ENGG 2420

Complex Analysis & Differential Equations for Engineers

Part 3: Partial Differential Equations

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Partial Differential Equations

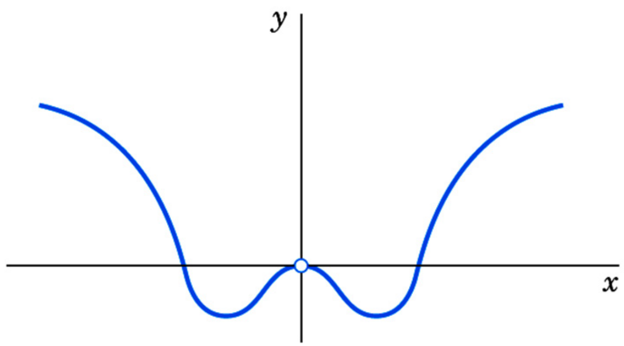
Partial Differential Equations – 1…

1. Even and odd functions, periodic functions
2. Fourier series of a periodic function
3. Fourier series: Half-range expansions
4. Concepts of partial differential equations
5. Heat equation (or diffusion equation)
6. Wave equation

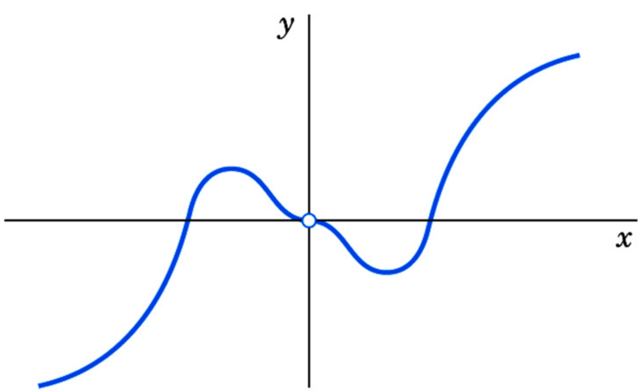
Even and Odd Functions

✓ During Fourier series analysis, it is useful to distinguish two classes of functions for which the Euler formulae for the coefficients can be simplified.

✓ The two classes are even and odd functions, which are characterized geometrically by the property of symmetry with respect to the y-axis and the origin, respectively.



even



odd

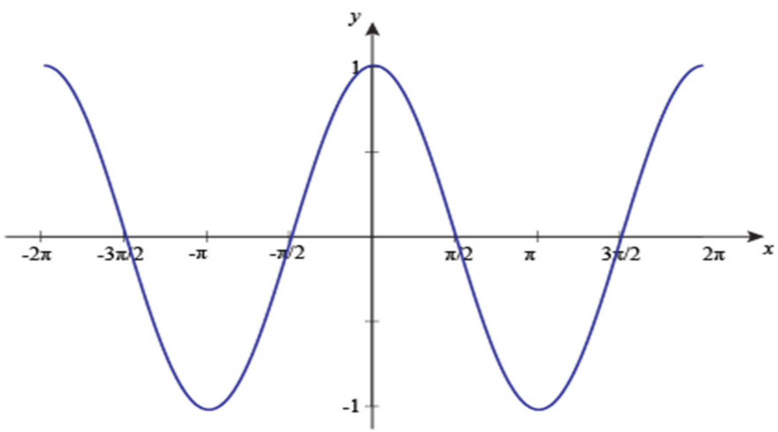
Definition of Even and Odd Functions

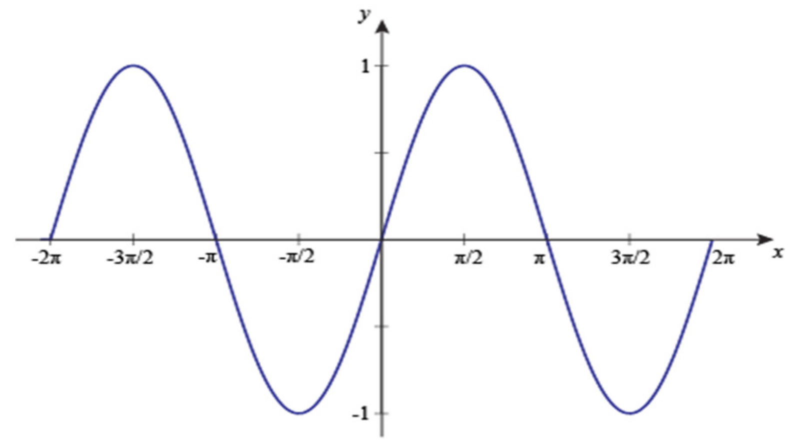
Let f be defined on an x interval, finite or infinite, that is centered at $x = 0$. We say that $f(x)$ is an **even function** if

$$f(-x) = f(x)$$ (i.e. the graph of f is symmetric about the y-axis.)

and an **odd function** if

$$f(-x) = -f(x)$$ (i.e. the graph of f is anti-symmetric about the origin.)





Even function examples:

$1,\ x^2,\ \cos x,\ |x|,\ 5x^6 - 2\sin^2 x$

Odd function examples:

$x,\ x^3,\ \sin x,\ 5x^7 - 2\sin^3 x$

Properties of Even and Odd Functions

- $even + even = even$

  i.e., the sum (difference) of two even functions is even.

- $even \times even = even$

  i.e., the product (quotient) of two even functions is even.

- $odd + odd = odd$

  i.e., the sum (difference) of two odd functions is odd.

- $odd \times odd = even$

  i.e., the product (quotient) of two odd functions is even.

- $even \times odd = odd$

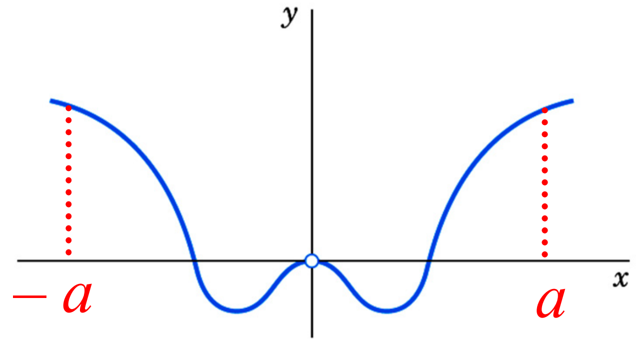
  i.e., the product (quotient) of an odd and an even function is odd.

Note: These properties can be verified directly from the definitions.

Two Useful Integral Properties

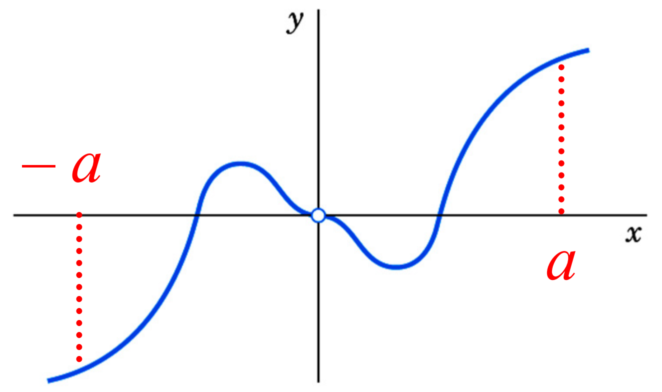
- If f is an even function, then

$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$$



- If f is an odd function, then

$$\int_{-a}^{a} f(x)\,dx = 0$$



Note: A given function is not necessarily even or odd. Every function can be uniquely decomposed into the sum of an even function f_e, and an odd function f_o as

$$f(x)=\frac{f(x)+f(-x)}{2}+\frac{f(x)-f(-x)}{2}=f_e(x)+f_o(x)$$

$$f_e(x)=\frac{f(x)+f(-x)}{2} \quad \Rightarrow \quad f_e(-x)=\frac{f(-x)+f(x)}{2}=f_e(x)$$

~ even ~

$$f_o(x)=\frac{f(x)-f(-x)}{2} \quad \Rightarrow \quad f_o(-x)=\frac{f(-x)-f(x)}{2}=-f_o(-x)$$

~ odd ~

Example

Decompose a function into the sum of an even function and an odd function

- ✓ Since $f(x) = e^x$ is neither symmetric nor anti-symmetric about $x = 0$.

  $\Rightarrow$ it is neither even nor odd.

- ✓ Putting $f(x) = e^x$ and $f(-x) = e^{-x}$ into

$$f(x) = \frac{f(x) + f(-x)}{2} + \frac{f(x) - f(-x)}{2} = f_e(x) + f_o(x)$$

gives

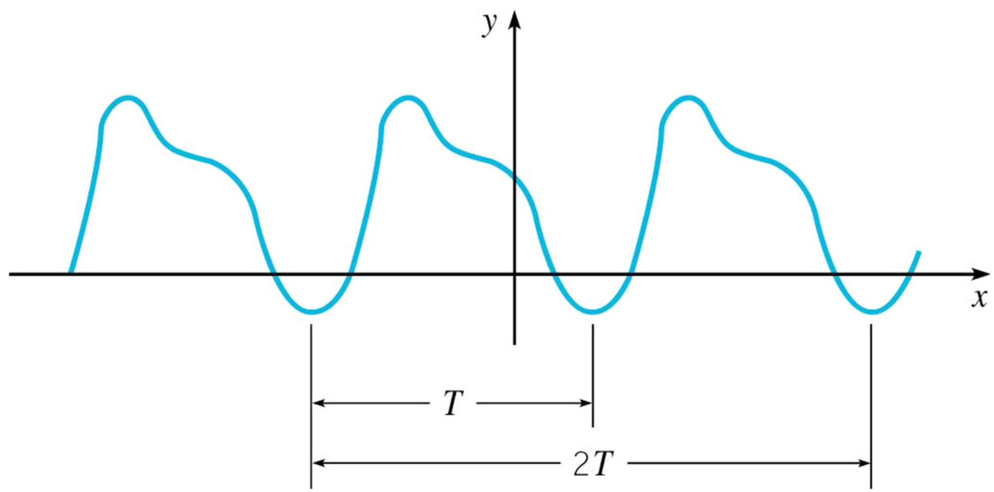
$$\underbrace{f_e(x) = \frac{e^x + e^{-x}}{2} = \cosh x}_{\text{even}} \qquad \underbrace{f_o(x) = \frac{e^x - e^{-x}}{2} = \sinh x}_{\text{odd}}$$

Periodic Functions

Given a function f defined over I, if there exists a positive constant T such that

$$f(x+T) = f(x)$$

for all real $x \in I$. Then, f is a **periodic function** of x with **period** T.



Fundamental Period

- For a periodic function of period T, $f(x+T) = f(x)$ for all x.
- Note that $2T$ is also a period, and so is any multiple of T.
- Of all these possible periods, if there exists a **smallest** one, that period is called **fundamental period** of f.
- **Example:**

$$\sin\left(\frac{m\pi x}{L}\right) = \sin(\omega x), \quad \cos\left(\frac{m\pi x}{L}\right) = \cos(\omega x)$$

are periodic with fundamental period $f = \dfrac{\omega}{2\pi} = \dfrac{m}{2L} \;\Rightarrow\; T = \dfrac{1}{f} = \dfrac{2L}{m}.$

Partial Differential Equations – 2…

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4. Concepts of partial differential equations
5. Heat equation (or diffusion equation)
6. Wave equation

Fourier Series of a Periodic Function

Given a periodic function $f(x)$ with fundamental period 2ℓ, we can define a trigonometric series as follows

$$\text{FS } f = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{n\pi x}{\ell}\right) + b_n \sin\left(\frac{n\pi x}{\ell}\right)\right)$$

This series is called **Fourier Series** of $f(x)$.

where the coefficients are given by the **Euler formulas**

$$a_0 = \frac{1}{2\ell}\int_{-\ell}^{\ell} f(x)dx,$$

$$a_n = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\cos\left(\frac{n\pi x}{\ell}\right)dx, \quad n = 1,2,\cdots$$

$$b_n = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\sin\left(\frac{n\pi x}{\ell}\right)dx, \quad n = 1,2,\cdots$$

and are known as the **Fourier coefficients** of $f(x)$.

Joseph Fourier
(1768–1830)
French Mathematician

Piecewise Continuous Functions

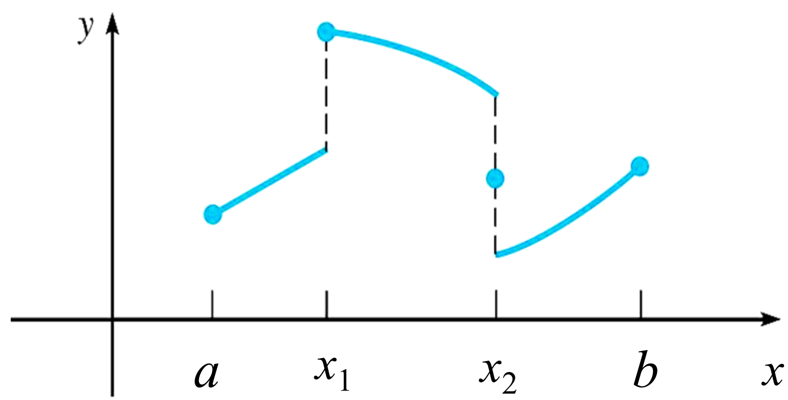
A function $f(x)$ is piecewise continuous on an interval $[a, b]$ if this interval can be partitioned by a finite number of points

$a = x_0 < x_1 < \ldots < x_n = b$ such that

(1) $f(x)$ is continuous on each (x_k, x_{k+1})

(2) $$\left| \lim_{x \to x_k^+} f(x) \right| < \infty, \quad k = 0, \ldots, n-1$$

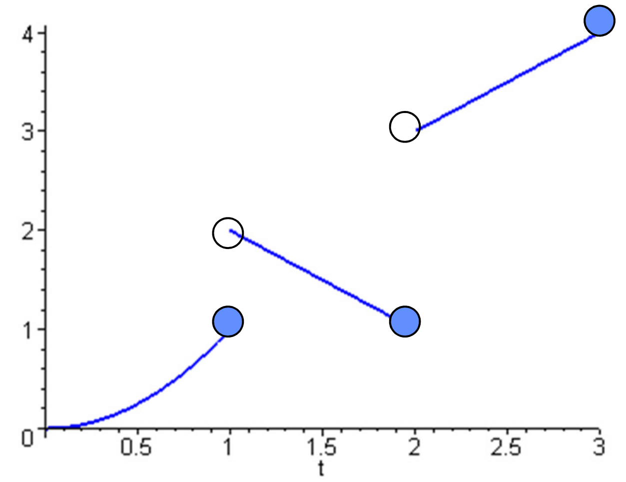
(3) $$\left| \lim_{x \to x_{k+1}^-} f(x) \right| < \infty, \quad k = 0, \ldots, n-1$$



In other words, a function $f(x)$ is piecewise continuous on $[a, b]$ if it is continuous on $[a, b]$ except for a finite number of jump discontinuities.

For example, consider the following piecewise-defined function $f(x)$,

$$f(x) = \begin{cases} x^2, & 0 \le x \le 1 \\ 3 - x, & 1 < x \le 2 \\ x + 1 & 2 < x \le 3 \end{cases}$$



Obviously, $f(x)$ is piecewise continuous on [0, 3].

Left- and right-hand limits of f:

$$f(x^-) \equiv \lim_{h \to 0} f(x - h); \quad f(x^+) \equiv \lim_{h \to 0} f(x + h)$$

where $h \to 0$ through positive values.

Convergence of Fourier Series

<u>Theorem</u>: Let $f(x)$ be 2ℓ -periodic, and let $f(x)$ and $f'(x)$ be **piecewise continuous** on $[-\ell, \ell]$. Then, for any x in $(-\ell, \ell)$, we have

$$a_0 + \sum_{n=1}^{\infty} \left\{ a_n \cos\frac{n\pi x}{\ell} + b_n \sin\frac{n\pi x}{\ell} \right\} = \frac{1}{2}\left\{ f(x^+) + f(x^-) \right\} ,$$

where the a_n's and b_n's are given by the Fourier coefficients of $f(x)$.

It converges to the average value of the **left- and right-hand limits** of $f(x)$.

Remark: If $f(x)$ is continuous, then

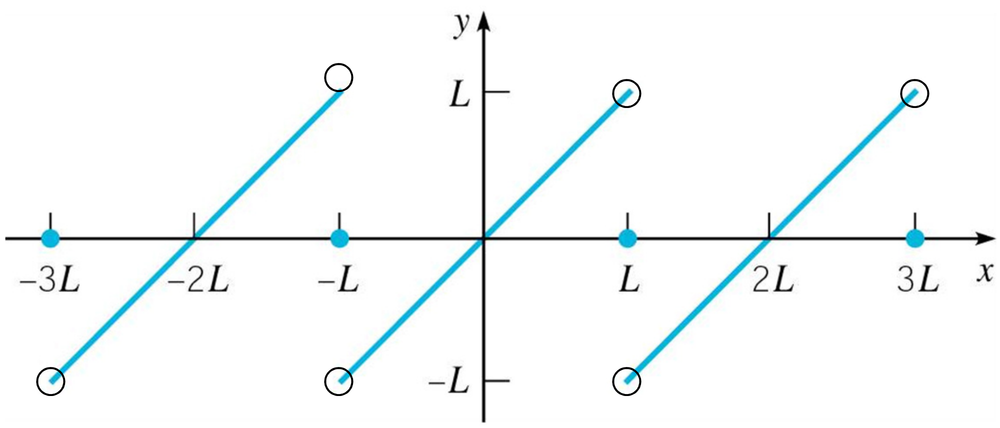
$$a_0 + \sum_{n=1}^{\infty} \left\{ a_n \cos\frac{n\pi x}{\ell} + b_n \sin\frac{n\pi x}{\ell} \right\} = \frac{1}{2}\left\{ f(x^+) + f(x^-) \right\} = f(x)$$

Example

Q1 (*Saw-tooth Wave*). Consider the function below.

$$f(x)=\begin{cases} x, & -L<x<L \\ 0, & x=\pm L \end{cases}, \qquad f(x+2L)=f(x)$$

This function represents a saw tooth wave, and is periodic with fundamental period *T = 2L.*



Find the Fourier series representation for this function.

Soln.: Fourier Series Representation

Given a periodic function $f(x)$ with fundamental period 2ℓ, the trigonometric Fourier series representation:

$$\text{FS } f = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{n\pi x}{\ell}\right) + b_n \sin\left(\frac{n\pi x}{\ell}\right)\right)$$

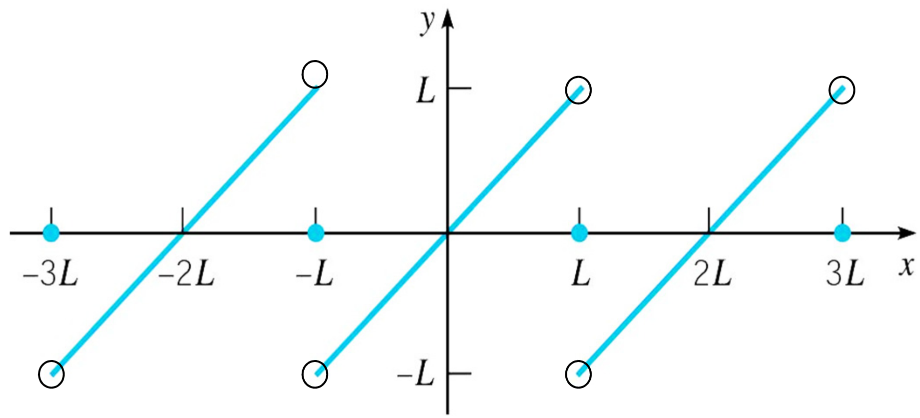
where

$$a_0 = \frac{1}{2\ell}\int_{-\ell}^{\ell} f(x)dx,$$

$$a_n = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\cos\left(\frac{n\pi x}{\ell}\right)dx, \quad n = 1,2,\cdots$$

$$b_n = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\sin\left(\frac{n\pi x}{\ell}\right)dx, \quad n = 1,2,\cdots$$

$$a_0 = \frac{1}{2\ell}\int_{-\ell}^{\ell} f(x)dx$$



$$a_0 = \frac{1}{2L}\int_{-L}^{L} xdx = 0$$

$$a_n = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\cos\left(\frac{n\pi x}{\ell}\right)dx, \;\; n = 1, 2, \cdots$$

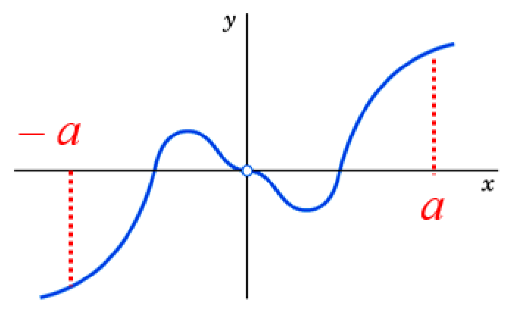
Because $x\cos\frac{n\pi x}{L}$ is an odd function

➢ $even \times odd = odd$

$$a_n = \frac{1}{L}\int_{-L}^{L} x\cos\frac{n\pi x}{L}\,dx = 0, \quad n = 1, 2, \cdots$$

➢ If f is an odd function, then

$$\int_{-a}^{a} f(x)\,dx = 0$$



$$b_n = \frac{1}{L}\int_{-L}^{L} x \sin\frac{n\pi x}{L}\,dx = \frac{1}{L}\int_{-L}^{L} x \frac{-L}{n\pi}\,d\cos\frac{n\pi x}{L}$$

$$= -\frac{1}{n\pi}\left(x\cos\frac{n\pi x}{L}\Big|_{-L}^{L} - \int_{-L}^{L}\cos\frac{n\pi x}{L}\,dx \right)$$

$$= -\frac{1}{n\pi}\left(2L\cos n\pi - \frac{L}{n\pi}\sin\frac{n\pi x}{L}\Big|_{-L}^{L} \right)$$

$$= -\frac{1}{n\pi}\left(2L(-1)^n - 0\right)$$

$$= \frac{2L}{n\pi}(-1)^{n+1}, \quad n = 1, 2, \cdots$$

➢ $odd \times odd = even$

It follows that the Fourier series of f is

$$\text{FS } f = \cancel{a_0} + \sum_{n=1}^{\infty}\left(\cancel{a_n}\cos\left(\frac{n\pi x}{\ell}\right) + b_n \sin\left(\frac{n\pi x}{\ell}\right)\right)$$

$$b_n = \frac{2L}{n\pi}(-1)^{n+1}, \quad n = 1,2,\ldots$$

$$f(x) = \frac{2L}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\frac{n\pi x}{L}$$

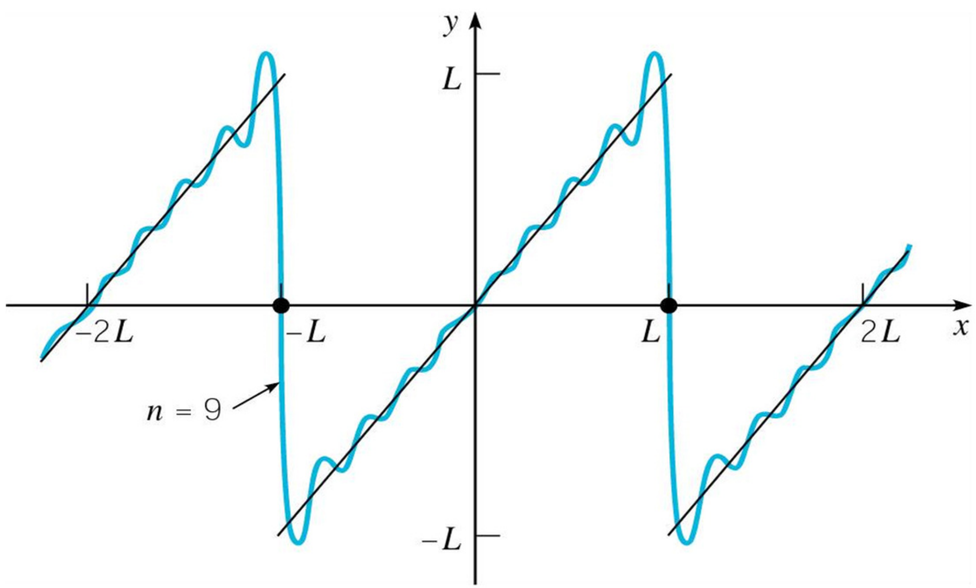
Note: Later we will find that f is an odd periodic function with period $2L$

$$\Rightarrow \quad a_n = 0$$

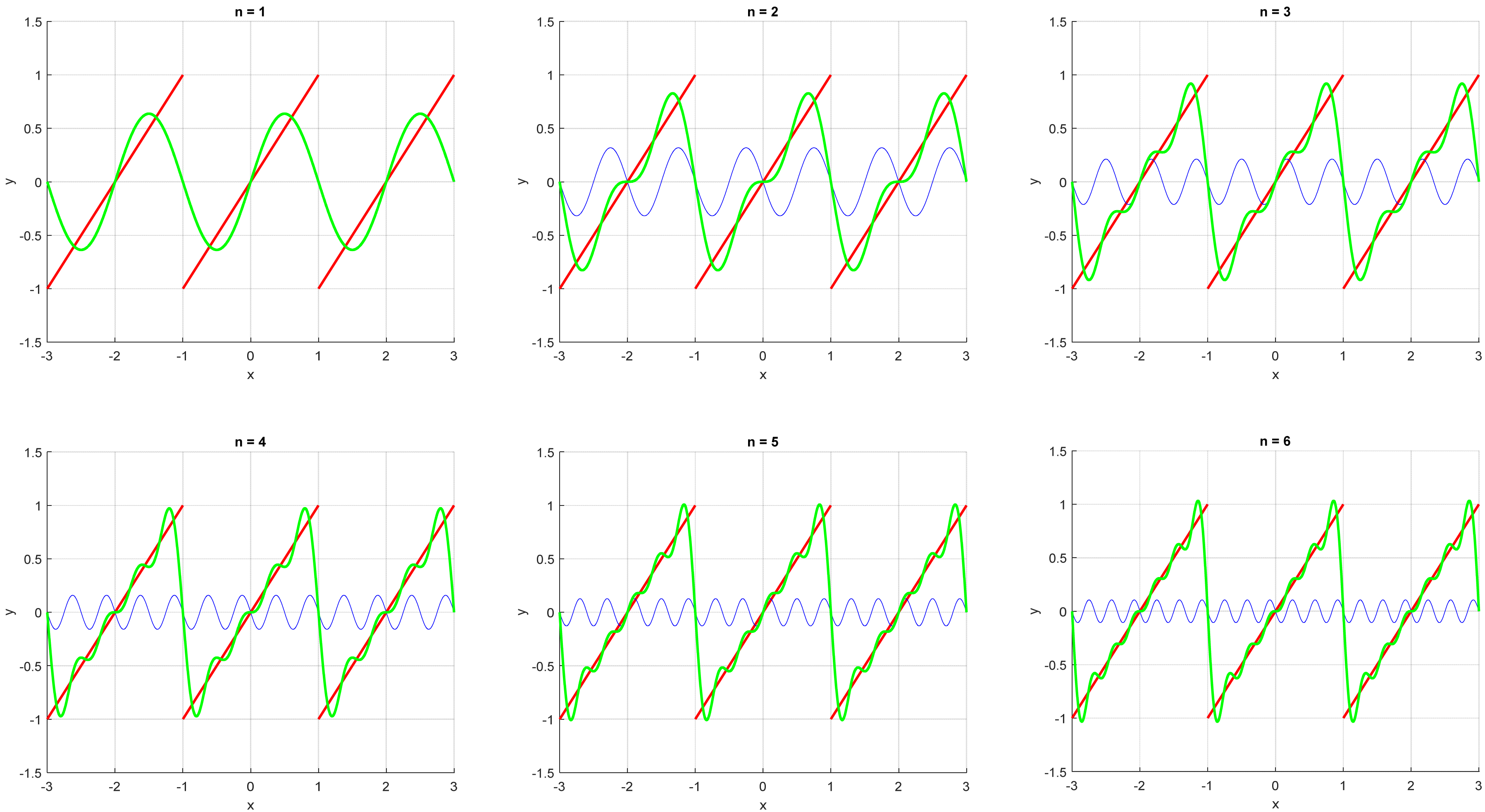
$$f(x)=\frac{2L}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\frac{n\pi x}{L}$$

The graphs of the partial sum $f_9(x)$ and f are given below

$$f_9(x)=\frac{2L}{\pi}\sum_{n=1}^{9}\frac{(-1)^{n+1}}{n}\sin\frac{n\pi x}{L}$$

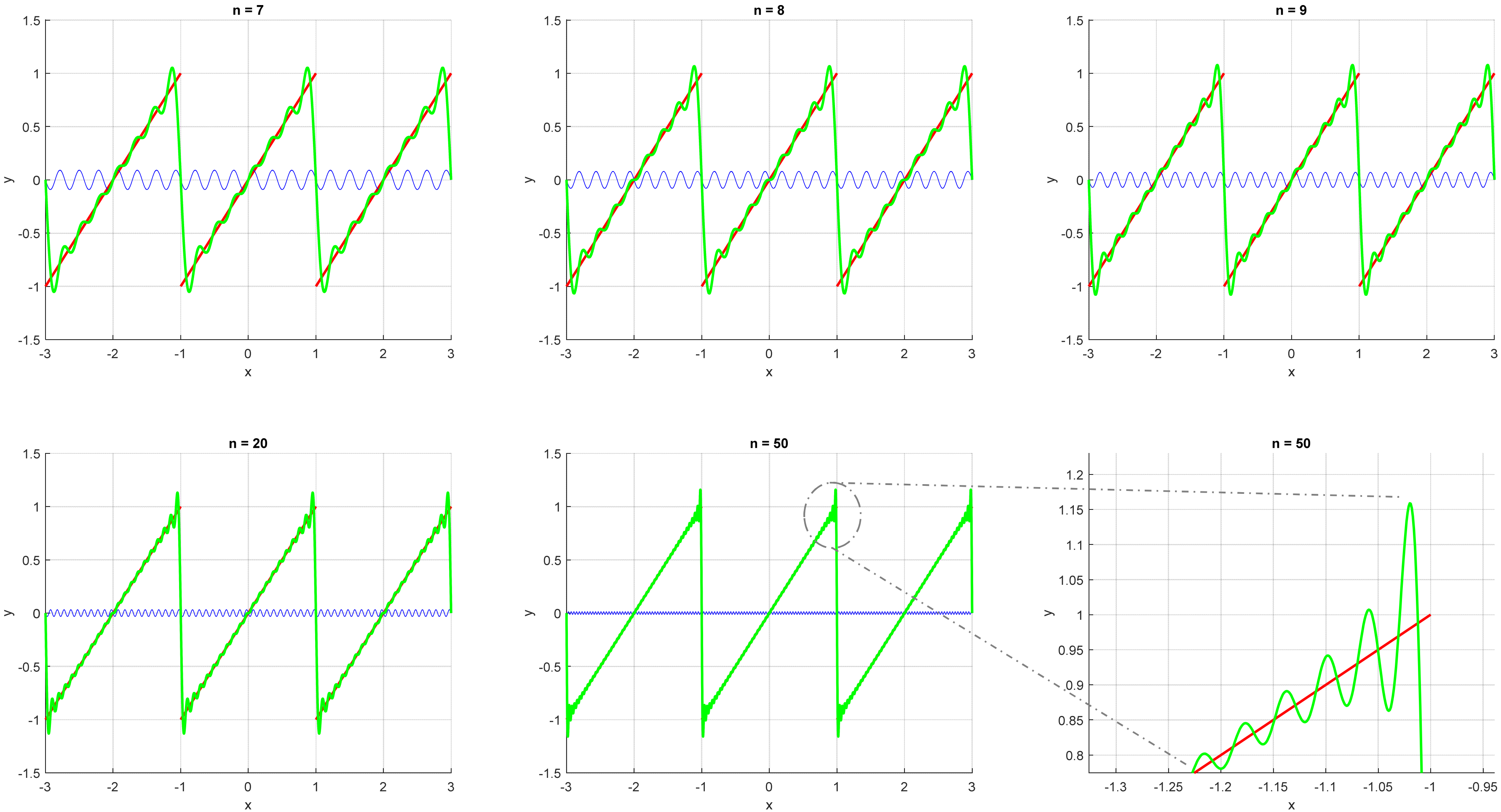


$$f(x)=\frac{2L}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\frac{n\pi x}{L}$$



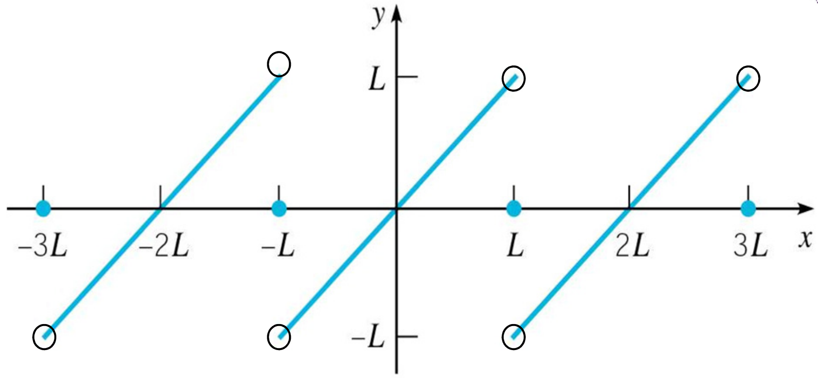
$$f(x)=\frac{2L}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\frac{n\pi x}{L}$$

sawtooth



Remarks

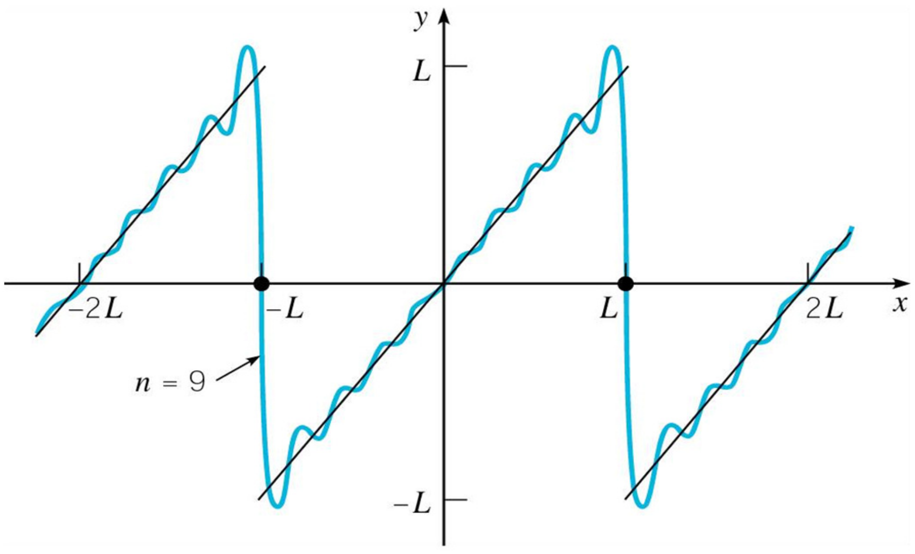
- f is discontinuous at $x = \pm(2n+1)L$,



- f is discontinuous at $x = \pm(2n+1)L$, and at these points the series converges to the average of the left and right limits, which is zero.

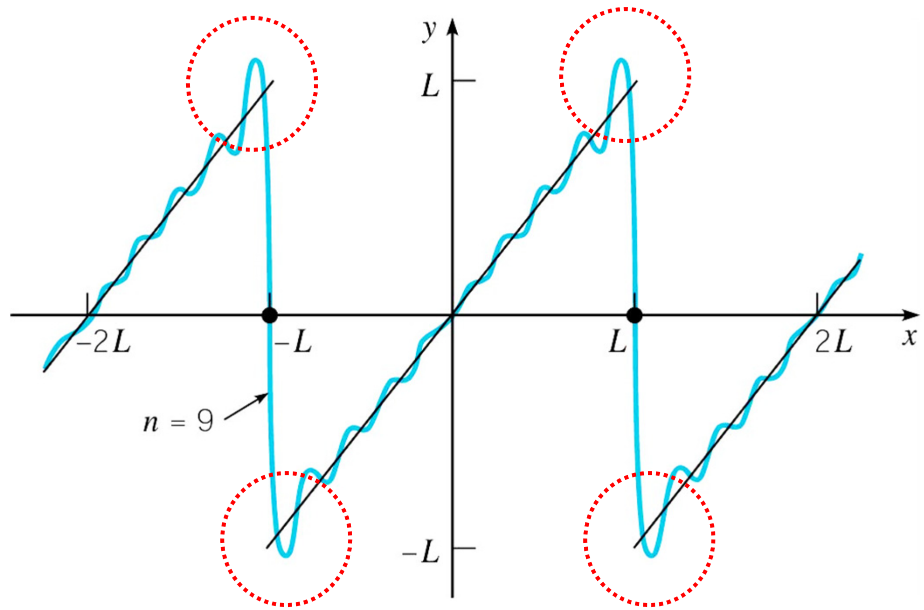
At $x = \pm(2n+1)L$

$$f(x) = \frac{f(x+) + f(x-)}{2} = \frac{-L+L}{2} = 0$$



Gibbs Phenomenon (occurs near the discontinuities)

- The partial sums appear to converge to f at points of continuity while they tend to overshoot f near points of discontinuity.
- This behavior is typical of Fourier series at points of discontinuity and is known as Gibbs phenomena.

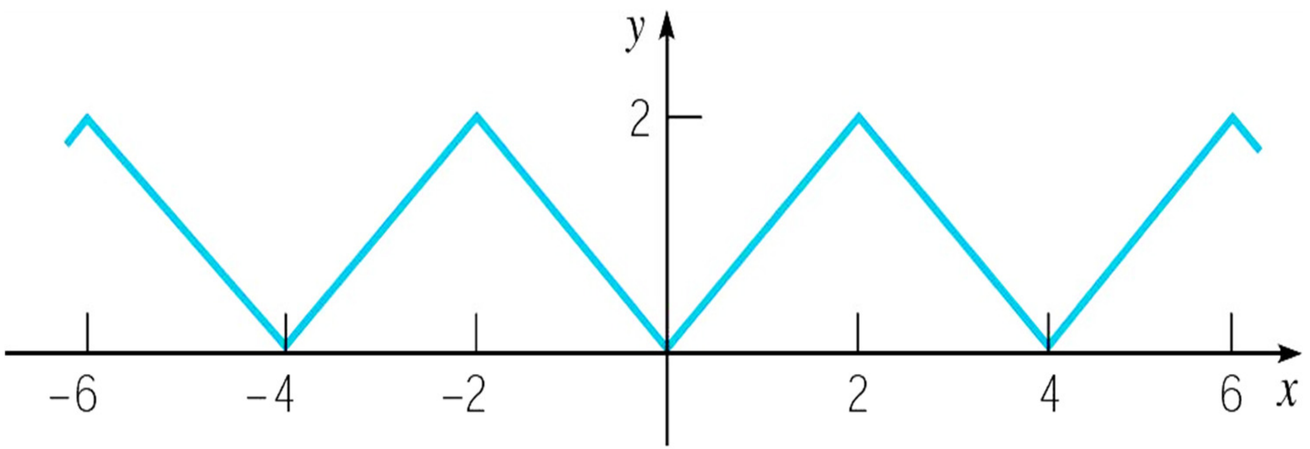


Q2 (Triangular Wave)

A triangular wave given below is periodic with fundamental period $2\ell = 4$.

Find the Fourier coefficients a_n and b_n of $f(x)$:

$$f(x) = \begin{cases} -x, & -2 \le x < 0 \\ x, & 0 \le x < 2 \end{cases}$$



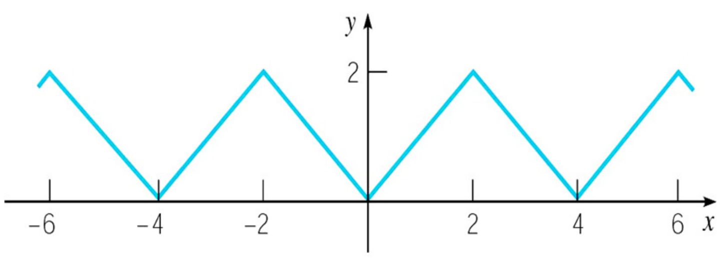
Solution:

$$\text{FS } f = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{n\pi x}{\ell}\right) + b_n \sin\left(\frac{n\pi x}{\ell}\right)\right)$$

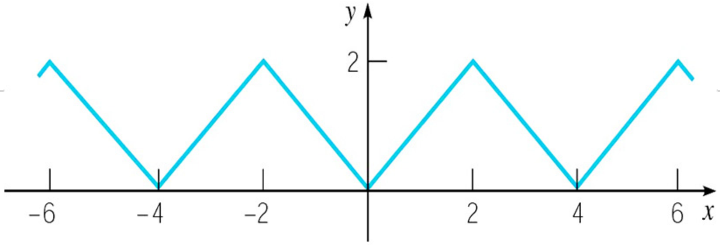
$$a_0 = \frac{1}{2\ell}\int_{-\ell}^{\ell} f(x)dx,$$

$$a_n = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\cos\left(\frac{n\pi x}{\ell}\right)dx, \ n = 1,2,\cdots$$

$$b_n = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\sin\left(\frac{n\pi x}{\ell}\right)dx, \ n = 1,2,\cdots$$



$$\Rightarrow \quad a_0 = \frac{1}{4}\int_{-2}^{0}(-x)\,dx + \frac{1}{4}\int_{0}^{2} x\,dx = 1$$



$$a_n = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\cos\left(\frac{n\pi x}{\ell}\right)dx, \quad n = 1,2,\cdots$$

$$\Rightarrow a_n = \frac{1}{2}\int_{-2}^{0}(-x)\cos\frac{n\pi x}{2}dx + \frac{1}{2}\int_{0}^{2} x\cos\frac{n\pi x}{2}dx$$

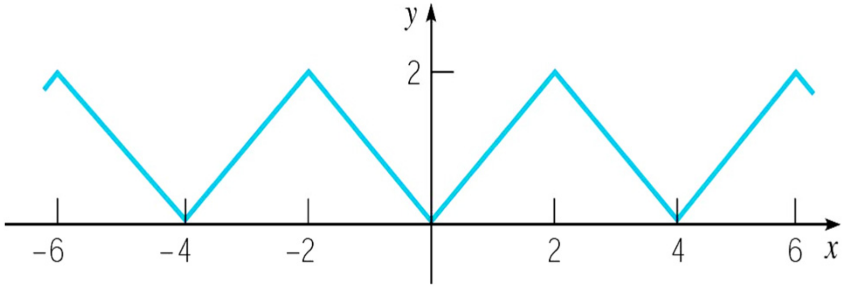
$$= \frac{-1}{2}\frac{2}{n\pi}\int_{-2}^{0} x\, d\sin\frac{n\pi x}{2} + \frac{1}{2}\frac{2}{n\pi}\int_{0}^{2} x d\sin\frac{n\pi x}{2}$$

$$= \frac{1}{n\pi}\left(-x\sin\frac{n\pi x}{2}\bigg|_{-2}^{0} + \int_{-2}^{0}\sin\frac{n\pi x}{2}dx + x\sin\frac{n\pi x}{2}\bigg|_{0}^{2} - \int_{0}^{2}\sin\frac{n\pi x}{2}dx\right)$$

$$= \frac{1}{n\pi}\left(4\sin(n\pi) + \int_{-2}^{0}\sin\frac{n\pi x}{2}dx - \int_{0}^{2}\sin\frac{n\pi x}{2}dx\right)$$

$$= \frac{1}{n\pi}\left(4\sin(n\pi) - \frac{2}{n\pi}\cos\frac{n\pi x}{2}\bigg|_{-2}^{0} + \frac{2}{n\pi}\cos\frac{n\pi x}{2}\bigg|_{0}^{2}\right)$$

$$= \frac{1}{n\pi}\left(4\sin(n\pi) - \frac{4}{n\pi}(1-\cos(n\pi))\right) = \begin{cases} 0, & n \text{ even} \\ -8/(n\pi)^2, & n \text{ odd} \end{cases}$$



$$\Rightarrow \quad b_n = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\sin\left(\frac{n\pi x}{\ell}\right)dx = 0$$

- even × odd = odd

- If f is an odd function, then $\int_{-a}^{a} f(x)\,dx = 0$

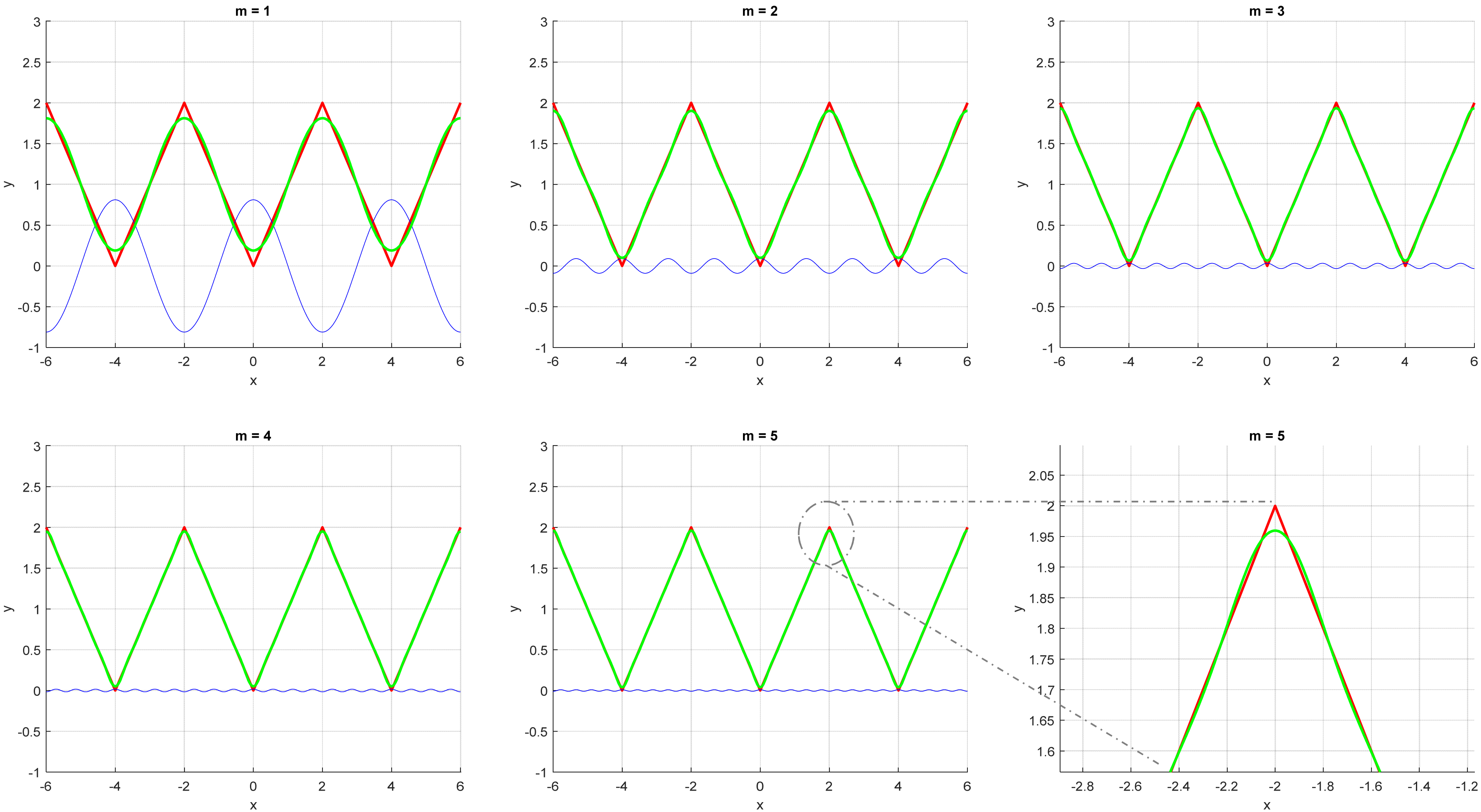
$$\Rightarrow \quad f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{\ell} + b_n \sin\frac{n\pi x}{\ell}\right)$$

$$= 1 - \frac{8}{\pi^2}\left(\cos\frac{\pi x}{2} + \frac{1}{3^2}\cos\frac{3\pi x}{2} + \frac{1}{5^2}\cos\frac{5\pi x}{2} + \cdots\right)$$

$$= 1 - \frac{8}{\pi^2}\sum_{n=1,3,5,\ldots}^{\infty}\frac{1}{n^2}\cos\frac{n\pi x}{2} = \boxed{1 - \frac{8}{\pi^2}\sum_{m=1}^{\infty}\frac{1}{(2m-1)^2}\cos\frac{(2m-1)\pi x}{2}}$$

$$f(x) = 1 - \frac{8}{\pi^2} \sum_{m=1}^{\infty} \frac{1}{(2m-1)^2} \cos \frac{(2m-1)\pi x}{2}$$

triangular



Fourier Series for Even and Odd Functions

Recall the Fourier series of $f(x)$ with period 2ℓ be

$$\text{FS} \qquad f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{\ell}\right) + b_n \sin\left(\frac{n\pi x}{\ell}\right) \right)$$

$$a_0 = \frac{1}{2\ell} \int_{-\ell}^{\ell} f(x)dx,$$

$$a_n = \frac{1}{\ell} \int_{-\ell}^{\ell} f(x)\cos\left(\frac{n\pi x}{\ell}\right)dx, \quad n = 1, 2, \cdots$$

$$b_n = \frac{1}{\ell} \int_{-\ell}^{\ell} f(x)\sin\left(\frac{n\pi x}{\ell}\right)dx, \quad n = 1, 2, \cdots$$

Case 1: If $f(x)$ is an even function, then

$$b_n = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\sin\left(\frac{n\pi x}{\ell}\right)dx = 0$$

- even × odd = odd

- If f is an odd function, then $\int_{-a}^{a} f(x)\,dx = 0$

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{\ell}\right)\right]$$ is a Fourier cosine series…

Case 2: If $f(x)$ is an odd function, then

$$a_0 = \frac{1}{2\ell}\int_{-\ell}^{\ell} f(x)dx = 0, \quad a_n = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\cos\left(\frac{n\pi x}{\ell}\right)dx = 0$$

- odd × even = odd

- If f is an odd function, then $\int_{-a}^{a} f(x)\,dx = 0$

$$f(x) = \sum_{n=1}^{\infty}\left[b_n \sin\left(\frac{n\pi x}{\ell}\right)\right]$$ is a Fourier sine series…

Exercise: Compute the Fourier series for

1. $$f(x) = \begin{cases} 0, & -\pi < x \le 0 \\ x, & 0 < x < \pi \end{cases}$$

2. $$g(x) = |x| , \quad -1 < x < 1$$

3. $$h(x) = x^3, \quad -\alpha < x < \alpha$$

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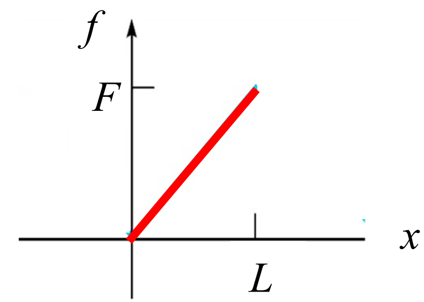
6. Wave equation

Problem: needs to expand a given function $f(x)$ in a Fourier series, where $f(x)$ is defined only on a finite interval, *e.g.*, $0 < x < L$.

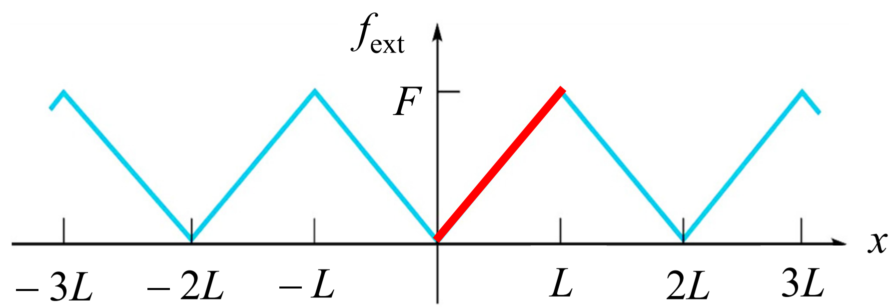


$f(x)$ is non-periodic, how to expand it in a Fourier series?

Half-range expansions: f is given only on half the range, half the interval of periodicity of length $2L$.

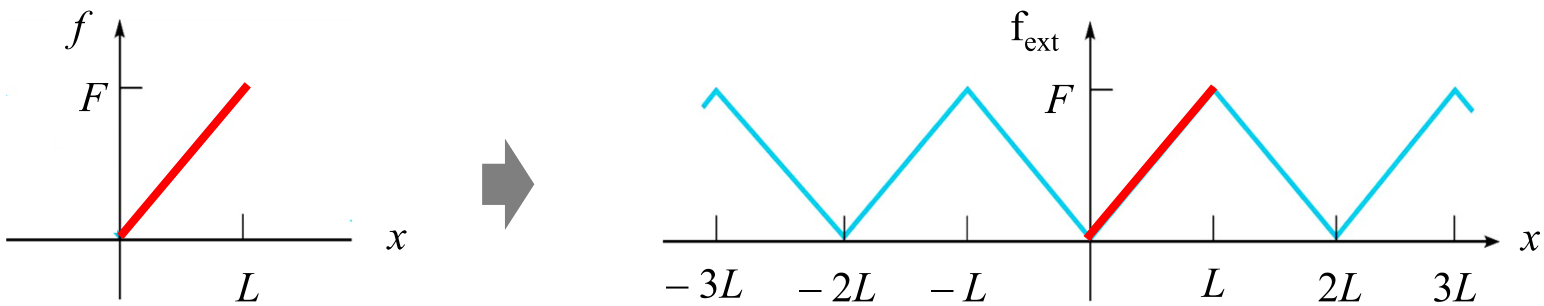
Extend the domain of definition of $f(x)$ to $-\infty < x < \infty$ by defining an *extended periodic function* f_{ext} of period $2L$ such that

$$f_{\text{ext}}(x) = f(x), \quad 0 < x < L$$



Two Ways of Extensions of $f(x)$

1. Extend $f(x)$ into an even periodic function (even extension)



In this case,

$$f_{\text{ext}}(x) = \begin{cases} f(x) & \text{for } 0 \le x \le L \\ f(-x) & \text{for } -L \le x < 0 \end{cases} \qquad \text{and} \quad f_{\text{ext}}(x+2L) = f_{\text{ext}}(x)$$

Note: $f_{\text{ext}}(x)$ is symmetric about $x = 0$ $\Rightarrow$ $f_{\text{ext}}(x)$ is an even periodic function

$\Rightarrow$ Its Fourier series contains only cosines (no sines).

Half-Range Cosine Expansion $f_{\text{ext}}(x) = a_0 + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L} \right) \right]$

Half-Range Cosine Expansion

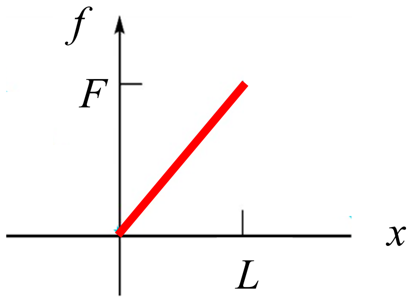
$$f(x) = a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{L}\right)\right], \qquad (0 < x < L)$$

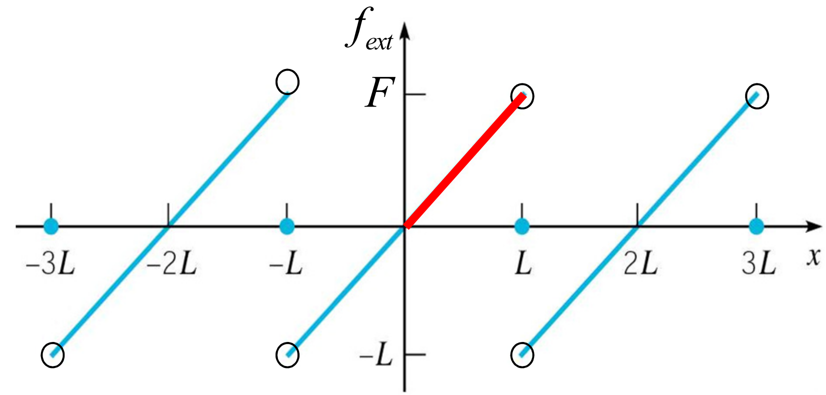
with

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f_{ext}(x)dx = \frac{1}{L}\int_0^L f(x)dx$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f_{ext}(x)\cos\left(\frac{n\pi x}{L}\right)dx = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

2. Extend $f(x)$ into an odd periodic function (odd extension)





In this case,

$$f_{\text{ext}}(x) = \begin{cases} f(x) & \text{for } 0 \le x \le L \\ -f(-x) & \text{for } -L \le x < 0 \end{cases} \quad \text{and} \quad f_{\text{ext}}(x+2L) = f_{\text{ext}}(x)$$

<u>Note:</u> $f_{\text{ext}}(x)$ is anti-symmetric about $x = 0$ $\Rightarrow$ $f_{\text{ext}}(x)$ is an odd periodic function

$\Rightarrow$ Its Fourier series contains only sines (no cosines).

Half-Range Sine Expansion $f_{\text{ext}}(x) = \sum_{n=1}^{\infty} \left[b_n \sin\left(\frac{n\pi x}{L} \right) \right]$

Half-Range Sine Expansion

Similarly, the half-range sine expansion of $f(x)$ can be expressed as

$$f(x) = \sum_{n=1}^{\infty} \left[b_n \sin\left(\frac{n\pi x}{L}\right) \right], \qquad (0 < x < L)$$

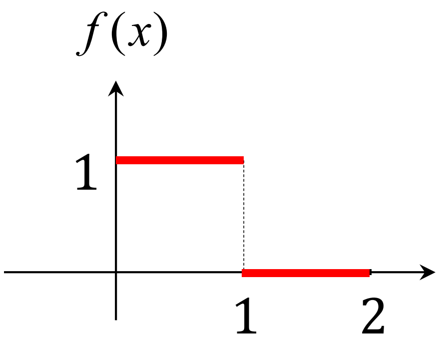
where

$$b_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

Example

Q1.: Find the Fourier cosine series for the function

$$f(x)=\begin{cases}1 & if \quad 0<x\le 1\\ 0 & if \quad 1<x<2\end{cases}$$



Solution. $L=2,\ b_n=0,$

$$a_0=\frac{1}{L}\int_0^L f(x)dx=\frac{1}{2}\int_0^1 1\,dx=\frac{1}{2}$$

$$a_n=\frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx=\int_0^1\cos\left(\frac{n\pi x}{2}\right)dx=\frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right)$$

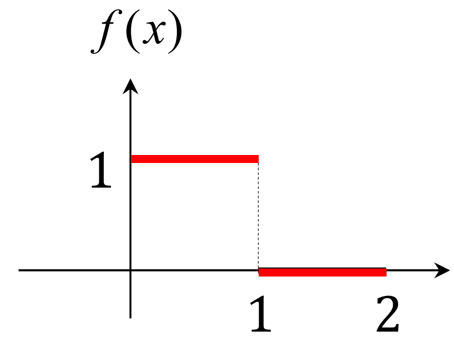
$$\Rightarrow f(x)=\frac{1}{2}+\sum_{n=1}^{\infty}\frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right)\cos\left(\frac{n\pi x}{2}\right)$$

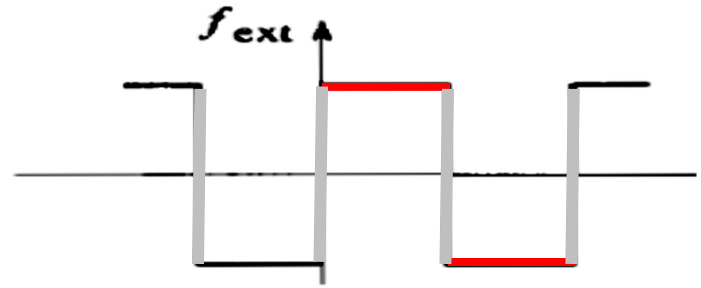
$$=\frac{1}{2}+\frac{2}{\pi}\left[\cos\left(\frac{\pi x}{2}\right)-\frac{1}{3}\cos\left(\frac{3\pi x}{2}\right)+\frac{1}{5}\cos\left(\frac{5\pi x}{2}\right)-\cdots\right]$$

Example

Q2.: Find the Fourier sine series for $f(x)$

$$f(x)=\begin{cases}1 & if \quad 0<x\le 1\\ 0 & if \quad 1<x<2\end{cases}$$



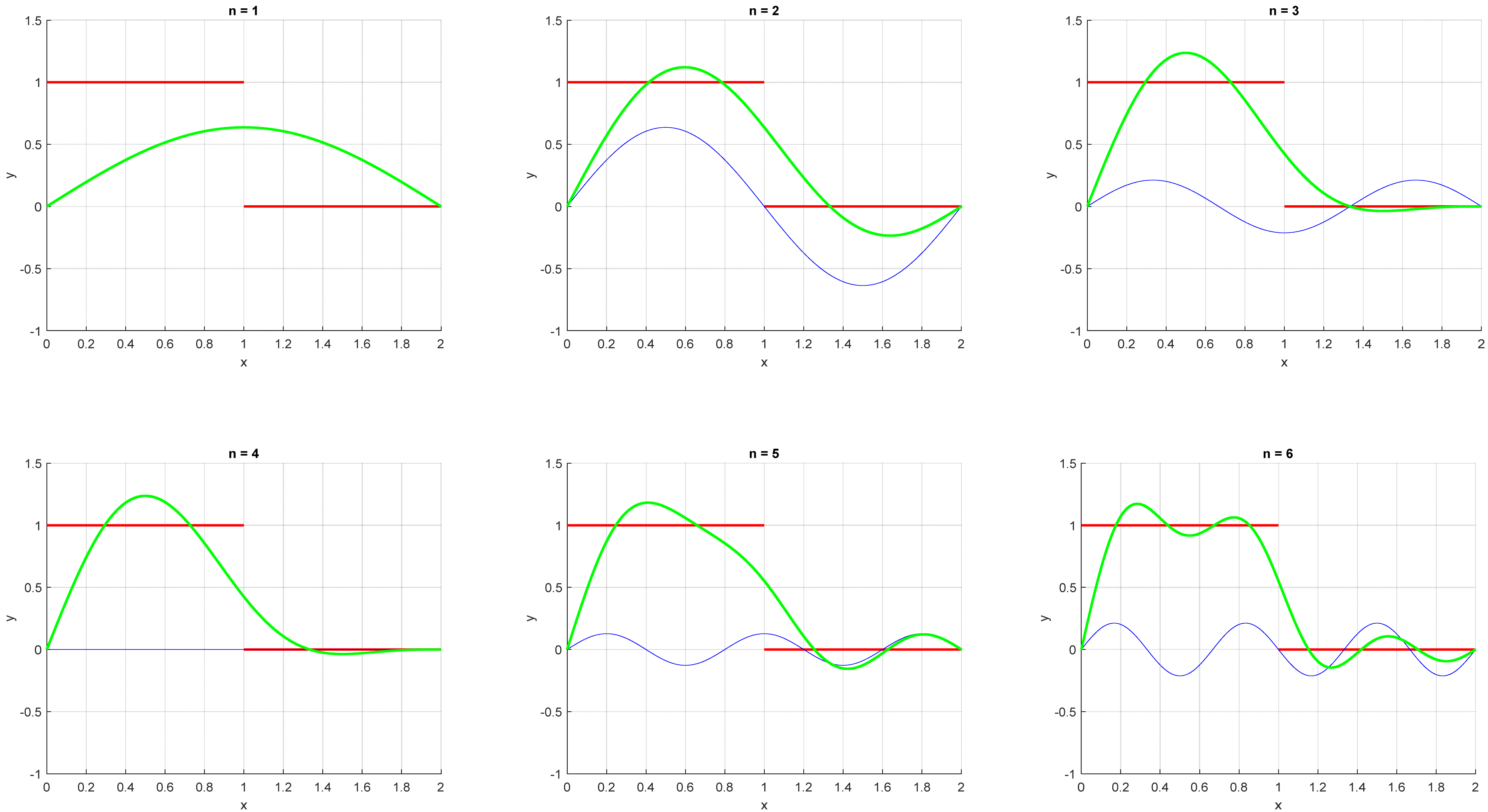


Solution: L = 2 and a_n = 0,

$$b_n=\frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx=\int_0^1\sin\left(\frac{n\pi x}{2}\right)dx=\frac{2}{n\pi}\left[1-\cos\left(\frac{n\pi}{2}\right)\right]$$

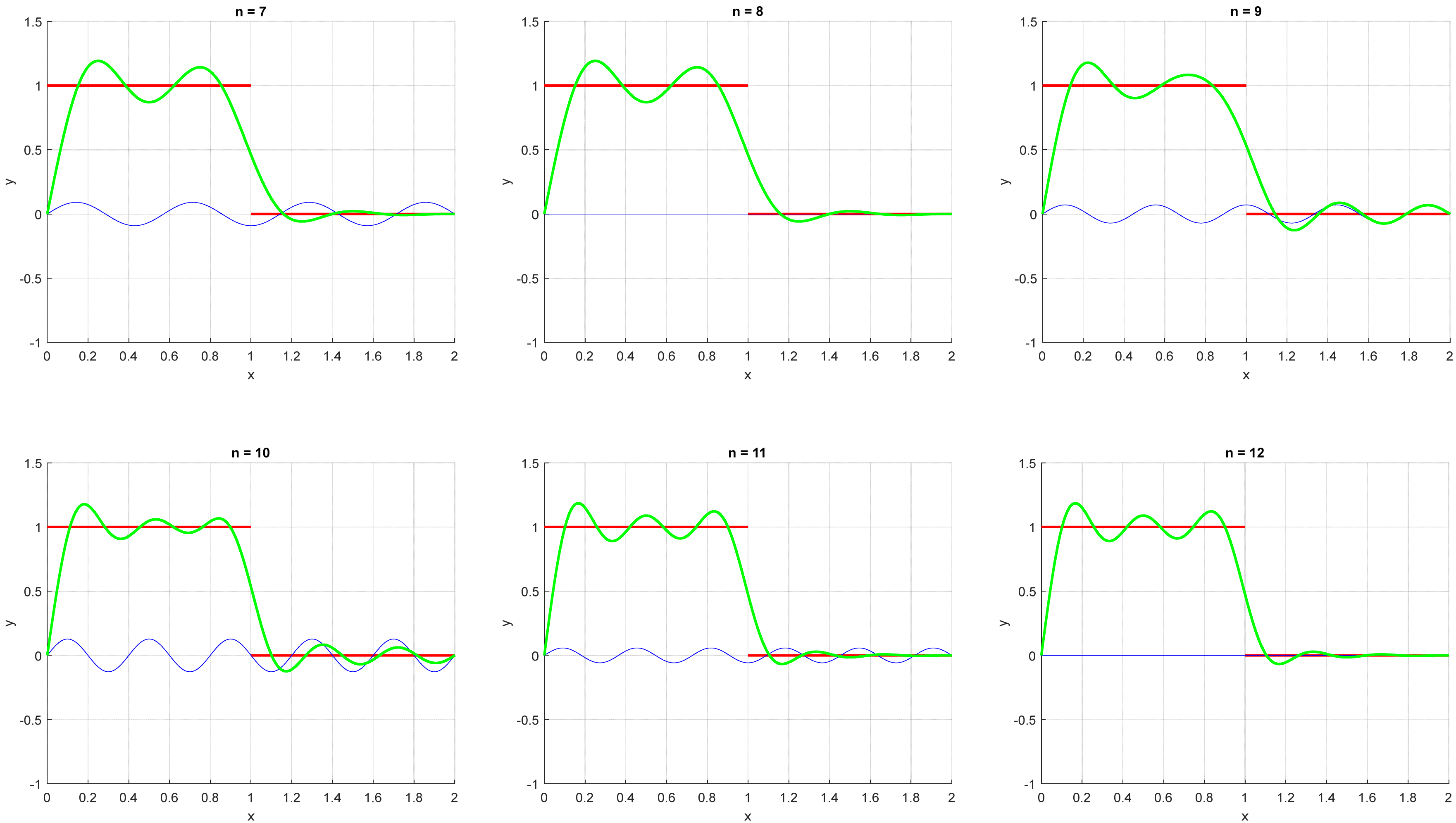
$$f(x)=\sum_{n=1}^{\infty}\frac{2}{n\pi}\left[1-\cos\left(\frac{n\pi}{2}\right)\right]\sin\left(\frac{n\pi x}{2}\right)=\frac{2}{\pi}\left[\sin\left(\frac{\pi x}{2}\right)+\sin(\pi x)+\frac{1}{3}\sin\left(\frac{3\pi x}{2}\right)+\ldots\right]$$

$$f(x)=\sum_{n=1}^{\infty}\frac{2}{n\pi}\left[1-\cos\left(\frac{n\pi}{2}\right)\right]\sin\left(\frac{n\pi x}{2}\right)=\frac{2}{\pi}\left[\sin\left(\frac{\pi x}{2}\right)+\sin\left(\pi x\right)+\frac{1}{3}\sin\left(\frac{3\pi x}{2}\right)+\ldots\right]$$



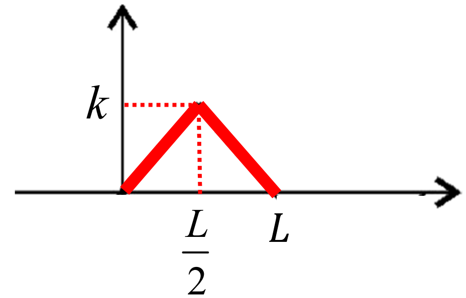
$$f(x)=\sum_{n=1}^{\infty}\frac{2}{n\pi}\left[1-\cos\left(\frac{n\pi}{2}\right)\right]\sin\left(\frac{n\pi x}{2}\right)=\frac{2}{\pi}\left[\sin\left(\frac{\pi x}{2}\right)+\sin\left(\pi x\right)+\frac{1}{3}\sin\left(\frac{3\pi x}{2}\right)+\dots\right]$$

halfsinstep



Exercise. Find the two half-range expansions of

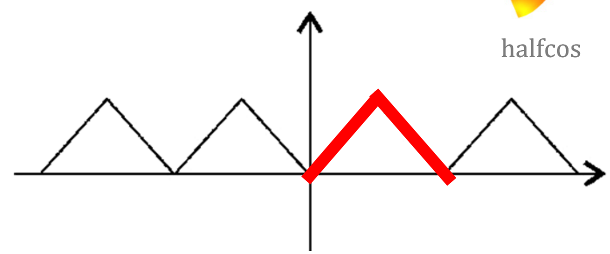
$$f(x)=\begin{cases}\dfrac{2kx}{L} & \text{if} \quad 0<x<\dfrac{L}{2}\\[2ex] \dfrac{2k(L-x)}{L} & \text{if} \quad \dfrac{L}{2}<x<L\end{cases}$$



1. Half-range cosine expansion (answer)

halfcos

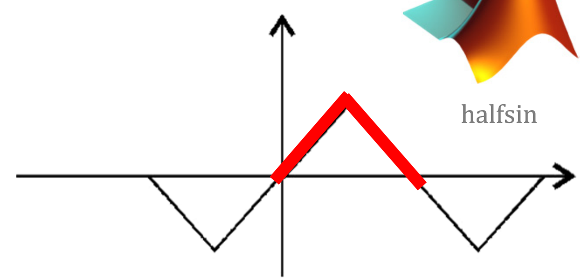
$$f(x)=\frac{k}{2}+\frac{4k}{\pi^2}\sum_{n=1}^{\infty}\left(\frac{2\cos\frac{n\pi}{2}-(-1)^n-1}{n^2}\right)\cos\left(\frac{n\pi x}{L}\right)$$



2. Half-range sine expansion (answer)

halfsin

$$f(x)=\frac{8k}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{2}\sin\left(\frac{n\pi x}{L}\right)$$



Partial Differential Equations – 4…

1. Even and odd functions, periodic functions
2. Fourier series of a periodic function
3. Fourier series: Half-range expansions
4. Concepts of partial differential equations
5. Heat equation (or diffusion equation)
6. Wave equation

Notation: Given a function $u(x,y)$, we denote

$$u_x = \frac{\partial u}{\partial x}$$

$$u_y = \frac{\partial u}{\partial y}$$

$$u_{xx} = \frac{\partial^2 u}{\partial x^2}$$

$$u_{yy} = \frac{\partial^2 u}{\partial y^2}$$

$$u_{xy} = \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x} = u_{yx}$$

We will mainly focus on second order PDEs…

Order of Partial Differential Equations

- The order of a differential equation is the order of the highest derivative of the unknown function that appears in the equation.

  Examples:

$$y' + 3y = 0 \qquad \text{1st Order}$$

$$y'' + 3y' - 2t = 0 \qquad \text{2nd Order}$$

$$\frac{d^4y}{dt^4} - \frac{d^2y}{dt^2} + 1 = e^{2t} \qquad \text{4th Order}$$

- Second-order PDEs will be the most important ones in applications.

  Example:

$$u_{xx} - 5u_y + u = xy^2(y - 15) \qquad \leftarrow \textbf{2nd order}$$

Linearity

- A PDE is linear if and only if it is a linear equation in the unknown function u and all of its partial derivatives.

  Otherwise, it is called nonlinear, for example, if it contains terms such as u^2, u^3, $(u_{xx})^2$, $(u_{xy})^3$, …, etc.

Homogeneous

A linear PDE is homogeneous if each term contains either u or one of its partial derivatives. Otherwise, it is non-homogeneous.

Example:

$$2(x-y)u_{xy} + y u_y - 3xu = e^{x+y}$$

is a 2nd order linear PDE, u is the dependent variable, while x and y are the independent variables. It is non-homogeneous.

Some Important 2nd order PDEs

- One dimensional heat Equation: $u_t = \alpha^2 u_{xx}$
- One-dimensional wave Equation: $u_{tt} = c^2 u_{xx}$
- Two-dimensional Laplace Equation: $u_{xx} + u_{yy} = 0$
- Two-dimensional Possion's Equation: $u_{xx} + u_{yy} = f(x,y)$

where α and c are positive constants, t is time, x and y are Cartesian coordinates.

Which of the above PDEs are homogeneous and which are not?

2nd Order Linear PDEs and Their Classification

Consider a linear second-order PDE

$$A u_{xx} + 2 B u_{xy} + C u_{yy} + D u_x + E u_y + F u = f$$

Classified by three types:

Parabolic: if $B^2 - AC = 0$

Hyperbolic: if $B^2 - AC > 0$

Elliptic: if $B^2 - AC < 0$

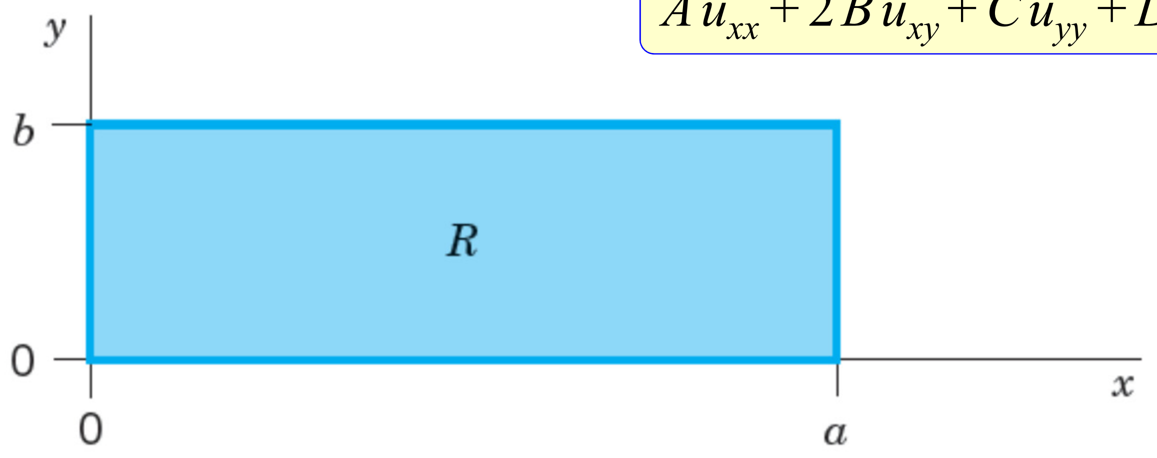
Solutions to Differential Equations

- A solution to a PDE in some region R :

  ✓ A function $u(x,y)$ that has all the partial derivatives appearing in the PDE in some domain D containing R, and satisfies the PDE everywhere in R.

  ✓ The function $u(x,y)$ is continuous on the boundary of R, and has those partial derivatives in the interior of R.

$$A u_{xx} + 2B u_{xy} + C u_{yy} + D u_x + E u_y + F u = f$$



- In general, the totality of solutions of a PDE is very large.

- For example, $u = x^2 - y^2$, $u = e^x \cos y$, $u = \sin x \cosh y$, $u = \ln(x^2 + y^2)$ are all solutions of

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Check for $u = x^2 - y^2$,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 (x^2 - y^2)}{\partial x^2} + \frac{\partial^2 (x^2 - y^2)}{\partial y^2} = \frac{\partial (2x)}{\partial x} + \frac{\partial (-2y)}{\partial y} = 2 - 2 = 0$$

Check for $u = e^x \cos y$,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 \left(e^x \cos y\right)}{\partial x^2} + \frac{\partial^2 \left(e^x \cos y\right)}{\partial y^2}$$

$$= \frac{\partial}{\partial x}\left(e^x \cos y\right) + \frac{\partial}{\partial y}\left(-e^x \sin y\right) = e^x \cos y - e^x \cos y = 0$$

- The unique solution of a PDE will be obtained by the use of additional conditions arising from the problem.

  ◊ Boundary conditions

  ◊ Initial conditions

- Two methods for solving PDEs:

  1. Using ordinary methods (as ODEs)

  2. Using the separation of variables technique

Remarks:

Three important questions in the study of differential equations:

- Is there a solution? (Existence)
- If there is a solution, is it unique? (Uniqueness)
- If there is a solution, how do we find it?

Analytical Solution, Numerical Approximation, …

Solve PDEs using ordinary methods (as ODEs)

Problem: Find all solutions of the PDE: $u_{xy} = 0$

Solution:

✓ Integrate with respect to y to get: $u_x = f(x)$

✓ Integrate with respect to x to get: $u = X(x) + Y(y)$

$u = \sin x + e^y$ and $u = 3x + x^2 + \cos y$ are solutions of the above PDE.

$$u_{xy} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\left(\sin x + e^y\right)\right) = \frac{\partial}{\partial x}\left(e^y\right) = 0$$

$$u_{xy} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\left(3x + x^2 + \cos y\right)\right) = \frac{\partial}{\partial x}\left(-\sin y\right) = 0$$

Note

✓ Solutions of PDEs involve arbitrary functions (instead of arbitrary constants).

✓ Those arbitrary functions can be found from some given boundary (or initial) conditions.

Method for solving separable equation

Example: Use the separation of variables technique to solve

$$3u_x + 2u_y = 0 \quad \text{with} \quad u(x,0) = 4e^{-x}$$

Soln. Assume $u(x,y) = X(x) \cdot Y(y)$. Then PDE becomes

$$3X'Y + 2XY' = 0 \;\Rightarrow\; \frac{X'}{X} = -\frac{2}{3}\frac{Y'}{Y} = k = \text{constant}$$

$$\Rightarrow \begin{cases} X' - kX = 0 \;\Rightarrow\; X(x) = c_1 e^{kx} \\ Y' + \frac{3}{2}kY = 0 \;\Rightarrow\; Y(y) = c_2 e^{-\frac{3}{2}ky} \end{cases}$$

$$\Rightarrow \quad u(x,y) = X(x)Y(y) = c_1 e^{kx} c_2 e^{-\frac{3}{2}ky} = c e^{\frac{k}{2}(2x-3y)}$$

Given that $4e^{-x} = u(x,0) = ce^{kx} \;\Rightarrow\; c = 4, k = -1$, we have

$$u(x,y) = 4e^{-\frac{1}{2}(2x-3y)}$$

To verify if $u(x, y) = 4e^{-\frac{1}{2}(2x-3y)}$ is indeed a solution of

$$3u_x + 2u_y = 0$$

we check

$$\begin{aligned}
3u_x + 2u_y &= 3\frac{\partial}{\partial x}\left(4e^{-\frac{1}{2}(2x-3y)}\right) + 2\frac{\partial}{\partial y}\left(4e^{-\frac{1}{2}(2x-3y)}\right) \\
&= 12\frac{\partial}{\partial x}\left(e^{-x}e^{\frac{3}{2}y}\right) + 8\frac{\partial}{\partial y}\left(e^{-x}e^{\frac{3}{2}y}\right) \\
&= -12e^{-x}e^{\frac{3}{2}y} + 8\cdot\frac{3}{2}e^{-x}e^{\frac{3}{2}y} \\
&= 0
\end{aligned}$$

Checked! It is indeed a solution. The separation of variables technique works for solving this problem…

Remarks:

- ✓ Separation method is a very effective way in solving many PDEs, e.g., heat equations, wave equations, Laplace equations, which are to be studied next.
- ✓ All PDEs we are going to tackle in the coming sections are to be solved using the separation method with some mathematical tricks.

Is the separation of variables method a universal technical for solving PDE problems?

The answer is No!

Example: It can be verified that $u(x,y)=x^2-y^2$ is a solution to

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \quad \text{with} \quad u(x,0)=x^2$$

However, this solution can never be expressed as the product of two separate functions of $X(x)$ and $Y(y)$, i.e., $u(x,y)=x^2-y^2 \neq X(x)\cdot Y(y)$.

The separation of variables technique is effective, but not universal…

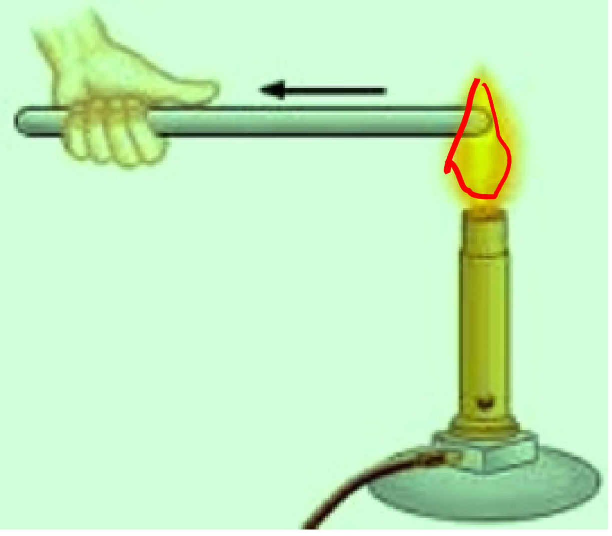
Partial Differential Equations – 5…

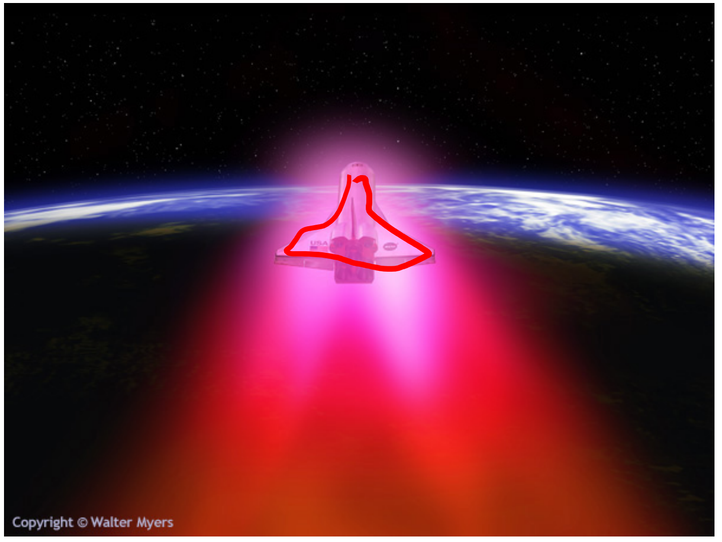
1. Even and odd functions, periodic functions
2. Fourier series of a periodic function
3. Fourier series: Half-range expansions
4. Concepts of partial differential equations
5. Heat equation (or diffusion equation)
6. Wave equation

Heat Equation and Modeling

One of the classical partial differential equation of mathematical physics is the equation describing the **conduction of heat** in a solid body (originated in the 18th century).

A modern one: **space vehicle reentry problem** – to analyze transfer and dissipation of heat generated by the friction with earth's atmosphere.



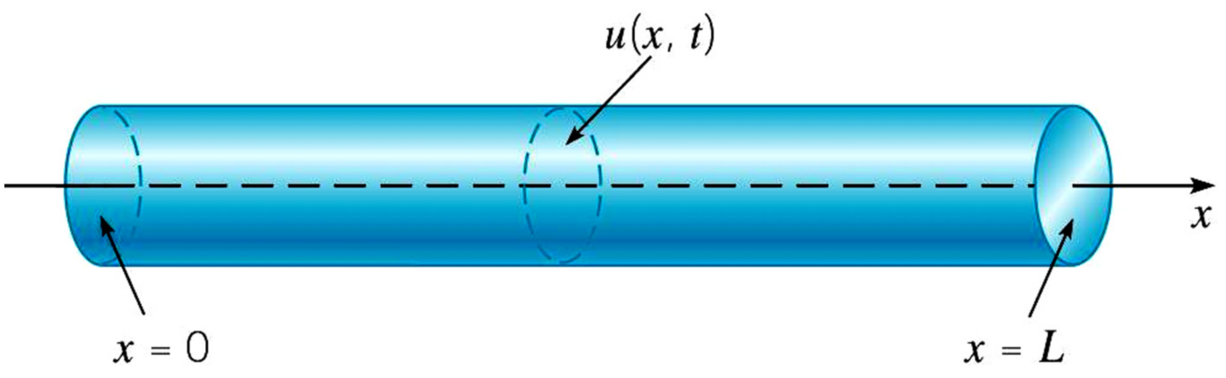


Heat Equation

- Describe the temperature $u(x,t)$ in a solid body as a function of position x and time t

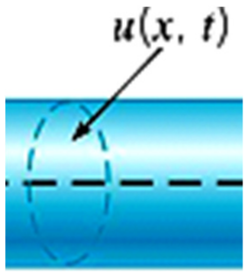
- **Assumptions…**

  Consider a straight bar with uniform cross-section and homogeneous material. We wish to develop a model for heat flow through the bar.

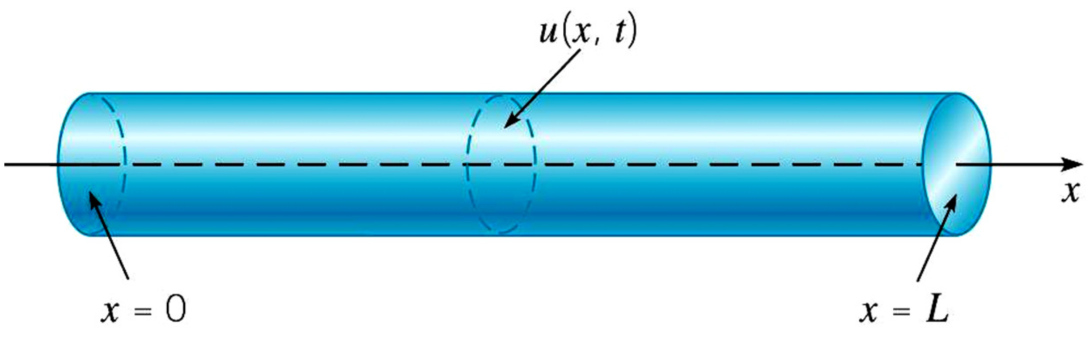


Suppose that the sides of the bar are perfectly insulated so that no heat passes through them.

Assume the cross-sectional dimensions are so small that the temperature u can be considered constant on the cross sections.



Then u is a function only of the axial coordinate x and time t .



- Let $u(x,t)$ be the temperature on a cross section located at x and at time t. We shall follow some **basic principles** of physics:

(i). The amount of heat per unit time flowing through a unit of cross-sectional area is proportional to $\partial u/\partial x$ with constant of proportionality k called the **thermal conductivity** of the material.

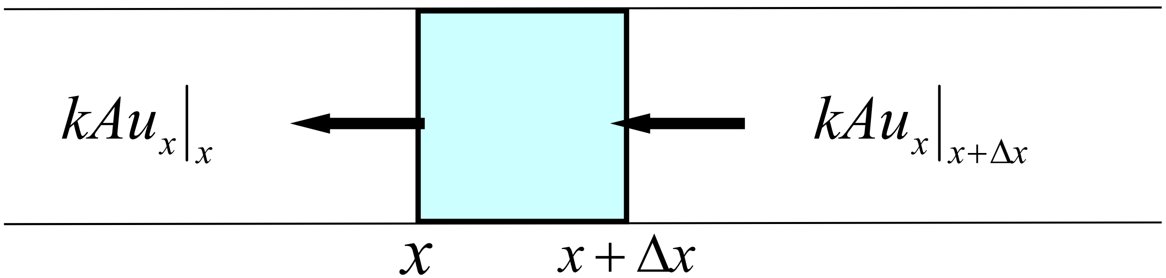
(ii). Heat flow is always from points of higher temperature to points of lower temperature.

$$\frac{\partial}{\partial t}(\text{amount of heat}) \,/\, A \;\propto\; k\frac{\partial u}{\partial x}$$

(iii). The net heat influx into the element per unit time is

$$k\,A\,u_x\Big|_{x+\Delta x} - k\,A\,u_x\Big|_{x}$$

which must be equal to the rate of change of the heat (mcu) contained in the element, where c is the specific heat capacity and $m = A\,\Delta x\,\sigma$, with σ the mass density of the material.



i.e.,

$$k\,Au_x\,\big|_{x+\Delta x} - k\,Au_x\,\big|_{x} = \frac{\partial}{\partial t}(mcu) = \frac{\partial}{\partial t}(A\Delta x\sigma cu)$$

$$\Rightarrow \quad k\,A\left(u_x\,\big|_{x+\Delta x} - u_x\,\big|_{x}\right) = kA\Delta u_x = \frac{\partial}{\partial t}(A\Delta x\sigma cu)$$

Dividing on both sides of $kA\Delta u_x = \frac{\partial}{\partial t}(A\Delta x\sigma cu)$ by $A\Delta x c\sigma$, i.e.,

$$\frac{k\cancel{A}(\Delta u_x)}{\cancel{A}\Delta x c\sigma} = \frac{\partial}{\partial t}\frac{(\cancel{A}\cancel{\Delta x}\cancel{\sigma}\cancel{c}u)}{\cancel{A}\cancel{\Delta x}\cancel{c}\cancel{\sigma}}$$

$$\Rightarrow \quad \frac{k}{c\sigma}\left(\frac{\Delta u_x}{\Delta x}\right) = \frac{\partial}{\partial t}(u) = u_t$$

Taking $\Delta x \to 0$, we have

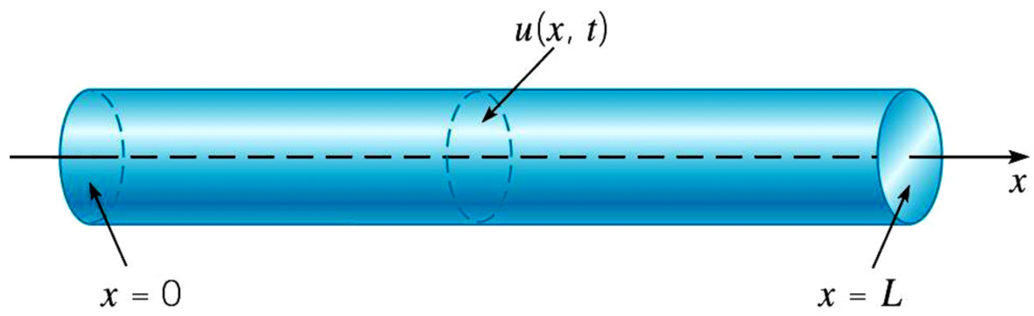
Thermal diffusivity

$$\boxed{\frac{k}{c\sigma}} u_{xx} = u_t$$

1D Heat Equation

Solution to the Heat Equation

Problem: Consider a long bar of constant cross section and homogeneous heat conducting material.



The temperature $u(x, t)$ is described by 1D heat equation

$$\alpha^2 u_{xx} = u_t, \qquad 0 < x < L, \;\; 0 < t < \infty, \;\; \alpha^2 = \frac{k}{c\sigma} \qquad \text{(6-1a)}$$

with boundary conditions: $u(0,t) = u_1, \;\; u(L,t) = u_2, \;\; 0 < t < \infty$ (6-1b)

where u_1 and u_2 are given constants, and initial condition:

$$u(x,0) = f(x), \;\; 0 < x < L, \qquad \text{(6-1c)}$$

The problem is to find **a solution** $u(x, t)$ that satisfies (6-1a) to (6-1c)…

Solution by the Separation of Variables Technique

<u>Step 1:</u> Assume that the solution $u(x,t)$ has the form

$$u(x,t) = X(x)\cdot T(t) \qquad (6\text{-}2)$$

Substituting this form into the partial differential equation

$$\alpha^2 u_{xx} = u_t \quad \Rightarrow \quad \alpha^2 \frac{\partial^2}{\partial x^2}\big(X(x)\cdot T(t)\big) = \frac{\partial}{\partial t}\big(X(x)\cdot T(t)\big)$$

yields

$$\alpha^2 X'' T = X T' \qquad (6\text{-}3a)$$

or

$$\frac{X''}{X} = \frac{1}{\alpha^2}\frac{T'}{T} \qquad (6\text{-}3b)$$

Since the left side depends only on x and the right side only on t, both sides of this equation must equal the same constant, say $-\kappa^2$ $(\kappa \geq 0)$. **If they were variable, the changing t or x would affect only the left or the right side, respectively.** Then, we have

$$\frac{X''}{X} = \frac{1}{\alpha^2}\frac{T'}{T} = -\kappa^2 \quad \Rightarrow \quad \begin{cases} X'' + \kappa^2 X = 0 & \text{(6-4a)} \\ T' + \kappa^2 \alpha^2 T = 0 & \text{(6-4b)} \end{cases}$$

Thus, solving the partial differential equation is replaced by solving two ODEs.

Remark: Motivation for having the constant non-positive, i.e., $-\kappa^2$, will be explained later. If we choose κ^2 instead of $-\kappa^2$, we will end up with a meaningless solution.

Step 2: i). When $\kappa = 0$,

$$X'' + \kappa^2 X = 0 \quad \Rightarrow \quad X'' = 0$$

$$T' + \kappa^2 \alpha^2 T = 0 \quad \Rightarrow \quad T' = 0$$

Thus, we have

$$X(x) = D + Ex, \qquad T(t) = G \tag{6-5}$$

where D, E and G are constants. (Why?)

The characteristic equation associated with X is given by

$$\lambda^2 = 0 \quad \Rightarrow \quad \lambda_{1,2} = 0 \quad \Rightarrow \quad X(x) = (D + Ex)e^{0 \cdot x} = D + Ex$$

and that associated with T is given by

$$\lambda = 0 \quad \Rightarrow \quad T(t) = G e^{0 \cdot t} = G$$

Step 2: ii). When $\kappa \neq 0$, solving the above two equations gives

$$X(x) = A\cos(\kappa x) + B\sin(\kappa x), \tag{6-6}$$

$$T(t) = Fe^{-\kappa^2\alpha^2 t}$$

where A, B and F are constants. (Why?)

The characteristic equation associated with X is given by

$$\lambda^2 + \kappa^2 = 0 \quad \Rightarrow \quad \lambda_{1,2} = \pm i\kappa \quad \Rightarrow \quad X(x) = A\cos(\kappa x) + B\sin(\kappa x)$$

and that associated with T is given by

$$\lambda + \kappa^2\alpha^2 = 0 \quad \Rightarrow \quad \lambda = -\kappa^2\alpha^2 \quad \Rightarrow \quad T(t) = Fe^{-\kappa^2\alpha^2 t}$$

Thus,

$$u = X(x)T(t) = (D + Ex)G \qquad \text{for } \kappa = 0, \text{ and} \tag{6-7a}$$

$$u = X(x)T(t) = \left[A\cos(\kappa x) + B\sin(\kappa x)\right] F e^{-\kappa^2\alpha^2 t}$$

$$\text{for any } \kappa \neq 0. \tag{6-7b}$$

Let $DG = H$, $EG = I$, $AF = J$ and $BF = K$, then,

$$u = H + Ix \quad \text{for } \kappa = 0, \text{ and} \tag{6-8a}$$

$$u = \left[J\cos(\kappa x) + K\sin(\kappa x)\right] e^{-\kappa^2\alpha^2 t} \quad \text{for any } \kappa \neq 0. \tag{6-8b}$$

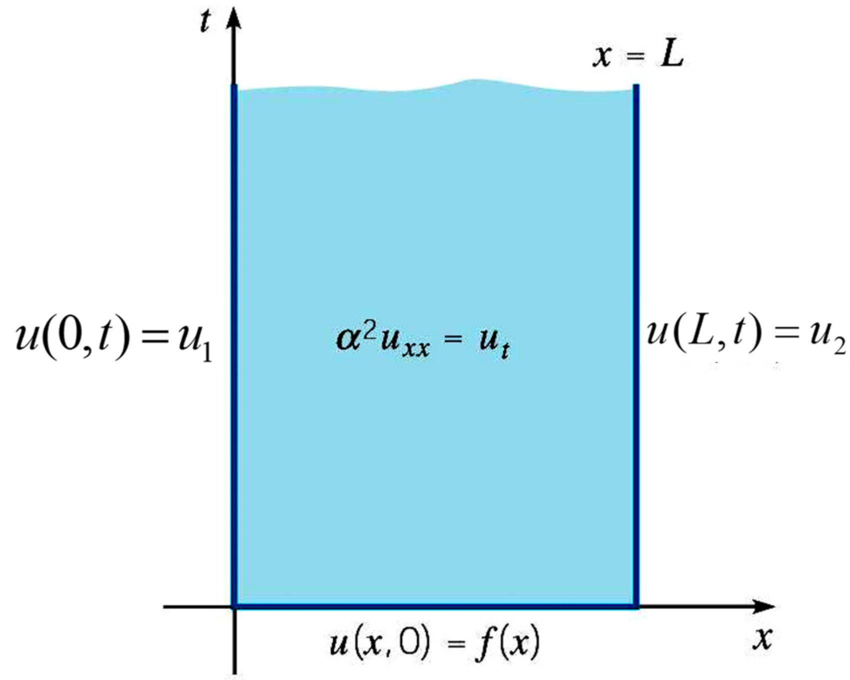
Note: If we choose κ^2 instead $-\kappa^2$ earlier, we will have a solution with $e^{\kappa^2\alpha^2 t} \to \infty$ as $t \to \infty$, which cannot happen in real-life.

Since Equation (6-1a) is linear, the sum of these solutions must also be a solution,

$$u(x,t) = H + I\,x + \left[J\cos(\kappa x) + K\sin(\kappa x) \right] e^{-\kappa^2\alpha^2 t} \qquad (6\text{-}9)$$

How to determine *H, I, J* and *K*?

Boundary Conditions

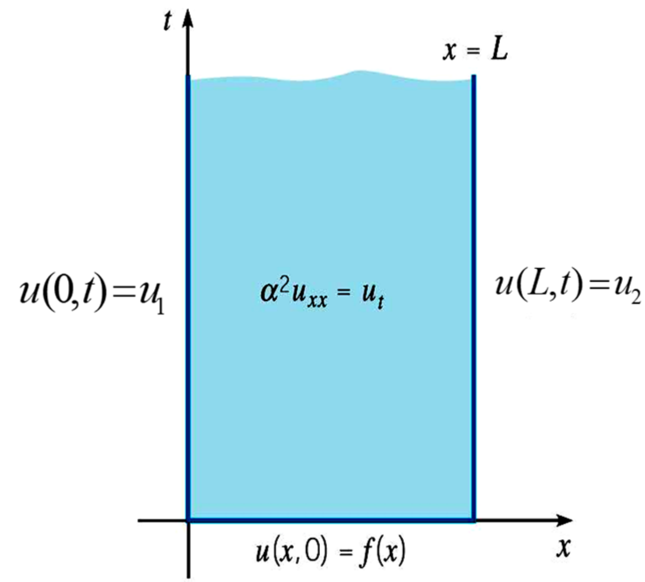


Substituting $u(x,t)$ of Equation (6-9) into the boundary conditions

$$u(0,t) = u_1$$

yields

$$u(0,t) = u_1 = H + Je^{-\kappa^2\alpha^2 t} \quad (0 < t < \infty)$$



i.e., $(H - u_1)(1) + Je^{-\kappa^2\alpha^2 t} = 0$ (6-10)

Since the functions 1 and $e^{-\kappa^2\alpha^2 t}$ are LI on the t interval (why?),

$$H - u_1 = 0 \ (\Rightarrow H = u_1) \quad \text{and} \quad J = 0$$

$$u(x,t) = H + I\,x + \left[J\cos(\kappa x) + K\sin(\kappa x)\right]e^{-\kappa^2\alpha^2 t} \qquad (6\text{-}9)$$

Side Note:

The functions $y_1(t)=1$ and $y_2(t)=e^{-\kappa^2\alpha^2 t}$ are linearly independent because their Wronskian determinant

$$\begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} = \begin{vmatrix} 1 & e^{-\kappa^2\alpha^2 t} \\ 0 & -\kappa^2\alpha^2 e^{-\kappa^2\alpha^2 t} \end{vmatrix} = -\kappa^2\alpha^2 e^{-\kappa^2\alpha^2 t} \neq 0$$

By definition, the linear independence of y_1 and y_2 implies

$$a_1 y_1(t) + a_2 y_2(t) = 0 \quad \Leftrightarrow \quad a_1 = 0, \; a_2 = 0.$$

Thus, Equation (6-9) is simplified to

$$u(x,t) = u_1 + Ix + K\sin(\kappa x)e^{-\kappa^2\alpha^2 t} \tag{6-11}$$

Applying the boundary condition $u(L,t) = u_2$ into Eq. (6-11) gives

$$u(L,t) = u_2 = u_1 + I\cdot L + K\sin(\kappa L)e^{-\kappa^2\alpha^2 t}$$

or

$$(I\cdot L + u_1 - u_2)(1) + K\sin(\kappa L)e^{-\kappa^2\alpha^2 t} = 0 \tag{6-12}$$

$\Rightarrow$ $\boxed{(I\cdot L + u_1 - u_2) = 0 \;\Downarrow\; I = (u_2 - u_1)/L}$ and $\boxed{K\sin(\kappa L) = 0 \;\Downarrow\; K = 0 \text{ or } \sin(\kappa L) = 0}$ (6-13)

If we choose $K = 0$, then

$$u(x,t) = u_1 + Ix + K\sin(\kappa x)e^{-\kappa^2\alpha^2 t}$$

$$= u_1 + \frac{u_2 - u_1}{L}x$$

which is independent of t. Obviously, this cannot be the case.

$K = 0$ is not a valid choice!

Thus, we have to settle with

$$\sin(\kappa L) = 0 \quad \Rightarrow \quad \kappa = \frac{n\pi}{L} \qquad n = 1, 2, \ldots \tag{6-14}$$

Substituting $I = (u_2 - u_1)/L$ and (6-14) into (6-11), we obtain

$$u(x,t) = u_1 + \frac{(u_2 - u_1)x}{L} + K_n \sin\frac{n\pi x}{L} e^{-(\pi\alpha/L)^2 t}, \quad n = 1, 2, \cdots$$

By superposition principle,

$$\begin{aligned} u(x,t) &= u_1 + \frac{(u_2 - u_1)x}{L} + K_1 \sin\frac{\pi x}{L} e^{-(\pi\alpha/L)^2 t} + K_2 \sin\frac{2\pi x}{L} e^{-(2\pi\alpha/L)^2 t} + \cdots \\ &= u_1 + \frac{(u_2 - u_1)x}{L} + \sum_{n=1}^{\infty} K_n \sin\frac{n\pi x}{L} e^{-(n\pi\alpha/L)^2 t} \end{aligned} \tag{6-15}$$

which satisfies Equation (6-1a) and boundary condition (6-1b).

Step 3. In order to satisfy the initial condition

$$u(x,0) = f(x), \quad 0 \le x \le L,$$

the coefficients K_n should be determined by

$$u(x,0) = u_1 + \frac{(u_2 - u_1)x}{L} + \sum_{n=1}^{\infty} K_n \sin\frac{n\pi x}{L} = f(x) \tag{6-16}$$

or

$$f(x) - u_1 - \frac{(u_2 - u_1)x}{L} = \sum_{n=1}^{\infty} K_n \sin\frac{n\pi x}{L} \tag{6-17}$$

Denote

$$F(x) = f(x) - u_1 - \frac{(u_2 - u_1)x}{L} \tag{6-18}$$

$$\Rightarrow \quad F(x) = \sum_{n=1}^{\infty} K_n \sin\frac{n\pi x}{L} \qquad (0 < x < L) \tag{6-19}$$

$$F(x) = \sum_{n=1}^{\infty} K_n \sin\frac{n\pi x}{L} \qquad (0 < x < L) \tag{6-19}$$

Have we seen this type of equation before? It is an ...

half-range Fourier sine series expansion of $F(x)$ in the interval $(0, L)$

FS $\quad f(x) = \sum_{n=1}^{\infty}\left[b_n \sin\left(\frac{n\pi x}{L}\right)\right], \qquad (0 < x < L)$

where

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

Thus,

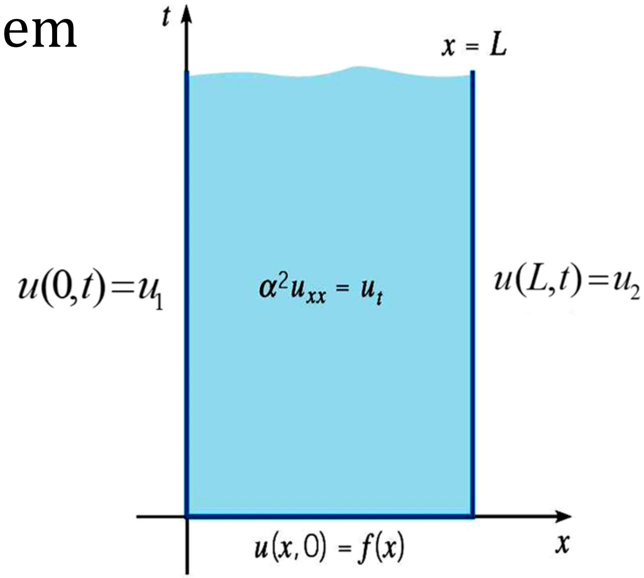
$$F(x)=\sum_{n=1}^{\infty}K_n \sin\frac{n\pi x}{L} \qquad (0<x<L) \tag{6-19}$$

$$\boxed{K_n=\frac{2}{L}\int_0^L F(x)\sin\frac{n\pi x}{L}dx} \tag{6-20}$$

The solution to the **1D heat problem** is thus given by

$$\boxed{u(x,t)=u_1+\frac{(u_2-u_1)x}{L}+\sum_{n=1}^{\infty}K_n \sin\frac{n\pi x}{L}e^{-(n\pi\alpha/L)^2 t}} \tag{6-21}$$

Example (a): Solve the following heat flow problem



$$\frac{\partial u}{\partial t}=2\frac{\partial^2 u}{\partial x^2},\qquad 0<x<3,\quad 0<t<\infty$$

$$u(0,t)=u(3,t)=0,\qquad 0<t<\infty$$

$$u(x,0)=5\sin 4\pi x-3\sin 8\pi x+2\sin 10\pi x,\ 0<x<3$$

Solution: Recall the solution to the heat equation

$$u(x,t)=u_1+\frac{(u_2-u_1)x}{L}+\sum_{n=1}^{\infty}K_n\sin\frac{n\pi x}{L}e^{-(n\pi\alpha/L)^2 t}$$

For the given problem, we have

$$u_1=u_2=0$$

$$F(x)=f(x)-u_1-\frac{(u_2-u_1)x}{L}=5\sin 4\pi x-3\sin 8\pi x+2\sin 10\pi x,\ L=3$$

Thus,

$$u(x,t)=\sum_{n=1}^{\infty}K_n\sin\frac{n\pi x}{L}e^{-(n\pi\alpha/L)^2 t}$$

with $L = 3$, $\alpha^2 = 2$ and

$$K_n=\frac{2}{L}\int_0^L F(x)\sin\frac{n\pi x}{L}dx$$

$$\Downarrow$$

$$\begin{aligned}K_n&=\frac{2}{3}\int_0^3 \left(5\sin 4\pi x-3\sin 8\pi x+2\sin 10\pi x\right)\sin\frac{n\pi x}{3}dx, \quad n=1,2,\cdots\\ &=\frac{2}{3}\int_0^3\left(5\sin\frac{12\pi x}{3}-3\sin\frac{24\pi x}{3}+2\sin\frac{30\pi x}{3}\right)\sin\frac{n\pi x}{3}dx, \quad n=1,2,\cdots\end{aligned}$$

Based on the orthogonality property of cosine and sine functions, i.e.,

$$\int_0^L \sin\frac{m\pi x}{L}\sin\frac{n\pi x}{L}\,dx = \frac{1}{2}\int_0^L\left[\cos\left(\frac{m\pi x}{L}-\frac{n\pi x}{L}\right)-\cos\left(\frac{m\pi x}{L}+\frac{n\pi x}{L}\right)\right]dx$$

$$=\frac{1}{2}\left[\int_0^L \cos\frac{(m-n)\pi x}{L}\,dx-\int_0^L \cos\frac{(m+n)\pi x}{L}\,dx\right]$$

$$=\begin{cases}\dfrac{1}{2}\left[\dfrac{L}{(m-n)\pi}\sin\dfrac{(m-n)\pi x}{L}\Bigg|_0^L-\dfrac{L}{(m+n)\pi}\sin\dfrac{(m+n)\pi x}{L}\Bigg|_0^L\right]=0, & \text{if } m\neq n\\[2ex] \dfrac{L}{2}, & \text{if } m=n\end{cases}$$

$$\sin\alpha\cdot\sin\beta=\frac{\cos(\alpha-\beta)-\cos(\alpha+\beta)}{2}$$

$$K_n = \frac{2}{3}\int_0^3 \left(5\sin\frac{12\pi x}{3} - 3\sin\frac{24\pi x}{3} + 2\sin\frac{30\pi x}{3}\right)\sin\frac{n\pi x}{3}\,dx, \quad n = 1, 2, \cdots$$

$$= \frac{2}{3}\int_0^3 \left(5\sin\frac{12\pi x}{3}\cdot\sin\frac{n\pi x}{3} - 3\sin\frac{24\pi x}{3}\cdot\sin\frac{n\pi x}{3} + 2\sin\frac{30\pi x}{3}\cdot\sin\frac{n\pi x}{3}\right)dx$$

$$\Rightarrow \quad K_{12} = \frac{2}{3}\int_0^3 \left(5\sin\frac{12\pi x}{3}\cdot\sin\frac{12\pi x}{3} - 3\sin\cancel{\frac{24\pi x}{3}\cdot\sin}\frac{12\pi x}{3} + 2\sin\cancel{\frac{30\pi x}{3}\cdot\sin}\frac{12\pi x}{3}\right)dx$$

$$= \frac{2}{3}\cdot 5\cdot\frac{L}{2} = \frac{2}{3}\cdot 5\cdot\frac{3}{2} = 5$$

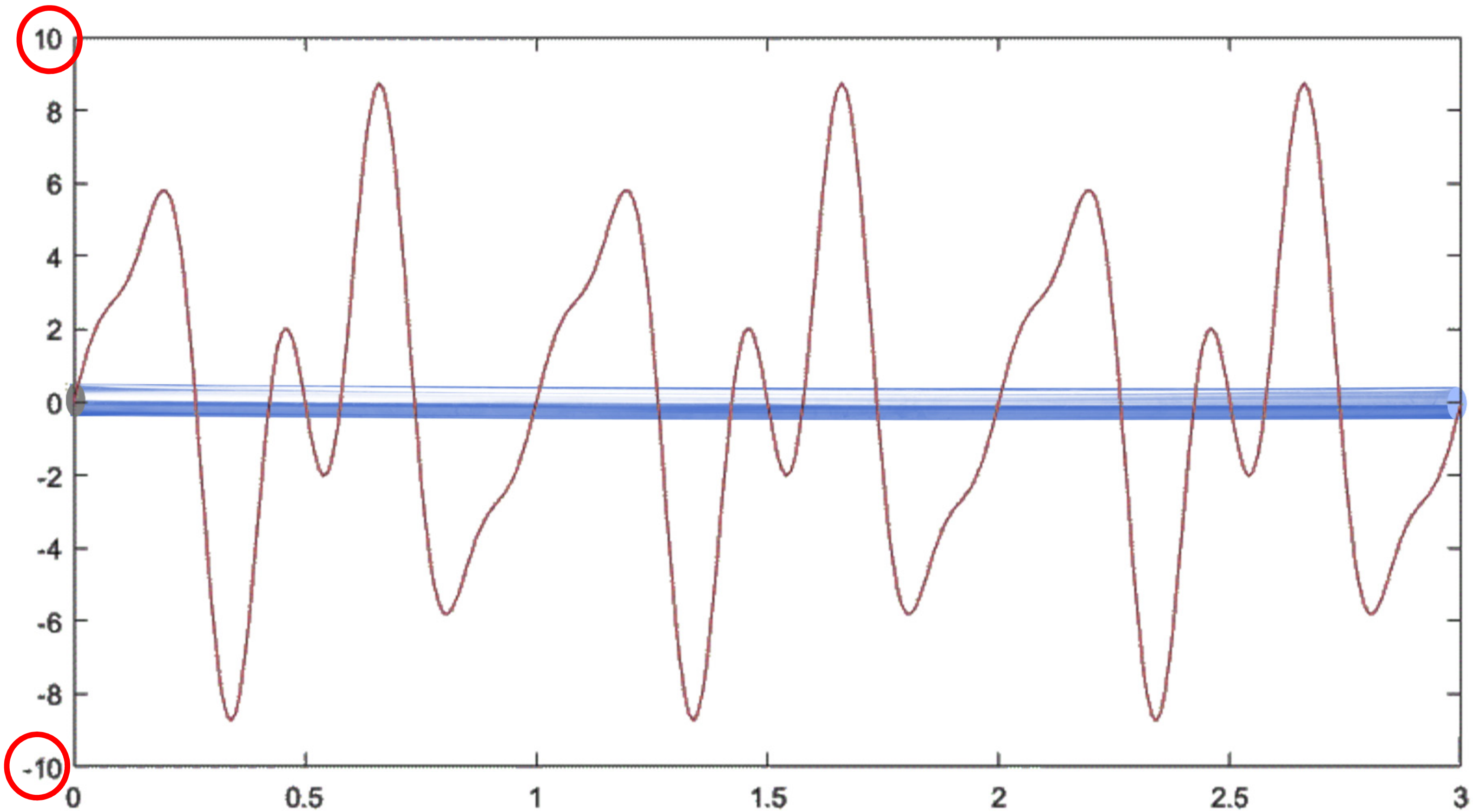
$$K_{24} = -3, \quad K_{30} = 2, \quad K_n = 0, \text{ otherwise}$$

The solution to the given problem in Example (a) is thus given by

$$u(x,t) = 5\sin(4\pi x)e^{-32\pi^2 t} - 3\sin(8\pi x)e^{-128\pi^2 t} + 2\sin(10\pi x)e^{-200\pi^2 t}$$

Graphically, the temperature $u(x, t)$ at $t = 0\ldots$

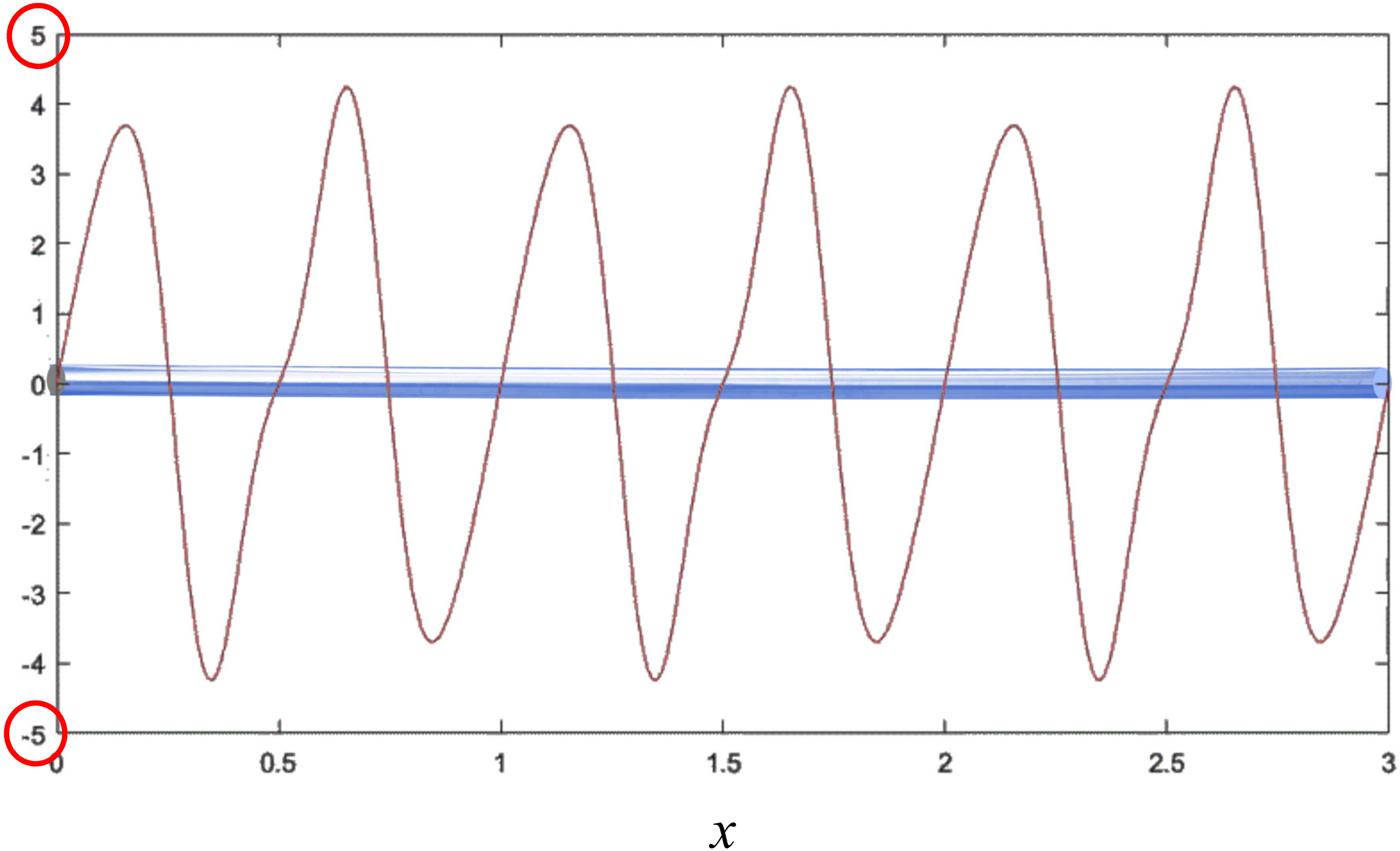
$$u(x,0) = 5\sin(4\pi x) - 3\sin(8\pi x) + 2\sin(10\pi x)$$



x

Graphically, the temperature $u(x, t)$ at $t = 0.001\ldots$

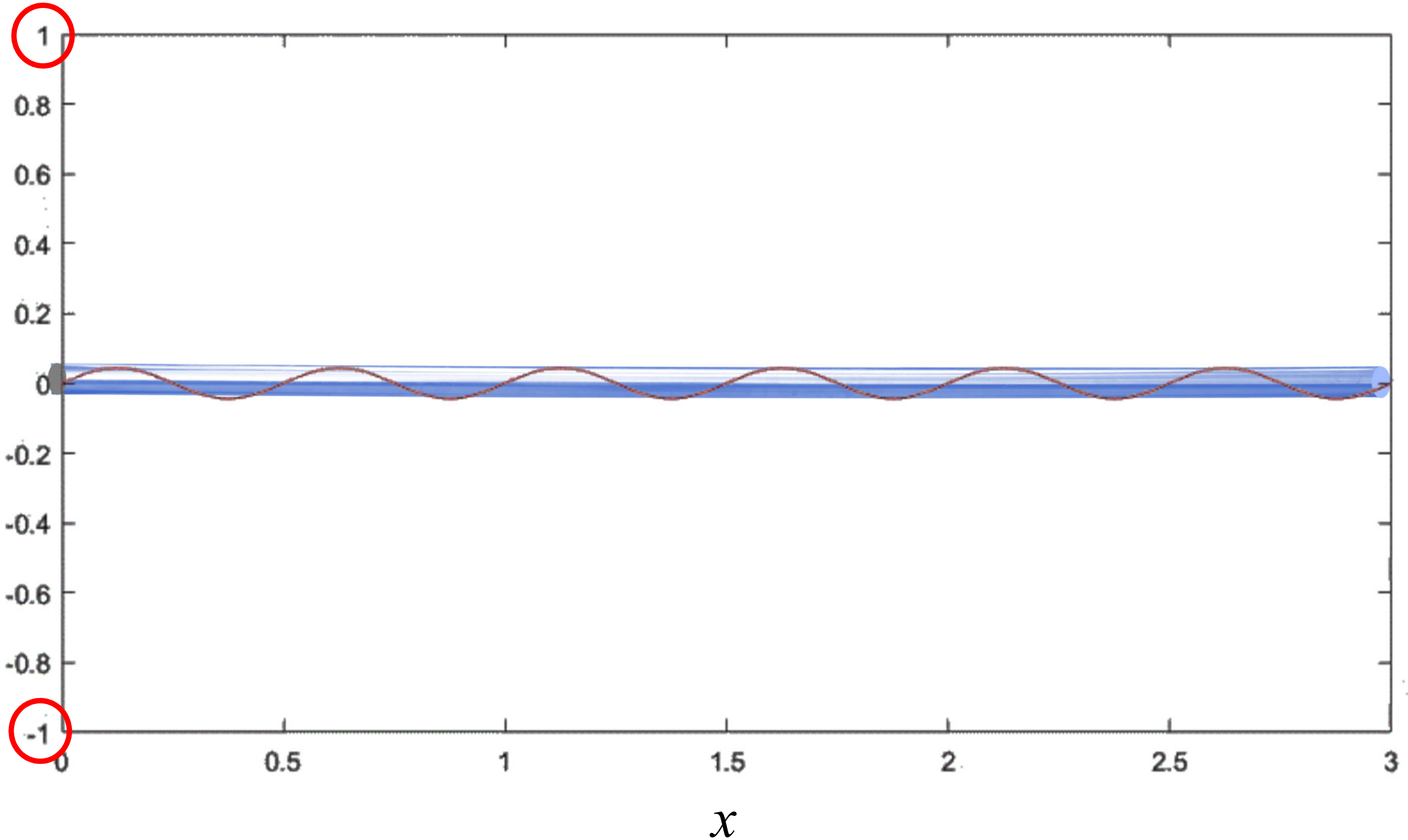
$$u(x,0.001)=5\sin\left(4\pi x\right)e^{-0.032\pi^2}-3\sin\left(8\pi x\right)e^{-0.128\pi^2}+2\sin\left(10\pi x\right)e^{-0.2\pi^2}$$



Graphically, the temperature $u(x, t)$ at $t = 0.015\ldots$

heat

$$u(x,0.015) = 5\sin\left(4\pi x\right)e^{-0.48\pi^2} - 3\sin\left(8\pi x\right)e^{-1.92\pi^2} + 2\sin\left(10\pi x\right)e^{-3\pi^2}$$



Example (b)

Solve the following heat equation by the method of separation of variables:

$$\begin{cases} u_t = u_{xx}, & 0 < x < 1, t > 0, \\ u(0,t) = u(1,t) = 0, & t > 0, \\ u(x,0) = f(x), \end{cases}$$

where

$$f(x) = \begin{cases} x & \text{for } 0 < x < \frac{1}{2}, \\ 1 - x & \text{for } \frac{1}{2} \le x < 1. \end{cases}$$

Solution: $\alpha = 1, \quad L = 1, \quad u_1 = u_2 = 0$ and thus

$$u(x,t) = u_1 + \frac{(u_2 - u_1)x}{L} + \sum_{n=1}^{\infty} K_n \sin\frac{n\pi x}{L} e^{-\left(\frac{n\pi\alpha}{L}\right)^2 t}$$

$$= \sum_{n=1}^{\infty} K_n \sin(n\pi x) e^{-n^2\pi^2 t},$$

$$F(x)=f(x)-u_1-\frac{(u_2-u_1)x}{L}=f(x)$$

$$\Downarrow$$

$$K_n=\frac{2}{L}\int_0^L F(x)\sin\frac{n\pi x}{L}\,dx=2\int_0^L f(x)\sin(n\pi x)\,dx$$

$$\Downarrow$$

$$K_n=2\left[\int_0^{1/2} x\sin(n\pi x)dx+\int_{1/2}^{1}(1-x)\sin(n\pi x)dx\right]$$

$$=2\left[-\frac{1}{n\pi}\int_0^{1/2} x\,d\cos(n\pi x)-\frac{1}{n\pi}\cos(n\pi x)\Big|_{1/2}^{1}+\frac{1}{n\pi}\int_{1/2}^{1} x\,d\cos(n\pi x)\right]$$

$$=\frac{2}{n\pi}\left[\left(-x\cos(n\pi x)\Big|_0^{1/2}+\int_0^{1/2}\cos(n\pi x)\,dx\right)-\cos(n\pi)+\cos\frac{n\pi}{2}+\left(x\cos(n\pi x)\Big|_{1/2}^{1}-\int_{1/2}^{1}\cos(n\pi x)\,dx\right)\right]$$

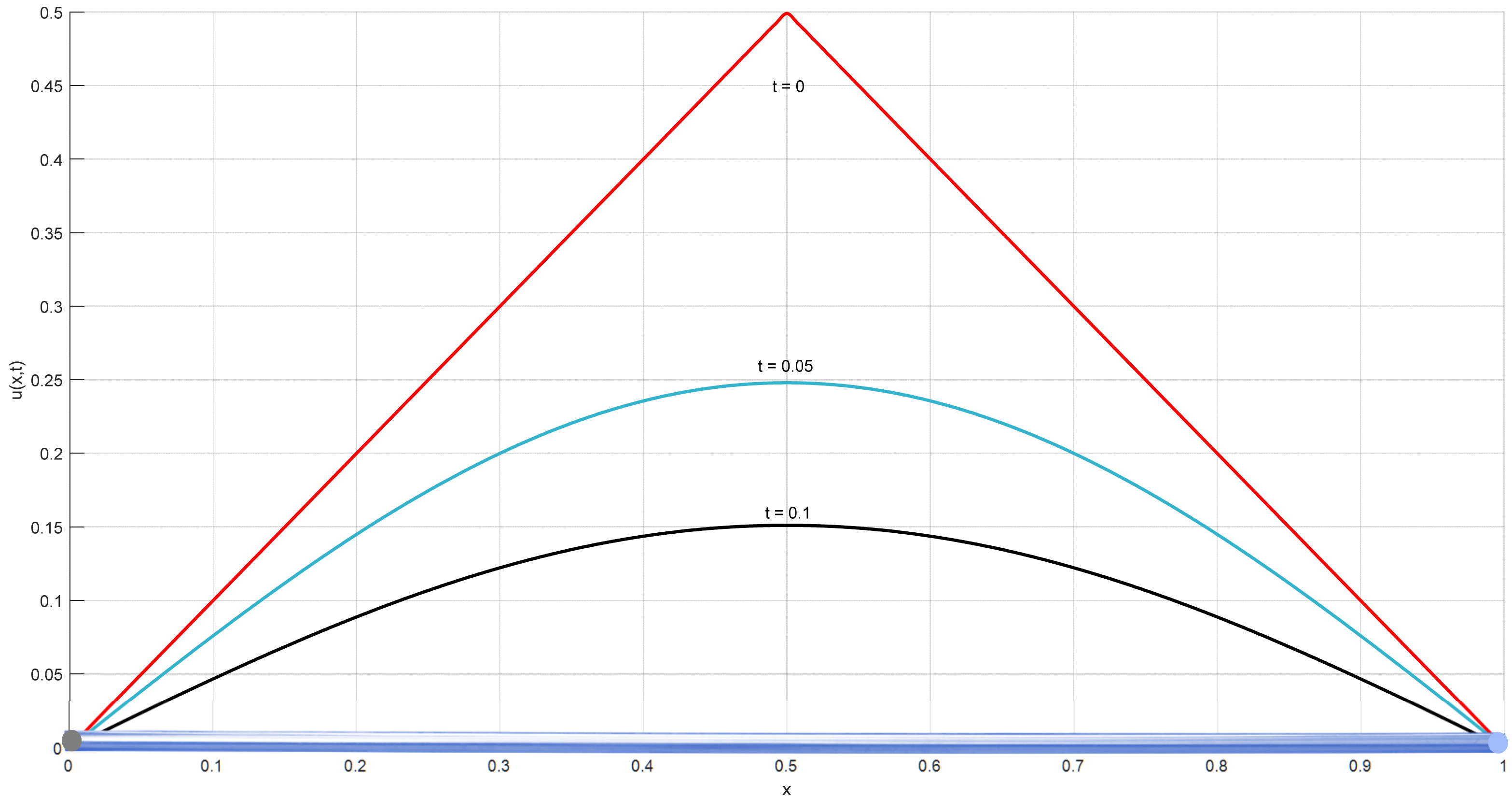
$$=\frac{2}{n\pi}\left[\cancel{-\frac{1}{2}\cos\frac{n\pi}{2}}+0+\frac{1}{n\pi}\sin(n\pi x)\Big|_0^{1/2}\cancel{-\cos(n\pi)}+\cancel{\cos\frac{n\pi}{2}}+\cancel{\cos(n\pi)}\cancel{-\frac{1}{2}\cos\frac{n\pi}{2}}-\frac{1}{n\pi}\sin(n\pi x)\Big|_{1/2}^{1}\right]$$

$$=\frac{2}{n\pi}\left[\frac{1}{n\pi}\sin\frac{n\pi}{2}+\frac{1}{n\pi}\sin\frac{n\pi}{2}\right]=\frac{4}{n^2\pi^2}\sin\frac{n\pi}{2}$$

heat2

The solution to the problem is thus given by

$$u(x,t)=\sum_{n=1}^{\infty}K_n\sin(n\pi x)e^{-n^2\pi^2 t}=\frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{2}\sin(n\pi x)e^{-n^2\pi^2 t}$$



Exercise

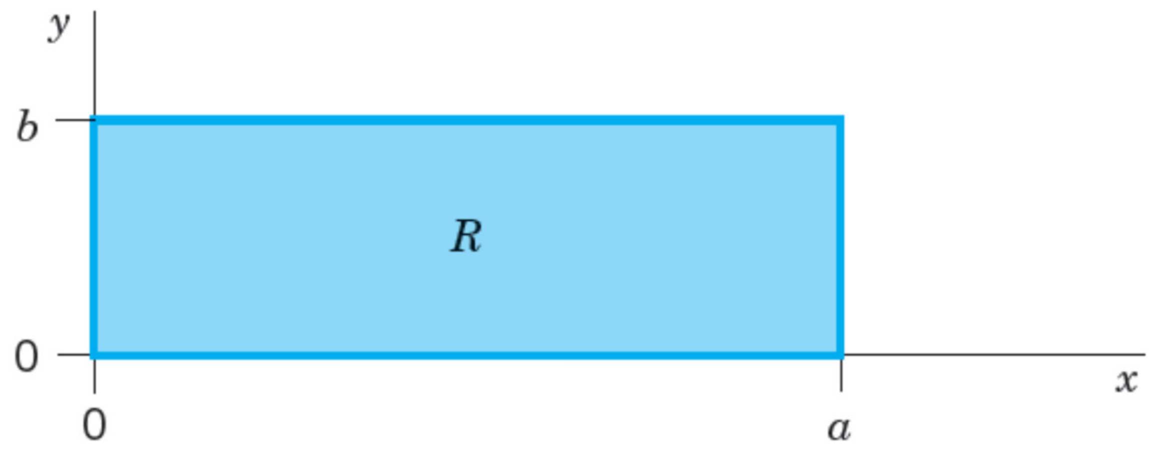
Solve $\dfrac{\partial u}{\partial t} = 7\dfrac{\partial^2 u}{\partial x^2}\,, \qquad 0 < x < \pi, \quad 0 < t < \infty$

Given $u(0,t) = u(\pi,t) = 0\,, \quad 0 < t < \infty$

$$u(x,0) = 3\sin 2x - 6\sin 5x, \quad 0 < x < \pi$$

Steady 2D Heat Problems and Dirichlet Problem

We now consider a 2D heat problems as depicted in the figure below:



2D Heat Problem on a region *R*

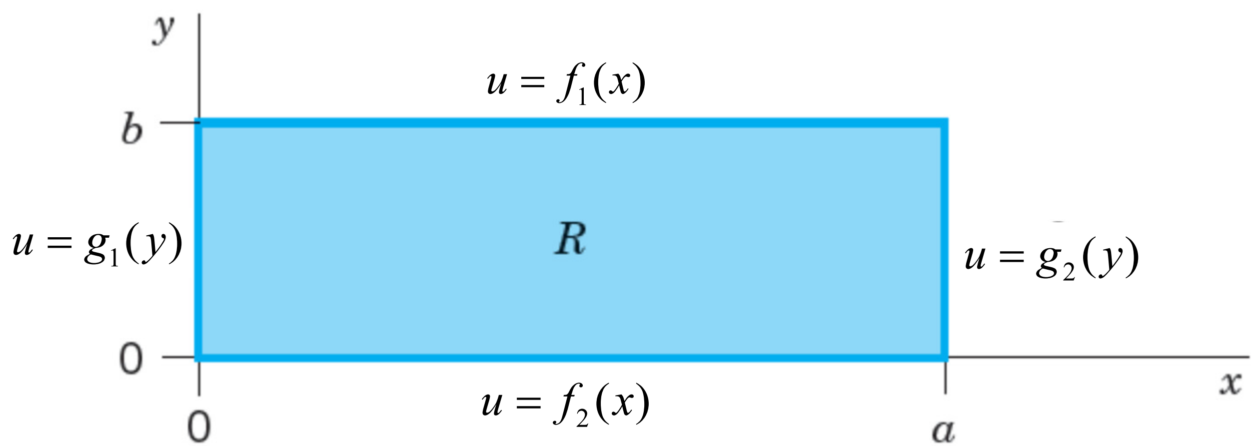
It can be showed that the temperature u on R is characterized by

$$\frac{\partial u}{\partial t} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$$

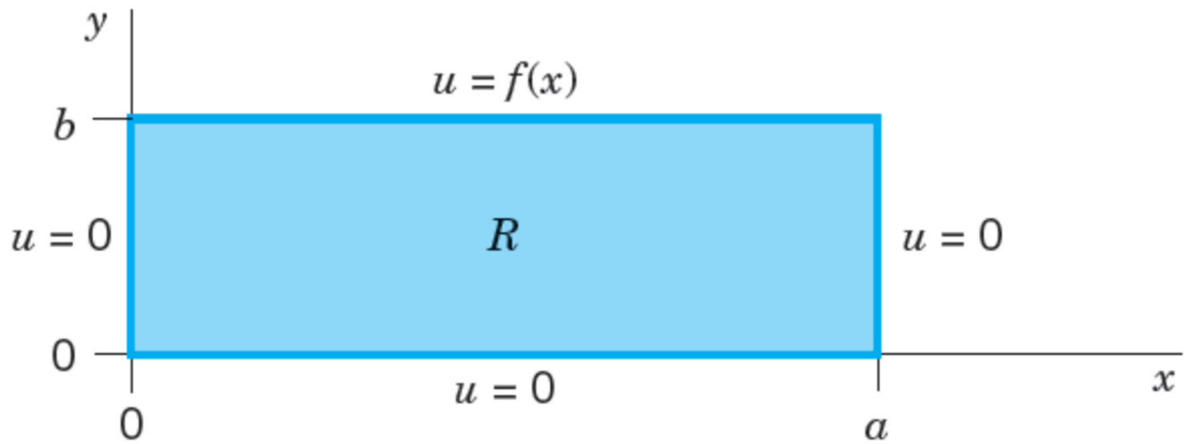
For the steady (*that is, time-independent*) problems, i.e., $\partial u/\partial t = 0$, the heat equation reduces to **Laplace's equation**:

$$\frac{\partial u}{\partial t} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \quad \Rightarrow \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

A heat problem consists of this PDE in region R of the xy-plane and a given boundary condition on the boundary curve C of R is called a **Dirichlet Problem** if u is prescribed on C.



We consider a Dirichlet problem in a rectangle R, assuming that the temperature $u(x, y)$ equals a given function $f(x)$ on the upper side and 0 on the other three sides of the rectangle:



We solve this problem by separating variables. Substituting $u(x, y) = F(x)\, G(y)$ into

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \Rightarrow \quad \frac{\partial^2 \left(F(x) G(y) \right)}{\partial x^2} + \frac{\partial^2 \left(F(x) G(y) \right)}{\partial y^2} = \frac{d^2 F}{dx^2} G + \frac{d^2 G}{dy^2} F = 0$$

$$\frac{1}{F}\cdot\frac{d^2F}{dx^2} = -\frac{1}{G}\cdot\frac{d^2G}{dy^2} = -k \quad (k>0)$$

$$\Rightarrow \quad \frac{d^2F}{dx^2} + kF = 0$$

i). When $k=0$,

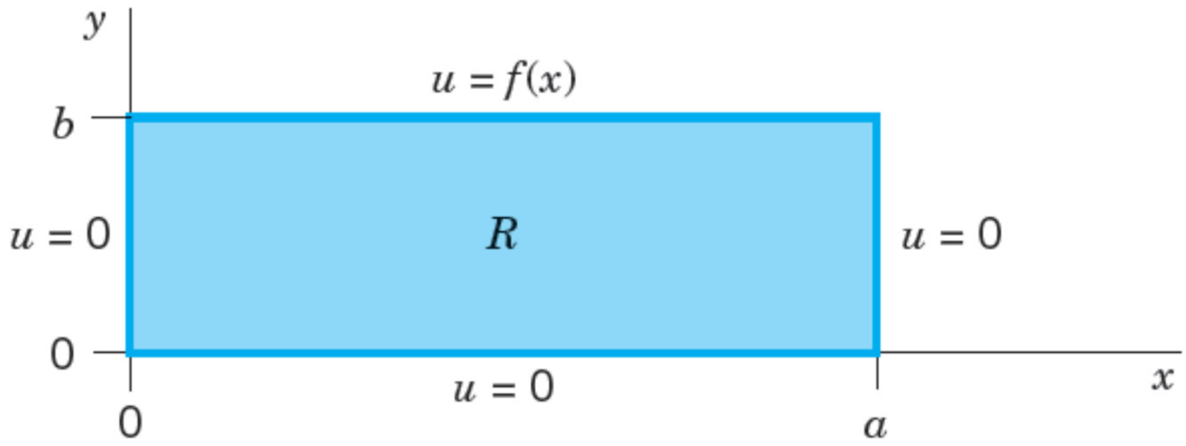
$$F'' + kF = 0 \quad \Rightarrow \quad F'' = 0 \quad \Rightarrow \quad F(x) = D + Ex \qquad (\clubsuit)$$

where D and E are constants.

ii). When $k \neq 0$, solving the ODE gives

$$F(x) = A\cos(\sqrt{k}\,x) + B\sin(\sqrt{k}\,x), \qquad (\clubsuit\clubsuit)$$

where A and B are constants.



From the above boundary conditions, we obtain

$$F(0)=0 \quad \text{and} \quad F(a)=0$$

and (♣) implies

$$F(0)=D=0, \quad F(a)=D+Ea=Ea=0 \quad \Rightarrow \quad E=0$$

$$F(x)=D+Ex \qquad (♣)$$

(♣♣) implies

$$F(0) = A\cos(\sqrt{k}) = 0 \quad \Rightarrow \quad A = 0$$

$$F(a) = A\cos(\sqrt{k}\,a) + B\sin(\sqrt{k}\,a) = B\sin(\sqrt{k}\,a) = 0$$

Since we cannot choose $B = 0$ (why?), we have

$$\sin(\sqrt{k}\,a) = 0 \quad \Rightarrow \quad k = \left(\frac{n\pi}{a}\right)^2$$

and the corresponding nonzero solution

$$F(x) = F_n(x) = B_n \sin\frac{n\pi x}{a}$$

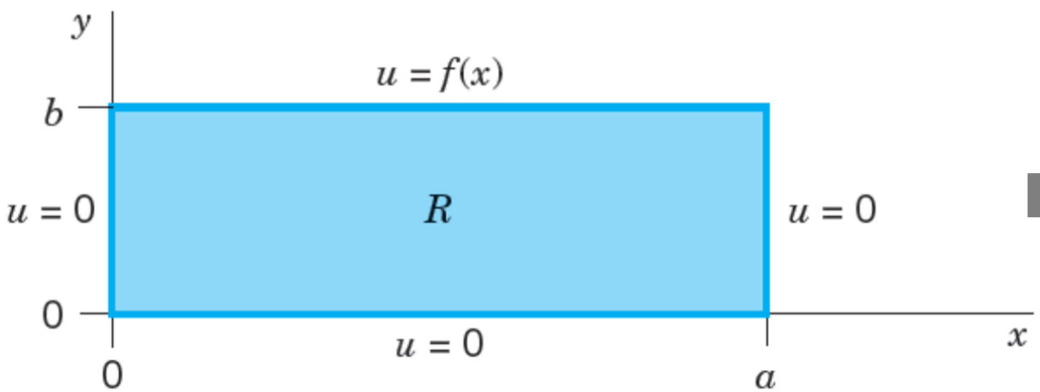
$$F(x) = A\cos(\sqrt{k}\,x) + B\sin(\sqrt{k}\,x), \qquad (♣♣)$$

Recall

$$\frac{1}{F}\cdot\frac{d^2F}{dx^2} = -\frac{1}{G}\cdot\frac{d^2G}{dy^2} = -k \quad\Rightarrow\quad \frac{d^2G}{dx^2} - \left(\frac{n\pi}{a}\right)^2 G = 0$$

We then have

$$G(y) = G_n(y) = C_n e^{n\pi y/a} + E_n e^{-n\pi y/a}$$



$$G(0) = C_n + E_n = 0$$

$$\Downarrow$$

$$C_n = -E_n$$

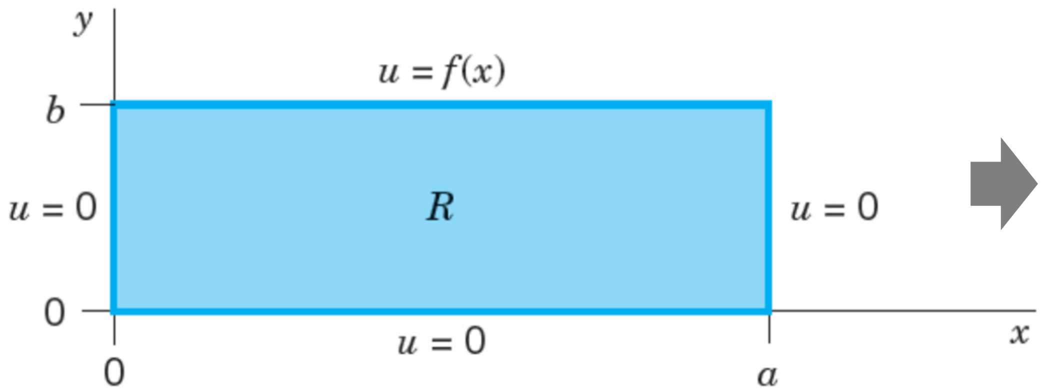
$$\Rightarrow \quad G(y) = G_n(y) = C_n e^{n\pi y/a} - C_n e^{-n\pi y/a} = 2C_n \frac{e^{n\pi y/a} - e^{-n\pi y/a}}{2} = 2C_n \sinh\frac{n\pi y}{a}$$

We thus obtain a solution

$$u_n(x,y) = F_n(x)G_n(y) = (B_n C_n)\sin\frac{n\pi x}{a}\sinh\frac{n\pi y}{a} = D_n \sin\frac{n\pi x}{a}\sinh\frac{n\pi y}{a}$$

which is called an **eigenfunction** of the problem. By superposition, we have obtained a solution to the Dirichlet problem

$$u(x,y) = \sum_{n=1}^{\infty} u_n(x,y) = \sum_{n=1}^{\infty} D_n \sin\frac{n\pi x}{a}\sinh\frac{n\pi y}{a}$$



$$u(x,b) = f(x)$$

$$= \sum_{n=1}^{\infty} D_n \sin\frac{n\pi x}{a}\sinh\frac{n\pi b}{a}$$

$$\Rightarrow \quad u(x,b) = f(x) = \sum_{n=1}^{\infty} \left(D_n \sinh \frac{n\pi b}{a} \right) \sin \frac{n\pi x}{a}$$

By half-range sine expansion,

$$b_n = D_n \sinh \frac{n\pi b}{a} = \frac{2}{a} \int_0^a f(x) \sin \frac{n\pi x}{a} dx$$

Thus, the solution to the Dirichlet problem is given as

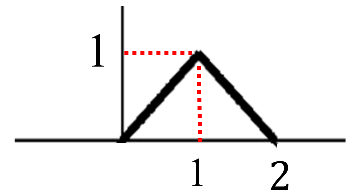
$$u(x,y) = \sum_{n=1}^{\infty} D_n \sin \frac{n\pi x}{a} \sinh \frac{n\pi y}{a}$$

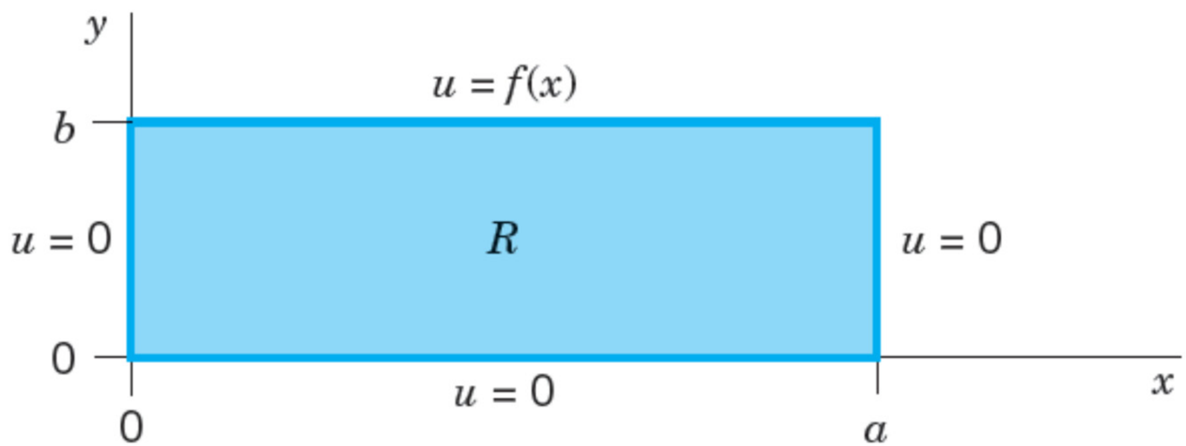
where

$$D_n = \frac{2}{a \sinh(n\pi b / a)} \int_0^a f(x) \sin \frac{n\pi x}{a} dx$$

Example: Solve the Dirichlet problem with $a = 2$, $b = 1$ and

$$f(x) = \begin{cases} x & \text{if} \quad 0 < x < 1 \\ (2-x) & \text{if} \quad 1 < x < 2 \end{cases}$$





Solution: The solution is given by

$$u(x, y) = \sum_{n=1}^{\infty} D_n \sin \frac{n\pi x}{a} \sinh \frac{n\pi y}{a}$$

with $D_n = \dfrac{2}{a\sinh(n\pi b/a)}\displaystyle\int_0^a f(x)\sin\frac{n\pi x}{a}dx = \frac{1}{\sinh(n\pi/2)}\int_0^2 f(x)\sin\frac{n\pi x}{2}dx$

$$\int_0^2 f(x)\sin\frac{n\pi x}{2}dx = \int_0^1 x\sin\frac{n\pi x}{2}dx + \int_1^2 (2-x)\sin\frac{n\pi x}{2}dx$$

$$= -\frac{2}{n\pi}\int_0^1 x\, d\cos\frac{n\pi x}{2} - \frac{2}{n\pi}\int_1^2 (2-x)\, d\cos\frac{n\pi x}{2}$$

$$= -\frac{2}{n\pi}\left[x\cos\frac{n\pi x}{2}\bigg|_0^1 - \int_0^1 \cos\frac{n\pi x}{2}dx \right] - \frac{2}{n\pi}\left[(2-x)\cos\frac{n\pi x}{2}\bigg|_1^2 - \int_1^2 \cos\frac{n\pi x}{2}d(2-x) \right]$$

$$= -\frac{2}{n\pi}\left[\cancel{\cos\frac{n\pi}{2}} - \frac{2}{n\pi}\sin\frac{n\pi x}{2}\bigg|_0^1 \right] - \frac{2}{n\pi}\left[-\cancel{\cos\frac{n\pi}{2}} + \int_1^2 \cos\frac{n\pi x}{2}dx \right]$$

$$= \left[\left(\frac{2}{n\pi}\right)^2 \sin\frac{n\pi}{2} - 0 \right] - \left(\frac{2}{n\pi}\right)^2 \sin\frac{n\pi x}{2}\bigg|_1^2$$

$$= \left(\frac{2}{n\pi}\right)^2 \sin\frac{n\pi}{2} - \left[0 - \left(\frac{2}{n\pi}\right)^2 \sin\frac{n\pi}{2} \right] = \boxed{\frac{8}{n^2\pi^2}\sin\frac{n\pi}{2}}$$

$$\Rightarrow \quad D_n = \frac{1}{\sinh\left(n\pi/2\right)} \int_0^2 f(x)\sin\frac{n\pi x}{2}\,dx$$

$$= \frac{8}{n^2\pi^2 \sinh\frac{n\pi}{2}} \cdot \sin\frac{n\pi}{2}$$

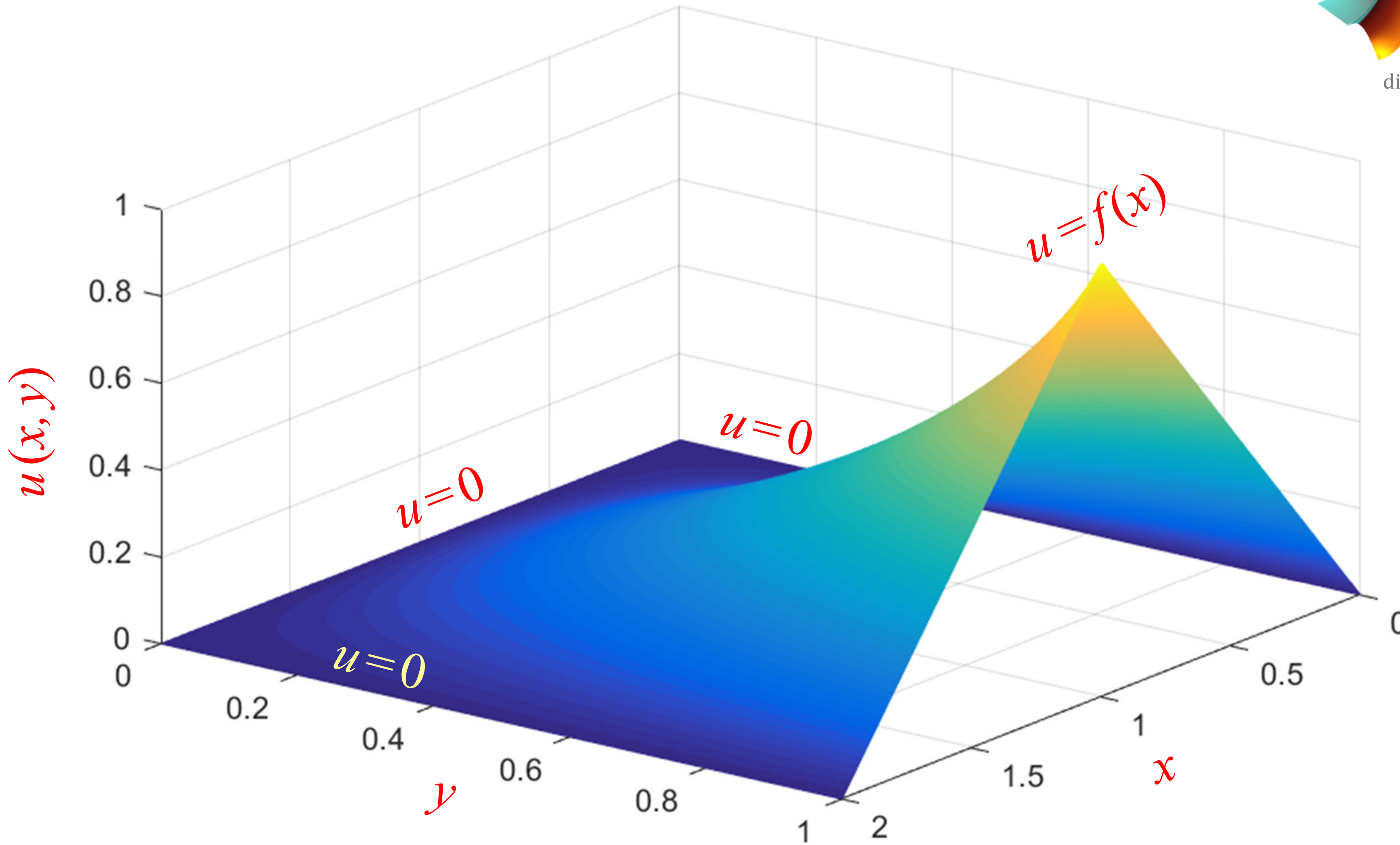
The solution to the Dirichlet problem is given by

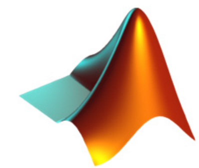
$$u(x,y) = \sum_{n=1}^{\infty} D_n \sin\frac{n\pi x}{a} \sinh\frac{n\pi y}{a}$$

$$= \sum_{n=1}^{\infty} \frac{8}{n^2\pi^2 \sinh\frac{n\pi}{2}} \cdot \sin\frac{n\pi}{2} \sin\frac{n\pi x}{2} \sinh\frac{n\pi y}{2}$$

$$\Rightarrow \quad u(x,y) = \frac{8}{\pi^2} \sum_{n=1}^{\infty} \left(\frac{\sin\frac{n\pi}{2}}{n^2 \sinh\frac{n\pi}{2}} \right) \sin\frac{n\pi x}{2} \sinh\frac{n\pi y}{2}$$

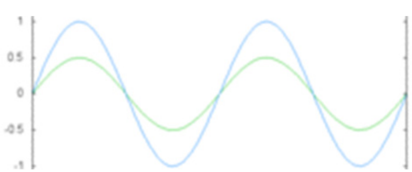
Graphically,





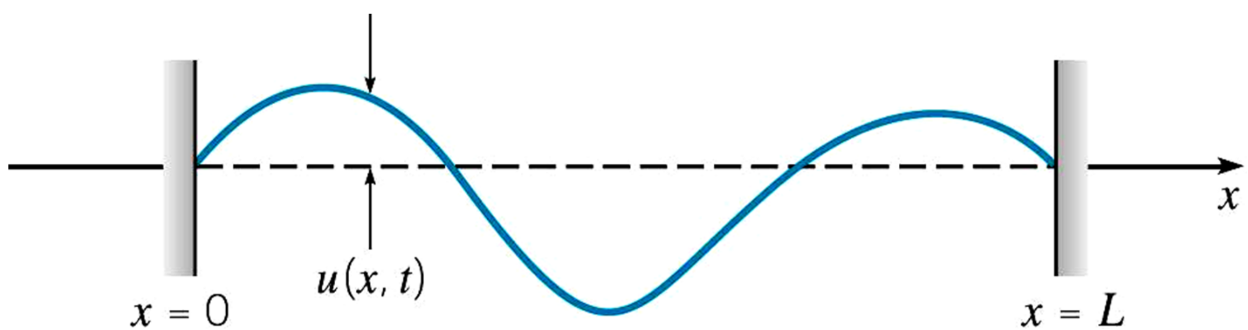
Partial Differential Equations – 6…

1. Even and odd functions, periodic functions
2. Fourier series of a periodic function
3. Fourier series: Half-range expansions
4. Concepts of partial differential equations
5. Heat equation (or diffusion equation)
6. Wave equation



1D Wave Equation: Vibrations of an Elastic String

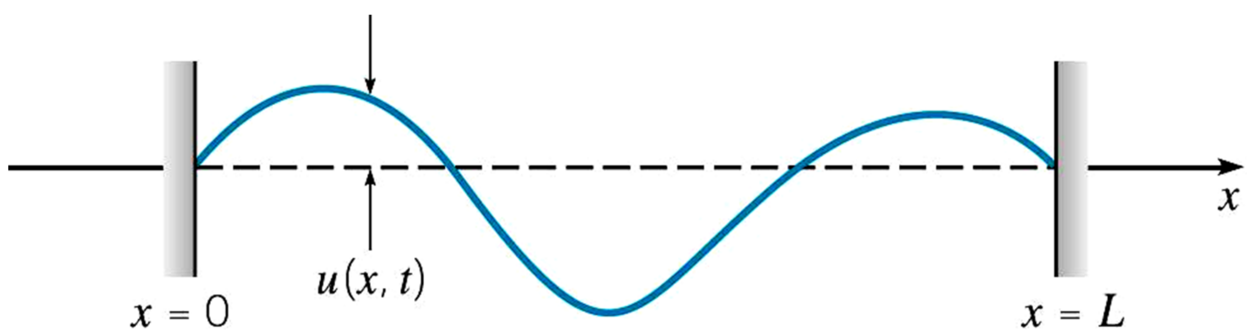
- Suppose that an elastic string of length L is tightly stretched between two supports at the same horizontal level.
- Let the x-axis be chosen to lie along the axis of the string, and let $x = 0$ and $x = L$ denote the ends of the string.
- Suppose that the string is set in motion so that it vibrates in a vertical plane, and let $u(x,t)$ denote the vertical deflection experienced by the string at the point x at time t.



- Assume that damping effects, such as air resistance, can be neglected, and that the amplitude of motion is not too large.
- It can be showed that under these assumptions, the string vibration is governed by the one-dimensional wave equation, and has the form

$$u_{tt} = c^2 u_{xx}, \qquad 0 < x < L, \quad t > 0$$

where the constant coefficient $c^2 = T/\rho$ with T being the tension, while ρ the mass per unit length of the string material.



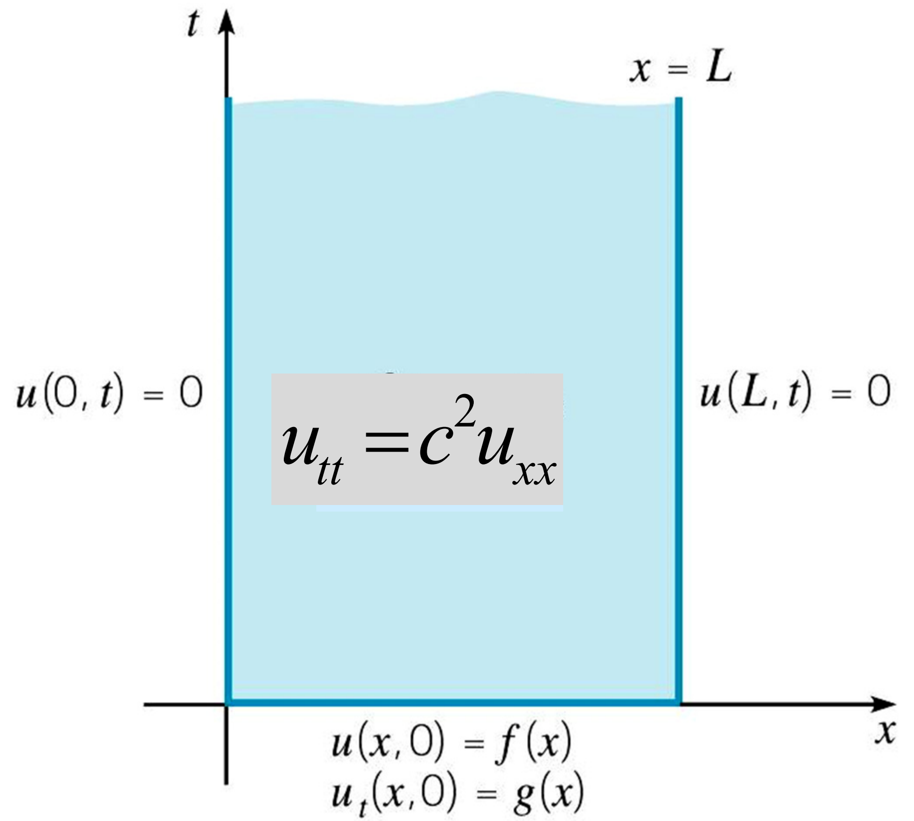
Initial and Boundary Conditions

- Assume that the ends of the string remain fixed,

$$u(0,t)=0, \quad u(L,t)=0, \qquad t \ge 0$$

Since the wave equation is of second order with respect to t, it is plausible to prescribe two initial conditions, the initial position of the string, and its initial velocity:

$$u(x,0)=f(x),$$
$$u_t(x,0)=g(x), \qquad 0<x<L$$



Wave Equation

Thus the wave equation problem is

$$u_{tt} = c^2 u_{xx}, \qquad 0 < x < L, \qquad t > 0$$

$$\text{Boundary conditions}: \quad u(0,t) = 0, \;\; u(L,t) = 0, \quad t > 0$$

$$\text{Initial conditions}: \quad u(x,0) = f(x), \quad u_t(x,0) = g(x), \qquad 0 < x < L$$

- This is an initial value problem with respect to t, and a boundary value problem with respect to x.

- The wave equation governs a large number of other wave problems besides the transverse vibrations of an elastic string.

Solution by Separating Variables

Problem: Find the solution of the following 1D wave equation

$$u_{tt} = c^2 u_{xx}, \qquad 0 < x < L, \qquad t > 0 \tag{1}$$

with the boundary conditions

$$u(0, t) = 0 \;\text{ and }\; u(L, t) = 0 \quad \text{for all } t > 0, \tag{2}$$

and initial conditions are

$$u(x, 0) = f(x) \quad \text{and} \quad u_t(x, 0) = g(x) \quad (0 < x < L) \tag{3}$$

where $f(x)$ denote the initial deflection and $g(x)$ the initial velocity.

Solution in Three Steps

Step 1.

Set $u(x, t) = F(x)G(t) \quad \Rightarrow$ Obtain two ODEs, one for $F(x)$ and the other one for $G(t)$.

- The method of separating variables

Step 2.

Determine solutions of these ODEs that satisfy the boundary conditions (2).

Step 3.

Compose the solutions gained in Step 2 using Fourier series $\Rightarrow$ Obtain a solution of (1) satisfying both (2) and (3).

Step 1. Method of Separating Variables

- Determine solutions of Eq.(1) in the form

$$u(x,\ t) = F(x)G(t) \tag{4}$$

which are a product of two functions, each depending only on one of the variables x and t.

- Differentiating (4), we get

$$\frac{\partial^2 u}{\partial t^2} = F\ddot{G} \qquad \text{and} \qquad \frac{\partial^2 u}{\partial x^2} = F''G$$

where dots denote derivatives with respect to t and primes derivatives with respect to x.

- By inserting them into Eq. (1), we have $F\ddot{G} = c^2F''G$.

- Dividing by c^2FG gives $\dfrac{\ddot{G}}{c^2G} = \dfrac{F''}{F}$

The variables are now separated, the left side depending only on t and the right side only on x. Since both sides must be equal to some common constants, say, k,

$$\frac{\ddot{G}}{c^2G} = \frac{F''}{F} = k.$$

- Multiplying by the denominators gives two ODEs

$$F''-kF=0 \qquad (5)$$

and

$$\ddot{G}-c^2kG=0 \qquad (6)$$

Step 2. Satisfying the Boundary Conditions

Determine solutions F and G so that $u = FG$ satisfies the boundary conditions, i.e., for all t

$$u(0, t) = F(0)G(t) = 0 \quad \Rightarrow \quad F(0) = 0 \tag{7}$$

$$u(L,t) = F(L)G(t) = 0 \quad \Rightarrow \quad F(L) = 0 \tag{8}$$

Reason: If $G \equiv 0 \Rightarrow u = FG \equiv 0$, which is trivial solution.

Hence $G \neq 0$.

Solving $F(x)$ from $F'' - kF = 0$ (5)

- k must be negative.

Why? For $k = 0$, the general solution of (5) is $F = a\,x + b$, and from (7) (i.e., $F(0) = 0$) and (8) (i.e., $F(L) = 0$), we obtain $a = b = 0$, so that $F(x) \equiv 0$ and $u = FG \equiv 0$, which is trivial solution.

- For positive $k = \lambda^2 > 0$ a general solution of (5) is

$$F(x) = Ae^{\lambda x} + Be^{-\lambda x}$$

and from (7) (i.e., $F(0) = 0$) and (8) (i.e., $F(L) = 0$), we obtain

$$F(0) = A + B = 0 \quad \Rightarrow \quad B = -A$$

$$F(L) = Ae^{\lambda L} + Be^{-\lambda L} = A(e^{\lambda L} - e^{-\lambda L}) = 0 \quad \Rightarrow \quad A = 0,\ B = 0$$

which imply $F(x) \equiv 0$ once again.

- We have to choose k as $k = -\lambda^2$ ($\lambda > 0$). Then, Eq. (5) can be rewritten as $F'' + \lambda^2 F = 0$, which has a general solution

$$F(x) = A_1 \cos(\lambda x) + B_1 \sin(\lambda x)$$

Using Eqns (7) (i.e., $F(0) = 0$) and (8) (i.e., $F(L) = 0$), we have

$$F(0) = A_1 = 0 \quad \Rightarrow \quad F(L) = B_1 \sin(\lambda L) = 0$$

Since $B_1 \neq 0$ (otherwise $F \equiv 0$) $\Rightarrow \sin(\lambda L) = 0$. Thus

$$\lambda L = n\pi \quad \Rightarrow \lambda = \frac{n\pi}{L} \quad \text{for } n = 1, 2, \ldots \tag{9}$$

This results in infinitely many solutions to Eq. (5) given by

$$F_n(x) = B_{1n} \sin \frac{n\pi x}{L}, \qquad n = 1, 2, \cdots \tag{10}$$

Solving $G(t)$

$$\ddot{G} - c^2 k G = 0 \qquad (6)$$

✓ Solve (6) with $k = -\lambda^2 = -(n\pi/L)^2$, that is,

$$\ddot{G} + c^2 \lambda^2 G = 0 \quad \text{with} \quad \lambda_n = \frac{n\pi}{L}$$

✓ A general solution is

$$G_n(t) = A_{2n} \cos \lambda_n c t + B_{2n} \sin \lambda_n c t \qquad (11)$$

Solving $u_n(x, t)$

- Solutions of (1) satisfying (2) are

$$u_n(x, t) = F_n(x)G_n(t)$$

$$u_n(x,t) = (A_n \cos \lambda_n ct + B_n \sin \lambda_n ct) \sin \lambda_n x \qquad (12)$$

for all $n = 1, 2, \ldots$ and $\lambda_n = \dfrac{n\pi}{L}$.

This is an **eigenfunction** for solving the problem.

Step 3. The solutions of $u(x, t)$

➢ By the **superposition principle,** the general solution of (1) is

$$u(x,t)=\sum_{n=1}^{\infty}u_n(x,t)$$

$$=\sum_{n=1}^{\infty}(A_n\cos\lambda_n ct+B_n\sin\lambda_n ct)\sin\lambda_n x \qquad (13)$$

where A_n and B_n*'s* are determined using the initial conditions.

<u>Case 1:</u> When initial deflection is given: $u(x,0) = f(x)$

$$t = 0 \;\Rightarrow\; f(x) = u(x,0) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L} \tag{14}$$

By Fourier Series:

$$A_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx, \quad n = 1, 2, \cdots \tag{15}$$

<u>Note:</u>

A_n's are the Fourier coefficients in the half-range Fourier sine series expansion of $f(x)$ in the interval $(0, L)$.

Case 2: When initial velocity is given: $u_t(x,0) = g(x)$

Differentiate (13) *w.r.t. t*,

$$\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty} (-A_n \lambda_n c \sin \lambda_n ct + B_n \lambda_n c \cos \lambda_n ct) \sin \lambda_n x$$

$$t = 0 \;\Rightarrow\; g(x) = \left.\frac{\partial u}{\partial t}\right|_{t=0} = \sum_{n=1}^{\infty} \left(B_n \lambda_n c\right) \sin \lambda_n x \qquad (16)$$

$$\Rightarrow \quad B_n \lambda_n c = \frac{2}{L} \int_0^L g(x) \sin \frac{n\pi x}{L} dx$$

Since

$$\lambda_n = \frac{n\pi}{L} \quad \Rightarrow \quad B_n = \frac{2}{cn\pi} \int_0^L g(x) \sin \frac{n\pi x}{L} dx \qquad (17)$$

The general solution of 1-D wave equation:

The general solution of the 1D wave equation (1) with boundary conditions (2) and initial conditions (3) is

$$u(x,t) = \sum_{n=1}^{\infty} \left[A_n \cos\frac{n\pi ct}{L} + B_n \sin\frac{n\pi ct}{L} \right] \sin\frac{n\pi x}{L} \qquad (13)$$

where A_n and B_n*'s* are given by (15) and (17), i.e.,

$$A_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx, \quad n = 1, 2, \cdots \qquad (15)$$

$$B_n = \frac{2}{cn\pi}\int_0^L g(x)\sin\frac{n\pi x}{L}\,dx, \quad n = 1, 2, \cdots \qquad (17)$$

Corollary 1

When only deflection is non-zero, *i.e.*, $u(x,0) = f(x) \neq 0$ and $u_t(x,0) = g(x) = 0$, in which case all B_n's are zero. Then the general solution is

$$u(x,t) = \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi ct}{L}\right) \sin\left(\frac{n\pi x}{L}\right)$$

with A_n's being given by

$$A_n = \frac{2}{L} \int_0^L f(x) \sin\frac{n\pi x}{L}\, dx, \quad n = 1, 2, \cdots$$

Corollary 2

When only initial velocity is non-zero, *i.e.*, $g(x) \neq 0$ and $f(x) = 0$, then the general solution is

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi ct}{L}\right) \sin\left(\frac{n\pi x}{L}\right)$$

with B_n's being given by

$$B_n = \frac{2}{cn\pi} \int_0^L g(x) \sin\frac{n\pi x}{L} dx, \; n = 1, 2, \cdots$$

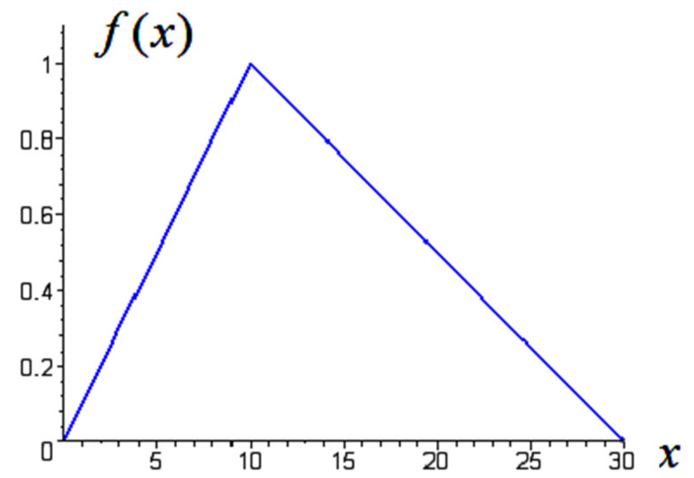
Examples

Q1. (**Vibrating String Problem**). Consider the vibrating string problem of the form

$$4u_{xx} = u_{tt}, \quad 0 < x < 30, \qquad t > 0$$
$$u(0,t) = 0, \; u(30,t) = 0, \qquad t > 0$$
$$u(x,0) = f(x), \quad u_t(x,0) = 0, \qquad 0 < x < 30$$

where

$$f(x) = \begin{cases} x/10, & 0 \le x \le 10 \\ (30 - x)/20, & 10 < x \le 30 \end{cases}$$



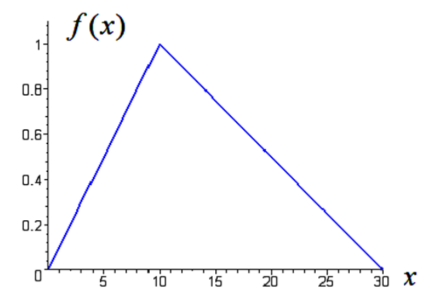
Solve $u(x,t)$.

Solution.

Since $u(x,0) = f(x) \neq 0, u_t(x,0) = 0,$ according to **Corollary 1**, the solution to the vibrating string problem satisfies

$$u(x,t) = \sum_{n=1}^{\infty} A_n \cos\frac{2n\pi t}{30} \sin\frac{n\pi x}{30} \qquad c=2,\ L=30$$

$$A_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx$$



$$= \frac{2}{30}\int_0^{10}\frac{x}{10}\sin\frac{n\pi x}{30}dx + \frac{2}{30}\int_{10}^{30}\frac{30-x}{20}\sin\frac{n\pi x}{30}dx$$

$$= -\frac{2\times 30}{30\times 10 n\pi}\int_0^{10} x\, d\cos\frac{n\pi x}{30} + \frac{2\times 30}{30\times 20}\int_{10}^{30}\sin\frac{n\pi x}{30}dx + \frac{2\times 30}{30\times 20 n\pi}\int_{10}^{30} x\, d\cos\frac{n\pi x}{30}$$

$$= -\frac{1}{5n\pi}\left(x\cos\frac{n\pi x}{30}\Big|_0^{10} - \int_0^{10}\cos\frac{n\pi x}{30}dx \right) - \frac{3}{n\pi}\cos\frac{n\pi x}{30}\Big|_{10}^{30}$$

$$+\frac{1}{10n\pi}\left(x\cos\frac{n\pi x}{30}\Big|_{10}^{30} - \int_{10}^{30}\cos\frac{n\pi x}{30}dx \right)$$

$$A_n = \cdots = -\frac{1}{5n\pi}\left(10\cos\frac{n\pi}{3} - \frac{30}{n\pi}\sin\frac{n\pi x}{30}\bigg|_0^{10}\right) - \frac{3}{n\pi}\cos n\pi + \frac{3}{n\pi}\cos\frac{n\pi}{3}$$

$$+\frac{1}{10n\pi}\left(30\cos n\pi - 10\cos\frac{n\pi}{3} - \frac{30}{n\pi}\sin\frac{n\pi x}{30}\bigg|_{10}^{30}\right)$$

$$= -\cancel{\frac{2}{n\pi}\cos\frac{n\pi}{3}} + \frac{6}{n^2\pi^2}\sin\frac{n\pi}{3} - \cancel{\frac{3}{n\pi}\cos n\pi} + \cancel{\frac{3}{n\pi}\cos\frac{n\pi}{3}}$$

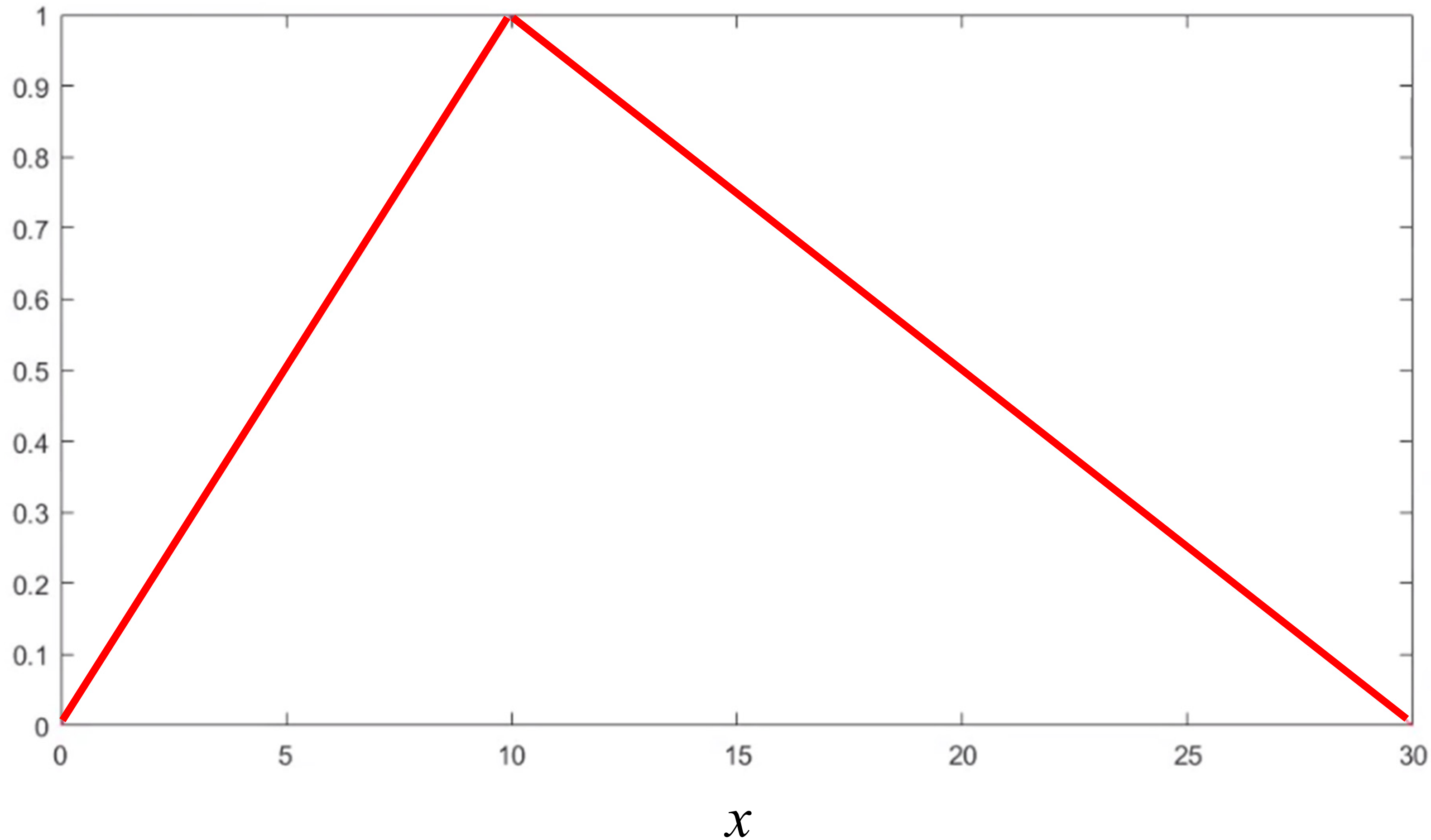
$$+\cancel{\frac{3}{n\pi}\cos n\pi} - \cancel{\frac{1}{n\pi}\cos\frac{n\pi}{3}} - 0 + \frac{3}{n^2\pi^2}\sin\frac{n\pi}{3}$$

$$= \frac{9}{n^2\pi^2}\sin\frac{n\pi}{3}, \qquad n = 1, 2, \cdots$$

$$\Rightarrow \boxed{u(x,t) = \sum_{n=1}^{\infty}\frac{9}{n^2\pi^2}\sin\frac{n\pi}{3}\sin\frac{n\pi x}{30}\cos\frac{2n\pi t}{30}}$$

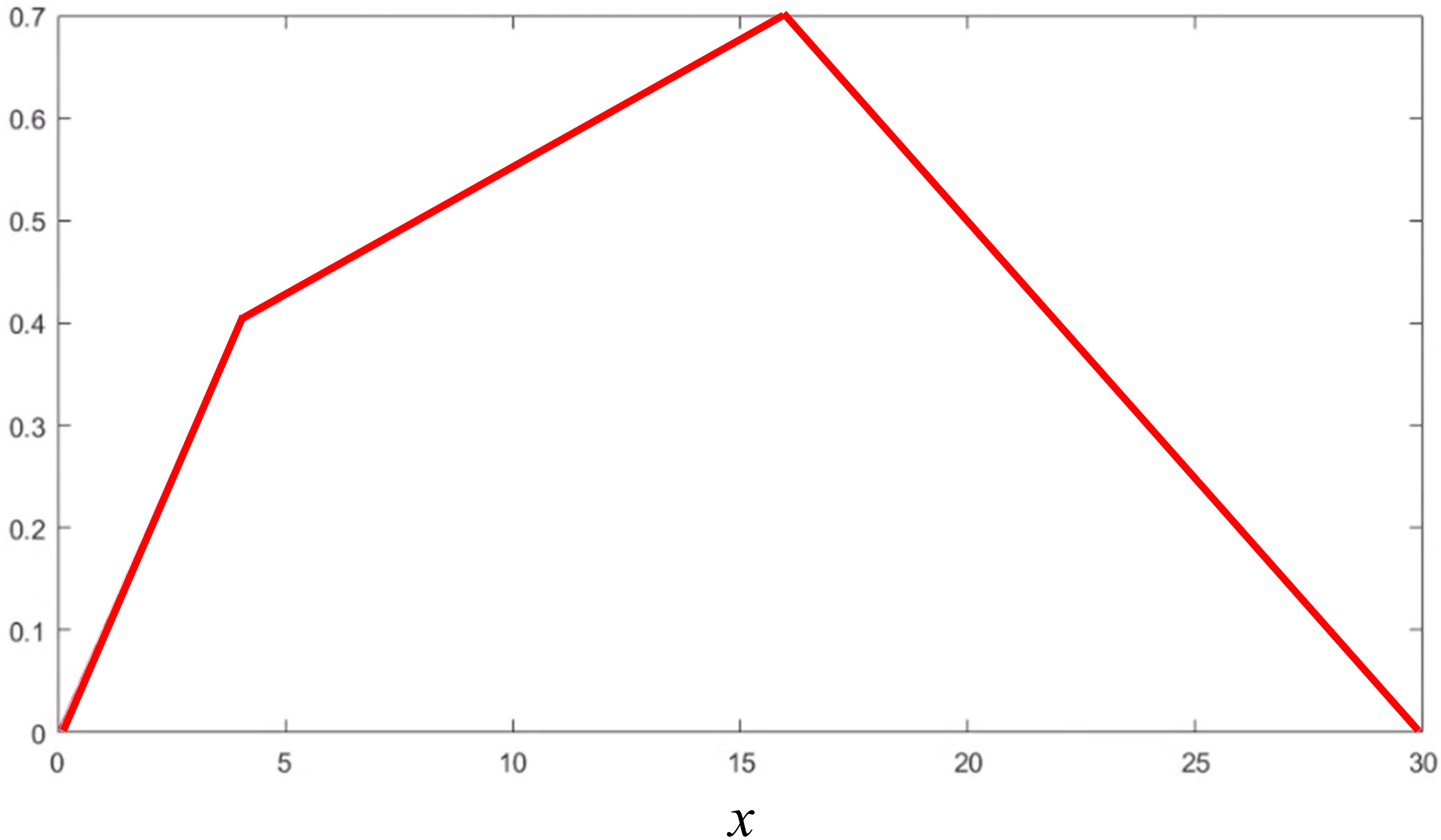
Graphically, the displacement $u(x, t)$ at $t = 0$…

$$u(x,t) = \sum_{n=1}^{\infty} \frac{9}{n^2\pi^2} \sin\frac{n\pi}{3} \sin\frac{n\pi x}{30}$$



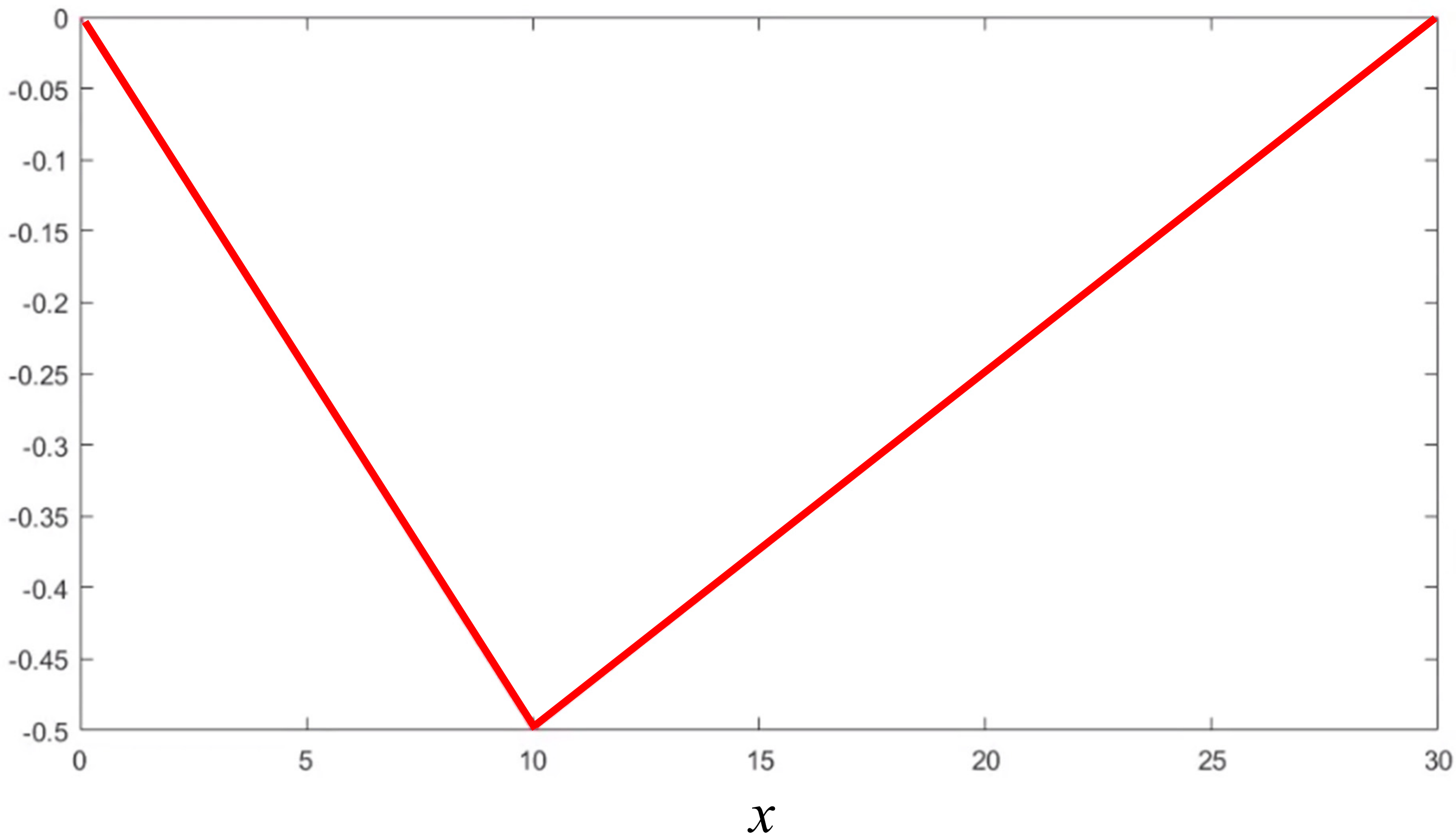
Graphically, the displacement $u(x, t)$ at $t = 3$…

$$u(x,3)=\sum_{n=1}^{\infty}\frac{9}{n^2\pi^2}\sin\frac{n\pi}{3}\sin\frac{n\pi x}{30}\cos\frac{n\pi}{5}$$



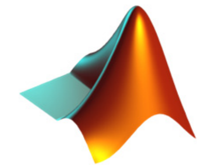
Graphically, the displacement $u(x, t)$ at $t = 10$…

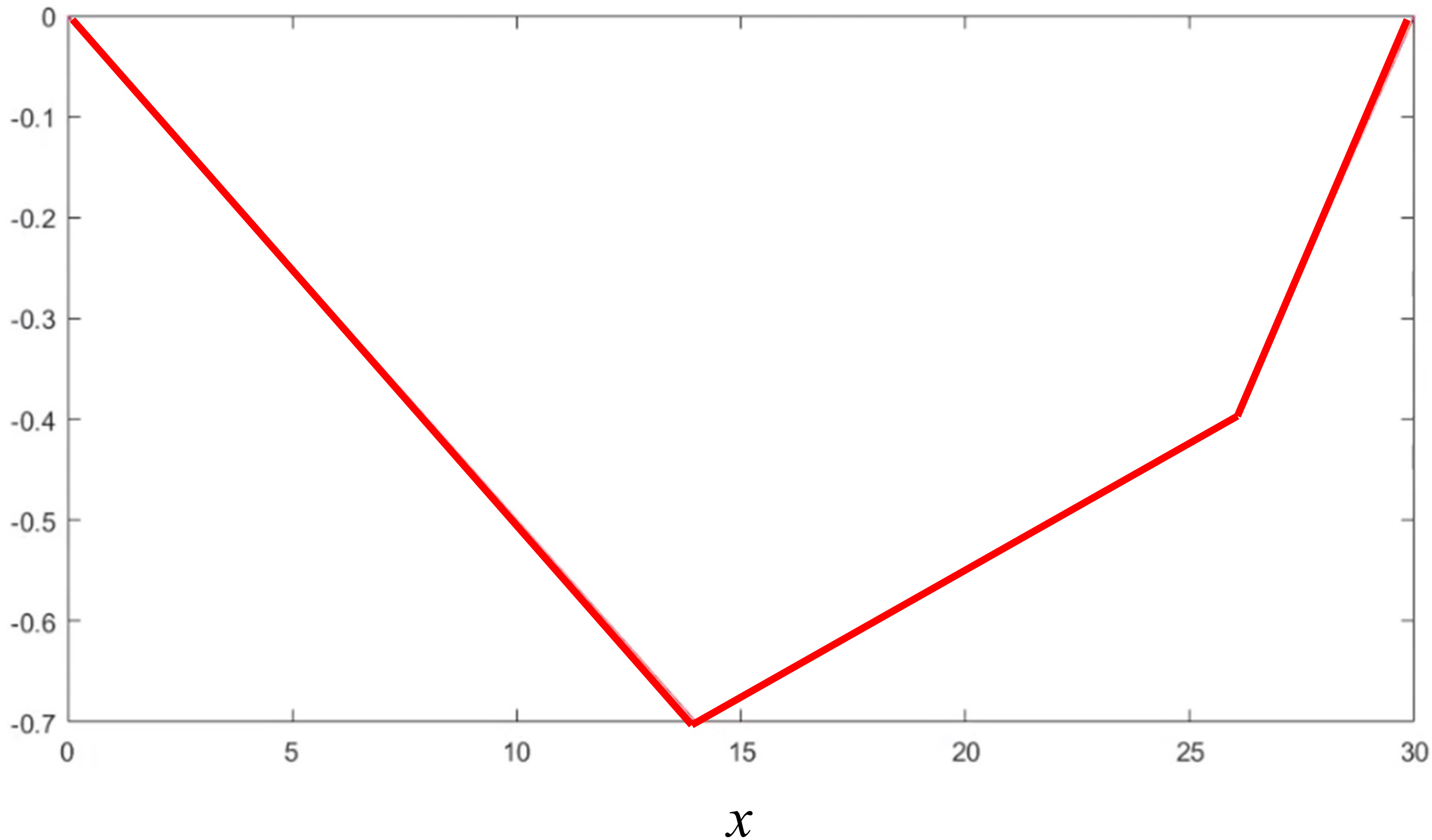
$$u(x,10) = \sum_{n=1}^{\infty} \frac{9}{n^2\pi^2} \sin\frac{n\pi}{3} \sin\frac{n\pi x}{30} \cos\frac{2n\pi}{3}$$



Graphically, the displacement $u(x, t)$ at $t = 18$…

$$u(x,18)=\sum_{n=1}^{\infty}\frac{9}{n^2\pi^2}\sin\frac{n\pi}{3}\sin\frac{n\pi x}{30}\cos(1.2n\pi)$$





Q2.

Solve the following wave equation by the method of separation of variables:

$$\begin{cases} u_{tt} = c^2 u_{xx}, \quad 0 < x < L, t > 0, \\ u(0,t) = u(L,t) = 0, \quad t > 0, \\ u(x,0) = 0, u_t(x,0) = \sin \frac{2\pi x}{L} + \sin \frac{3\pi x}{L} + 4 \sin \frac{8\pi x}{L}, \quad 0 < x < L. \end{cases}$$

Solution. By Corollary 2,

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi ct}{L} \sin \frac{n\pi x}{L}, \text{ where}$$

$$\begin{aligned} B_n &= \frac{2}{cn\pi} \int_0^L g(x) \sin \frac{n\pi x}{L}\, dx \\ &= \frac{2}{cn\pi} \int_0^L \left(\sin \frac{2\pi x}{L} + \sin \frac{3\pi x}{L} + 4 \sin \frac{8\pi x}{L} \right) \sin \frac{n\pi x}{L}\, dx. \end{aligned}$$

Based on the orthogonality property of cosine and sine functions, i.e.,

$$\int_0^L \sin\frac{m\pi x}{L}\sin\frac{n\pi x}{L}\,dx = \frac{1}{2}\int_0^L\left[\cos\left(\frac{m\pi x}{L}-\frac{n\pi x}{L}\right)-\cos\left(\frac{m\pi x}{L}+\frac{n\pi x}{L}\right)\right]dx$$

$$=\frac{1}{2}\left[\int_0^L \cos\frac{(m-n)\pi x}{L}\,dx-\int_0^L \cos\frac{(m+n)\pi x}{L}\,dx\right]$$

$$=\frac{1}{2}\left[\frac{L}{(m-n)\pi}\sin\frac{(m-n)\pi x}{L}\bigg|_0^L-\frac{L}{(m+n)\pi}\sin\frac{(m+n)\pi x}{L}\bigg|_0^L\right]=0,\quad \text{if } m\neq n$$

$$=\frac{L}{2},\quad \text{if } m=n$$

$$\Rightarrow\quad \int_0^L \sin\frac{m\pi x}{L}\sin\frac{n\pi x}{L}\,dx=\begin{cases}0, & \text{if } m\neq n\\ \dfrac{L}{2}, & \text{if } m=n\end{cases}$$

$$B_n = \frac{2}{cn\pi}\int_0^L \left(\sin\frac{2\pi x}{L} + \sin\frac{3\pi x}{L} + 4\sin\frac{8\pi x}{L} \right) \sin\frac{n\pi x}{L}\, dx.$$

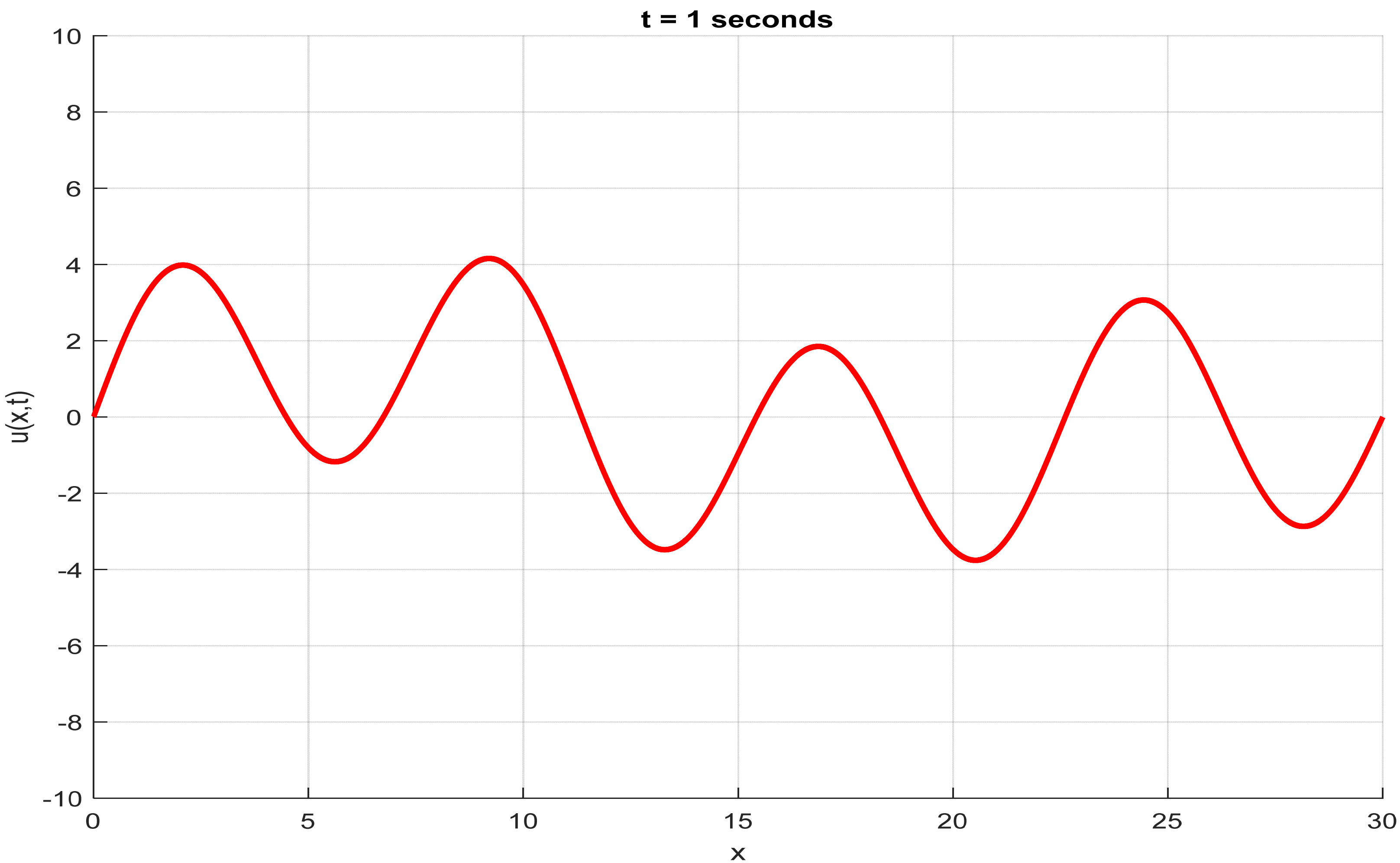
Now, by orthogonality property of cosine and sine functions, we have

$$B_2 = \frac{L}{2c\pi}, \quad B_3 = \frac{L}{3c\pi}, \quad B_8 = \frac{L}{2c\pi}, \quad \text{otherwise } B_n = 0.$$

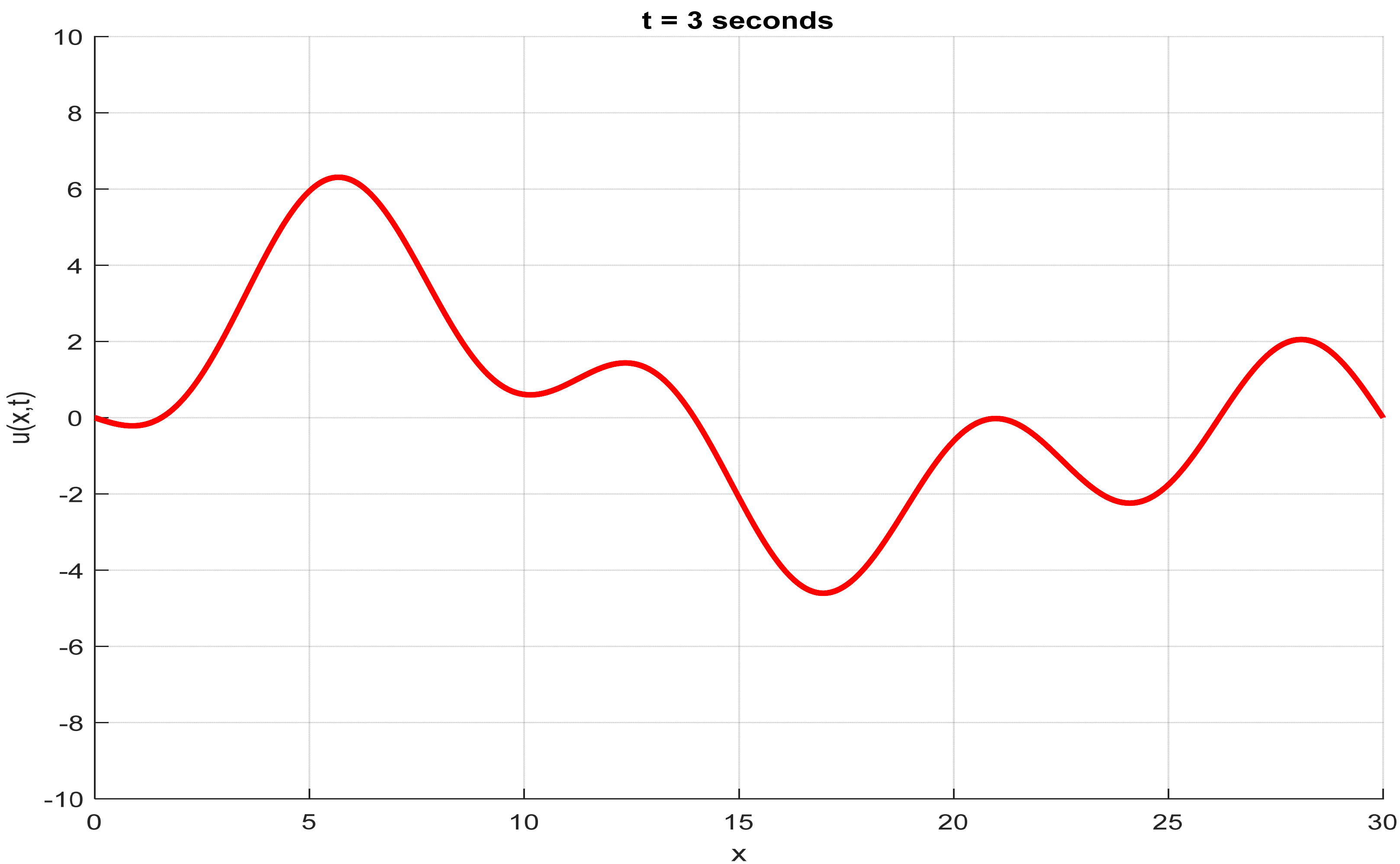
Hence,

$$u(x,t) = \frac{L}{2c\pi}\sin\frac{2\pi ct}{L}\sin\frac{2\pi x}{L} + \frac{L}{3c\pi}\sin\frac{3\pi ct}{L}\sin\frac{3\pi x}{L} + \frac{L}{2c\pi}\sin\frac{8\pi ct}{L}\sin\frac{8\pi x}{L}.$$

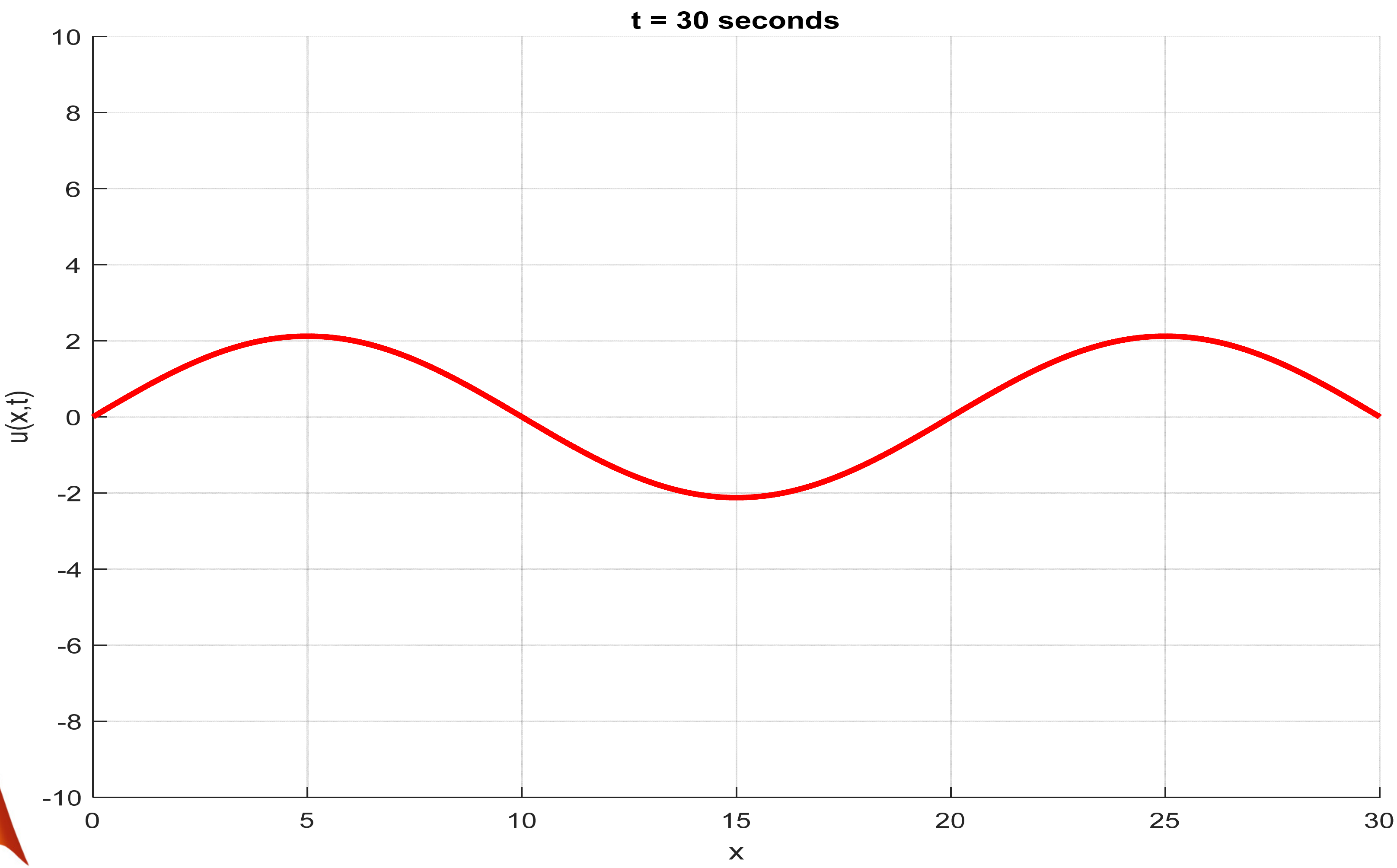
Graphically, the displacement $u(x, t)$ with $c = 1.5$ and $L = 30$ at $t = 1$…



Graphically, the displacement $u(x, t)$ with $c = 1.5$ and $L = 30$ at $t = 3$…



Graphically, the displacement $u(x, t)$ with $c = 1.5$ and $L = 30$ at $t = 30$…



wave2

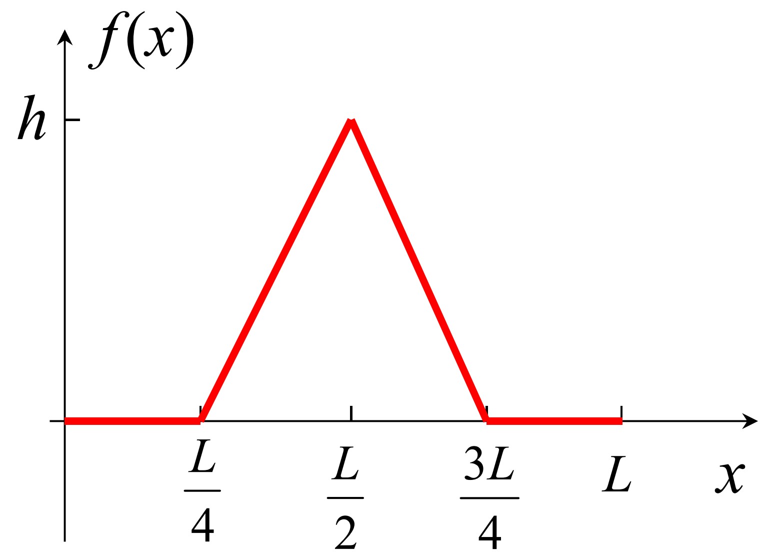
Exercise

A string of length L is stretched and fastened to two fixed points. Find the solution of the wave equation

$$u_{tt} = c^2 u_{xx}$$

with $u(0,t) = 0,\ u(L,t) = 0$ and $u(x,0) = f(x),\ u_t(x,0) = 0$

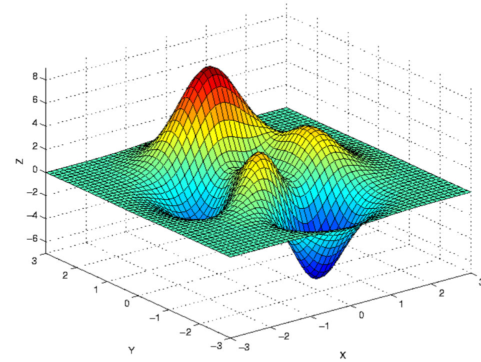
$$f(x) = \begin{cases} 0 & \text{if } 0 < x \le \frac{L}{4} \\ 4h\left(\frac{x}{L} - \frac{1}{4}\right) & \text{if } \frac{L}{4} < x \le \frac{L}{2} \\ 4h\left(\frac{3}{4} - \frac{x}{L}\right) & \text{if } \frac{L}{2} < x \le \frac{3L}{4} \\ 0 & \text{if } \frac{3L}{4} < x \le L \end{cases}$$



wave3

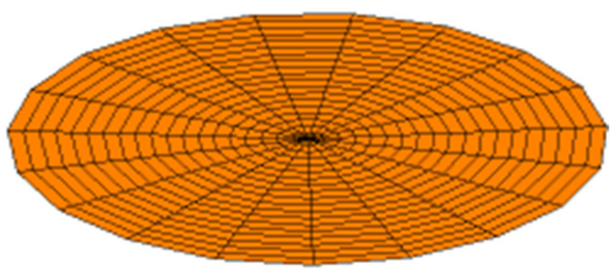
Answer: $A_n = \dfrac{8h}{n^2\pi^2}\left(2\sin\dfrac{n\pi}{2} - \sin\dfrac{n\pi}{4} - \sin\dfrac{3n\pi}{4}\right)$

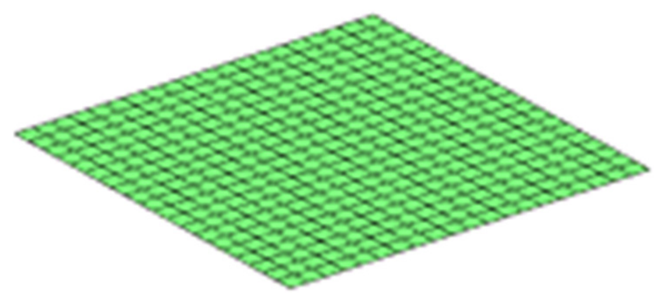
Just for fun…



Solutions to some 2D wave equations…

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$$





Homework Assignment No: 5

Due Date: 6:00pm, 5 December 2019

Please place your assignment to Assignment Box 3 outside PC Lab (ERB 218)

Summary of Partial Differential Equations…

Given a periodic function $f(x)$ with fundamental period 2ℓ and $f(x)$ and $f'(x)$ being piecewise continuous, we can express it as a **Fourier series**

$$\text{FS } f = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{n\pi x}{\ell}\right) + b_n \sin\left(\frac{n\pi x}{\ell}\right)\right)$$

with

$$a_0 = \frac{1}{2\ell}\int_{-\ell}^{\ell} f(x)dx,$$

$$a_n = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\cos\left(\frac{n\pi x}{\ell}\right)dx, \quad n = 1,2,\cdots$$

$$b_n = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\sin\left(\frac{n\pi x}{\ell}\right)dx, \quad n = 1,2,\cdots$$

Summary of Partial Differential Equations (cont.)…

Given a function $f(x)$ defined only on a finite interval, $0 < x < L$, and with $f(x)$ and $f'(x)$ being piecewise continuous, we can either a **Half-Range Cosine Expansion**

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{L}\right)\right], \qquad (0 < x < L)$$

with

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f_{ext}(x)dx = \frac{1}{L}\int_{0}^{L} f(x)dx$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f_{ext}(x)\cos\left(\frac{n\pi x}{L}\right)dx = \frac{2}{L}\int_{0}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

Summary of Partial Differential Equations (cont.)…

or a **Half-Range Sine Expansion**

$$f(x)=\sum_{n=1}^{\infty}\left[b_n \sin\left(\frac{n\pi x}{L}\right)\right], \qquad (0<x<L)$$

with

$$b_n=\frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

Summary of Partial Differential Equations (cont.)…

Temperature $u(x, t)$ distributed on a long bar is described by a **1D heat equation**

$$\alpha^2 u_{xx} = u_t, \qquad 0 < x < L, \quad 0 < t < \infty$$

with boundary conditions:

$$u(0,t) = u_1, \quad u(L,t) = u_2, \quad 0 < t < \infty$$

And initial condition:

$$u(x,0) = f(x), \quad 0 < x < L.$$

The solution to this heat equation:

$$u(x,t) = u_1 + \frac{(u_2 - u_1)x}{L} + \sum_{n=1}^{\infty} K_n \sin\frac{n\pi x}{L} e^{-(n\pi\alpha/L)^2 t}$$

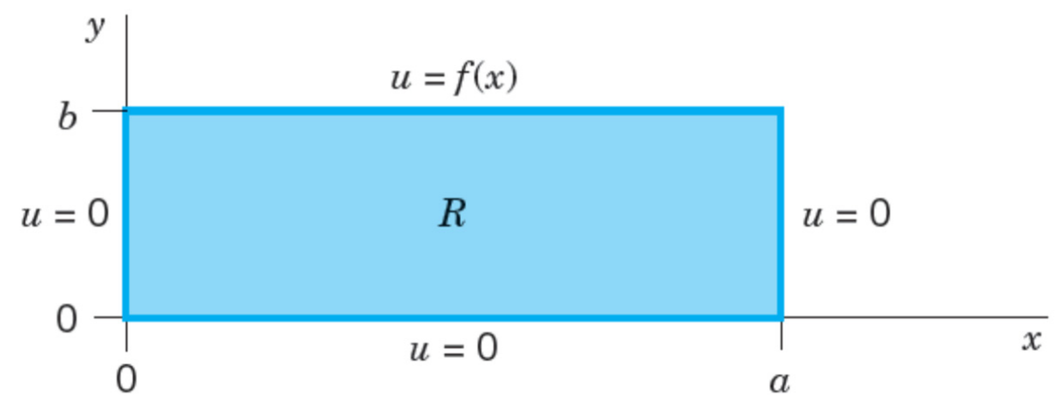
with

$$K_n = \frac{2}{L}\int_0^L F(x)\sin\frac{n\pi x}{L}dx = \frac{2}{L}\int_0^L \left(f(x) - u_1 - \frac{(u_2 - u_1)x}{L} \right)\sin\frac{n\pi x}{L}dx$$

Temperature $u(x, t)$ distributed on a steady **2D heat problem (**more specifically, the **Dirichlet problem**) is characterized by a **Laplace equation**:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

with boundary conditions:



The solution to this 2D heat equation is given by…

$$u(x, y) = \sum_{n=1}^{\infty} D_n \sin\frac{n\pi x}{a} \sinh\frac{n\pi y}{a} \quad \text{with} \quad D_n = \frac{2}{a\sinh(n\pi b/a)} \int_0^a f(x)\sin\frac{n\pi x}{a}\,dx$$

Summary of Partial Differential Equations (cont.)…

The vibrations of an elastic string can be characterized by a **wave equation**

$$u_{tt} = c^2 u_{xx}, \qquad 0 < x < L, \qquad t > 0$$

with boundary conditions:

$$u(0,t) = 0, \;\; u(L,t) = 0, \quad t > 0$$

And initial condition:

$$u(x,0) = f(x), \quad u_t(x,0) = g(x), \qquad 0 < x < L$$

The solution to this heat equation:

$$u(x,t) = \sum_{n=1}^{\infty} \left[A_n \cos\frac{n\pi ct}{L} + B_n \sin\frac{n\pi ct}{L} \right] \sin\frac{n\pi x}{L}$$

with

$$A_n = \frac{2}{L} \int_0^L f(x) \sin\frac{n\pi x}{L} dx, \quad n = 1, 2, \cdots$$

$$B_n = \frac{2}{cn\pi} \int_0^L g(x) \sin\frac{n\pi x}{L} dx, \; n = 1, 2, \cdots$$

Acknowledgement

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That's all, folks!

Thank You!

Bugs Bunny
1940–