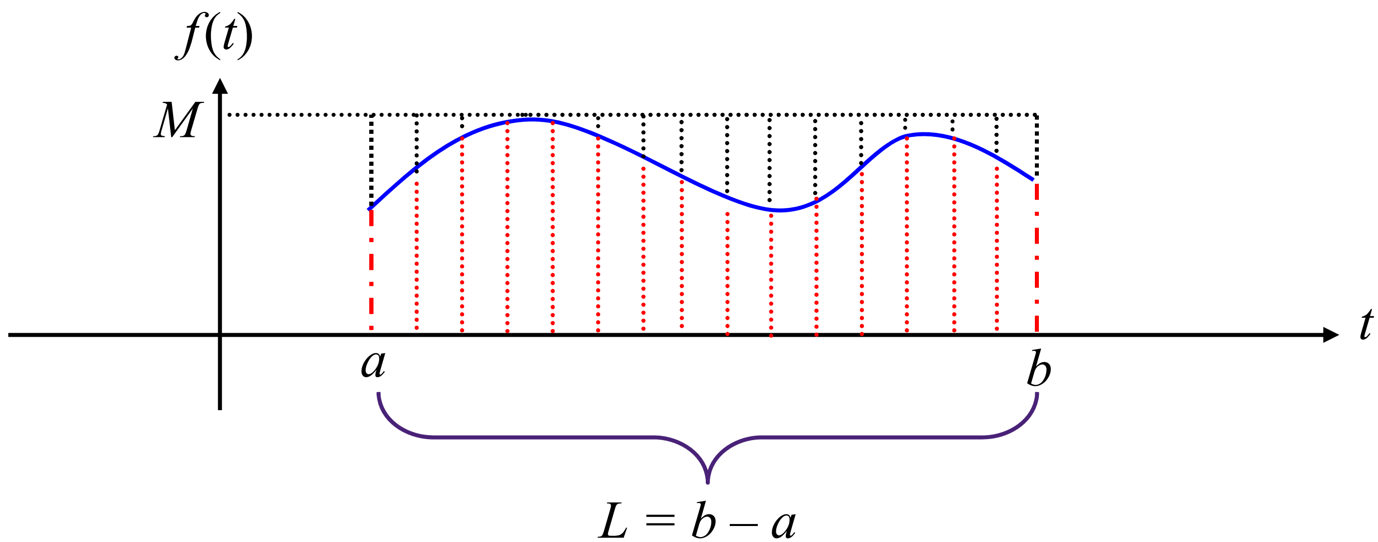
$$\left|\int_C f(z)dz\right| \le ML$$

The *ML*-Inequality

where L is the length of the curve C, i.e., $L = \int_\alpha^\beta |z'(t)|\,dt$.

Estimation of Complex Integral – An Illustration

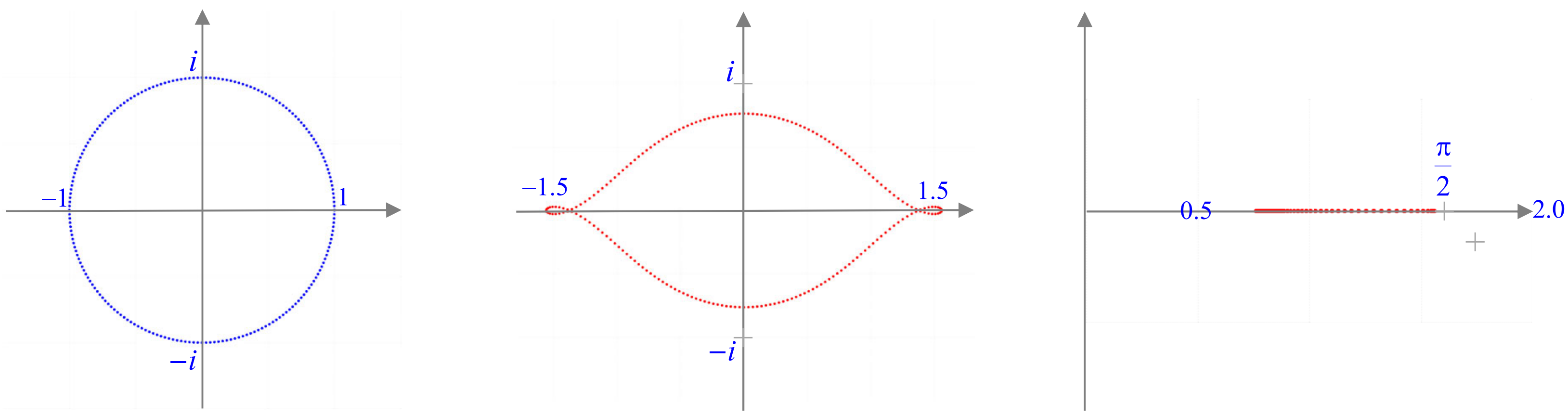
Graphically, take real integration as an example,



$$\left|\int_a^b f(t)dt\right| = \text{shaded area with red lines} \le M \cdot L$$

Estimation of Complex Integral – An Illustrative example

For complex cases, for example, we take $f(z)=\dfrac{\tan z}{z^2}$ and C to be a unit circle



z $\qquad$ $\dfrac{\tan z}{z^2}$ $\qquad$ $\left|\dfrac{\tan z}{z^2}\right| < \dfrac{\pi}{2}$

$$\left|\int_C f(z)dz\right| = \left|\int_C \frac{\tan z}{z^2}dz\right| \le ML = \frac{\pi}{2}\cdot 2\pi = \pi^2$$

$$\int_C \frac{\tan z}{z^2}dz = 2\pi i \Rightarrow \left|\int_C \frac{\tan z}{z^2}dz\right| = 2\pi < \pi^2 \ \checkmark$$

Evaluation Method: Indefinite Integration and Substitution of Limits

remarks

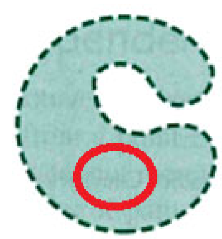
Theorem (4.1 Indefinite Integration of Analytical Functions)

Let $f(z)$ be analytic in a simply connected domian D. Then there exists an indefinite integral of $f(z)$ in the domain D, that is, an analytic function $F(z)$ such that $F'(z) = f(z)$ in D, and for all paths in D joining two points z_0 and z_1 in D we have

$$\int_{z_0}^{z_1} f(z)\,dz = F(z_1) - F(z_0) \qquad [F'(z) = f(z)] \tag{4.5}$$

(Note that we can write z_0 and z_1 instead of C, since we get the same value for all those C from z_0 and z_1.)

* An **indefinite integral** is a function whose derivative equals a given analytic function in a region.



simply connected

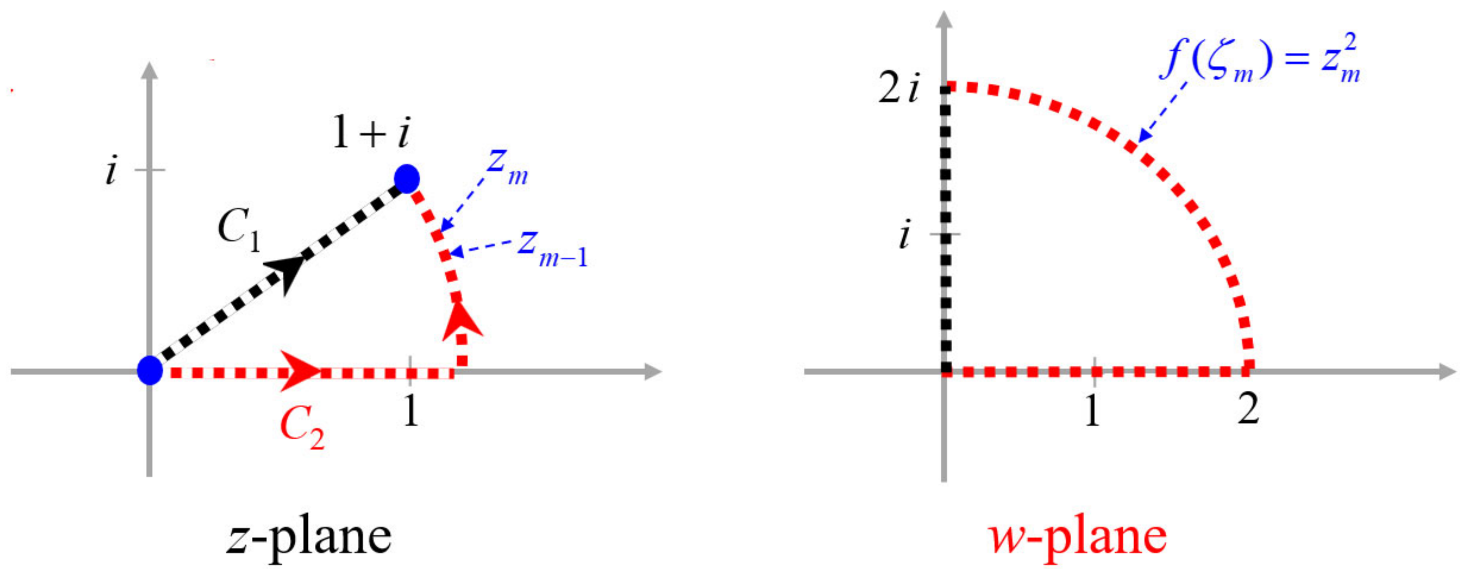
Example (4a)

plot_int

1, 2

$$\int_0^{1+i} z^2\,dz = \frac{1}{3}z^3\Big|_0^{1+i} = \frac{1}{3}(1+i)^3 = -\frac{2}{3} + \frac{2}{3}i$$

$$\int_C f(z)\,dz = \lim_{n\to\infty} S_n, \qquad S_n = \sum_{m=1}^{n} f(\zeta_m)(z_m - z_{m-1})$$



Evaluation Method: Use of a Representation of a Path

Theorem (4.2 Integration by the Use of the Path)

Let C be a piecewise smooth path, represented by $z = z(t)$, where $t \in [a, b]$. Let $f(z)$ be a continuous function on C. Then

$$\boxed{\int_C f(z)\,dz = \int_a^b f(z(t))z'(t)\,dt.} \tag{4.6}$$

By Theorem 4.2, the complex integral might depend on the path/curve chosen.

This is generally true for non-analytic functions.

Complex integral is independent of paths for analytic functions.

Example (4b)

- Show that by integrating $\frac{1}{z}$ counterclockwise around the unit circle (the circle of radius 1 and center 0)

$$\boxed{\oint_C \frac{dz}{z} = 2\pi i} \qquad (4.7)$$

- (a). Represent the unit circle C by

$$z(t) = \cos t + i \sin t = e^{it} \qquad 0 \leq \ t \leq 2\pi$$

so that counterclockwise integration corresponds to an increase of $t \in [0\ 2\pi]$.

- (b). Differentiation gives $\dot{z}(t) = ie^{it}$.
- (c). By substitution, $f(z(t)) = \frac{1}{z(t)} = e^{-it}$.
- (d). From Eq.(4.6), we obtain

$$\oint_C \frac{dz}{z} = \int_0^{2\pi} e^{-it}\ ie^{it}\ dt = i\int_0^{2\pi} dt = 2\pi i.$$

(Check this result by using $z(t) = \cos t + i \sin t$)

Theorem (4.1 Indefinite Integration of Analytical Functions)

Let $f(z)$ be analytic in a simply connected domian D. Then there exists an indefinite integral of $f(z)$ in the domain D, that is, an analytic function $F(z)$ such that $F'(z) = f(z)$ in D, and for all paths in D joining two points z_0 and z_1 in D we have

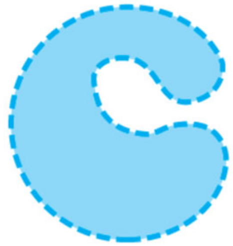
$$\int_{z_0}^{z_1} f(z)\,dz = F(z_1) - F(z_0) \qquad [F'(z) = f(z)] \qquad (4.5)$$

(Note that we can write z_0 and z_1 instead of C, since we get the same value for all those C from z_0 and z_1.)

Some remarks on Theorem 4.1...

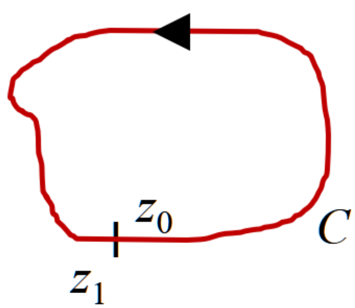
$$\int_{z_0}^{z_1} f(z)dz = F(z_1) - F(z_0), \qquad F'(z) = f(z) \qquad (4.5)$$

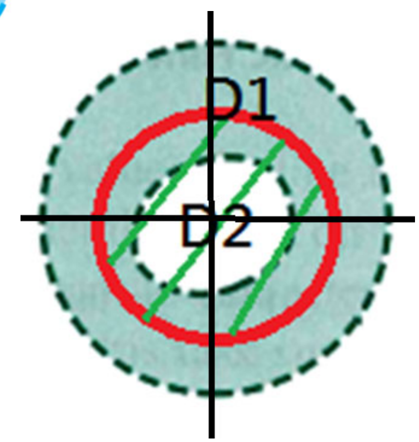
- $f(z)$ is analytic in a simply connected domain is essential in Theorem 4.1.



Simply connected

- Eq. (4.5) in Theorem 4.1 gives 0 for any closed path because then $z_1 = z_0$ and thus $F(z_1) - F(z_0) = 0$.





- Now, $\frac{1}{z}$ is not analytic at $z = 0$. But any simply connected domain containing the unit circle must contain $z = 0$, so that Theorem 4.1 does not apply.
- It is not enough that $\frac{1}{z}$ is analytic in an annulus, say, $\frac{1}{2} < |z| < \frac{3}{2}$, because an annulus is not simply connected!

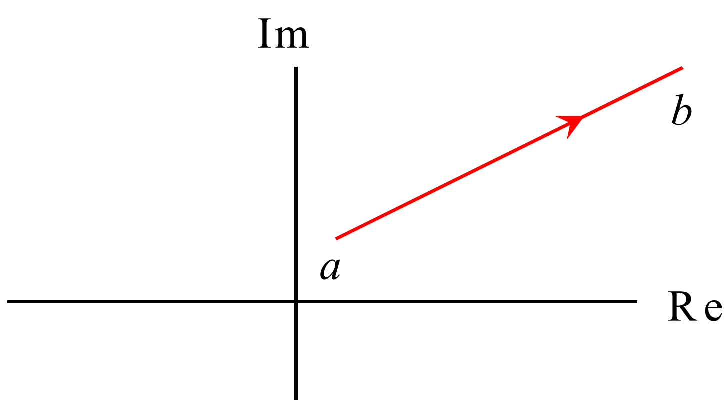
Example (4c)

- Suppose we want to evaluate the complex integral

$$\int_C |z|^2 \, dz,$$

where C is a straight line from $z = 0$ to $z = 1 + i$.

- Parametrize C as $z(t) = t + ti, t \in [0, 1]$, then $z'(t) = 1 + i$. Hence, by property (4.6),



$$z(t)=(b-a)t+a, \quad t\in[0,1]$$

$$\begin{aligned}
\int_C |z|^2 \, dz &= \int_0^1 |t + ti|^2 (1 + i) \, dt \\
&= \int_0^1 (2t^2)(1 + i) \, dt \\
&= (1 + i) \frac{2t^3}{3} \Big|_{t=0}^{1} \\
&= \frac{2}{3}(1 + i).
\end{aligned}$$

Example (4d)

- Evaluate

$$I = \oint_C (z-a)^n \, dz,$$

where a is a given complex number, n is any integer and C is a circle of radius R, centered at a and oriented in an anticlockwise direction as follows.

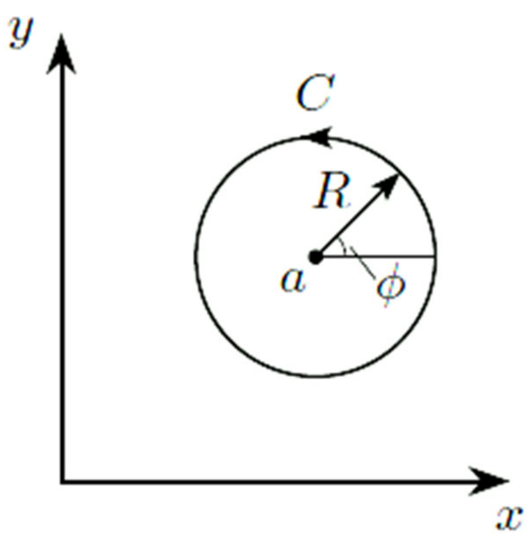
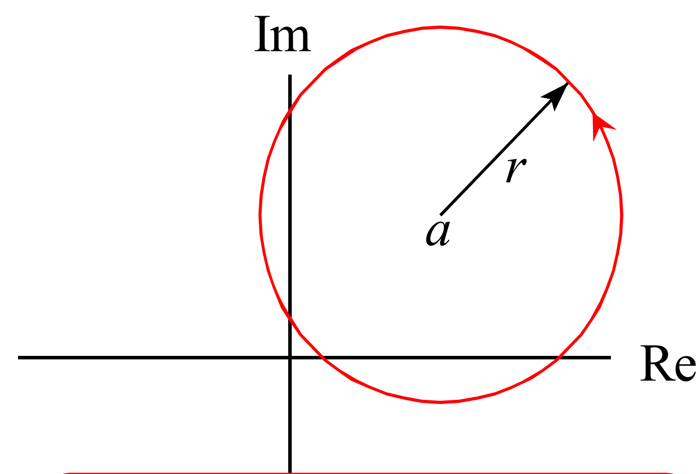


Fig. 4.3. The closed contour C.



$$z(t) = a + re^{it}, \quad t \in [0, 2\pi]$$

- Parametrize C as

$$z = a + Re^{i\phi}, \quad 0 \le \phi \le 2\pi.$$

Then, by property (4.6),

$$\begin{aligned} I &= \int_0^{2\pi} (Re^{i\phi})^n (Rie^{i\phi})\, d\phi \\ &= iR^{n+1} \int_0^{2\pi} e^{i(n+1)\phi}\, d\phi \\ &= \frac{R^{n+1}}{n+1} e^{i(n+1)\phi} \Big|_{\phi=0}^{2\pi} = 0, \end{aligned}$$

provided that $n \ne -1$.

- If $n = -1$, then $I = iR^0 \int_0^{2\pi} e^{i0\phi}\, d\phi = i \int_0^{2\pi} d\phi = 2\pi i$. Hence,

$$I = \oint_C (z-a)^n\, dz = \begin{cases} 2\pi i, & \text{if } n = -1, \\ 0, & \text{if } n \ne -1. \end{cases}$$

Homework Assignment No: 2 (Complete set due in one week)

Question 2.1: *What is an analytic function? What is the Cauchy-Rieman condition for? What is the derivative of a complex function? What is a curve? What is the length of a curve? How to characterize a circle and a straight line? What is a complex integral? What is the upper bound of a complex integral?*

Question 2.2: Find all the values of the given expressions:

(a) $\ln(1+i)$; (b) 1^{i}; (c) $(1+i)^{1+i}$

Question 2.3: Find and plot the domain of analyticity of the complex function $f(z) = \mathrm{Ln}\left(1+z^{-1}\right)$, where Ln denotes the principle value of the complex logarithm.

Question 2.4: Compute the derivatives of the following analytic functions:

(b) $\dfrac{iz+3}{z^2-(2+i)z+(4-3i)}$; (b) e^{z^2}; (c) $\dfrac{1}{e^{z}+e^{-z}}$

Question 2.5: Given the following complex functions, determine i) the domain of each function, and ii) which of them are analytic by using the Cauchy Riemann equations:

(a) $f(z) = f(x+i\,y) = (x^3 - 3xy^2 - x) + i(3x^2 y - y^3 - x)$

(b) $f(z) = f(x+i\,y) = x^2 + i\,y^2$

(c) $f(z) = f(x+i\,y) = \left(\sqrt{2}/2 - i \cdot \sqrt{2}/2 \right)^2 (x+i\,y)^2$

Question 2.6: Verify the following inequalities:

(a) $\left| \int_C z^2 dz \right| \le 2\sqrt{2}, \quad C: z(t) = t + i\,t, \quad 0 \le t \le 1$

(b) $\left| \int_C \frac{1}{z+1} dz \right| \le \frac{5\pi}{4}, \quad C: z(t) = 5\left(\cos t + i \sin t\right), \quad 0 \le t \le \pi$

Question 2.7: Let C be a circle with radius r and centred at the origin, i.e., C: $z(t) = a + re^{it}$, $t \in [0, 2\pi]$ and $f(z) = z^2$. Calculate the integral $\int_C z^2\, dz$.

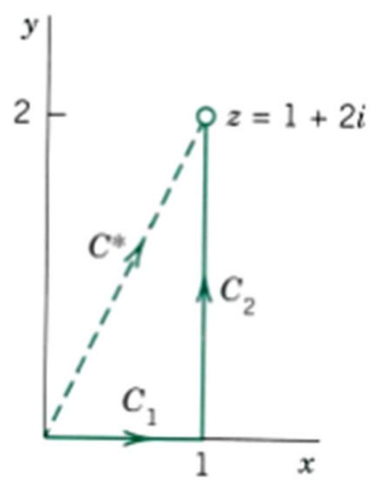
Different Paths different Values

- In general, a complex line integral depends not only on the end points of the path but also on the path itself.

Example (4e)

- Integrate $f(z) = \mathrm{Re}(z) = x$ from 0 to $1+2i$ (a) along C^*; (b) along C consisting of C_1 and C_2.
- (a). C^* can be represented by $z(t) = t + 2it \quad 0 \le t \le 1.$

$$\frac{dz(t)}{dt} = 1 + 2i \quad and \quad f[z(t)] = x(t) = t \quad on \quad C^*.$$



$$I^* = \int_{C^*} \mathrm{Re}\, z\; dz = \int_0^1 t(1+2i)dt$$

$$= \frac{1}{2}(1+2i) = \boxed{\frac{1}{2} + i}$$

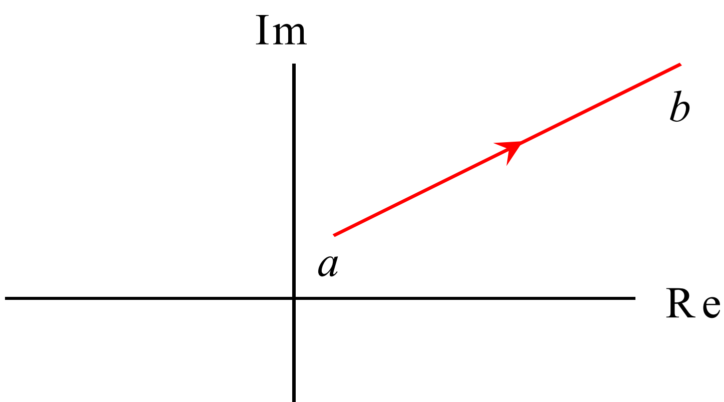
- (b).

$$C_1 : z(t) = t, \quad \dot{z}(t) = 1, \quad f(z(t)) = x(t) = t \quad 0 \le t \le 1$$

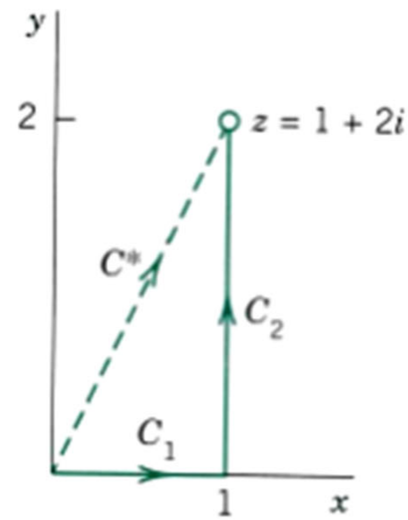
$$C_2 : \quad z(t) = 2it + 1, \quad \dot{z}(t) = 2i, \quad f(z(t)) = x(t) = 1, \quad 0 \le t < 1$$

$$I = \int_C \operatorname{Re} z dz = \int_{C_1} \operatorname{Re} z dz + \int_{C_2} \operatorname{Re} z dz = \int_0^1 t dt + \int_0^1 2i dt = \boxed{\frac{1}{2} + 2i}$$

- Note that this result differs from the result in (a).



$$z(t) = (b-a)t + a, \quad t \in [0,1]$$

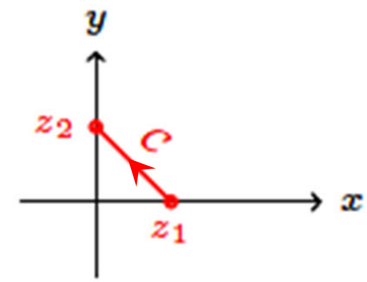


Path Dependent

Complex Integration: Path dependent

We want to compute the integral $\int_C \bar{z}\,\mathrm{d}z$ where C is the

- *line between* $z_1 = 1$ *and* $z_2 = i$

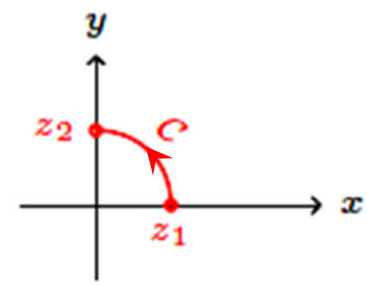


$$z(t) = 1 + t(i-1),\ t \in [0,1]$$

$$\text{Then } \int_C \bar{z}\,\mathrm{d}z = \int_0^1 \big(1 + t(-i-1)\big)(i-1)\,\mathrm{d}t$$

$$= (i-1)\Big(t - \frac{t^2}{2}(i+1)\Big)\Big|_0^1 = (i-1)\frac{1-i}{2} = i$$

- *arc of unit circle between* z_1 *and* $z_2 = i$



$$z(t) = \mathrm{e}^{it},\ t \in [0, \pi/2]$$

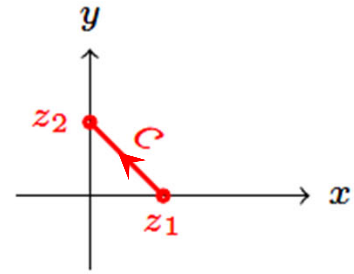
$$\text{Then } \int_C \bar{z}\,\mathrm{d}z = \int_0^{\pi/2} \mathrm{e}^{-it} i \mathrm{e}^{it}\,\mathrm{d}t$$

$$= it\Big|_0^{\pi/2} = i\pi/2$$

NOTE: The result of the integration is *path-dependent*. why?

We want to compute the integral $\int_{\mathcal{C}} z\,\mathrm{d}z$ where $\mathcal{C}$ is the

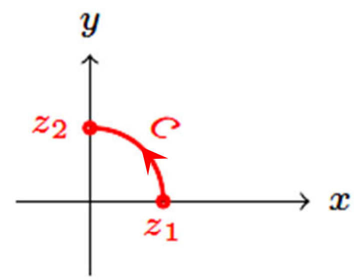
- *line between* $z_1 = 1$ *and* $z_2 = i$



$$z(t) = 1 + t(i-1),\ t \in [0,1]$$

Then $$\int_{\mathcal{C}} z\,\mathrm{d}z = \int_0^1 \big(1 + t(i-1)\big)(i-1)\,\mathrm{d}t$$
$$= (i-1)\left(t + \frac{t^2}{2}(i-1)\right)\Big|_0^1 = (i-1)\frac{1+i}{2} = -1$$

- *arc of unit circle between* z_1 *and* $z_2 = i$



$$z(t) = \mathrm{e}^{it},\ t \in [0, \pi/2]$$

Then $$\int_{\mathcal{C}} z\,\mathrm{d}z = \int_0^{\pi/2} \mathrm{e}^{it} i \mathrm{e}^{it}\,\mathrm{d}t$$
$$= \mathrm{e}^{2it}/2\Big|_0^{\pi/2} = \frac{e^{\pi i} - 1}{2} = \frac{\cos\pi - \sin\pi - 1}{2} = -1$$

NOTE: The result of the integration is the same for the two contours. Is it that the integral is *path-independent* and if so, why?

why?

Path Independent

Cauchy's Integral Theorem

Theorem (4.3 Cauchy's Integral Theorem)

If $f(z)$ is analytic in a simply connected domain D, then

$$\oint_C f(z)\,dz = 0 \tag{4.9}$$

for every simple closed path C in D.

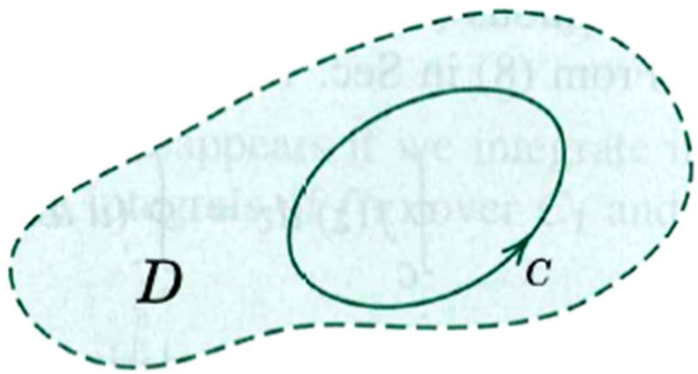


Fig. 4.4. Cauchy's integral theorem.

$$\int_{z_0}^{z_1} f(z)\,dz = F(z_1) - F(z_0) \qquad [F'(z) = f(z)] \tag{4.5}$$

Example (4g)

$$S_n = \sum_{m=1}^{n} f(\zeta_m)(z_m - z_{m-1})$$

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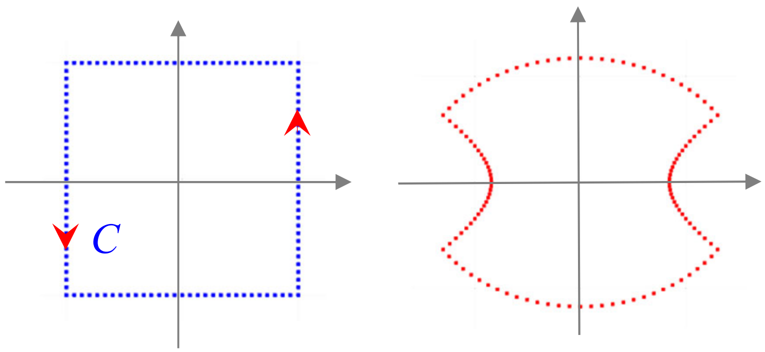
- Entire functions

$$\oint_C e^z dz = 0, \quad \oint_C cos z dz = 0, \quad \oint_C z^n dz = 0, \quad (n = 0, 1, \ldots)$$

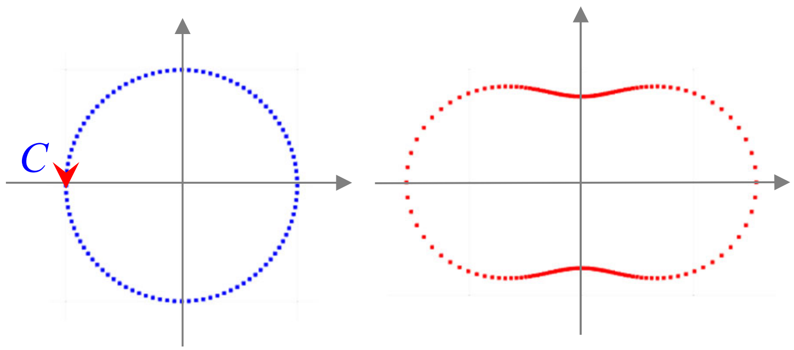
for any closed path, since these functions are entire (analytic for all z).

$$\oint_C \sin z \, dz = 0$$



z-plane w-plane

$$\oint_C \tan z \, dz = 0$$



z-plane w-plane

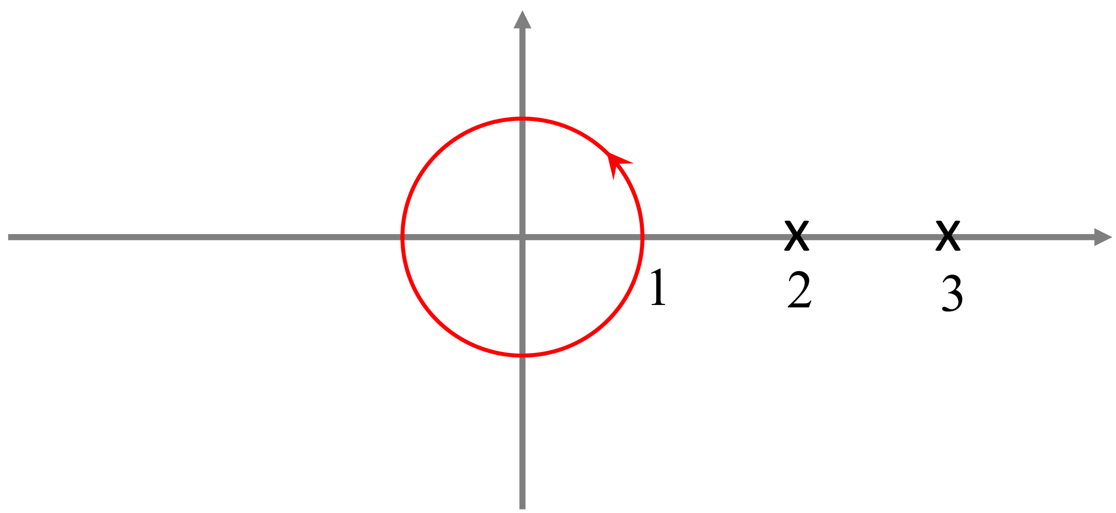
Example (4h)

- Consider

$$I = \oint_C \frac{dz}{z^2 - 5z + 6} = \oint_C \frac{dz}{(z-2)(z-3)},$$

where C is the unit circle $|z| = 1$ oriented in an anticlockwise direction.

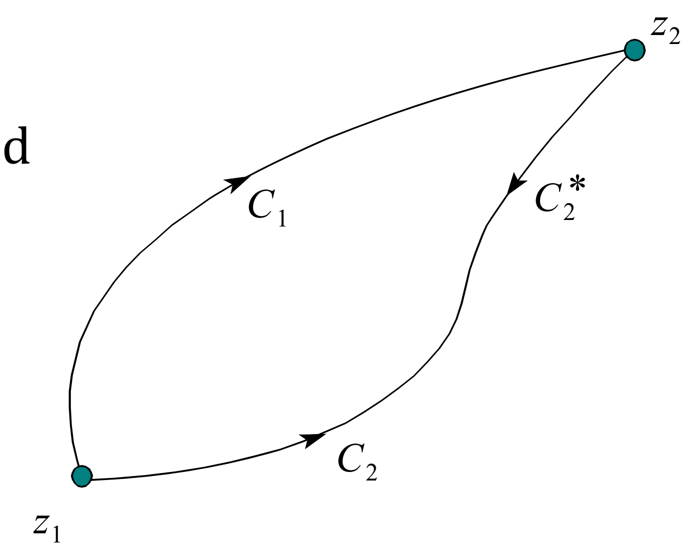
- Now, the integrand $f(z) = \frac{1}{(z-2)(z-3)}$ is analytic everywhere except at $z = 2$ and $z = 3$. Since the curve C does not enclose these two points, $I = 0$ by Cauchy's theorem (4.9).



Applications of Cauchy's Theorem

Applications:

1. If $f(z)$ is analytic in a simply connected domain D, then the integral of $f(z)$ is independent of path in D.



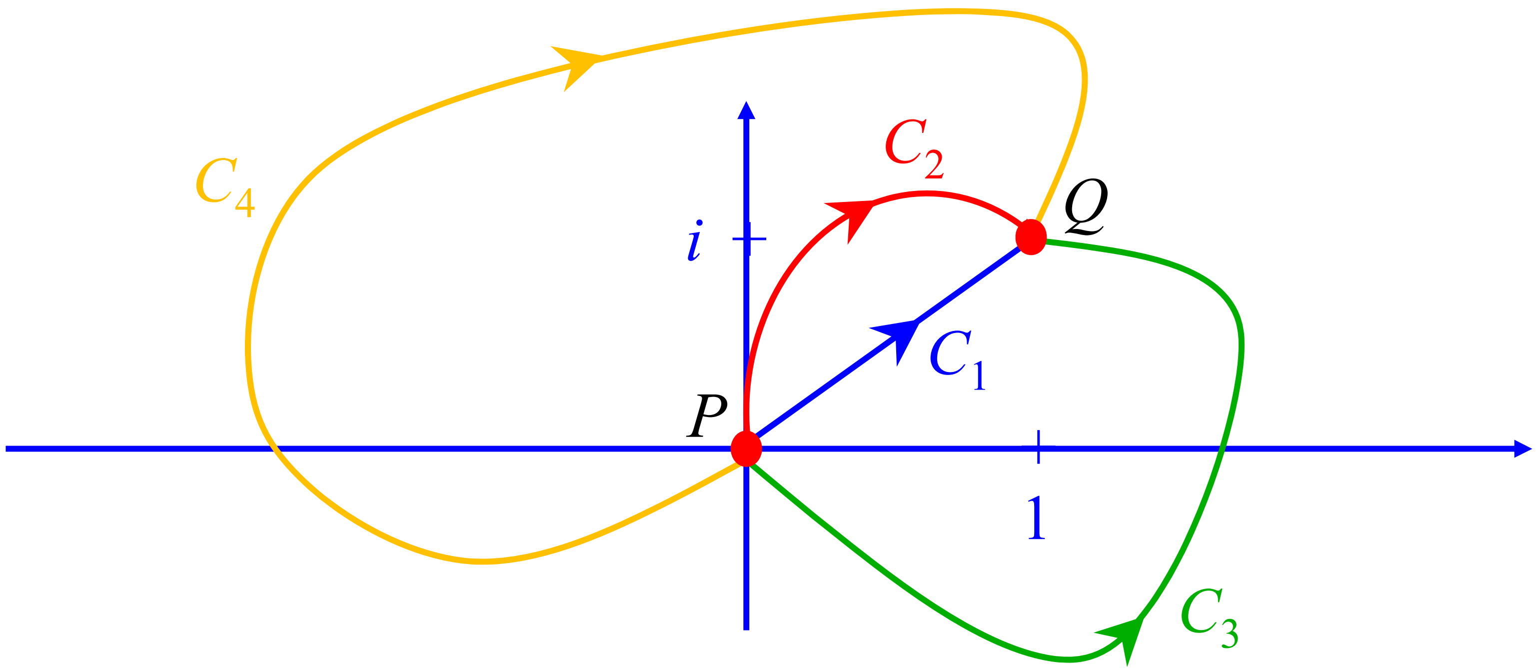
$$\int_{C_1} f(z)\,dz + \int_{C_2^*} f(z)\,dz = 0$$

$$\Rightarrow \quad \int_{C_1} f(z)\,dz = -\int_{C_2^*} f(z)\,dz = \int_{C_2} f(z)\,dz$$

Path Independence Theorem

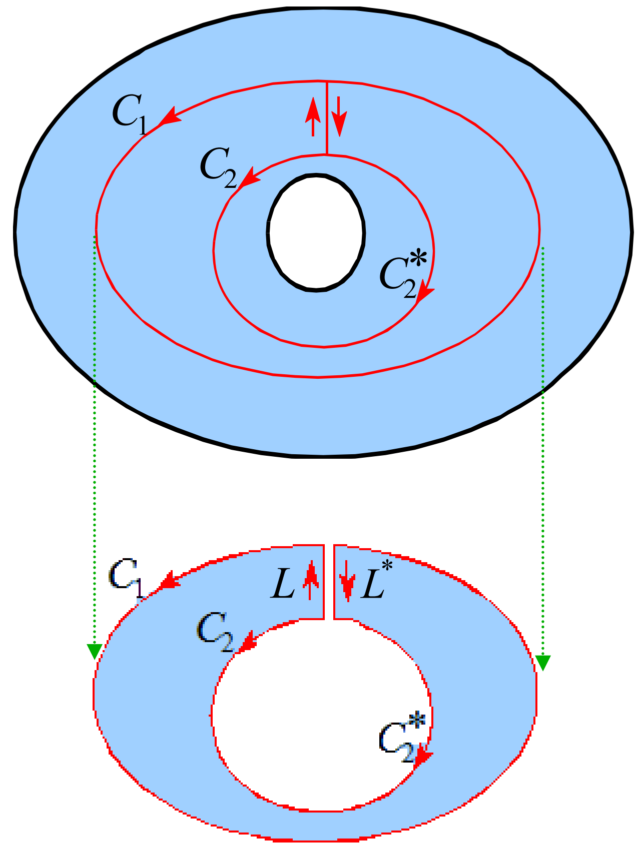
Theorem (Path Independence)

If $f(z)$ is analytic in a simply connected domain D, then $\int_C f(z)\,dz$ is independent of path in D. That is, given any initial point P in D and for any final point Q in D, the value of $\int_C f(z)\,dz$ is the same for every piecewise smooth path C, lying entirely within D, from P to Q.



Applications (cont.)

2. Consider a doubly connected domain D. If the function $f(z)$ is analytic in D, then the integral of $f(z)$ is the same around any closed path that encircles the opening.



$$\int_L f(z)\,dz + \int_{C_1} f(z)\,dz + \int_{L^*} f(z)\,dz + \int_{C_2^*} f(z)\,dz = 0$$

$$\Downarrow$$

$$\int_{C_1} f(z)\,dz + \int_{C_2^*} f(z)\,dz = 0$$

$$\Rightarrow \quad \int_{C_1} f(z)\,dz = -\int_{C_2^*} f(z)\,dz = \int_{C_2} f(z)\,dz$$

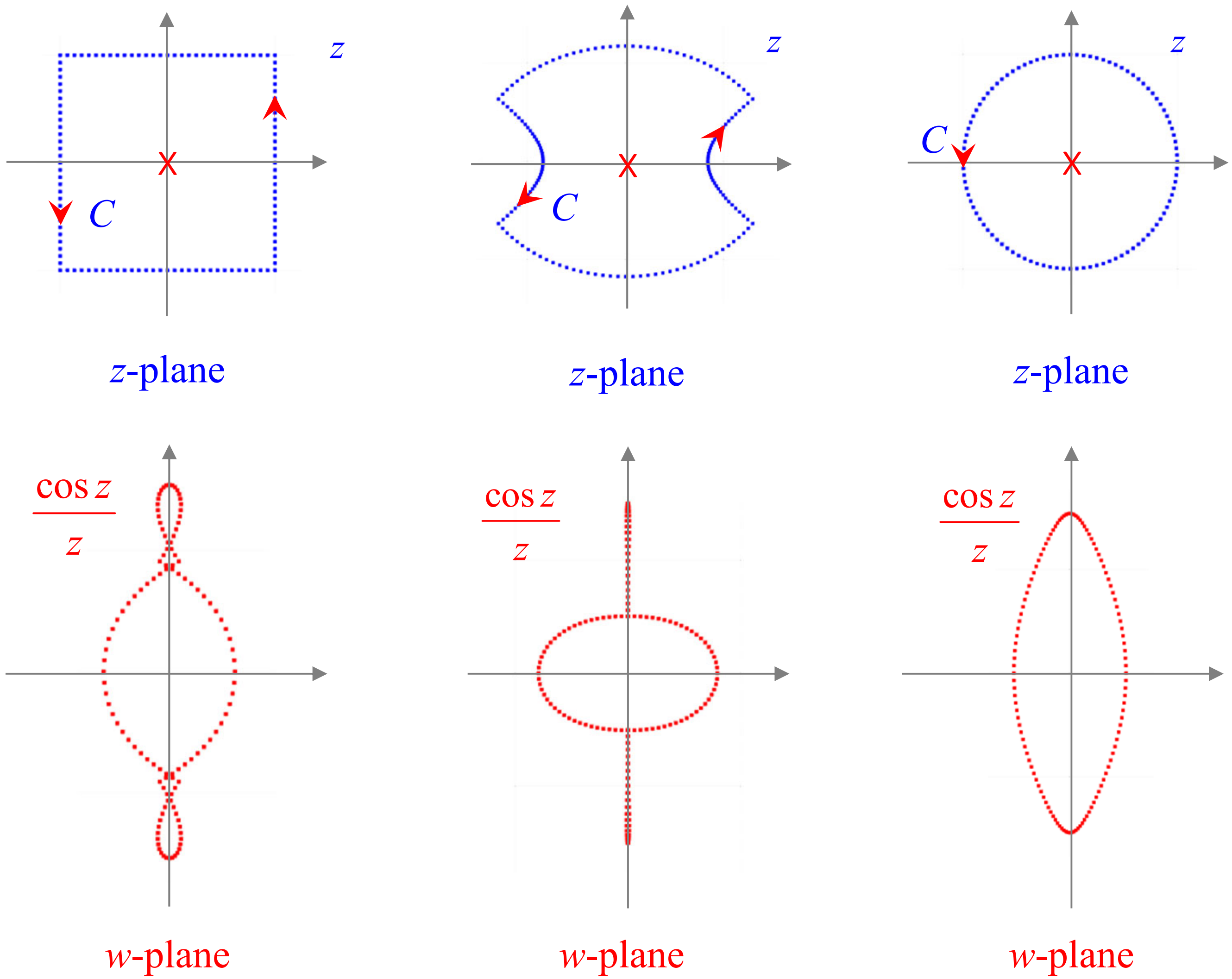
Note that we can always choose C_2 to be a circle…

Example…

$\oint_C f(z)\,dz$, with $f(z)=\dfrac{\cos z}{z}$ has a singular point at $z=0$.

$$\oint_C \frac{\cos z}{z}\,dz = 2\pi i$$

where C is any closed curve encircling the origin counter clockwise.



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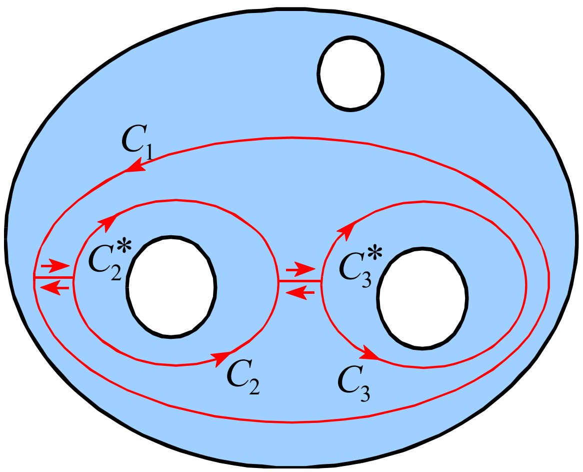
Applications (cont.)

3. The integral along a closed path C_1 of the function $f(z)$ which is analytic in the multiply connected domain D, is given by the sum of the integrals around paths which encircle all openings within the region bounded by C_1, e.g.

$$\int_{C_1} f(z)\,dz + \int_{C_2^*} f(z)\,dz + \int_{C_3^*} f(z)\,dz = 0$$

Thus, $$\int_{C_1} f(z)\,dz = -\int_{C_2^*} f(z)\,dz - \int_{C_3^*} f(z)\,dz = \int_{C_2} f(z)\,dz + \int_{C_3} f(z)\,dz$$



Note that we can always choose both C_2 and C_3 to be a circle…

Complex Analysis – 5…

1 Complex Numbers
2 Functions of One Complex Variable
3 Complex Differentiation
4 Complex Integration and Cauchy's Theorem
5 **Cauchy Integral Formula**
6 Complex Series, Power Series, Taylor Series, and Laurent Series
7 Residue Integration

Theorem (5.1 Cauchy Integral Formula)

Let $f(z)$ be analytic in a simply connected domain D. Then for any point z_0 in D and any simple closed path C in D that encloses z_0 (Fig.5.1)

$$\oint_C \frac{f(z)}{z-z_0}\,dz = 2\pi i\, f(z_0). \qquad (5.1)$$

Generalized Cauchy's integral formula…

$$\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz = \frac{2\pi i}{n!} f^{(n)}(z_0), \qquad (5.2)$$

Theorem (5.2 Derivatives of Analytic Function)

If $f(z)$ is analytic in a domain D, then it has derivatives of all orders in D, which are then also analytic functions in D. The values of these derivatives at a point z_0 in D are given by the formulas

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz$$

Recap…

Cauchy's Integral Theorem

Theorem (4.3 Cauchy's Integral Theorem)

If $f(z)$ is analytic in a simply connected domain D, then

$$\oint_C f(z)\,dz = 0 \tag{4.9}$$

for every simple closed path C in D.

Fig. 4.4. Cauchy's integral theorem.

Cauchy Integral Formula

Theorem (5.1 Cauchy Integral Formula)

Let $f(z)$ be analytic in a simply connected domain D. Then for any point z_0 in D and any simple closed path C in D that encloses z_0 (Fig.5.1)

$$\oint_C \frac{f(z)}{z - z_0} dz = 2\pi i\, f(z_0). \tag{5.1}$$

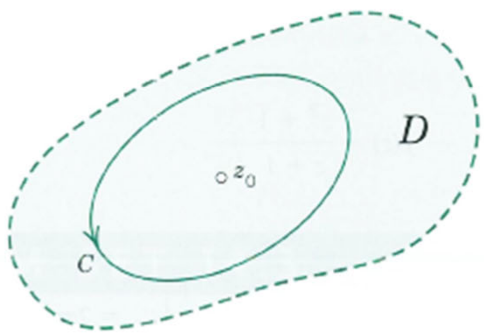


Fig. 5.1. Cauchy's integral formula.

Proof of Cauchy's Integral Formula*** ...

By addition and subtraction, $f(z) = f(z_0) + [f(z) - f(z_0)]$. Inserting this into (5.1) on the left and taking the constant factor $f(z_0)$ out from under the integral sign, we have

$$(2) \qquad \oint_C \frac{f(z)}{z - z_0}\, dz = f(z_0) \oint_C \frac{dz}{z - z_0} + \oint_C \frac{f(z) - f(z_0)}{z - z_0}\, dz.$$

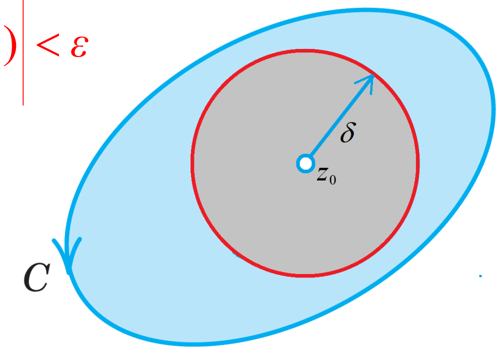
The first term on the right equals $f(z_0) \cdot 2\pi i$. Noting that

$$\frac{f(z) - f(z_0)}{z - z_0}$$

is analytic except at z_0, by Application 2 of Cauchy's Theorem, we can replace C by a small circle K of radius ρ (to be determined) and center at z_0 as follows:

Since $f(z)$ is analytic, it is differentiable. Hence, an $\epsilon > 0$ being given, we can find a $\delta > 0$ such that for all z in $0 < |z - z_0| < \delta$,

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \Rightarrow \left| \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \right| < \varepsilon$$



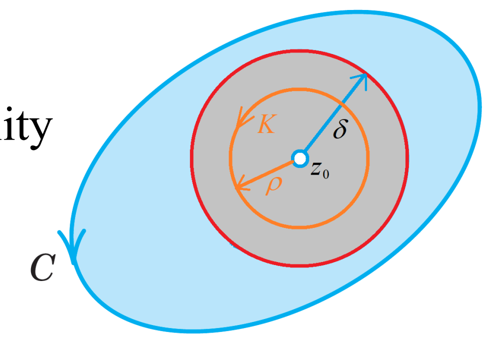
Thus, $$\left| \frac{f(z) - f(z_0)}{z - z_0} \right| = \left| \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) + f'(z_0) \right|$$

$$\le \left| \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \right| + |f'(z_0)| < \varepsilon + |f'(z_0)| = M$$

for all z in $0 < |z - z_0| < \delta$. Let us choose a ρ with $0 < \rho < \min\left\{ \delta, \varepsilon / M \right\}$. We have

$$\left| \frac{f(z) - f(z_0)}{z - z_0} \right| \le M < \frac{\varepsilon}{\rho} \qquad \left(\because \rho < \frac{\varepsilon}{M} \right)$$

at each point of K. The length of K is $2\pi\rho$. Hence, by the ML-inequality



$$\left| \oint_K \frac{f(z) - f(z_0)}{z - z_0} dz \right| < \frac{\epsilon}{\rho} 2\pi\rho = 2\pi\epsilon.$$

Since ϵ (> 0) can be chosen arbitrarily small, it follows that the last integral in (2) must have the value zero, and the theorem is proved. ■

Example (5)

- Evaluate

$$I = \oint_C \frac{e^z}{(z-2)(z+4)}\, dz,$$

where C is a counterclockwise circle of radius 3, centered at the origin.

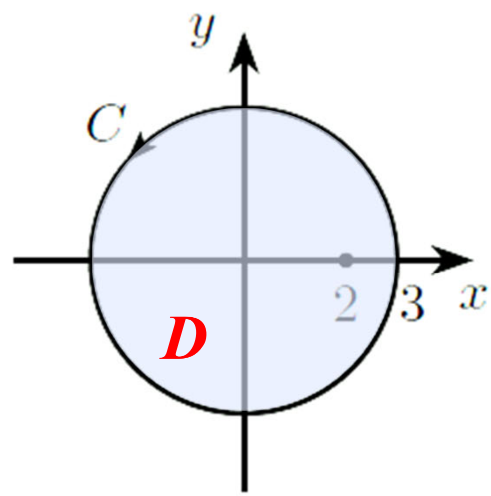


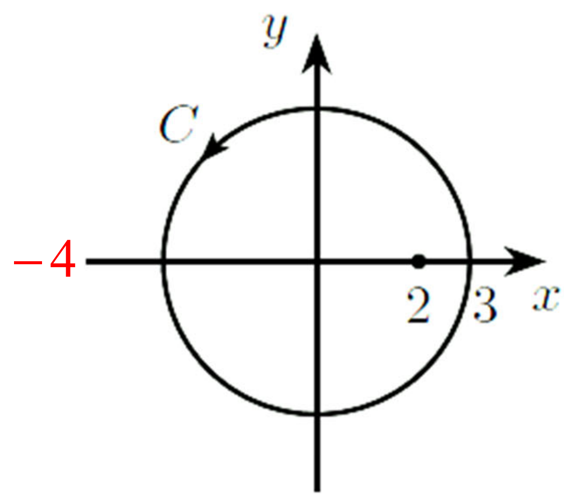
Fig. 5.3. The contour C.

Example (5)

- Evaluate

$$I = \oint_C \frac{e^z}{(z-2)(z+4)}\,dz,$$

where C is a counterclockwise circle of radius 3, centered at the origin.



- Let $f(z) = \dfrac{e^z}{z+4}$ and $z_0 = 2$, then $f(z)$ is analytic inside C. Hence, by the Cauchy integral formula (5.1),

$$I = \oint_C \frac{f(z)}{z - z_0}\,dz = 2\pi i\, f(z_0) = 2\pi i \left(\frac{e^2}{6}\right) = \frac{\pi e^2 i}{3}.$$

- The Cauchy integral formula enables us to evaluate any integral where the integrand has a "first order singularity" at some point $z = z_0$ within the contour C. If the singularity is second order or higher, then we have the **generalized Cauchy integral formula**

$$\boxed{\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\, dz = \frac{2\pi i}{n!} f^{(n)}(z_0),} \tag{5.2}$$

where $n = 0, 1, 2, \ldots$ if we have the same assumption as in Theorem 5.1.

$$\boxed{\oint_C \frac{f(z)}{z-z_0}\, dz = 2\pi i\, f(z_0).}$$

From the generalized Cauchy integral formula…

$$\boxed{\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz = \frac{2\pi i}{n!} f^{(n)}(z_0),} \qquad (5.2)$$

- **Remark:** Observe that having assumed only that $f(z)$ is analytic (once differentiable), one finds with no further assumption that $f(z)$ possesses derivatives of all orders:

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz.$$

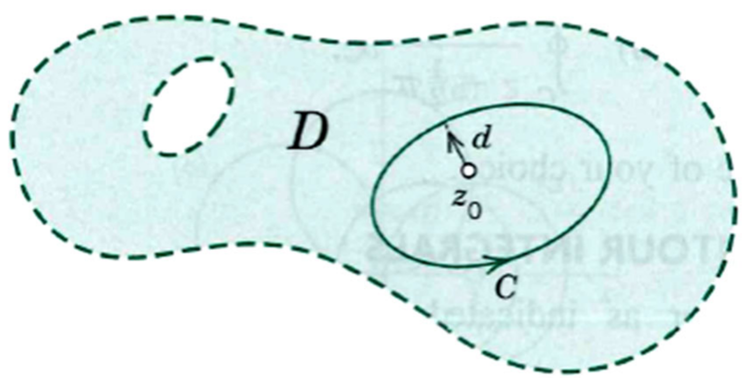
Derivatives of Analytic Functions

Theorem (5.2 Derivatives of Analytic Function)

If $f(z)$ is analytic in a domain D, then it has derivatives of all orders in D, which are then also analytic functions in D. The values of these derivatives at a point z_0 in D are given by the formulas

$$f'(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^2} \, dz.$$

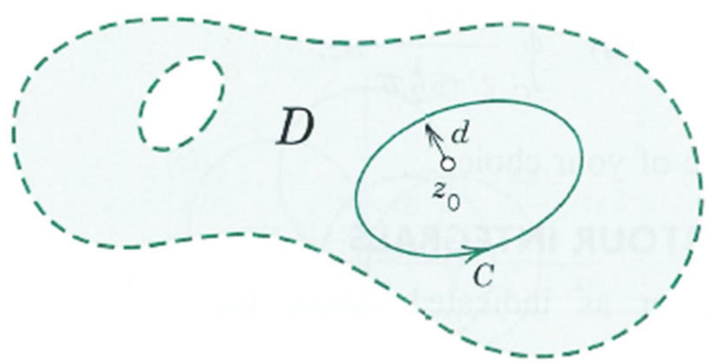
$$f''(z_0) = \frac{2!}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^3} \, dz.$$



or in general

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}} \, dz. \qquad (5.3)$$

Here C is any simple closed path in D that encloses z_0 and whose full interior belongs to D; and we integrate counterclockwise around C.



Example (5a)

- Evaluate

$$I = \oint_C \frac{e^z}{z^3}\, dz,$$

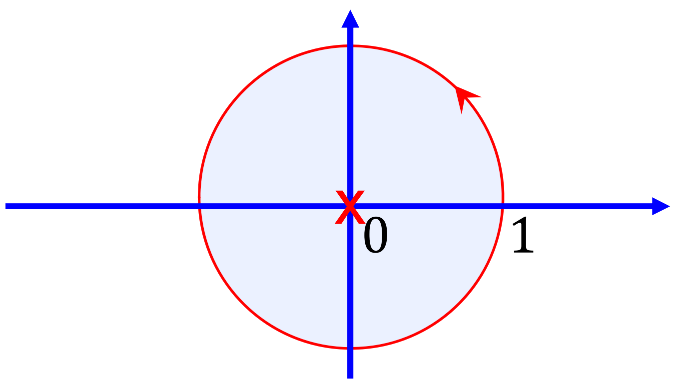
where C is the unit circle $|z| = 1$ oriented in an anticlockwise direction.

- Rewrite I as follows

$$I = \oint_C \frac{e^z}{(z-0)^3}\, dz$$

for comparison with the generalized Cauchy integral formula (5.2). It can be seen that $n = 2$, $z_0 = 0$, and $f(z) = e^z$ so (5.2) gives

$$I = \frac{2\pi i}{2!} \left(\frac{d^2}{dz^2} e^z \right) \Big|_{z=0} = \pi i.$$



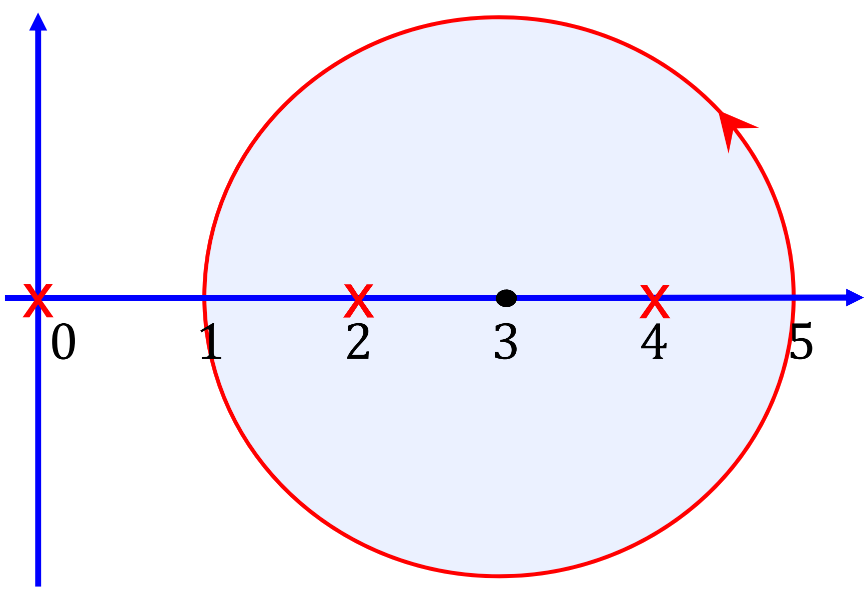
$$\boxed{\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\, dz = \frac{2\pi i}{n!} f^{(n)}(z_0),} \qquad (5.2)$$

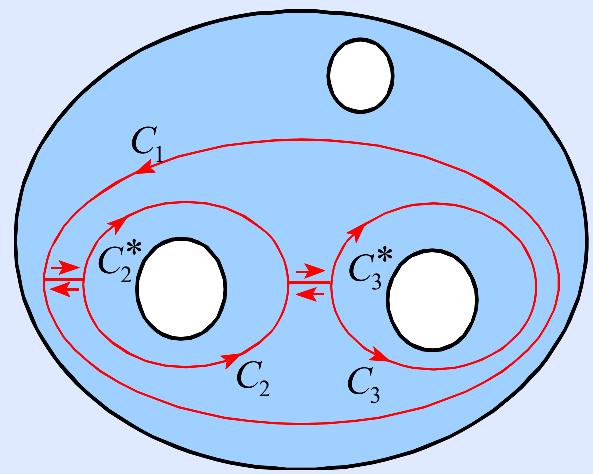
Example (5b)

- Evaluate

$$I = \oint_C \frac{z+1}{z(z-2)(z-4)^3}\, dz,$$

where C is the circle $|z-3| = 2$ oriented in an anticlockwise direction.



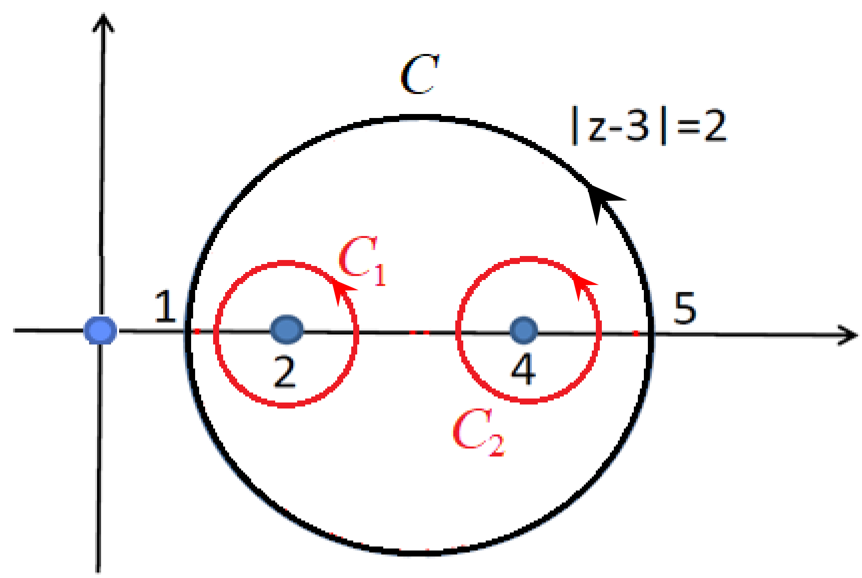


$$\oint_{C_1} f(z)dz = \oint_{C_2} f(z)dz + \oint_{C_3} f(z)dz$$

- The integrand has singularities at $z = 0, 2$ and 4, of which 2 and 4 fall within the contour C. If we deform C into two closed contours C_1 and C_2 so that 2 lies only within C_1 and 4 lies only within C_2, then the generalized Cauchy integral formula (5.2) gives

$$\begin{aligned} I &= \oint_{C_1} \left[\frac{z+1}{z(z-4)^3}\right] \frac{dz}{z-2} + \oint_{C_2} \left[\frac{z+1}{z(z-2)}\right] \frac{dz}{(z-4)^3} \\ &= 2\pi i \left[\frac{z+1}{z(z-4)^3}\right]\Bigg|_{z=2} + \frac{2\pi i}{2!}\frac{d^2}{dz^2}\left[\frac{z+1}{z(z-2)}\right]\Bigg|_{z=4} \\ &= -\frac{3\pi i}{8} + \frac{23\pi i}{64} \\ &= -\frac{\pi i}{64}. \end{aligned}$$

$$\boxed{\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz = \frac{2\pi i}{n!} f^{(n)}(z_0),}$$



Homework Assignment No: 3 (Due in one week)

Question 3.1: Find the values of $\int_C (x - y + i\,x^2)\,dz$, where $z = x + i\,y$ and C is the following:

(a) The straight line joining 0 to $1+i$

(b) The imaginary axis from 0 to i

(c) The line parallel to the real axis from i to $1+i$

Question 3.2: Let $C_1 :\ z(t) = 2 + 2e^{it},\ 0 \le t \le 2\pi$ and $C_2 :\ z(t) = i + e^{-it},\ 0 \le t \le \pi/2$

(a) Draw the path C_1 and C_2.

(b) Calculate (i) $\int_{C_1} \frac{dz}{z-2}$; and (ii) $\int_{C_2} \frac{dz}{(z-i)^3}$

Question 3.3: Let $C_1 :\ z(t) = -1 + \frac{1}{2}e^{it},\ 0 \le t \le 2\pi$ and $C_2 :\ z(t) = 1 + \frac{1}{2}e^{it},\ 0 \le t \le 2\pi$ and $C :\ z(t) = 2e^{it},\ 0 \le t \le 2\pi$. Also let $f(z) = \frac{1}{z^2 - 1}$. Use the Cauchy Integral Theorem to deduce that

$$\int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz$$

Question 3.4: Find the upper bound for the absolute value of $\int_C z\,dz$ where C is the half-circle, i.e.,

$z(t) = e^{it}, \quad t \in [0, \pi]$, as shown in Figure 1.

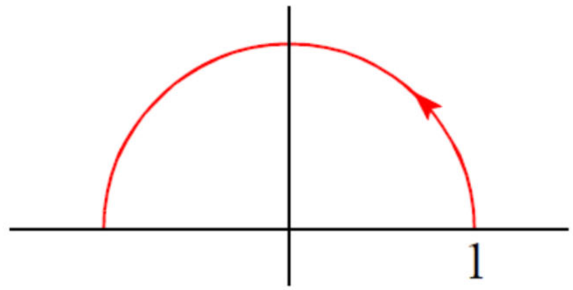


Figure 1: $z(t) = e^{it}, \qquad t \in [0, \pi]$

Question 3.5: Calculate the integral $\oint_{|z|=2} \frac{2z-1}{z^2 - z}\,dz$. Hint: You can try Cauchy's integral theorem here and note the poles inside. Draw a graph of the circle and indicate the poles.

Question 3.6: Calculate $\oint_{|z|=2} \frac{z^2+1}{z^2-1}\,dz$.

Question 3.7: Calculate $\oint_{|z|=2} \frac{1}{z^3(z+4)}\,dz$.

Connections to some engineering topics and problems… (***)

Laplace transform: Given a time domain function, $f(t)$, its Laplace transform is defined as follows:

$$F(s) = \mathcal{L}\{f\} = \int_0^{\infty} f(t) \cdot e^{-st} dt$$

where $s = \sigma + i\,\omega$ (with σ and ω being real) is a complex frequency parameter. $F(s)$ is regarded as a mapping of $f(t)$ in the frequency domain.

Inverse Laplace transform: On the other hand, one could transform a frequency domain function, $F(s)$, back to its time domain counterpart, $f(t)$, as the following:

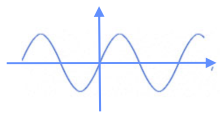
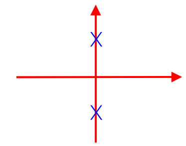
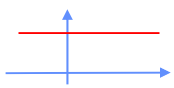
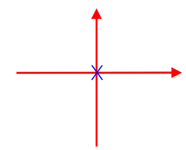
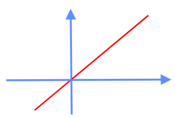
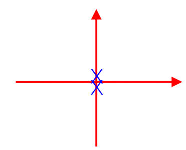
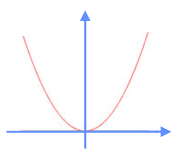
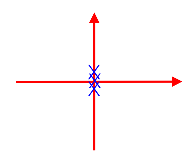
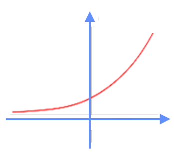
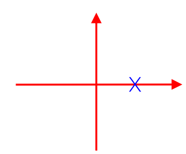
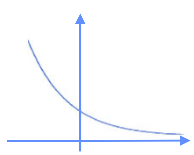
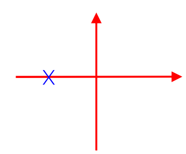
$$f(t) = \mathcal{L}^{-1}\{F\} = \frac{1}{2\pi i} \int_{\sigma_1 - i\infty}^{\sigma_1 + i\infty} F(s) \cdot e^{st} ds$$

where σ_1 is chosen such that the above integration exists.

Roughly, by setting $s = i\,\omega$, the Laplace transform becomes a so-called ***Fourier transform*** with ω being the real frequency parameter, *i.e.*, the term of frequency we use everyday.

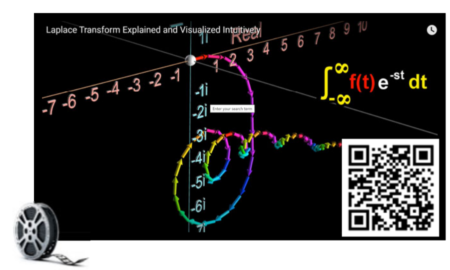
Many engineering problems become rather straightforward in the frequency domain!...

Some commonly used Laplace transform pairs… (***)

Waveforms *(vs time)*	$f(t)$	$\Leftrightarrow$	$F(s)$	*Singular points* *(z-plane)*
	$\sin \omega t$	$\Leftrightarrow$	$\dfrac{\omega}{s^2+\omega^2}$	
	$1(t)$	$\Leftrightarrow$	$\dfrac{1}{s}$	
	t	$\Leftrightarrow$	$\dfrac{1}{s^2}$	
	t^2	$\Leftrightarrow$	$\dfrac{2}{s^3}$	
	e^{at}	$\Leftrightarrow$	$\dfrac{1}{s-a}$	
		$a>0$		
	e^{-at}	$\Leftrightarrow$	$\dfrac{1}{s+a}$	

Pierre-Simon Laplace
(1749–1827)
French Scholar

Joseph Fourier
(1768–1830)
French Scholar

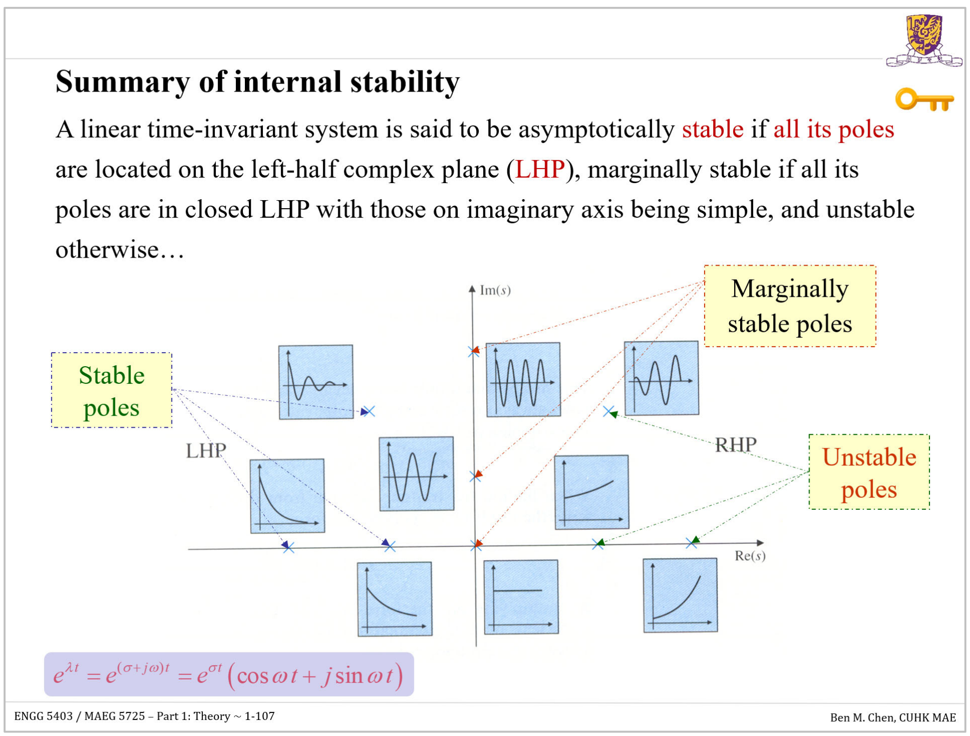


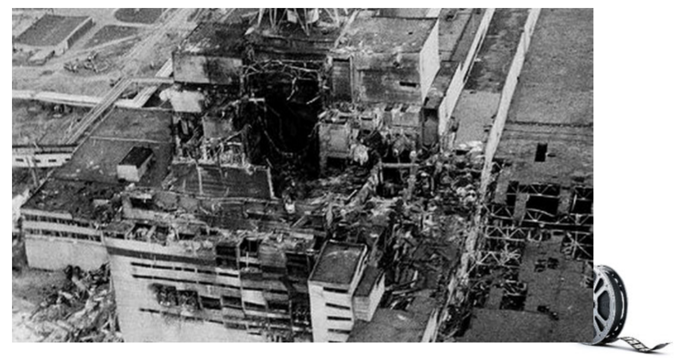
Some real engineering problems… (***)

Generally, a linear physical system can be represented by a complex rational function:

$$f(z)=\frac{b_1 z^{m-1}+\cdots+b_{m-1}z+b_m}{z^n+a_1 z^{n-1}+\cdots+a_{n-1}z+a_n}=\frac{b_1 z^{m-1}+\cdots+b_{m-1}z+b_m}{(z-\lambda_1)(z-\lambda_2)\cdots(z-\lambda_n)}$$

which is usually derived by using the Laplace transform. The stability of the system is then fully characterized by the singular points (or poles) of the complex function $f(z)$, which are the roots of the denominator, i.e., $z^n+a_1 z^{n-1}+\cdots+a_{n-1}z+a_n=0.$





Complex Analysis – 6…

1 Complex Numbers

2 Functions of One Complex Variable

3 Complex Differentiation

4 Complex Integration and Cauchy's Theorem

5 Cauchy Integral Formula

6 **Complex Series, Power Series, Taylor Series, Laurent Series**

7 Residue Integration

Material flow...

$z_1, z_2, z_3, \cdots$ ⇨ Sequence ~ *Convergence, divergence*

$$\sum_{n=0}^{\infty} z_n = z_0 + z_1 + z_2 + \cdots$$

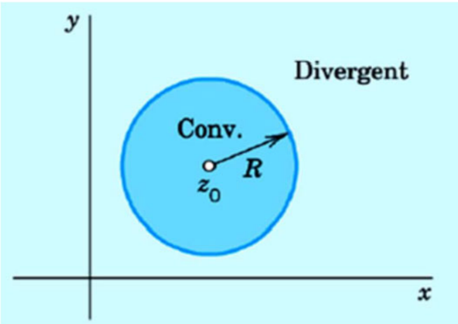
⇨ Series ~ *Convergence, divergence, absolute convergence, root and* ***ratio tests***

$$\lim_{n\to\infty}\left|\frac{z_{n+1}}{z_n}\right| = L \Rightarrow \begin{cases}\text{the series is absolutely convergent if } L<1 \\ \text{the series is divergent if } L>1\end{cases}$$

$$\sum_{n=0}^{\infty} a_n (z-z_0)^n$$

⇨ Power series ~ *Region of convergence, ratio tests,* ***radius of convergence***

$$R = \frac{1}{L^*} = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| \Rightarrow \begin{cases}\text{it is convergent for } |z-z_0|<R \\ \text{it is divergent for } |z-z_0|>R\end{cases}$$

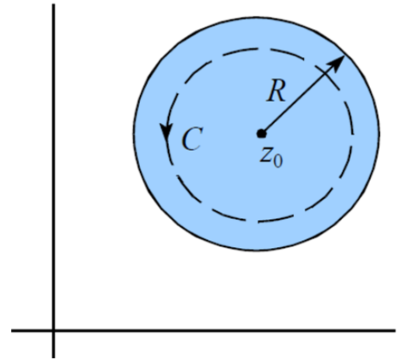


Material flow (cont.)...

$$f(z)=\sum_{n=0}^{\infty} a_n (z-z_0)^n$$

Taylor series expansion for analytic functions in a disc

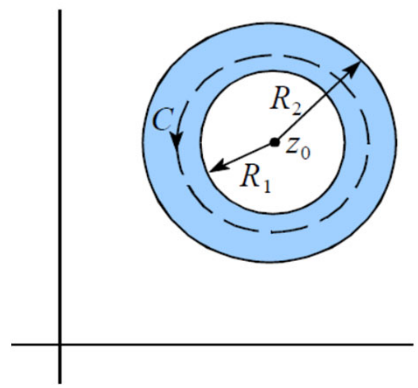
$$a_n=\frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz=\frac{f^{(n)}(z_0)}{n!}$$



$$f(z)=\sum_{n=-\infty}^{\infty} a_n (z-z_0)^n$$

Laurent series expansion for analytic functions in a ring

$$a_n=\frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz$$



Sequences

- A **sequence** is obtained by assigning to each positive integer n a number z_n, called a **term** of the sequence, and is written

$$z_1, z_2, \ldots, \quad or \quad \{z_1, z_2, \ldots\} \quad or \quad \{z_n\}$$

- A **real sequence** is one whose terms are real.

Examples…

$$1, 2, 3, \ldots, n, \ldots\ldots$$

and

$$2+i, (2+i)^2, (2+i)^3, \ldots, (2+i)^n, \ldots$$

- A **convergent sequence** $z_1, z_2, \ldots$ is one that has a limit c, written

$$\lim_{n \to \infty} z_n = c \quad \textit{or simply} \quad z_n \to c.$$

By definition of **limit**, this means that for every $\epsilon > 0$ we can find an N such that $|z_n - c| < \epsilon \quad \textit{for all } n > N$. Geometrically, all term z_n with $n > N$ lie in the open disk of radius ϵ and center c, and only finitely many terms do not lie in that disk.

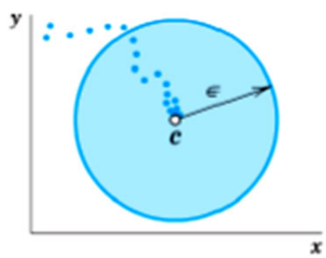


Fig. 6.1. Convergent complex sequence.

- A **divergent sequence** is one that does not converge.

Example (6a)

- The sequence

$$\{\frac{i^n}{n}\} = \{i, -\frac{1}{2}, -\frac{i}{3}, \frac{1}{4}, \ldots\}$$

is convergent with limit 0.

- The sequence

$$\{z_n\} = \{(1+i)^n\}$$

is divergent.

Theorem (6.1 Sequences of the Real and the Imaginary Parts)

A sequence $z_1, z_2, \ldots, z_n, \ldots$ *of complex numbers* $z_n = x_n + iy_n$ *(where* $n = 1, 2, \ldots$*) converges to* $c = a + ib$ *if and only if the sequence of the real parts* $x_1, x_2, \ldots$ *converges to* a *and the sequence of the imaginary parts* $y_1, y_2, \ldots$ *converges to* b.

Series

- Given a sequence $z_1, z_2, \ldots, z_m, \ldots$, we may form the sequence of the sums

$$s_1 = z_1, \quad s_2 = z_1 + z_2, \quad s_3 = z_1 + z_2 + z_3, \ldots$$

and in general

$$s_n = z_1 + z_2 + \ldots + z_n \qquad (n = 1, 2, \ldots).$$

s_n is called the **nth partial sum** of the **series**

$$\sum_{m=1}^{\infty} z_m = z_1 + z_2 + \ldots.$$

The $z_1, z_2, \ldots$ are called the **terms** of the series.

- A **convergent series** is one whose sequence of partial sums converges, i.e.,

$$\lim_{n\to\infty} s_n = s. \quad \textit{Then we write} \quad s = \sum_{m=1}^{\infty} z_m = z_1 + z_2 + \dots.$$

  and call s the **sum** of the series. A series that is not convergent is called **divergent series**.

Theorem (6.2 Real and the Imaginary Parts)

A series $\sum_{m=1}^{\infty} z_m = z_1 + z_2 + \dots.$ *with* $z_m = x_m + iy_m$ *converges and has the sum* $s = u + iv$ *if and only if* $x_1 + x_2 + \dots$ *converges and has the sum* u *and* $y_1 + y_2 + \dots$ *converges and has the sum* v.

Theorem (6.3 Divergence)

A necessary condition…

If a series $z_1 + z_2 + \ldots.$ converges, then $\lim_{m\to\infty} z_m = 0$. Hence if this does not hold, the series diverges.

- For a simple test, the series $z_1 + z_2 + \ldots. + z_n + \ldots$ converges only if $z_n \to 0$ as $n \to \infty$. On the other hand, if a complex series does not converge, it **diverges**.

Example (6b)

- Determine the convergence or divergence of the series

$$\sum_{n=0}^{\infty} \left(\frac{3+n}{4+n}\right)^{100}.$$

- Since $\left(\frac{3+n}{4+n}\right)^{100} = \left(\frac{\frac{3}{n}+1}{\frac{4}{n}+1}\right)^{100} \to 1$ as $n \to \infty$, the series diverges.

Just for fun... Consider the following series,

$$\sum_{n=1}^{\infty} n = 1+2+3+4+5+6+7+8+9+10+\cdots$$

which is obviously divergent. Assuming $S = \sum_{n=1}^{\infty} n$, we have

$$\begin{aligned} S &= 1+2+3+4+5+6+7+8+9+10+\cdots \\ &= 1+\underbrace{(2+3+4)}+\underbrace{(5+6+7)}+\underbrace{(8+9+10)}+\cdots \\ &= 1 + 9 + 18 + 27 + \cdots \\ &= 1+9\times(1+2+3+4+5+6+\cdots) \\ &= 1+9\cdot S \end{aligned}$$

We then have $S = 1+9\cdot S \;\Rightarrow\; S = -\frac{1}{8}$, i.e.,

$$S = 1+2+3+4+\cdots = -\frac{1}{8} \qquad \text{(what happens?)}$$

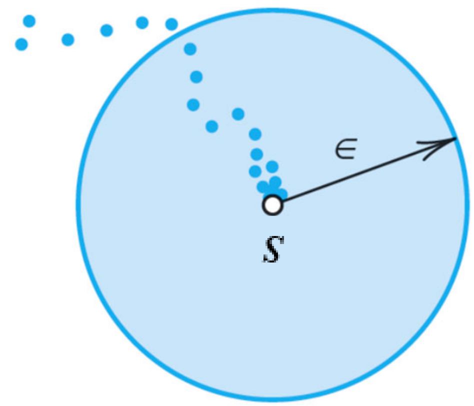
Theorem (6.4 Cauchy's Convergence Principle for Series)

A series $z_1 + z_2 + \dots.$ *is convergent if and only if for every given* $\epsilon > 0$ *(no matter how small) we can find an* N *(which depends on* ϵ*, in general) such that*

$$|z_{n+1} + z_{n+2} + \dots + z_{n+p}| < \epsilon \qquad \textit{for every } n > N \textit{ and } p = 1, 2, \dots$$

$$z_1 + z_2 + \cdots + z_N \;\; + z_{N+1} + z_{N+2} + \cdots$$

the tail of the series ➡ $\left| z_{N+1} + z_{N+2} + \cdots \right| < \varepsilon$



Example: Consider the following series,

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \ln 2 = 0.6931$$

which is known as the alternating harmonic series.

series_tail.m

For $\varepsilon = 0.01$, we have $N = 50$, i.e., for $N > 50$,

$$\left| z_{N+1} + z_{N+2} + \cdots \right| < 0.01.$$

- **Absolute Convergence**: A series $z_1 + z_2 + \ldots.$ is called **Absolutely Convergent** if the series of the absolute values of the terms

$$\sum_{m=1}^{\infty} |z_m| = |z_1| + |z_2| + \ldots$$

is convergent.

If $z_1 + z_2 + \ldots.$ converges but $|z_1| + |z_2| + \ldots$ diverges, then the series $z_1 + z_2 + \ldots.$ is called **Conditionally Convergent**.

Example (6c) (A conditionally Convergent Series)

The alternating harmonic series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + - \ldots \qquad = \ln 2$$

converges, but only conditionally since the harmonic series diverges.

Harmonic series… $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots.$

Theorem (6.5 Comparison Test)

If a series $z_1 + z_2 + \ldots.$ is given and we can find a convergent series $b_1 + b_2 + \ldots.$ which nonnegative real terms such that $|z_1| \le b_1, |z_2| \le b_2, \ldots,$ then the given series converges, even absolutely.

- A good comparison series is the geometric series, which behaves as follows.

Theorem (6.6 Geometric Series) !

The geometric series

$$\sum_{m=0}^{\infty} q^m = 1 + q + q^2 + \ldots$$

converges with the sum $\frac{1}{1-q}$ if $|q| < 1$ and diverges if $|q| \ge 1$.

Example (6d)

- Determine the convergence or divergence of the series

$$\sum_{n=0}^{\infty} \frac{i^n}{n!} = 1 + i + \frac{i^2}{2!} + \frac{i^3}{3!} + \cdots .$$

- Now, $\left|\frac{i^n}{n!}\right| = \frac{1}{n!} < \frac{1}{2^n}$ for all $n \geq 4$, and $\sum_{n=0}^{\infty} \frac{1}{2^n} = \frac{1}{1-\frac{1}{2}} = 2$ is a convergent geometric series. By comparison test, the original series converges.

$$n! = \underbrace{1\cdot 2\cdot 3\cdot 4}_{24}\cdot 5\cdots n > \underbrace{2\cdot 2\cdot 2\cdot 2}_{16}\cdot 2\cdots 2 = 2^n \quad \Rightarrow \quad \frac{1}{n!} < \frac{1}{2^n}, \quad n \geq 4$$

The geometric series

$$\sum_{m=0}^{\infty} q^m = 1 + q + q^2 + \ldots$$

converges with the sum $\frac{1}{1-q}$ *if* $|q| < 1$ *and diverges if* $|q| \geq 1$.

Theorem (6.7 Ratio Test)

If a series $z_1 + z_2 + \ldots$ *with* $z_n \neq 0 \quad (n = 1, 2, \ldots)$ *has the property that for every* n *greater than some* N,

$$\left|\frac{z_{n+1}}{z_n}\right| \leq q < 1 \qquad (n > N) \tag{6.1}$$

(where $q < 1$ *is fixed), this series converges absolutely. If for every* $n > N$,

$$\left|\frac{z_{n+1}}{z_n}\right| \geq 1 \qquad (n > N) \tag{6.2}$$

the series diverges.

- The inequality Eq.(6.1) implies $\left|\frac{z_{n+1}}{z_n}\right| < 1$, but this does **not** imply convergence, as we see from the harmonic series, which satisfies $\frac{z_{n+1}}{z_n} = \frac{n}{n+1} < 1$ for all n but diverges.

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$$ *the harmonic series…*

Theorem (6.8 Ratio Test)

If a series $z_1 + z_2 + \ldots.$ *with* $z_n \neq 0 \quad (n = 1, 2, \ldots)$ *is such that* $\lim_{n\to\infty} \left| \frac{z_{n+1}}{z_n} \right| = L$*, then the series converges absolutely if* $L < 1$ *and diverges if* $L > 1$. *No information is obtained if* $L = 1$ *or if the limit does not exist.*

Example (6e)

- Determine the convergence or divergence of the series

$$\sum_{n=0}^{\infty} \frac{(1+i)^n}{n!}.$$

- Since

$$\lim_{n\to\infty} \left| \frac{z_{n+1}}{z_n} \right| = \lim_{n\to\infty} \left| \frac{\frac{(1+i)^{n+1}}{(n+1)!}}{\frac{(1+i)^n}{n!}} \right| = \lim_{n\to\infty} \left| \frac{1+i}{n+1} \right| = \sqrt{2} \lim_{n\to\infty} \left| \frac{1}{n+1} \right| = 0,$$

$L = 0$. By ratio test, the series converges.

Root Test 1

If a series $z_1 + z_2 + \cdots$ *is such that for every n greater than some N,*

$$\sqrt[n]{|z_n|} \leqq q < 1 \qquad (n > N)$$

(*where* $q < 1$ *is fixed*), *this series converges absolutely. If for infinitely many n,*

$$\sqrt[n]{|z_n|} \geqq 1,$$

the series diverges.

Root Test 2

If a series $z_1 + z_2 + \cdots$ *is such that* $\lim_{n\to\infty} \sqrt[n]{|z_n|} = L$, *then:*

(a) *The series converges absolutely if* $L < 1$.

(b) *The series diverges if* $L > 1$.

(c) *If* $L = 1$, *the test fails; that is, no conclusion is possible.*

Power Series

- Generally, the terms in a series may be some functions of z, then the series becomes

$$\sum_{n=0}^{\infty} f_n(z) = f_0(z) + f_1(z) + \cdots .$$

The set of all points in the z-plane for which the series converges is called the **region of convergence** of the series.

- If we let $f_n(z) = a_n(z - z_0)^n$, where a_n's and z_0 are some complex (or real) constants in general, then the resulting series

$$\boxed{\sum_{n=0}^{\infty} a_n(z - z_0)^n = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots ,} \qquad (6.5)$$

is called a **power series** (about the point $z = z_0$). a_n's are called coefficients of the power series (6.5).

Convergence Behavior of Power Series

- Power series have variable terms (functions of z), but if we fix z, then all the concepts for series with constant terms in the last section apply.
- A series with variable terms will converge for some z and diverge for others.
- For a power series, e.g., (6.5), it may converge in a disk with center z_0 or in the whole z-plane or only at z_0.

Example (6g)

The geometric series

$$\sum_{n=0}^{\infty} z^n = 1 + z + z^2 + \dots$$

converges absolutely if $|z| < 1$ and diverges if $|z| \geq 1$.

$$\lim_{n\to\infty}\left|\frac{z_{n+1}}{z_n}\right| = \lim_{n\to\infty}\left|\frac{z^{n+1}}{z^n}\right| = |z|$$

Example (6h)

The power series

$$\sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots$$

is absolutely convergent for every z. (Check by using the ratio test)

$$\lim_{n\to\infty}\left|\frac{z_{n+1}}{z_n}\right| = \lim_{n\to\infty}\left|\frac{\dfrac{z^{n+1}}{(n+1)!}}{\dfrac{z^n}{n!}}\right| = \lim_{n\to\infty}\left|\frac{z}{n+1}\right| = 0$$

Theorem (6.11 Convergence of a Power Series)

(a) *Every power series (6.5) converges at the center* z_0.

(b) *If (6.5) converges at a point* $z = z_1 \neq z_0$, *it converges absolutely for every* z *closer to* z_0 *than* z_1, *that is,* $|z - z_0| < |z_1 - z_0|$.

(c) *If (6.5) diverges at* $z = z_2$, *it diverges for every* z *fartheraway from* z_0 *than* z_2.

$$\sum_{n=0}^{\infty} a_n(z-z_0)^n = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots \qquad (6.5)$$

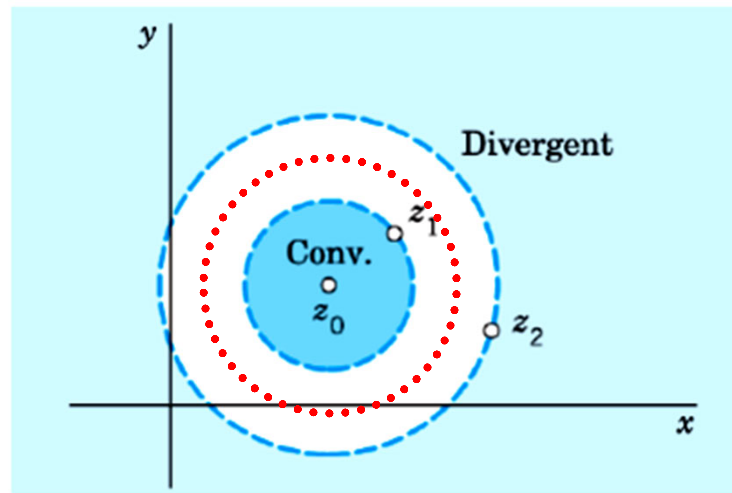


Fig. 6.2. Theorem 6.11.

Radius of Convergence of a Power Series

$$\sum_{n=0}^{\infty} a_n(z-z_0)^n = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots \qquad (6.5)$$

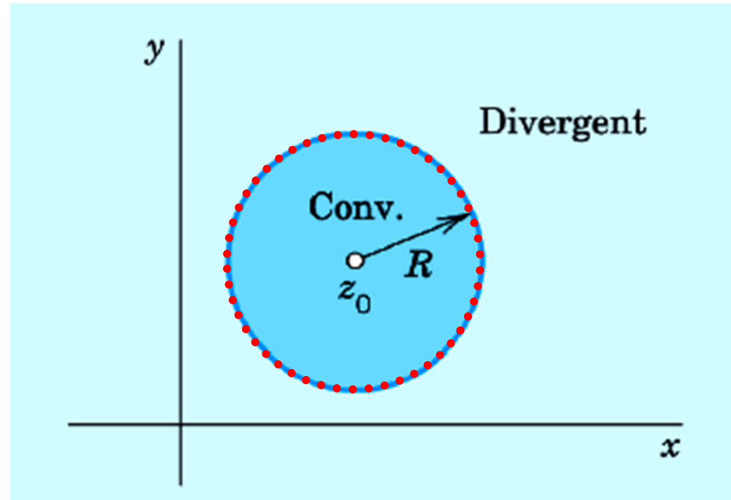


Fig. 6.3. Circle of convergence

Let R denote its radius, the circle,

$$|z - z_0| = R$$

is called the **circle of convergence** and its radius R the **radius of convergence** of (6.5), if power series (6.5) is convergent for all z inside the circle and divergent for all z outside.

Recap: the power series (6.5) is convergent for all z for which

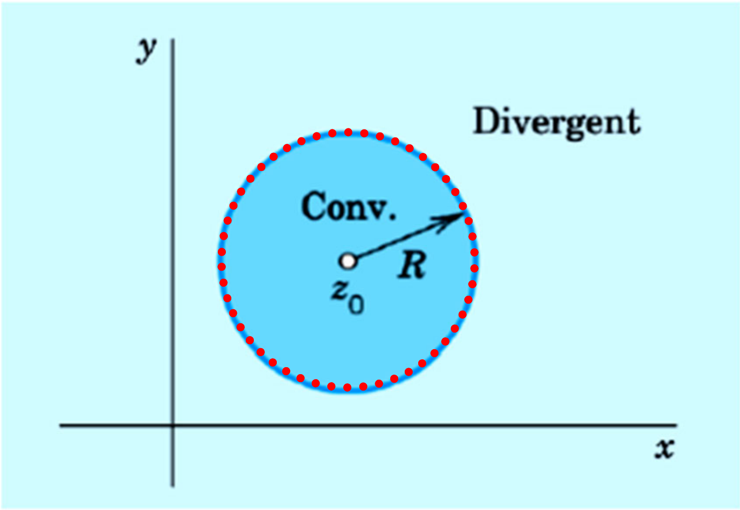
$$|z - z_0| < R.$$

It is divergent for all z for which

$$|z - z_0| > R.$$

No conclusion can be made about the convergence of the power series (6.5) on the circle of convergence.

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots \qquad (6.5)$$



Example (6i)

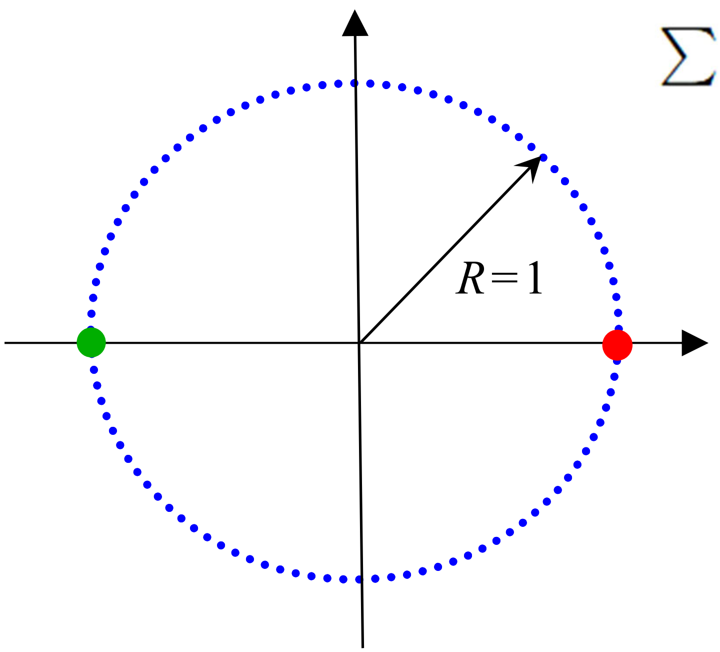
On the circle of convergence (radius $R = 1$ in all three series),

(a) $\sum \frac{z^n}{n^2}$ converges everywhere since $\sum \frac{1}{n^2}$ converges.

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

(b) $\sum z^n$ diverges everywhere.

(c) $\sum \frac{z^n}{n}$ converges at -1 but diverges at 1.



$\sum \frac{z^n}{n}$ with $z = -1$ gives an alternating harmonic series

$$-\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + - \dots\right) = -\ln 2$$

$\sum \frac{z^n}{n}$ with $z = 1$ gives a harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$$

Demonstration of $\sum \frac{z^n}{n}$ with $\left(z_1 = 1.01i,\ z_2 = 0.996 - 0.1677i,\ z_3 = -1 + 0.1i,\ z_4 = -1.01\frac{1+3i}{\sqrt{10}} \right)$

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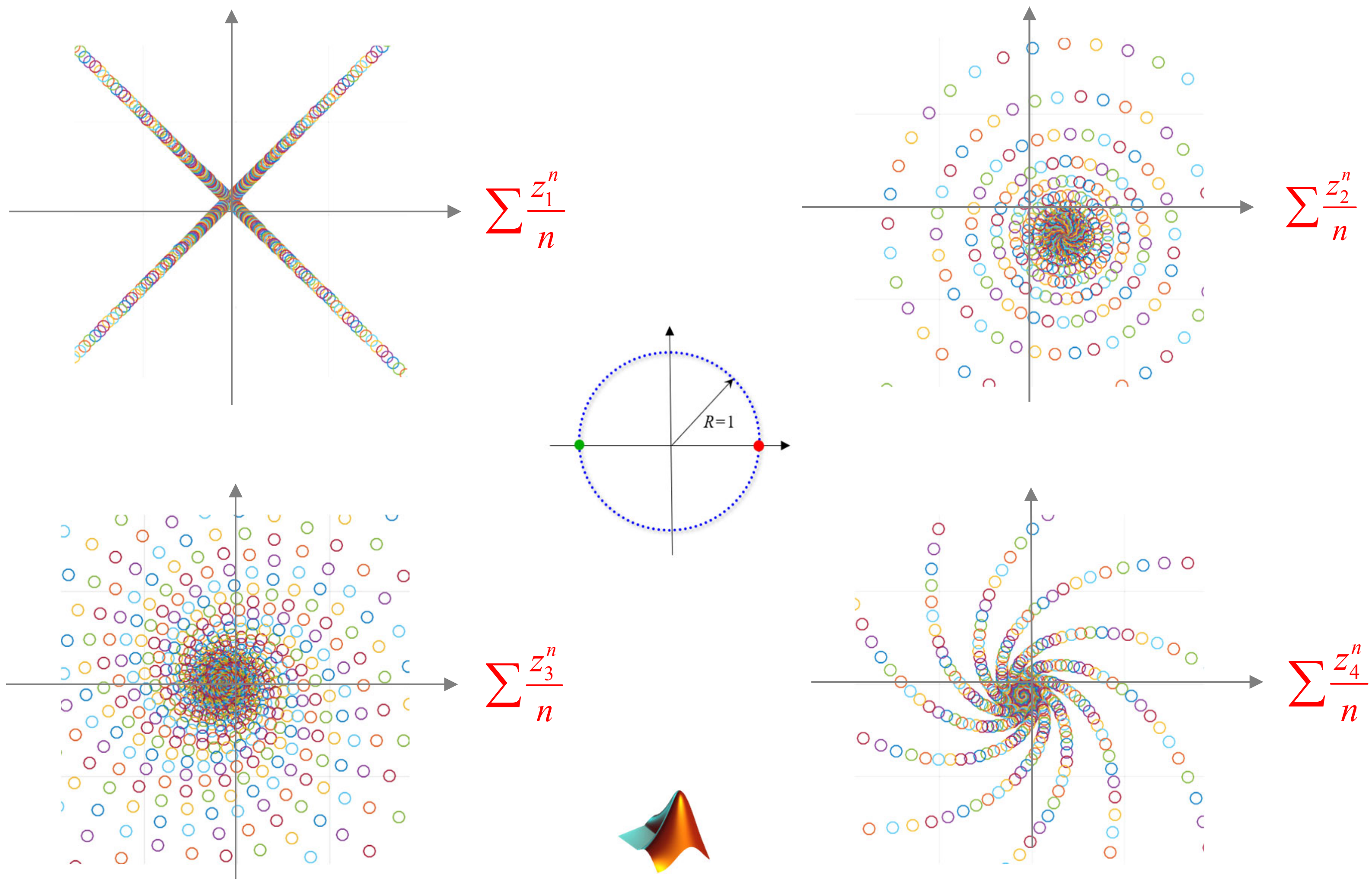

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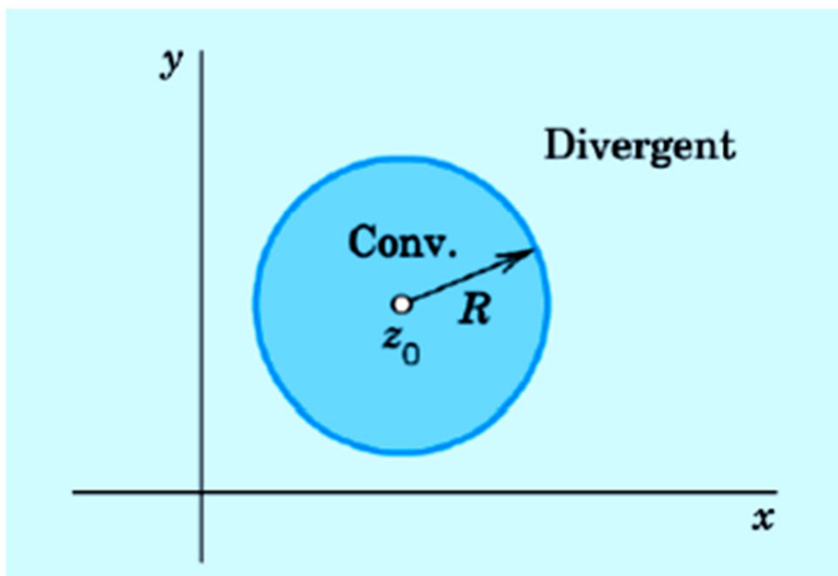
Determination of the Radius of Convergence from the Coefficients

Notations $R = \infty \; and \; R = 0$:

(a) $R = \infty$ if the series (6.5) converges for all z,

(b) $R = 0$ if (6.5) converges only at the center $z = z_0$.



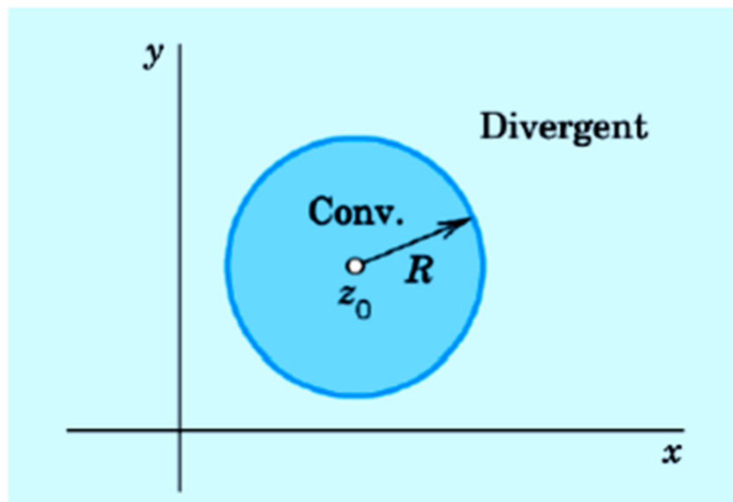
Theorem (6.12 Radius of Convergence R)

Suppose that the sequence $\left|\frac{a_{n+1}}{a_n}\right|$, $n = 1, 2, \ldots$, *converges with limit* L^*. *If* $L^* = 0$, *then* $R = \infty$; *that is, the power series (6.5) converges for all* z. *If* $L^* \neq 0$ $(hence\ L^* > 0)$, *then*

$$\boxed{R = \frac{1}{L^*} = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|} \tag{6.8}$$

If $\left|\frac{a_{n+1}}{a_n}\right| \to \infty$, *then,* $R = 0$ *(convergence only at the center* z_0*).*

$$\sum_{n=0}^{\infty} a_n(z-z_0)^n = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots \tag{6.5}$$



$$\sum_{n=0}^{\infty} a_n(z-z_0)^n = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots \qquad (6.5)$$

By Theorem 6.8 (ratio test), the above series converges when

$$\lim_{n\to\infty}\left|\frac{z_{n+1}}{z_n}\right| = \lim_{n\to\infty}\left|\frac{a_{n+1}(z-z_0)^{n+1}}{a_n(z-z_0)^n}\right| = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|\cdot|z-z_0| = L^*\cdot|z-z_0| < 1$$

$$\Downarrow$$

$$|z-z_0| < \frac{1}{L^*} = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = R$$

Theorem (6.8 Ratio Test)

If a series $z_1 + z_2 + \ldots.$ *with* $z_n \neq 0 \quad (n = 1, 2, \ldots)$ *is such that* $\lim_{n\to\infty}\left|\frac{z_{n+1}}{z_n}\right| = L$, *then the series converges absolutely if* $L < 1$ *and diverges if* $L > 1$. *No information is obtained if* $L = 1$ *or if the limit does not exist.*

Example (6j) (Radius of Convergence)

By Eq.(6.8), the radius of convergence of the power series $\sum_{n=0}^{\infty} \frac{(2n)!}{(n!)^2}(z-3i)^n$ is

$$R = \lim_{n\to\infty}\left[\frac{\frac{(2n!)}{(n!)^2}}{\frac{(2n+2)!}{((n+1)!)^2}}\right] = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$$

$$= \lim_{n\to\infty}\left[\frac{(2n!)}{(2n+2)!}\cdot\frac{((n+1)!)^2}{(n!)^2}\right] = \lim_{n\to\infty}\frac{(2n)!}{(2n)!(2n+1)(2n+2)}\cdot\frac{(n!)^2(n+1)^2}{(n!)^2}$$

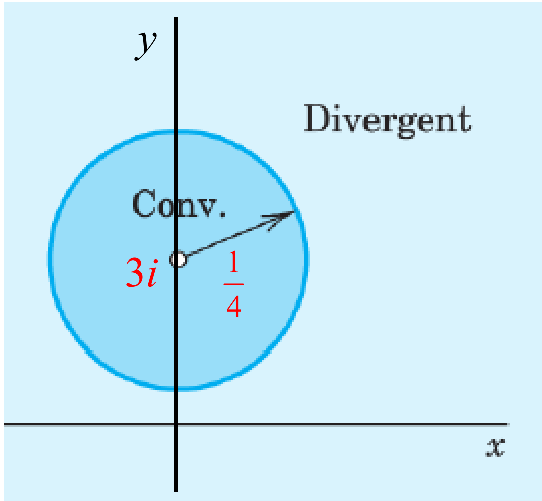
$$= \lim_{n\to\infty}\frac{(n+1)^2}{(2n+2)(2n+1)}$$

$$= \frac{1}{4}.$$

The series converges in the open disk

$|z-3i| < \frac{1}{4}$ of radius $\frac{1}{4}$

and center $3i$.



Functions Given by Power Series

- To simplify the formulas, we take $z_0 = 0$, and write Eq. (6.5) as

$$\boxed{\sum_{n=0}^{\infty} a_n z^n} \tag{6.9}$$

- If any given power series (6.9) has a nonzero radius of convergence R (thus $R > 0$), its sum is a function of z, say $f(z)$. Then we write

$$\boxed{f(z) = \sum_{n=0}^{\infty} a_n z^n = a_0 + a_1 z + a_2 z^2 + \dots} \tag{6.10}$$

we say that $f(z)$ is represented by the power series or that it is developed in the power series.

Uniqueness of a Power Series Representation

Theorem (6.13 Uniqueness of Power Series)

Let the power series $a_0 + a_1 z + a_2 z^2 + \dots$ *and* $b_0 + b_1 z + b_2 z^2 + \dots$ *both be convergent for* $|z| < R$, *where* R *is positive, and let them both have the same sum for all these* z, *Then the series are identical, that is,* $a_0 = b_0, a_1 = b_1, a_2 = b_2, \dots$. *Hence if a function* $f(z)$ *can be represented by a power series with any center* z_0, *this representation is* ***unique***.

- If a_n, b_n are coefficients of two power series and $a_n = b_n$, then it is sure that

$$\sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} b_n z^n, \tag{6.11}$$

i.e., the two power series are the same about the point $z = 0$. $\Rightarrow$

A function $f(z)$ cannot be represented by two different power series with the same center. That is, if $f(z)$ can at all be developed in a power series with center z_0, the development is unique.

Power Series Represent Analytic Functions

Theorem (6.15 Analytic Functions. Their Derivatives)

A power series with a nonzero radius of convergence R represents an analytic function at every point interior to its circle of convergence.

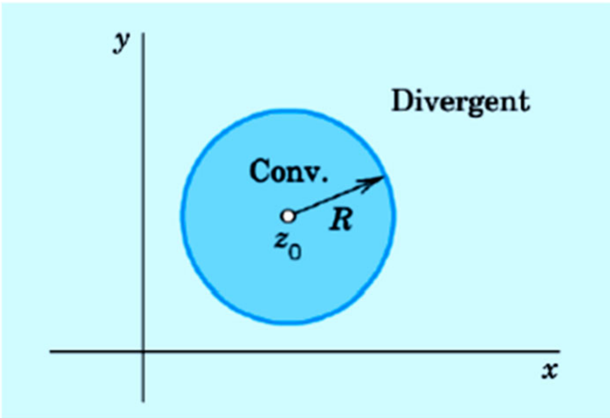
The derivatives of this function are obtained by differentiating the original series term by term.

All the series thus obtained have the same radius of convergence as the original series. **Why?**

Hence, by the first statement, each of them represents an analytic function.

$$f(z)=\sum_{n=0}^{\infty} a_n z^n = a_0 + a_1 z + a_2 z^2 + \dots$$

$$f'(z)=\left(\sum_{n=0}^{\infty} a_n z^n\right)' = 0 + a_1 + 2a_2 z + \cdots$$



Why? We note that

$$f'(z) = \left(\sum_{n=0}^{\infty} a_n z^n\right)' = \sum_{n=0}^{\infty} a_n n z^{n-1} = \sum_{m=0}^{\infty} a_{m+1}(m+1) z^m = \sum_{n=0}^{\infty} a_{n+1}(n+1) z^n$$

$$\Rightarrow \quad R = \lim_{n\to\infty} \left| \frac{a_{n+1}(n+1)}{a_{n+2}(n+2)} \right| = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_{n+2}} \right| \cdot \lim_{n\to\infty} \left| \frac{n+1}{n+2} \right| = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| \cdot 1 = \frac{1}{L^*}$$

For a convergent series, we have…

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots \quad \Rightarrow \quad \boxed{f(z_0) = a_0}$$

$$f'(z) = a_1 + 2a_2(z-z_0) + 3a_3(z-z_0)^2 + \cdots \quad \Rightarrow \quad \boxed{f'(z_0) = a_1}$$

$$f''(z) = 2a_2 + 3\times 2\times a_3(z-z_0) + \cdots \quad \Rightarrow \quad f''(z_0) = 2a_2$$

$$\Rightarrow \quad \boxed{a_2 = \frac{f''(z_0)}{2!}} \quad \ldots\ldots \quad \boxed{a_n = \frac{f^{(n)}(z_0)}{n!}}$$

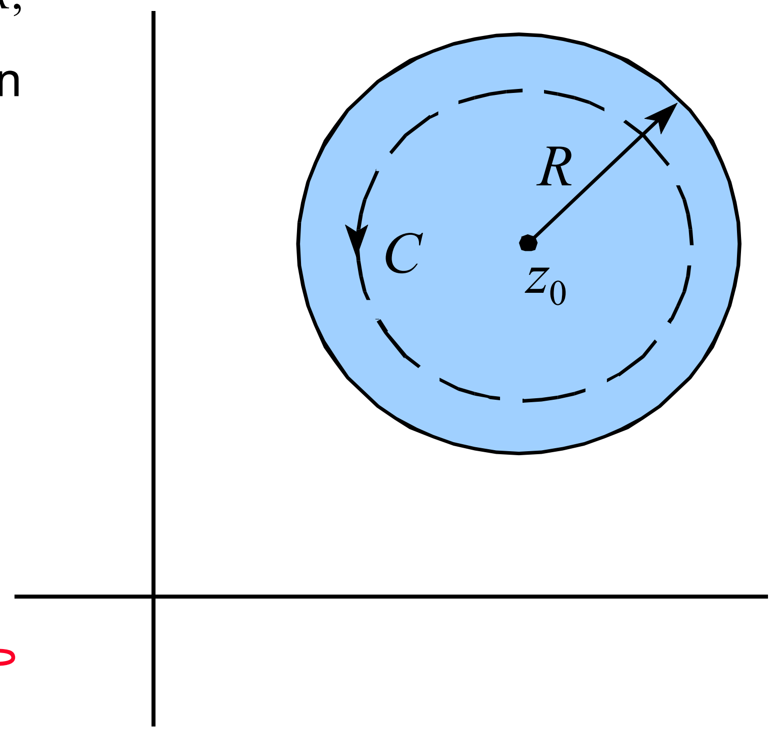
Laurent & Taylor Series Expansions of Complex Functions

If a function $f(z)$ is analytic for $|z - z_0| < R$, then $f(z)$ has a **Taylor series** expansion

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

where

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}} dz = \frac{f^{(n)}(z_0)}{n!}$$



$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}} dz. \quad (5.3)$$

Taylor Series

- If a function $f(z)$ is analytic at $z = z_0$, then it admits derivatives of all orders there by generalized Cauchy integral formula, i.e., $f^{(n)}(z_0)$ exist for any integer $n \geq 0$. If we let $a_n = \frac{f^{(n)}(z_0)}{n!}$ in the power series (6.5), we have

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n. \tag{6.13}$$

which is called the **Taylor series** of $f(z)$ about the point $z = z_0$.

Brook Taylor
(1685–1731)
English Mathematician

Colin Maclaurin
(1698–1746)
Scottish Mathematician

$$f(z)=\sum_{n=0}^{\infty}\frac{f^{(n)}(z_0)}{n!}(z-z_0)^n.$$

- or, by (5.3),

$$f^{(n)}(z_0)=\frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz$$

$\Longrightarrow$

$$a_n=\frac{1}{2\pi i}\oint_C \frac{f(z^*)}{(z^*-z_0)^{n+1}}\,dz^*. \qquad (6.14)$$

- If we let $z_0=0$ in Taylor series (6.13), then the Taylor series about $z=0$ is called a **Maclaurin series**, i.e.,

$$f(z)=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}z^n. \qquad (6.15)$$

Example (6l)

- Find the Maclaurin series of $\text{Ln}(1+z)$ and find its radius of convergence R.
- Let $f(z) = \text{Ln}(1+z)$. Since $f^{(n)}(z) = \frac{(-1)^{n+1}(n-1)!}{(1+z)^n}, n \geq 1$, we have

$$a_n = \frac{f^{(n)}(0)}{n!} = \frac{(-1)^{n+1}}{n}, \quad n \geq 1,$$

and $a_0 = \text{Ln}(1) = 0$. Now, $L^* = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{n}{n+1} \right| = 1$. Thus $R = \frac{1}{L^*} = 1$ and the Maclaurin series (6.15) is

$$\text{Ln}(1+z) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{z^n}{n} = z - \frac{z^2}{2} + \frac{z^3}{3} - \cdots, \quad |z| < 1.$$

It can be proved that the **alternating harmonic series**

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + - \ldots = \ln 2$$ by letting $z = 1$.

Important Special Taylor Series

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n = 1 + z + z^2 + \cdots \quad \text{in } |z| < 1,$$

$$\text{Ln}(1+z) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{z^n}{n} = z - \frac{z^2}{2} + \frac{z^3}{3} - \cdots \quad \text{in } |z| < 1,$$

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \cdots \quad \text{in } |z| < \infty,$$

$$\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots \quad \text{in } |z| < \infty,$$

$$\cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots \quad \text{in } |z| < \infty,$$

$$\sinh z = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!} = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots \quad \text{in } |z| < \infty,$$

$$\cosh z = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!} = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots \quad \text{in } |z| < \infty.$$

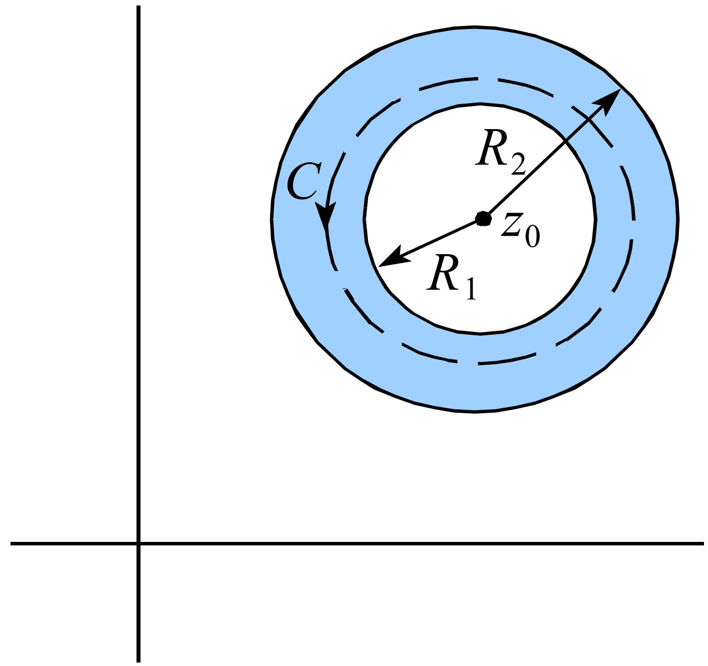
Laurent Series Expansions of Complex Functions

If a function $f(z)$ is analytic in the ring area $R_1 < |z - z_0| < R_2$, then $f(z)$ has a **Laurent series** expansion

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

where

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}} dz$$



Note that points where a function is not analytic are called **singularities**.

Note that for $n = -1$, we have $a_{-1} = \frac{1}{2\pi i} \oint_C f(z)\, dz \quad \Rightarrow \quad \oint_C f(z)\, dz = 2\pi i \cdot a_{-1}$

Example 10

(a) The functions e^z, $\sin z$ and $\cos z$ are analytic functions, and have Taylor series expansions with a centre $z_0 = 0$ of

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots \quad \Rightarrow \quad a_n = \frac{1}{n!} \quad \Rightarrow \quad R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|$$

$$= \lim_{n\to\infty} \frac{\frac{1}{n!}}{\frac{1}{(n+1)!}}$$

$$= \lim_{n\to\infty} \frac{(n+1)!}{n!}$$

$$= \lim_{n\to\infty} (n+1) = \infty$$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots$$

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots$$

Example 10 (cont.)

Pierre A. Laurent
(1813–1854)
French Mathematician

(b) The functions $\dfrac{e^z}{z}$ and $\dfrac{\sin z}{z^3}$ are not analytic in the point $z = 0$. In the region excluding the point $z_0 = 0$, these functions have Laurent series expansions of

$$\frac{e^z}{z} = \frac{1}{z} + 1 + \frac{z}{2!} + \frac{z^2}{3!} + \cdots \qquad \frac{\sin z}{z^3} = \frac{1}{z^2} - \frac{1}{3!} + \frac{z^2}{5!} - \frac{z^4}{7!} + \cdots$$

✦ About Homework Assignment No. 4……

You can start working on Problems 4.1 to 4.3 at this point.

Example 10 (cont.)

(c) The function $\frac{1}{1-z}$ is analytic for $|z|<1$. The Taylor series expansion with centre $z_0=0$ of this function is the geometric series, i.e.,

$$\frac{1}{1-z}=1+z+z^2+\cdots$$

(d)* Find Taylor series expansion of $f(z)=\frac{1}{z}$ at $z_0=3$ & its convergence radius.

$$f(z)=\frac{1}{z}=\frac{\frac{1}{3}}{1-\left(\frac{3-z}{3}\right)}=\frac{1}{3}\left[1+\frac{3-z}{3}+\left(\frac{3-z}{3}\right)^2+\left(\frac{3-z}{3}\right)^3+\cdots\right]=\frac{1}{3}-\frac{z-3}{3^2}+\frac{(z-3)^2}{3^3}-\frac{(z-3)^3}{3^4}+\cdots$$

The series converges for all $\left|\frac{3-z}{3}\right|<1 \;\;\Rightarrow\;\; |z-3|<3$. Thus, its $R=3$.

Complex Analysis – 7…

1 Complex Numbers

2 Functions of One Complex Variable

3 Complex Differentiation

4 Complex Integration and Cauchy's Theorem

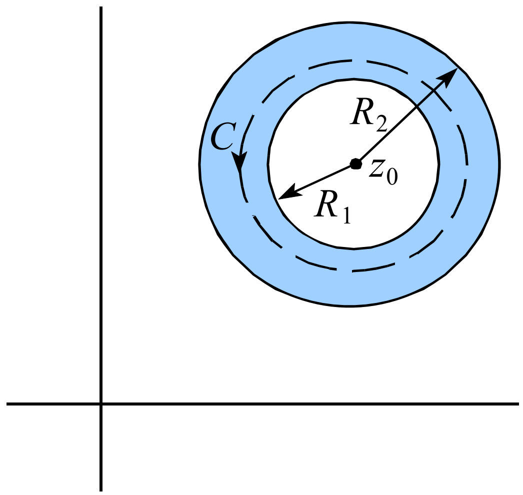
5 Cauchy Integral Formula

6 Complex Series, Power Series, Taylor Series, and Laurent Series

7 Residue Integration

Classification of Singularities

A complex function $f(z)$, which has a Laurent series expansion at $z_0\ldots$



$$f(z)=\sum_{n=-m}^{\infty} a_n (z-z_0)^n$$

$$=\frac{a_{-m}}{(z-z_0)^m}+\frac{a_{-m+1}}{(z-z_0)^{m-1}}+\cdots+\frac{a_{-1}}{z-z_0}+a_0+a_1(z-z_0)+\cdots$$

Then, it is said that $f(z)$ has a singular point (pole) of order m at z_0.

Residues

$$\operatorname{Res}(f,z_0):=a_{-1}=\frac{1}{2\pi i}\oint_C f(z)\,dz \quad\Rightarrow\quad \oint_C f(z)\,dz=2\pi i\operatorname{Res}(f,z_0)$$

Material flow (cont.)…

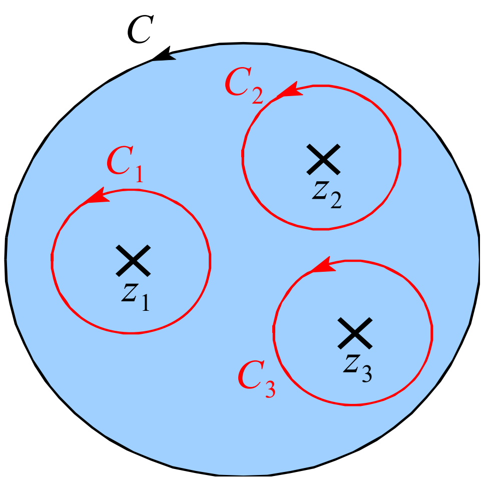
Residue integration

If $f(z)$ is not analytic in several points $z_1, z_2, \ldots z_n$, then

$$\oint_C f(z)\,dz = \oint_{C_1} f(z)\,dz + \oint_{C_2} f(z)\,dz + \cdots + \oint_{C_n} f(z)\,dz$$

$$= 2\pi i\,\mathrm{Res}(f, z_1) + 2\pi i\,\mathrm{Res}(f, z_2) + \cdots + 2\pi i\,\mathrm{Res}(f, z_n)$$

$$= 2\pi i \sum_{j=1}^{n} \mathrm{Res}(f, z_j)$$



$$\oint_C f(z)\,dz = 2\pi i \sum_{j=1}^{n} \mathrm{Res}(f, z_j)$$

Material flow (cont.)…

Calculation of Residues

1. $f(z)$ has a simple pole at $z = z_0$:

$$\mathrm{Res}(f, z_0) = \lim_{z \to z_0} (z - z_0) f(z)$$

2. $f(z)$ has an nth order pole at $z = z_0$:

$$\mathrm{Res}(f, z_0) = \frac{1}{(n-1)!} \lim_{z \to z_0} \left[\frac{d^{n-1}}{dz^{n-1}} \left[(z - z_0)^n f(z) \right] \right]$$

3. $f(z) = \dfrac{A(z)}{B(z)}$ where $B(z)$ has a simple zero at $z = z_0$, while $A(z_0) \neq 0$

and both A and B are differentiable at $z = z_0$:

$$\mathrm{Res}(f, z_0) = \frac{A(z_0)}{B'(z_0)}$$

Material flow (cont.)...

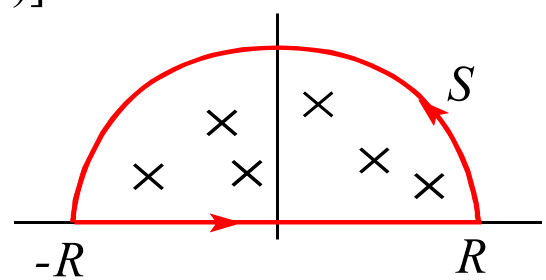
Real integral of the form $\int_0^{2\pi} f(\cos\theta, \sin\theta)\, d\theta$

$$\int_0^{2\pi} f\left(\cos\theta, \sin\theta\right) d\theta = \oint_{|z|=1} f\left[\frac{1}{2}\left(z+\frac{1}{z}\right), \frac{1}{2i}\left(z-\frac{1}{z}\right)\right]\frac{1}{iz}dz$$

Improper integrals of rational functions $\int_{-\infty}^{\infty} f(x)\, dx = \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}\, dx$

$Q(x) \neq 0$ for all x being real, degree$[Q(x)] \geq$ degree$[P(x)] + 2$

$$\int_{-\infty}^{\infty} f(x)\,dx = 2\pi i \sum_j \text{Res}(f, z_j)$$



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Material flow (cont.)…

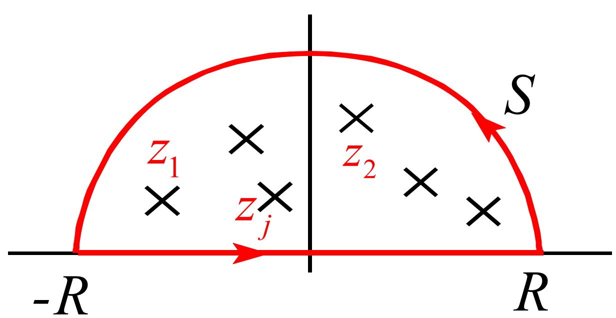
Improper integrals of Fourier-type $\int_{-\infty}^{+\infty} f(x)\cos mx dx$ or $\int_{-\infty}^{+\infty} f(x)\sin mx dx$

$$f(x)=\frac{P(x)}{Q(x)},\quad Q(x)\neq 0 \text{ for all } x \text{ being real,}$$

$$\text{degree}[Q(x)] \geq \text{degree}[P(x)]+1,\quad m>0$$

$$\int_{-\infty}^{\infty} f(x)\cos mx\, dx = \text{Re}\left[2\pi i\sum_j \text{Res}(f\, e^{imz}, z_j)\right]$$

$$\int_{-\infty}^{\infty} f(x)\sin mx\, dx = \text{Im}\left[2\pi i\sum_j \text{Res}(f\, e^{imz}, z_j)\right]$$



Classification of Singularities

Recall: Analytic Functions and Singularities...

Theorem (3.1 Cauchy-Riemann Equations)

Let $f(z) = u(x, y) + iv(x, y)$ be defined and continuous in some neighborhood of a point $z = x + iy$ and differentiable at z itself. Then, at that point, the first-order partial derivatives of u and v exist and satisfy the Cauchy-Riemann equations (3.6).

Hence, if $f(z)$ is analytic in a domain D, those partial derivatives exist and satisfy (3.6) at all points of D.

$$u_x = v_y, \quad u_y = -v_x. \tag{3.6}$$

Points where a function $f(z)$ is not analytic are called **singularities** or **poles** or **singular points**.

Classification of Singularities

Poles

Consider the Laurent expansion of different functions:

1. No negative powers of z in the expansion. For example $\dfrac{\sin z}{z}$ has a singularity at $z_0 = 0$.

$$\frac{\sin z}{z} = \frac{1}{z}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots\cdots \right) = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \frac{z^6}{7!} + \cdots\cdots$$

so that its Laurent expansion has no negative powers of $(z - z_0)$. The function is said to have a ***removable singularity*** at $z_0 = 0$.

Poles (cont.)

2. A finite number of negative powers of z in the expansion, e.g.,

$$\frac{e^z}{z^3} = \frac{1}{z^3}\left(1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \cdots\right) = \frac{1}{z^3} + \frac{1}{z^2} + \frac{1}{2!z} + \frac{1}{3!} + \frac{z}{4!}\cdots$$

The highest negative power is 3. This function is said to have a **3rd order pole** at $z_0 = 0$.

3. An infinite number of negative powers of z in the expansion, e.g.,

$$e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \cdots$$

This function is said to have an ***essential singularity*** at $z_0 = 0$.

Example 11

(a) $$f(z)=\frac{\cos z-1}{z}=\frac{1}{z}\left[\left(1-\frac{z^2}{2!}+\frac{z^4}{4!}-\frac{z^6}{6!}+\cdots\right)-1\right]=-\frac{z}{2!}+\frac{z^3}{4!}-\frac{z^5}{6!}+\cdots$$

The function $f(z)$ has a removable at $z_0=0$.

(b) $$f(z)=\frac{\sin z}{z^5}=\frac{1}{z^5}\left[z-\frac{z^3}{3!}+\frac{z^5}{5!}-\frac{z^7}{7!}+\cdots\right]=\frac{1}{z^4}-\frac{1}{3!z^2}+\frac{1}{5!}-\frac{z^2}{7!}+\cdots$$

The function $f(z)$ has a **4th order pole** at $z_0=0$.

Example 11 (cont.)

(c) $$f(z) = z^2 e^{1/z} = z^2\left[1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \cdots\right] = z^2 + z + \frac{1}{2!} + \frac{1}{3!z} + \cdots$$

The function $f(z)$ has an essential singularity at $z_0 = 0$.

(d) $$f(z) = \frac{z^2 - 2}{(z+1)^2} = \frac{z^2 + 2z + 1 - 2z - 1 - 2}{(z+1)^2} = \frac{(z+1)^2 - 2(z+1) - 1}{(z+1)^2}$$

$$= -\frac{1}{(z+1)^2} - \frac{2}{(z+1)} + 1$$

Thus, $f(z)$ has 2nd order pole at $z_0 = -1$.

Example 11 (cont.)

(e) $$f(z)=\frac{z^2-2}{z(z+1)}=\frac{z+2}{z+1}-\frac{2}{z}=1+\frac{1}{z+1}-\frac{2}{z}$$

It is clear that $f(z)$ has 2 singular points at $z_0=0$ & $z_0=-1$, respectively.

For $z_0=0$, we have the following Laurent series of $f(z)$ centered at $z_0=0$

$$f(z)=\frac{z^2-2}{z(z+1)}=1+\frac{1}{z+1}-\frac{2}{z}=1-\frac{2}{z}+\frac{1}{1-(-z)}$$

$$=-\frac{2}{z}+1+\left[1+(-z)+(-z)^2+(-z)^3+\cdots\right]$$

$$=-\frac{2}{z^1}+2-z+z^2-z^3+\cdots$$

Thus, the order of singularity of $f(z)$ at $z_0=0$ is 1.

Example 11 (cont.)

For $z_0 = -1$, we have the following Laurent series of $f(z)$ centered at $z_0 = -1$

$$f(z) = \frac{z^2 - 2}{z(z+1)} = 1 + \frac{1}{z+1} - \frac{2}{z} = 1 + \frac{1}{z+1} + \frac{2}{1-(z+1)}$$

$$= \frac{1}{z+1} + 1 + 2\cdot\left[1 + (z+1) + (z+1)^2 + (z+1)^3 + \cdots\right]$$

$$= \frac{1}{(z+1)^1} + 3 + 2(z+1) + 2(z+1)^2 + 2(z+1)^3 + \cdots$$

Thus, the order of singularity of $f(z)$ at $z_0 = -1$ is again equal to 1.

1st order poles** are also called **simple poles** or **simple singularities.

Zeros

If $g(z_0) = 0$, then the function $g(z)$ is said to have a zero or root at $z = z_0$.

If $g(z_0) = g'(z_0) = g''(z_0) = \cdots = g^{(n-1)}(z_0) = 0$ and $g^{(n)}(z_0) \neq 0$, then the function is said to have an n-th order zero at $z = z_0$.

Obviously, for a function $g(z)$ with an n-th order zero at $z = z_0$, its Taylor series expansion at z_0 can be written as

$$g(z) = 0 + \frac{g^{(n)}(z_0)}{n!}(z - z_0)^n + \frac{g^{(n+1)}(z_0)}{(n+1)!}(z - z_0)^{n+1} + \cdots$$

Theorem:

If the function $g(z)$ has an n-th order zero at $z = z_0$, then $f(z) = \dfrac{1}{g(z)}$ has an n-th order pole at $z = z_0$.

Example 12

a) Consider the function $f(z)=\dfrac{1}{(z-1)(e^z-e)}$, which has a singularity at $z_0=1$.

Let $g(z)=(z-1)(e^z-e)$

Then $g'(z)=e^z-e+(z-1)e^z, \quad g'(1)=0$

$$g''(z)=e^z+(z-1)e^z+e^z, \quad g''(1)=2e\neq 0$$

Therefore $g(z)$ has a 2nd order zero at $z_0=1$, and $f(z)$ has a 2nd order pole at $z_0=1$. Also, $g(z)$ has a Taylor series expansion at $z_0=1$ as follows

$$g(z)=(z-1)\left(e^z-e\right)=(z-1)\left(e\cdot e^{z-1}-e\right)=e(z-1)\left(e^{z-1}-1\right)$$

$$=e(z-1)\left(1+(z-1)+\frac{1}{2!}(z-1)^2+\frac{1}{3!}(z-1)^3+\cdots-1\right)$$

$$=e\cdot(z-1)^2+\frac{e}{2!}(z-1)^3+\frac{e}{3!}(z-1)^4+\cdots$$

Example 12 (cont.)

b) Consider $f(z) = \dfrac{1}{z - \sin z}$, which has a singularity at $z_0 = 0$.

Let $g(z) = z - \sin z$.

Then, $g'(z) = 1 - \cos z \qquad g'(0) = 0$

$$g''(z) = \sin z \qquad g''(0) = 0$$

$$g'''(z) = \cos z \qquad g'''(0) = 1 \neq 0$$

Therefore $g(z)$ has a 3rd order zero at $z_0 = 0$, and $f(z)$ has a 3rd order pole at $z_0 = 0$. Also, $g(z)$ has a Taylor series expansion at $z_0 = 0$ as follows

$$g(z) = z - \sin z = z - \left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots \right) = \frac{z^3}{3!} - \frac{z^5}{5!} + \frac{z^7}{7!} - \cdots$$

Residues

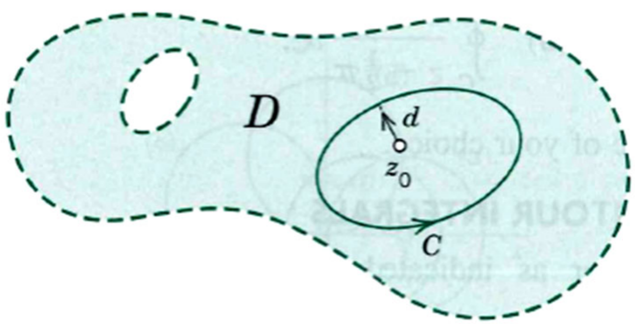
We know that if $f(z)$ is analytic in domain D except at the point z_0, it has a Laurent series expansion of

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

with

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}} dz$$

and C any closed curve in D which encloses z_0.



Residues

From last expression of a_n, it follows that

$$a_{-1} = \frac{1}{2\pi i} \oint_C f(z)\, dz = \text{Res}\,(f, z_0)$$

where $\text{Res}\,(f, z_0)$ is known as the residue of f at z_0. Thus

$$\oint_C f(z)\, dz = 2\pi i\, \text{Res}\,(f, z_0)$$

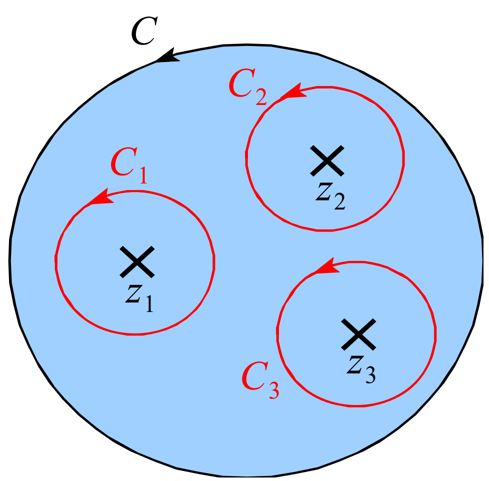
Residues

If $f(z)$ is not analytic in several points $z_1, z_2, \ldots z_n$, then

$$\oint_C f(z)\,dz = \oint_{C_1} f(z)\,dz + \oint_{C_2} f(z)\,dz + \cdots + \oint_{C_n} f(z)\,dz$$

$$= 2\pi i\,\text{Res}(f, z_1) + 2\pi i\,\text{Res}(f, z_2) + \cdots + 2\pi i\,\text{Res}(f, z_n)$$

$$= 2\pi i \sum_{j=1}^{n} \text{Res}(f, z_j)$$



$$\oint_C f(z)\,dz = 2\pi i \sum_{j=1}^{n} \text{Res}(f, z_j)$$

Calculation of Residues

Examples

1. $f(z)$ has a simple pole at $z = z_0$:
$$\text{Res}\,(f, z_0) = \lim_{z \to z_0} (z - z_0)\, f(z)$$

2. $f(z)$ has an n-th order pole at $z = z_0$:

$$\text{Res}\,(f, z_0) = \frac{1}{(n-1)!} \lim_{z \to z_0} \left[\frac{d^{n-1}}{dz^{n-1}} \left[(z - z_0)^n\, f(z) \right] \right]$$

3. $f(z) = \dfrac{A(z)}{B(z)}$ where $B(z)$ has a simple zero at $z = z_0$, while $A(z_0) \neq 0$

and both A and B are differentiable at $z = z_0$:
$$\text{Res}\,(f, z_0) = \frac{A(z_0)}{B'(z_0)}$$

Proof of… $\operatorname{Res}(f, z_0) = \lim_{z \to z_0} (z - z_0) f(z)$ and $\operatorname{Res}(f, z_0) = \dfrac{A(z_0)}{B'(z_0)}$

For a simple pole of $f(z)$ at $z = z_0$, its Laurent series can be written as follows

$$f(z) = \frac{a_{-1}}{z - z_0} + a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots \quad (0 < |z - z_0| < R)$$

Multiplying both sides by $z - z_0$ and then letting $z \to z_0$, we obtain

$$\lim_{z \to z_0} (z - z_0) f(z) = \lim_{z \to z_0} \left[a_{-1} + a_0(z - z_0) + a_1(z - z_0)^2 + a_2(z - z_0)^3 + \cdots \right] = a_{-1} \checkmark$$

This is exactly the 1st formula for calculating the residue.

For the 3rd formula, since $B(z)$ has a simple zero at z_0, its Taylor series can be written as

$$B(z) = B'(z_0)(z - z_0) + \frac{B''(z_0)}{2!}(z - z_0)^2 + \cdots$$

Then, it follows from the 1st formula,

$$\operatorname{Res}(f, z_0) = \lim_{z \to z_0} (z - z_0) \frac{A(z)}{B(z)}$$

$$= \lim_{z \to z_0} \frac{(z - z_0) A(z)}{B'(z_0)(z - z_0) + \dfrac{B''(z_0)}{2!}(z - z_0)^2 + \cdots} = \lim_{z \to z_0} \frac{A(z)}{B'(z_0) + \dfrac{B''(z_0)}{2!}(z - z_0) + \cdots} = \frac{A(z_0)}{B'(z_0)} \checkmark$$

Proof of… $\text{Res}(f, z_0) = \frac{1}{(n-1)!} \lim_{z \to z_0} \left[\frac{d^{n-1}}{dz^{n-1}} \left[(z-z_0)^n f(z) \right] \right]$

For $f(z)$ having an n-th order pole at $z = z_0$, its Laurent series can be written as follows

$$f(z) = \frac{a_{-n}}{(z-z_0)^n} + \cdots + \frac{a_{-1}}{z-z_0} + a_0 + a_1(z-z_0) + \cdots$$

Multiplying both sides by $(z-z_0)^n$ gives

$$(z-z_0)^n f(z) = a_{-n} + a_{-n+1}(z-z_0) + \cdots + a_{-1}(z-z_0)^{n-1} + a_0(z-z_0)^n + \cdots$$

Let $g(z) = (z-z_0)^n f(z)$. Then, we note that

$$g(z) = a_{-n} + a_{-n+1}(z-z_0) + \cdots + a_{-1}(z-z_0)^{n-1} + a_0(z-z_0)^n + \cdots$$

is a Taylor series expansion $g(z)$ and a_{-1} is the coefficient of its $(n-1)$-th term.

For the Taylor series expansion of $g(z)$, the coefficient of the $(n-1)$-th term is given by

$$\Rightarrow a_{-1} = \frac{g^{(n-1)}(z_0)}{(n-1)!} = \frac{1}{(n-1)!} \lim_{z \to z_0} \left[\frac{d^{n-1}}{dz^{n-1}} \left[(z-z_0)^n f(z) \right] \right] \quad ✓$$

Example 13

(a) Calculate $\oint_{|z|=2} \frac{4-3z}{z^2-z}\, dz.$

$$\text{Res}\,(f, z_0) = \lim_{z \to z_0} (z - z_0)\, f(z)$$

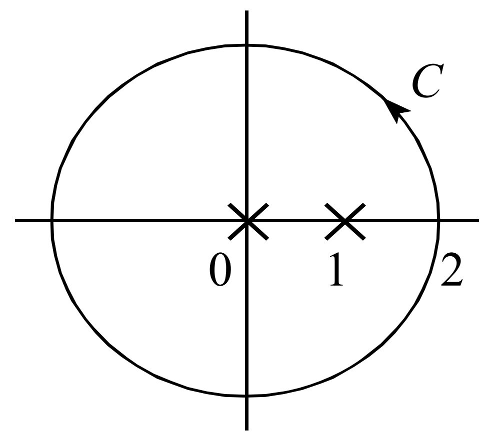
$f(z) = \frac{4-3z}{z^2-z} = \frac{4-3z}{z(z-1)}$ has simple poles in $z_0 = 0$ and $z_0 = 1$.

$$\oint_{|z|=2} \frac{4-3z}{z^2-z}\, dz = 2\pi i \left[\text{Res}(f,0) + \text{Res}(f,1)\right]$$

$$= 2\pi i \left[\lim_{z\to 0} \frac{4-3z}{z-1} + \lim_{z\to 1} \frac{4-3z}{z}\right]$$

$$= 2\pi i(-4+1)$$

$$= -6\pi i$$



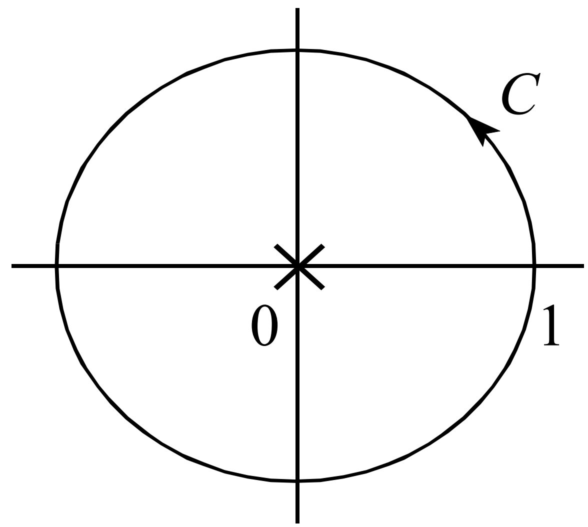
Example 13

(b) Compute $\oint_{|z|=1} \frac{e^z}{z} dz$

$$\oint_{|z|=1} \frac{e^z}{z} dz = 2\pi i \operatorname{Res}(f, 0)$$

$$= 2\pi i \lim_{z \to 0} e^z$$

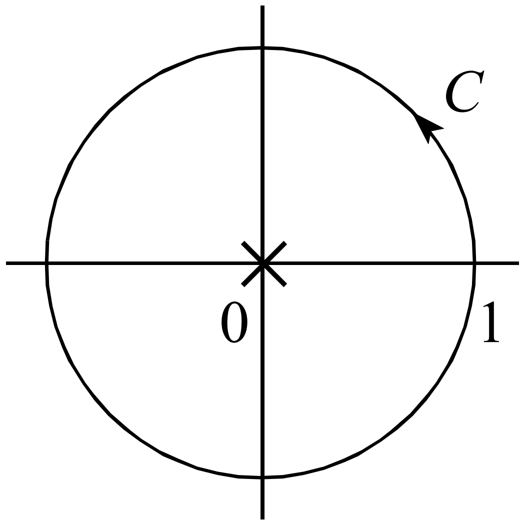
$$= 2\pi i$$



$$\operatorname{Res}(f, z_0) = \lim_{z \to z_0} (z - z_0) f(z)$$

Example 13

(c) Compute $\displaystyle\oint_{|z|=1} \frac{\sin z}{z^2}\, dz$



$f(z) = \dfrac{\sin z}{z^2}$ has a simple pole at $z_0 = 0$.

Thus

$$\oint_{|z|=1} \frac{\sin z}{z^2}\, dz = \oint_{|z|=1} \frac{\frac{\sin z}{z}}{z}\, dz$$

$$= 2\pi i \lim_{z\to 0} \frac{\sin z}{z}$$

$$= 2\pi i$$

$$\mathrm{Res}(f, z_0) = \lim_{z\to z_0} (z - z_0)\, f(z)$$

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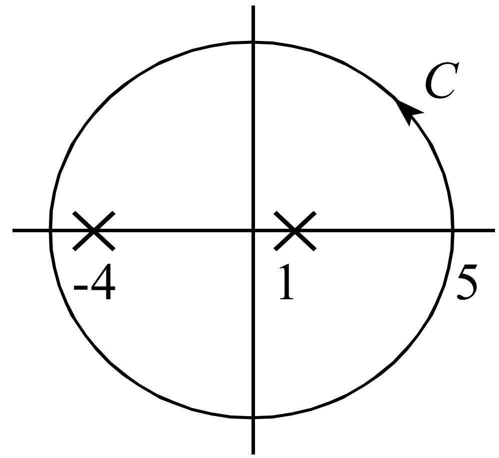
Example 13

$$\text{Res}(f, z_0) = \frac{1}{(n-1)!} \lim_{z \to z_0} \left[\frac{d^{n-1}}{dz^{n-1}} \left[(z - z_0)^n f(z) \right] \right]$$

(d) Compute $\oint_{|z|=5} \frac{2z}{(z+4)(z-1)^2} dz$



The function $f(z) = \frac{2z}{(z+4)(z-1)^2}$ has a simple pole at $z_0 = -4$ and a 2nd order pole at $z_0 = 1$.

$$\text{Res}(f, -4) = \lim_{z \to -4} \frac{2z}{(z-1)^2} = -\frac{8}{25}$$

$$\text{Res}(f, 1) = \frac{1}{1!} \lim_{z \to 1} \frac{d}{dz} \left[\frac{2z}{z+4} \right]$$

$$= \lim_{z \to 1} \frac{2(z+4) - 2z}{(z+4)^2} = \frac{8}{25}$$

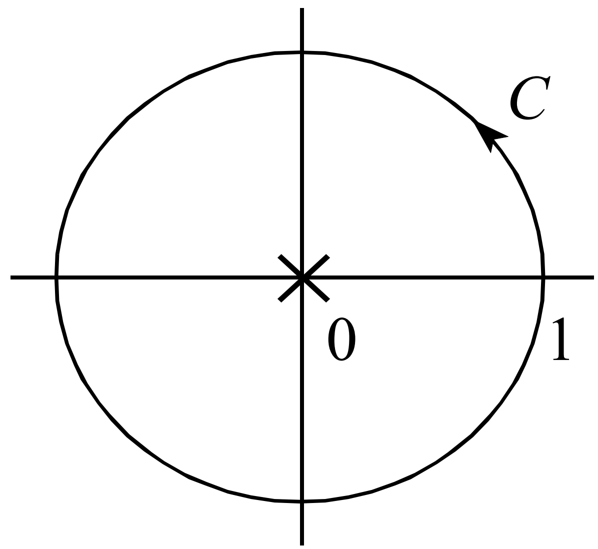
$$\oint_{|z|=5} \frac{2z}{(z+4)(z-1)^2} dz = 2\pi i \left[\text{Res}(f, -4) + \text{Res}(f, 1) \right] = 0$$

$$\text{Res}(f, z_0) = \lim_{z \to z_0} (z - z_0) f(z)$$

Example 17

Example 13

(e) Compute $\oint_{|z|=1} \frac{1}{1-e^z} dz$



$$\oint_{|z|=1} \frac{1}{1-e^z} dz = 2\pi i \operatorname{Res}(f,0)$$

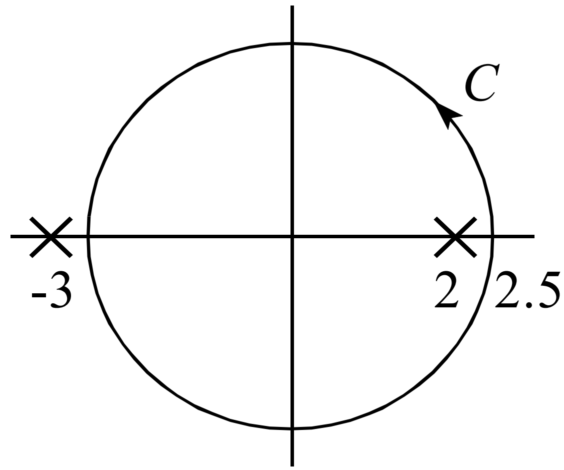
$$= 2\pi i \left[\frac{1}{-e^z} \right]_{z=0}$$

$$= -2\pi i$$

$$\operatorname{Res}(f, z_0) = \frac{A(z_0)}{B'(z_0)}$$

Example 13

(f) Compute $\oint_{|z|=2.5} \frac{2z+4}{z^2+z-6}\,dz$



$$f(z) = \frac{2z+4}{z^2+z-6} = \frac{2z+4}{(z+3)(z-2)}$$

has a simple pole at $z_0 = 2$ enclosed by C. Thus,

$$\oint_{|z|=2.5} \frac{2z+4}{z^2+z-6}\,dz = 2\pi i \operatorname{Res}(f,2)$$

$$= 2\pi i \lim_{z\to 2}\left[\frac{2z+4}{z+3}\right]$$

$$= \frac{16}{5}\pi i$$

$$\operatorname{Res}(f,z_0) = \lim_{z\to z_0}(z-z_0)\,f(z)$$

Homework Assignment No: 4 (Due in one week)

Question 4.1: *What is a singular point? What is the order of singularities? What is the Cauchy's integral theorem? What is the Cauchy's integral formula? What are the Taylor series expansion and Laurent series expansion of complex functions? What are the residues of complex functions?*

Question 4.2: Determine the convergence or divergence of the following series:

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{3^{1+2n}}$; (ii) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{e^n}$

Question 4.3: Consider the function $f(z) = \dfrac{1}{\cosh(z)}$.

(a) Let $f(z) = \displaystyle\sum_{n=0}^{\infty} a_n z^n$ be the Taylor series expansion of $f(z)$ around $z = 0$. Determine a_0, a_1, a_2.

(b) Let $f(z) = \displaystyle\sum_{n=0}^{\infty} b_n \left(z - \frac{i\pi}{4} \right)^n$ be the Taylor series expansion of $f(z)$ around $z = \dfrac{i\pi}{4}$. Determine b_0, b_1, b_2. Simplify the resulting expressions as much as possible.

Question 4.4: Find the singularities of $f(z)=\dfrac{e^z-\sin z-1}{z^2}$.

Question 4.5: Find the residue of $f(z)=\dfrac{e^z}{z^2-z^3}$ at $z=0$.

Question 4.6: Calculate $\displaystyle\oint_{|z|=2720}\left(\frac{e^z}{z^2-z^3}\right)dz$

Question 4.7: Calculate $\displaystyle\oint_{|z|=2}\frac{e^z}{z^2-1}\,dz$.

Question 4.8: Calculate $\displaystyle\oint_{|z|=1}\frac{z^2+1}{e^z\sin z}\,dz$.

Question 4.9: Calculate $\displaystyle\int_0^{2\pi}\frac{\sin^2\theta}{5+4\cos\theta}\,d\theta$.

Applications to solve real integration problems…

Real Integral of the form $\int_0^{2\pi} f(\cos\theta, \sin\theta)\, d\theta$

These integrals can be transformed to an integral of a complex function along the circle $|z| = 1$ counterclockwise.

The circle can be described by $z = e^{i\theta}, \quad 0 \le \theta \le 2\pi$, Then

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{1}{2}\left(z + \frac{1}{z}\right)$$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} = \frac{1}{2i}\left(z - \frac{1}{z}\right)$$

$$dz = \frac{dz}{d\theta} d\theta = i\, e^{i\theta} d\theta$$

$$\Rightarrow\ d\theta = \frac{1}{ie^{i\theta}} dz = \frac{1}{iz} dz$$

Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta,$

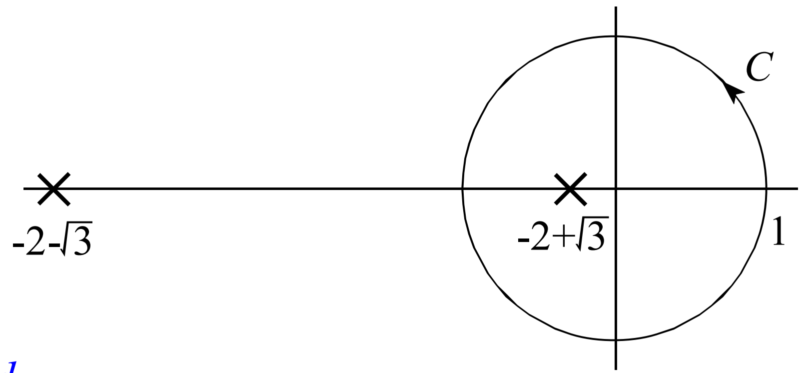
Real Integrals

Consequently, we have

$$\int_0^{2\pi} f(\cos\theta, \sin\theta)\, d\theta = \oint_{|z|=1} f\left[\frac{1}{2}\left(z+\frac{1}{z}\right), \frac{1}{2i}\left(z-\frac{1}{z}\right)\right]\cdot \frac{1}{iz}\, dz$$

Example 14

(a) Evaluate $\displaystyle\int_0^{2\pi} \frac{1}{\cos\theta + 2}\, d\theta$



$$\int_0^{2\pi} \frac{1}{\cos\theta + 2}\, d\theta = \oint_{|z|=1} \frac{1}{\frac{1}{2}\left(z+\frac{1}{z}\right)+2} \cdot \frac{1}{iz}\, dz$$

$$= -2i \oint_{|z|=1} \frac{1}{z^2 + 4z + 1}\, dz$$

$$= -2i \oint_{|z|=1} \frac{1}{\left(z-(-2+\sqrt{3})\right)\left(z-(-2-\sqrt{3})\right)}\, dz$$

$$= -2i \cdot 2\pi i \,\mathrm{Res}\left(f, -2+\sqrt{3}\right)$$

$$= \frac{2\pi}{\sqrt{3}}$$

$$\mathrm{Res}\,(f, z_0) = \lim_{z \to z_0} (z - z_0)\, f(z)$$

$$\int_0^{2\pi} f(\cos\theta, \sin\theta)\, d\theta = \oint_{|z|=1} f\left[\frac{1}{2}\left(z+\frac{1}{z}\right), \frac{1}{2i}\left(z-\frac{1}{z}\right)\right] \cdot \frac{1}{iz}\, dz$$

Example 14

(b) Compute $\int_0^{2\pi} \frac{1}{5+3\sin\theta}\, d\theta$

$$\mathrm{Res}\,(f, z_0) = \lim_{z\to z_0} (z-z_0)\, f(z)$$

$$\int_0^{2\pi} \frac{1}{5+3\sin\theta}\, d\theta = \oint_{|z|=1} \frac{1}{5+3\left[\frac{1}{2i}\left(z-\frac{1}{z}\right)\right]} \cdot \frac{1}{iz}\, dz$$

$$= 2\oint_{|z|=1} \frac{1}{3z^2+10iz-3}\, dz$$

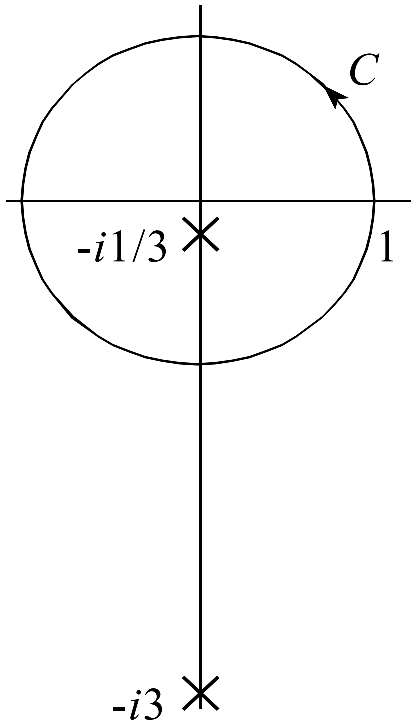
$$= \frac{2}{3}\oint_{|z|=1} \frac{1}{z^2+\frac{10}{3}iz-1}\, dz$$

$$= \frac{2}{3}\oint_{|z|=1} \frac{1}{\left(z+\frac{i}{3}\right)(z+i3)}\, dz$$

$$= \frac{2}{3}\, 2\pi i\, \mathrm{Res}\left(f, -\frac{i}{3}\right)$$

$$= \frac{4\pi i}{3} \lim_{z\to -i/3} \frac{1}{(z+i3)}$$

$$= \frac{\pi}{2}$$



$$\int_0^{2\pi} f(\cos\theta, \sin\theta)\, d\theta = \oint_{|z|=1} f\left[\frac{1}{2}\left(z+\frac{1}{z}\right), \frac{1}{2i}\left(z-\frac{1}{z}\right)\right] \cdot \frac{1}{iz}\, dz$$

Improper Integrals of Rational Functions

We now consider real integrals for which the interval of integration is not finite. These are called improper integrals, and are defined by

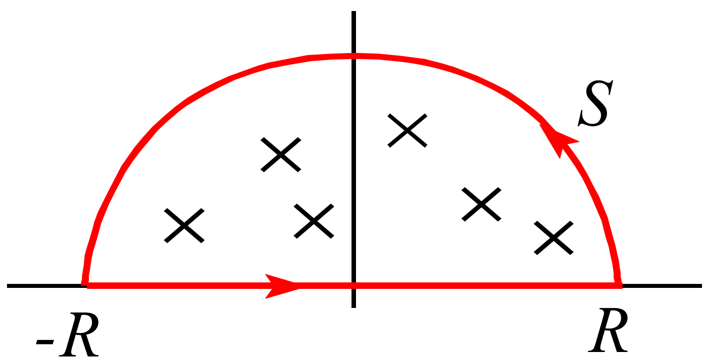
$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{R\to\infty} \int_{-R}^{R} f(x)\,dx$$

Assume that the $f(x) = \dfrac{P(x)}{Q(x)}$ is a real rational function with

- $Q(x) \neq 0$ for all real x (i.e., no real poles)
- $\text{degree}[Q(x)] \geq \text{degree}[P(x)] + 2$

Improper Integrals of Rational Functions (cont.)

Consider the complex integral $\oint_C f(z)\,dz$ with C as indicated in the figure below. Since $f(x)$ is rational, $f(z)$ will have a finite number of poles in the upper half-plane, and if we choose R large enough, C encloses all these poles.

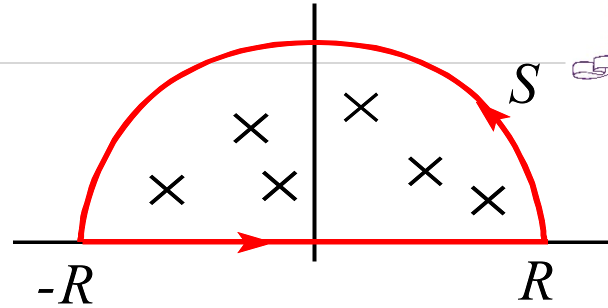


Note that C consists of a straight path from $-R$ to R and a half circle S on the upper plane.

Improper Integrals…

Then

$$\oint_C f(z)\,dz = \int_{-R}^{R} f(z)\,dz + \int_S f(z)\,dz \qquad (*)$$



The 2nd condition, i.e., degree $[Q(x)] \geq$ degree $[P(x)] + 2$, implies that if we set

$$Q(z) = q_n z^n + q_{n-1} z^{n-1} + \cdots + q_0, \quad P(z) = p_m z^m + p_{m-1} z^{m-1} + \cdots + p_0$$

then, $n \geq m + 2$. Also, for z on S when R is sufficiently large,

$$\left|z^2 f(z)\right| = \left|\frac{z^2 P(z)}{Q(z)}\right| = \left|\frac{p_m z^{m+2} + p_{m-1} z^{m+1} + \cdots}{q_n z^n + q_{n-1} z^{n-1} + \cdots}\right| = \left|\frac{p_m + p_{m-1} z^{-1} + \cdots}{z^{n-m-2}\left(q_n + q_{n-1} z^{-1} + \cdots\right)}\right|$$

$$\leq \frac{|p_m| + |p_{m-1}| \cdot |z|^{-1} + \cdots}{|z|^{n-m-2} \cdot \left||q_n| - |q_{n-1}| \cdot |z|^{-1} - \cdots\right|} = \frac{|p_m| + |p_{m-1}| \cdot R^{-1} + \cdots}{R^{n-m-2} \cdot \left||q_n| - |q_{n-1}| \cdot R^{-1} - \cdots\right|}$$

$$\leq \left|\frac{p_m}{q_n}\right| \cdot \frac{\alpha}{R^{n-m-2}} \leq k < \infty, \quad \text{for some } \alpha > 0 \text{ and } k > 0$$

$$\Rightarrow \quad |f(z)| \leq \frac{k}{|z^2|} = \frac{k}{R^2} = M$$

Improper Integrals of Rational Functions (cont.)

By *ML*-inequality,

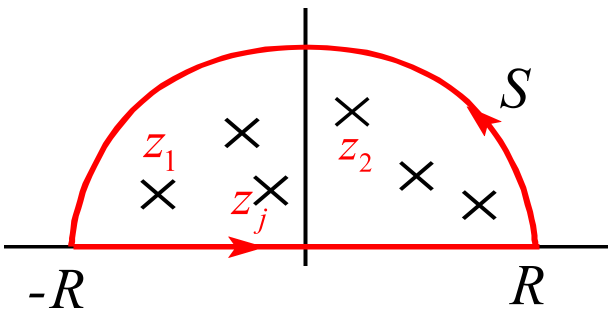
$$\left|\int_S f(z)\,dz\right| \le ML = \frac{k}{R^2}\pi R = \frac{k\pi}{R} \to 0 \quad \text{as} \quad R \to \infty \qquad \text{(Result S)}$$

Consequently, $\lim_{R\to\infty}\int_S f(z)\,dz = 0$. From the equation $(*)$ on the previous slide, we therefore have that

$$\lim_{R\to\infty}\int_{-R}^{R} f(z)\,dz = \oint_C f(z)\,dz$$

or

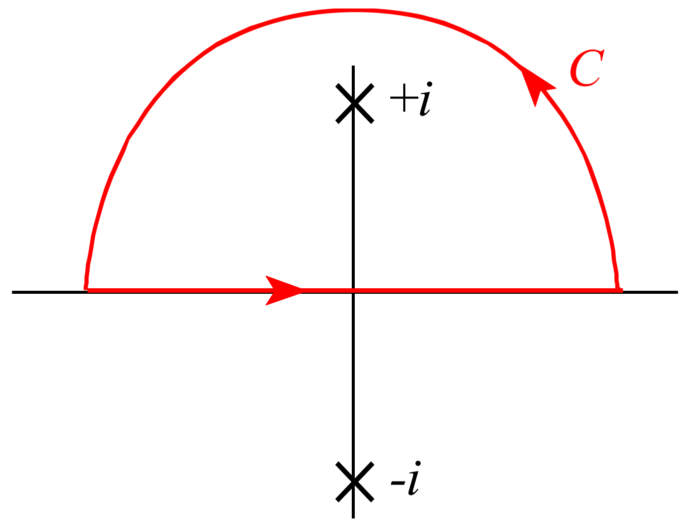
$$\int_{-\infty}^{\infty} f(x)\,dx = 2\pi i \sum_j \text{Res}(f, z_j)$$



where the sum is taken over all the poles in the upper half-plane.

Example 15

(a) Calculate $\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$



Let $f(z) = \frac{1}{1+z^2} = \frac{1}{(z+i)(z-i)}$

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = 2\pi i \operatorname{Res}(f, i)$$

$$= 2\pi i \frac{1}{2i}$$

$$= \pi$$

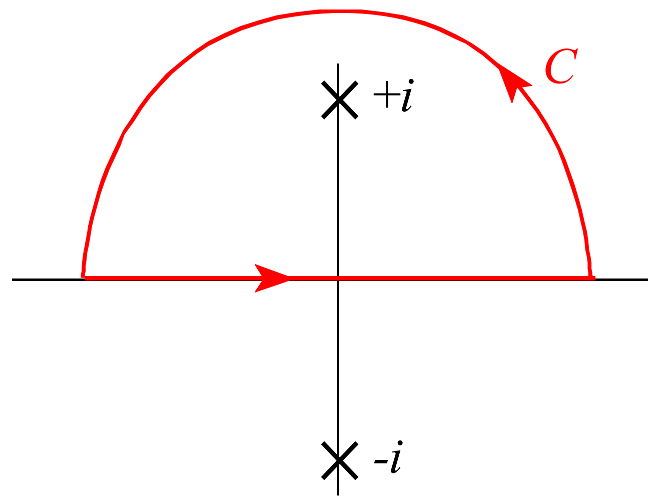
$$\operatorname{Res}(f, z_0) = \lim_{z \to z_0} (z - z_0) f(z)$$

Example 15

$$\text{Res}(f, z_0) = \frac{1}{(n-1)!} \lim_{z \to z_0} \left[\frac{d^{n-1}}{dz^{n-1}} \left[(z - z_0)^n f(z) \right] \right]$$

(b) Calculate $\displaystyle\int_{-\infty}^{\infty} \frac{1}{(1+x^2)^3} dx$

Let $f(z) = \dfrac{1}{(1+z^2)^3} = \dfrac{1}{(z+i)^3 (z-i)^3}$



$$\int_{-\infty}^{\infty} \frac{1}{(1+x^2)^3} dx = 2\pi i \,\text{Res}(f, i)$$

$$= \pi i \lim_{z \to i} \frac{12}{(z+i)^5}$$

$$= \pi i \left(\frac{-6i}{16} \right)$$

$$= \frac{3\pi}{8}$$

Example 15

(c) Calculate $\int_{-\infty}^{\infty} \frac{1}{4+x^4}\,dx$. Let $f(z)=\frac{1}{z^4+4}$ and its poles are given by

$$z^4+4=0 \Rightarrow z^4=-4=4\,e^{i(2n+1)\pi} \Rightarrow z=(4)^{1/4}\,e^{i(2n+1)\pi/4}$$

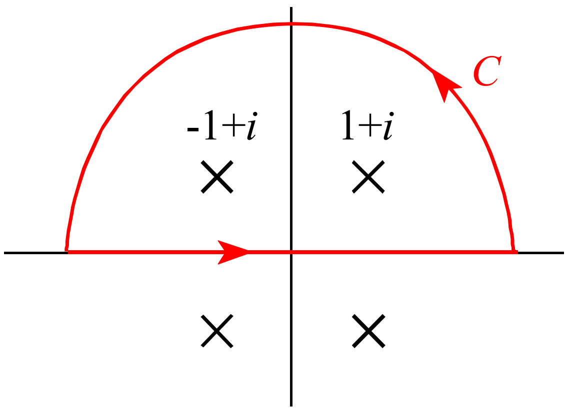
$$\Rightarrow z=\sqrt{2}\,e^{i(2n+1)\pi/4}, \qquad n=0,\,1,\,2,\,3,\cdots$$

$$\Rightarrow z_1=\sqrt{2}\,e^{i\pi/4} = 1+i,\ n=0$$

$$z_2=\sqrt{2}\,e^{i3\pi/4} = -1+i,\ n=1$$

$$z_3=\sqrt{2}\,e^{i5\pi/4} = -1-i,\ n=2$$

$$z_4=\sqrt{2}\,e^{i7\pi/4} = 1-i,\ n=3$$



Example 15

Then

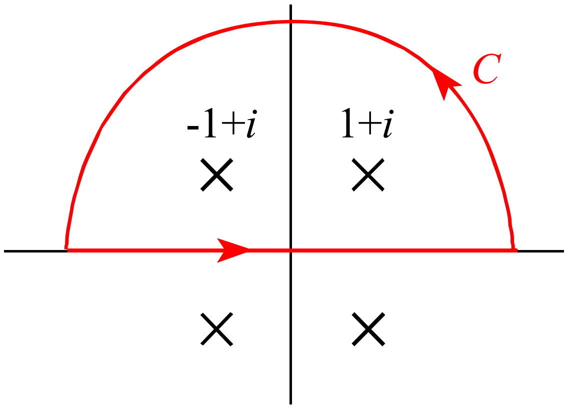
$$\int_{-\infty}^{\infty} \frac{1}{4+x^4}\,dx = 2\pi i\left[\mathrm{Res}(f,1+i)+\mathrm{Res}(f,-1+i)\right]$$

$$= 2\pi i\left\{\left[\frac{1}{4z^3}\right]_{z=1+i}+\left[\frac{1}{4z^3}\right]_{z=-1+i}\right\}$$

$$= 2\pi i\left[\frac{-1}{16}(1+i)+\frac{1}{16}(1-i)\right]$$

$$= 2\pi i\left(\frac{-i}{8}\right)$$

$$= \frac{\pi}{4}$$



$$\mathrm{Res}(f,z_0)=\frac{A(z_0)}{B'(z_0)}$$

Improper Integrals of Fourier-type Functions

Consider integrals of the form

$$\int_{-\infty}^{+\infty} f(x)\cos mx\,dx \qquad \text{or} \qquad \int_{-\infty}^{+\infty} f(x)\sin mx\,dx$$

Assume that $f(x) = P(x)/Q(x)$ is a real rational function with

- $Q(x) \neq 0$ for all real x (i.e., no real poles)
- degree $[Q(x)] \geq$ degree $[P(x)] + \mathbf{1}$
- $m > 0$

Improper Integrals of Fourier-type Functions

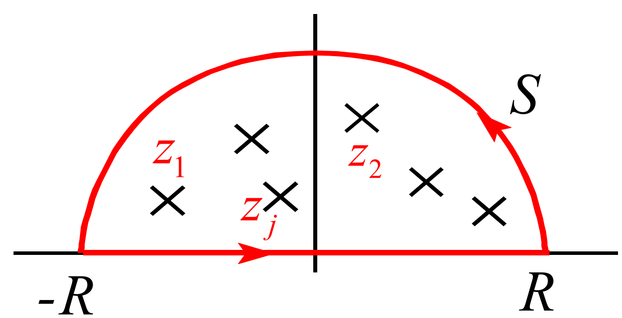
Consider the complex integral

$$\oint_C f(z)e^{imz}\,dz = \int_{-R}^{R} f(z)e^{imz}dz + \int_S f(z)e^{imz}dz = 2\pi i \sum_j \text{Res}(f\,e^{imz}, a_j)$$

We will show (i.e., Theorem X next page) that under the conditions $m > 0$ and degree $[Q(x)] \ge$ degree $[P(x)] + 1$, we have $\oint_S f(z)e^{imz}\,dz = 0$ as $R \to \infty$. Noting that $e^{imz} = \cos mz + i \sin mz$, it can be shown

$$\int_{-\infty}^{\infty} f(x)\cos mx\,dx = \text{Re}\left[2\pi i \sum_j \text{Res}(f\,e^{imz}, z_j)\right]$$

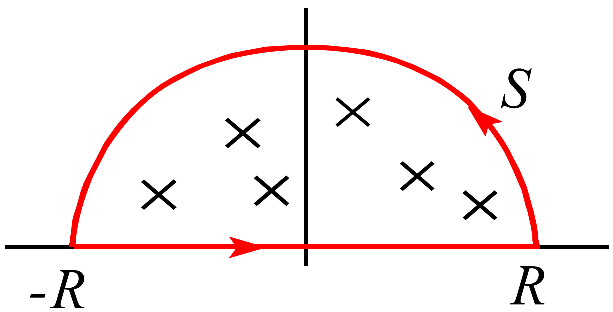
$$\int_{-\infty}^{\infty} f(x)\sin mx\,dx = \text{Im}\left[2\pi i \sum_j \text{Res}(f\,e^{imz}, z_j)\right]$$



Theorem X

If $g(z)=\dfrac{P(z)}{Q(z)}$ with $\text{degree}[Q(x)]\ge \text{degree}[P(x)]+1$ and $m>0$, then

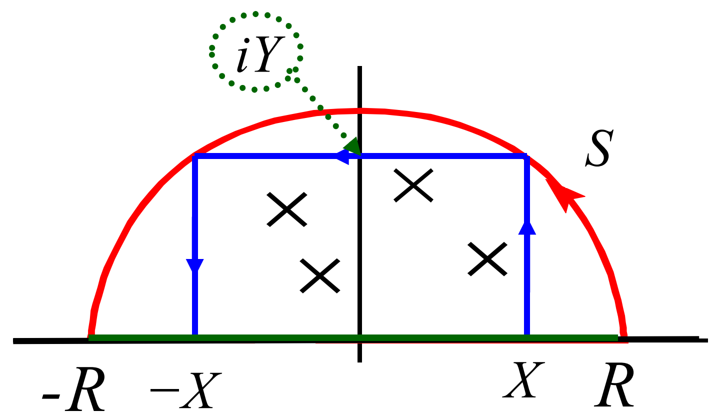
$$\lim_{R\to\infty}\int_S g(z)e^{imz}\,dz=0$$



There is no name for this theorem. As such, for easy references, we call it Theorem X. The result has been used earlier in deriving improper integrals of Fourier-type. It will be used later few more times.

Proof of Theorem X

Observing the curves on the right, if we let $X \to \infty$ ($\Rightarrow Y \to \infty$ and $R \to \infty$), the integral of the function along S is the same as it along the blue straight lines. Under that



$$\text{degree } [Q(x)] \ge \text{degree } [P(x)] + 1$$

and along the straight line from $X + i0$ to $X + iY$, we have

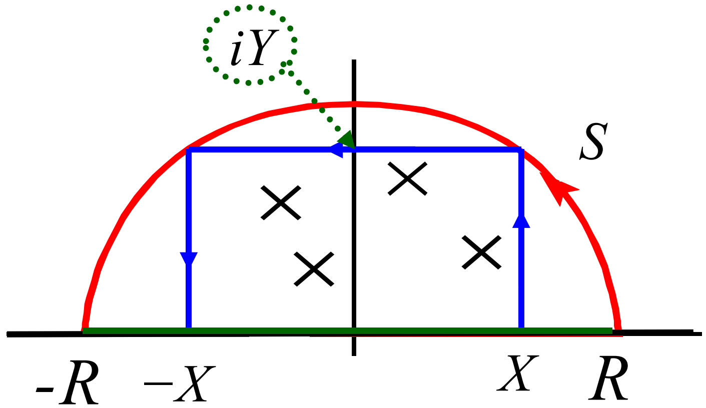
$$|zg(z)| \le K \quad \Rightarrow \quad |g(z)| \le \frac{K}{|z|} = \frac{K}{|X+iy|} \le \frac{K}{X}, \ |e^{imz}| = \left|e^{im(X+iy)}\right| = |e^{imX}| \cdot |e^{-my}| = e^{-my}$$

and thus

$$\left| \int_{X+i0}^{X+iY} g(z) e^{imz} dz \right| \le \frac{K}{X} \int_0^Y e^{-my} dy = \frac{K}{X} \cdot \frac{1}{m}\left(1 - e^{-mY}\right) < \frac{K}{mX} \to 0 \ \text{ as } \ X \to \infty$$

Proof of Theorem X (cont.)

Similarly, we can show the integral along the straight line from $-X+iY$ to $-X+i0$ has a same bound, i.e.,



$$\left|\int_{-X+iY}^{-X+i0} g(z)e^{imz}dz\right| < \frac{K}{mX} \to 0 \text{ as } X \to \infty$$

For the integration along the line from $X+iY$ to $-X+iY$, we have

$$|zg(z)| \le K \quad \Rightarrow \quad |g(z)| \le \frac{K}{|z|} = \frac{K}{|x+iY|} \le \frac{K}{Y}, \; |e^{imz}| = \left|e^{im(x+iY)}\right| = |e^{imx}| \cdot |e^{-mY}| = e^{-mY}$$

and thus

$$\left|\int_{X+iY}^{-X+iY} g(z)e^{imz}dz\right| \le \frac{Ke^{-mY}}{Y}\int_{-X}^{X} dx = 2K\left(\frac{X}{Y}e^{-mY}\right) \to 0 \text{ as } X \to \infty, Y \to \infty$$

QED

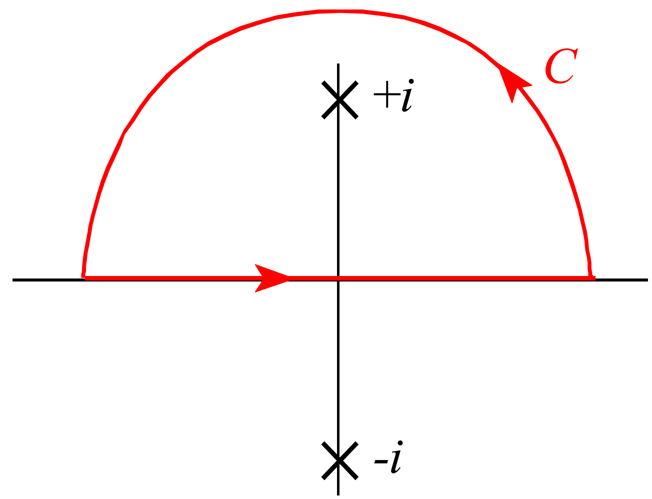
Example 16

(a) $$\int_{-\infty}^{\infty} \frac{x\cos x}{x^2+1}\,dx = \mathrm{Re}\left[2\pi i\,\mathrm{Res}\left(\frac{z\,e^{iz}}{z^2+1}, i\right)\right]$$

$$= \mathrm{Re}\left[2\pi i\left(\frac{z\,e^{iz}}{2z}\right)_{z=i}\right]$$

$$= \mathrm{Re}\left[\frac{\pi i}{e}\right]$$

$$= 0$$



$$\mathrm{Res}(f, z_0) = \frac{A(z_0)}{B'(z_0)}$$

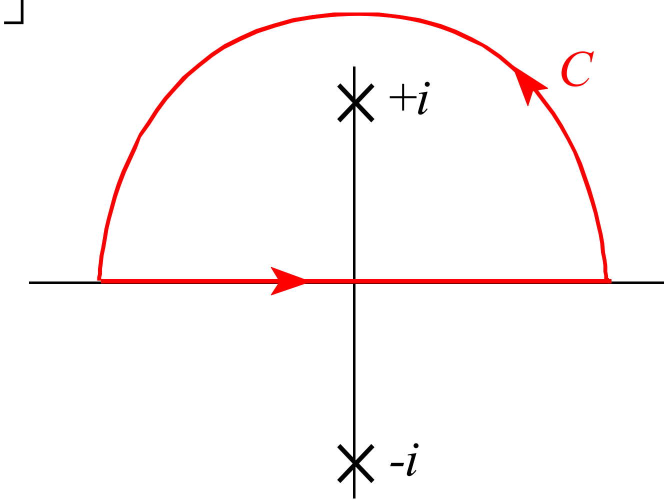
Example 16

(b) $$\int_{-\infty}^{\infty} \frac{x \sin x}{x^2+1}\,dx = \text{Im}\left[2\pi i\,\text{Res}\left(\frac{z\,e^{iz}}{z^2+1}, i\right)\right]$$

$$= \text{Im}\left[2\pi i\left[\frac{ze^{iz}}{2z}\right]_{z=i}\right]$$

$$= \text{Im}\left[\frac{\pi i}{e}\right]$$

$$= \frac{\pi}{e}$$



$$\text{Res}(f, z_0) = \frac{A(z_0)}{B'(z_0)}$$

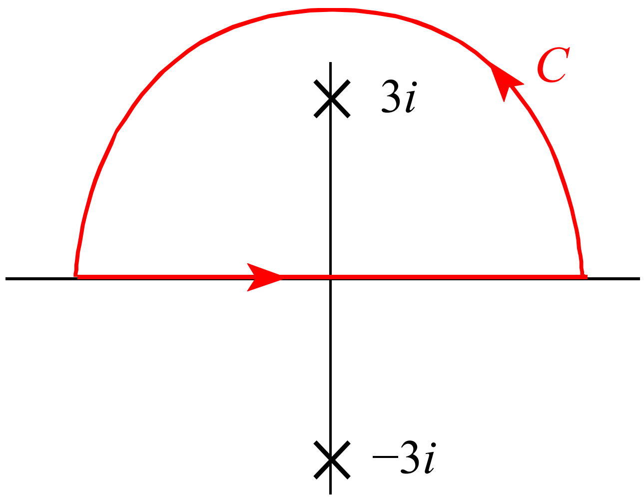
Example 16

(c) $$\int_{-\infty}^{\infty} \frac{\cos 2x}{9+x^2}\,dx = \mathrm{Re}\left[2\pi i\,\mathrm{Res}\left(\frac{e^{i2z}}{9+z^2}, 3i\right)\right]$$

$$= \mathrm{Re}\left[2\pi i\left(\frac{e^{i2z}}{2z}\right)_{z=3i}\right]$$

$$= \mathrm{Re}\left[\frac{\pi e^{-6}}{3}\right]$$

$$= \frac{\pi e^{-6}}{3}$$



$$\mathrm{Res}(f, z_0) = \frac{A(z_0)}{B'(z_0)}$$

Example 16

(d) $$\int_{-\infty}^{\infty} \frac{\sin x}{x^2+2x+2}\,dx = \mathrm{Im}\left[2\pi i \,\mathrm{Res}\left(\frac{e^{iz}}{z^2+2z+2}, -1+i\right)\right]$$

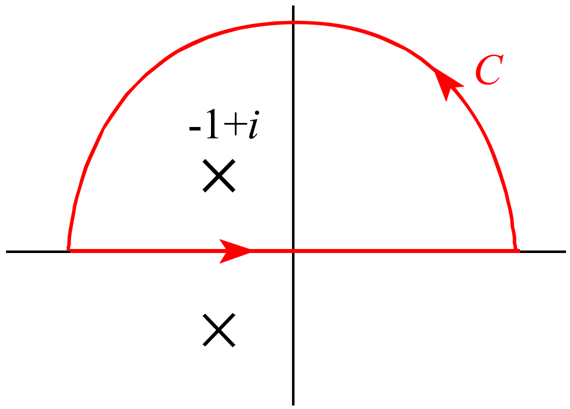
$$= \mathrm{Im}\left[2\pi i \left(\frac{e^{iz}}{2z+2}\right)_{z=-1+i}\right]$$

$$= \mathrm{Im}\left[2\pi i \left(\frac{e^{-i}\,e^{-1}}{2i}\right)\right]$$

$$= \mathrm{Im}\left[\frac{\pi e^{-i}}{e}\right]$$

$$= \mathrm{Im}\left[\frac{\pi(\cos 1 - i\sin 1)}{e}\right]$$

$$= -\frac{\pi}{e}\sin 1$$



$$\mathrm{Res}(f, z_0) = \frac{A(z_0)}{B'(z_0)}$$

*Integrals of Rational Functions (IRF-1)

Given a rational function of z, $f(z) = \dfrac{P(z)}{Q(z)}$,

with real or complex coefficients and with

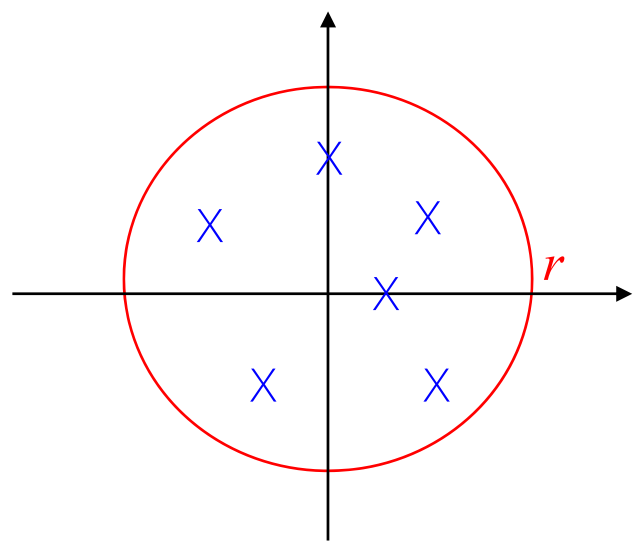
- $\text{degree}[Q(z)] \ge \text{degree}[P(z)] + 2$,

we then have

$$\oint_{|z|=r} f(z)\,dz = \oint_{|z|=r} \frac{P(z)}{Q(z)}\,dz = 0$$

for any $r > 0$ so long as $Q(z) = 0$ has all its roots (or equivalently $f(z)$ has all its singular points) inside the circle $|z| = r$.

Remark: The idea came from an example presented by Liu Haitong in an ESTR 2014 Interactive Tutorial Class…



Proof of IRF-1…

As in the proof of Result S, the condition, i.e., degree $[Q(x)] \geq$ degree $[P(x)] + 2$, implies that

$$Q(z) = q_n z^n + q_{n-1} z^{n-1} + \cdots + q_0, \quad P(z) = p_m z^m + p_{m-1} z^{m-1} + \cdots + p_0$$

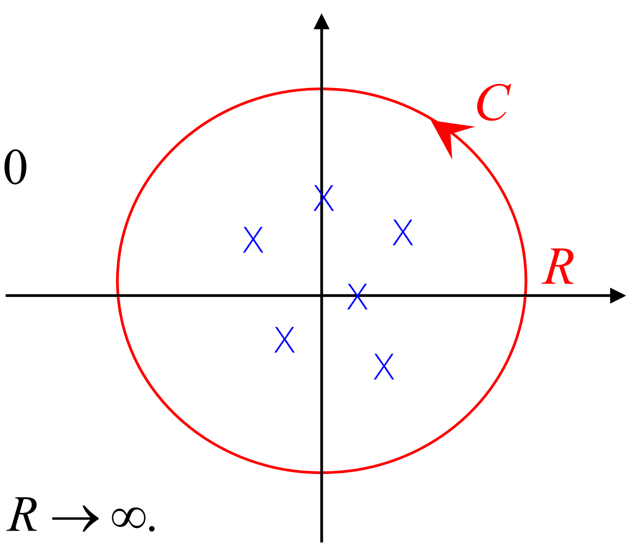
with $n \geq m + 2$. Then, for z on C when R is sufficiently large, we have

$$\left| z^2 f(z) \right| = \left| \frac{z^2 P(z)}{Q(z)} \right| = \left| \frac{p_m z^{m+2} + p_{m-1} z^{m+1} + \cdots}{q_n z^n + q_{n-1} z^{n-1} + \cdots} \right|$$

$$\leq \left| \frac{p_m}{q_n} \right| \cdot \frac{\alpha}{|z|^{n-m-2}} \leq k < \infty, \quad \text{for some } \alpha > 0,\ k > 0$$

$$\Rightarrow \quad \left| f(z) \right| \leq \frac{k}{|z^2|} = \frac{k}{R^2}$$

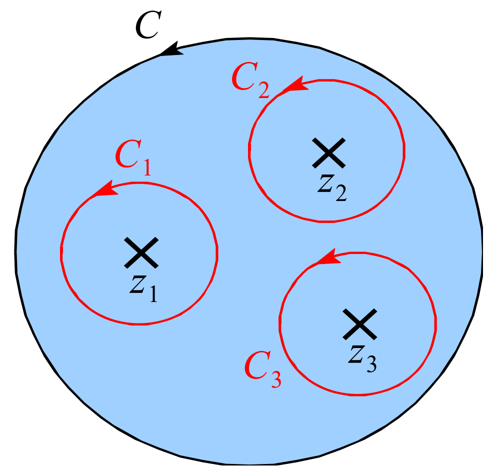
$$\Rightarrow \quad \left| \oint_{|z|=R} f(z)\, dz \right| \leq ML = \frac{k}{R^2} 2\pi R = \frac{2k\pi}{R} \to 0 \quad \text{as } R \to \infty.$$



Proof of IRF-1 (cont.)…

In view of the fact that

$$\int_C f(z)dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz + \int_{C_3} f(z)\,dz$$

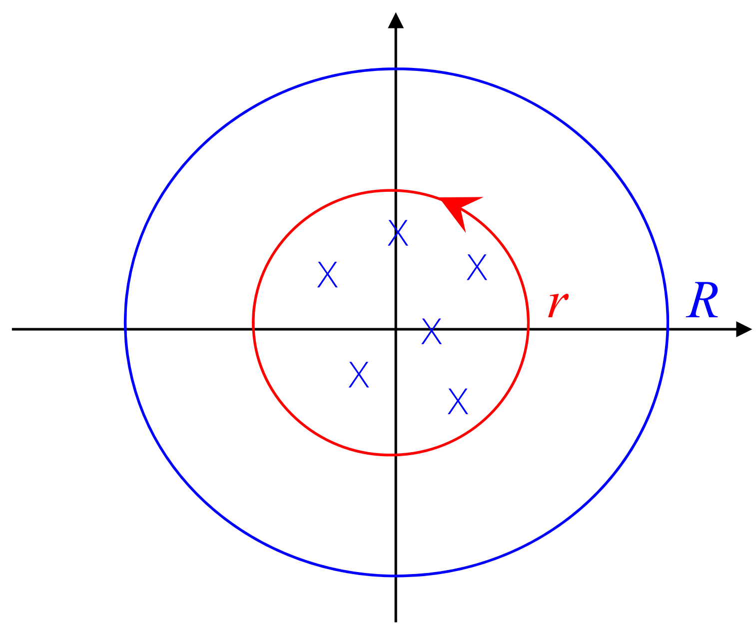


we have

$$\oint_{|z|=r} f(z)\,dz = \oint_{|z|=R} f(z)\,dz$$

$$= \oint_{|z|=R} \frac{P(z)}{Q(z)}dz = 0 \text{ as } R \to \infty,$$

for any $r > 0$ so long as $Q(z) = 0$ has all its roots (or equivalently $f(z)$ has all its singular points) inside the circle $|z| = r$.



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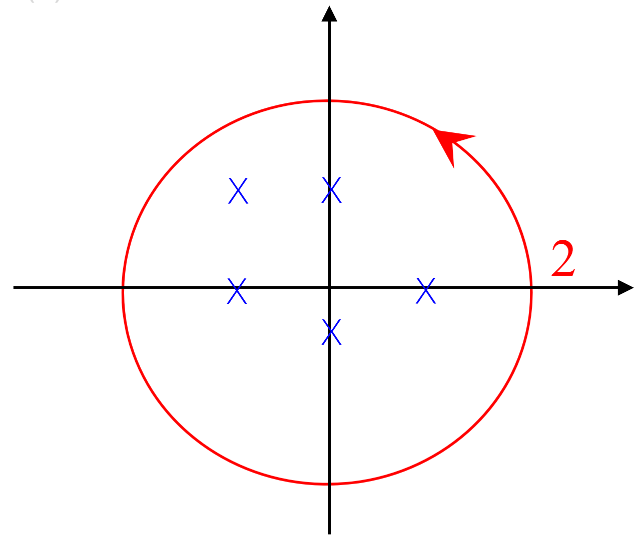
Example 17

Example 13(d)

Compute $\displaystyle\oint_{|z|=2} \frac{2z}{(z-1)(z+1)(z-i)(z+\frac{i}{2})(z+1-i)}\,dz$



By the result of IRF-1, we have

$$\oint_{|z|=2} \frac{2z}{(z-1)(z+1)(z-i)(z+\frac{i}{2})(z+1-i)}\,dz = 0.$$

We can verify this by finding all the residues (very tedious!), i.e.,

$$\text{Res}(f,1) = \left.\frac{2z}{(z+1)(z-i)(z+\frac{i}{2})(z+1-i)}\right|_{z=1} = \frac{1}{5}(1+i), \ \cdots\cdots$$

$$\text{Res}(f,-1) = \frac{1}{5}(1+3i),\ \text{Res}(f,i) = -\frac{2}{3},\ \text{Res}(f,-\tfrac{i}{2}) = -\frac{32}{195} - \frac{48}{195}i,\ \text{Res}(f,-1+i) = \frac{28}{65} - \frac{36}{65}i.$$

We have

$$\oint_{|z|=2} f(z)\,dz = 2\pi i\left[\overbrace{\left(\frac{39}{195}+\frac{39}{195}i\right)}^{\frac{1}{5}(1+i)} + \overbrace{\left(\frac{39}{195}+\frac{117}{195}i\right)}^{\frac{1}{5}(1+3i)} - \overbrace{\frac{130}{195}}^{\frac{2}{3}} - \left(\frac{32}{195}+\frac{48}{195}i\right) + \overbrace{\left(\frac{84}{195}-\frac{108}{195}i\right)}^{\frac{28}{65}-\frac{36}{65}i}\right] = 0. \checkmark$$

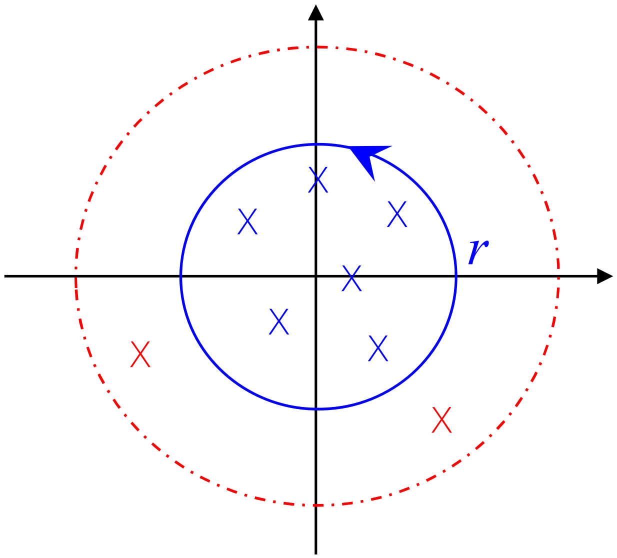
*Integrals of Rational Functions (IRF-2)

Given a rational function of z, $f(z)=\dfrac{P(z)}{Q(z)}$,

with real or complex coefficients and with

- $\text{degree}[Q(z)] \ge \text{degree}[P(z)] + 2$,

it then follows from the result of IRF-1,



$$\oint_{|z|=r} f(z)\,dz = 2\pi i \sum_{j=1}^{n_{\text{in}}} \text{Res}\,(f, z_j),$$ where z_j are all singular points inside $|z|=r$.

$$= -2\pi i \sum_{k=1}^{n_{\text{out}}} \text{Res}\,(f, z_k),$$ where z_k are all singular points outside $|z|=r$.

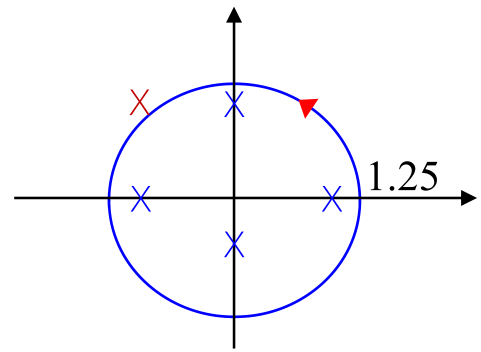
This result is particularly economical when n_{out} is significantly smaller than n_{in}.

It shows that the integration is somehow linked to the singular points outside the integration path as well.

Example 18

Compute

$$\oint_{|z|=1.25} \frac{2z}{(z-1)(z+1)(z-i)(z+\frac{i}{2})(z+1-i)}\,dz$$



By the result of IRF-2, we have

$$\oint_{|z|=1.25} f(z)dz = -2\pi i \cdot \mathrm{Res}(f,-1+i) = -2\pi i\left(\frac{28}{65}-\frac{36}{65}i\right) = \pi\left(\frac{72}{65}-\frac{56}{65}i\right)$$

where the residue was calculated in Example 17.

Instead of computing 4 residues for the singular points inside the path, we only need to work out one outside.

Homework Assignment No: 5 (Due in one week)

Question 5.1: Calculate $\oint\limits_{|z|=2} \frac{1}{z^2 + 2z + 2} dz$.

Question 5.2: Calculate $\oint\limits_{|z|=2} \frac{z}{z^2 + 2z + 2} dz$.

Question 5.3: Calculate $\oint\limits_{|z|=6} \frac{z^2 + 1}{(z+1)^5 (z-2)^4 (z+3)^3 (z-4)^2 (z+5)(z-7)}$

Question 5.4: Calculate $\int_{-\infty}^{\infty} \frac{1}{(x^2+1)(x^2+9)} dx$.

Question 5.5: Calculate $\int_{-\infty}^{\infty} \frac{1}{(x^2+1)^2} dx$.

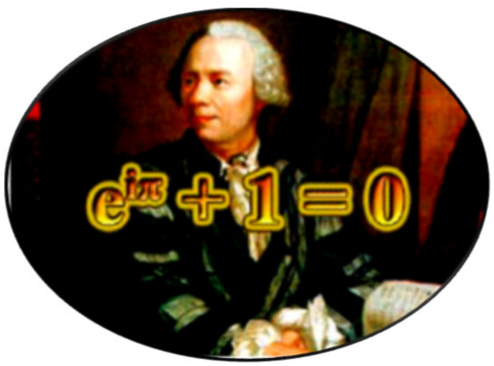
Question 5.6: Calculate $\int_{-\infty}^{\infty} \frac{x \sin \pi x}{x^2 + 2x + 5} dx$.

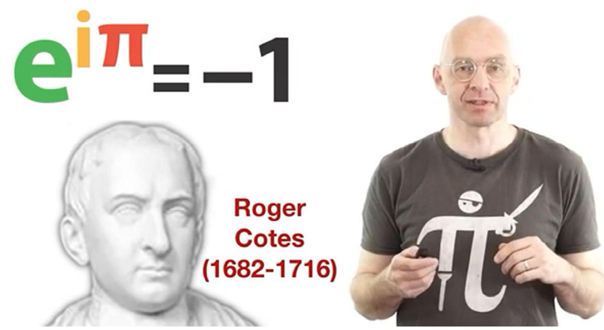
Question 5.7: Calculate $\int_{0}^{\infty} \frac{\cos x}{(x^2+1)^2} dx$.

Final Remarks on Complex Analysis…

In this course, we have learnt some special operations of -1, i.e., we start with $\sqrt{-1}$ and end up with the residue a_{-1}. If you know where this a_{-1} comes from, you know almost everything about what we have covered in the class.

Finally, also note that for a complex function, its singularity matters the most…





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This side of the story…

$$e^{ix} = \cos x + i \sin x$$

Another side of the story…

$$ix = \ln(\cos x + i \sin x)$$

That's all, folks!

Thank You!