ENGG 2420

Complex Analysis & Differential Equations for Engineers

Part 2: Ordinary Differential Equations

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Ordinary Differential Equations

Ordinary Differential Equations – 1…

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4 Separable Equations

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6 Linear Independence and Wronskian Determinants

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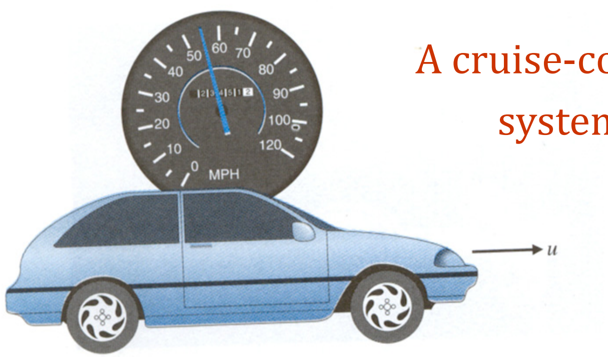
Introduction…

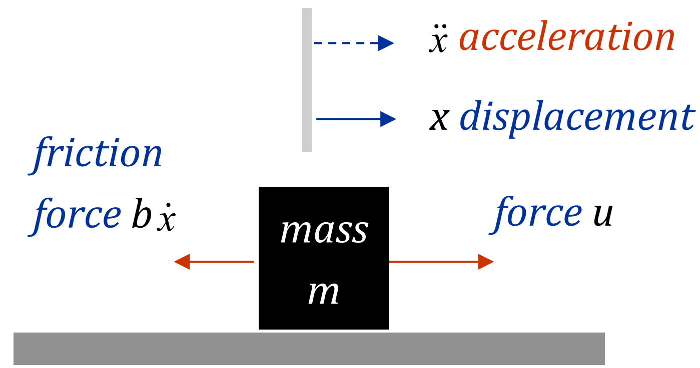
- Many problems in engineering can be formulated by mathematical equations. These equations usually involve derivatives of one or more unknown functions. Such equations are called **differential equations**.

Example… A cruise control system



A cruise-control system

⟺

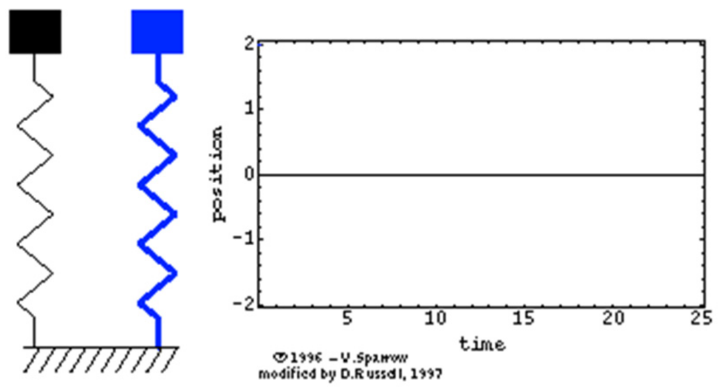


By the well-known Newton's Law of motion: $f = m\,a$, where f is the total force applied to an object with a mass m and a is the acceleration, we have

$$u - b\dot{x} = m\ddot{x} \quad \Leftrightarrow \quad \ddot{x} + \frac{b}{m}\dot{x} = \frac{u}{m} \quad \Leftrightarrow \quad \dot{v} + \frac{b}{m}v = \frac{u}{m}$$

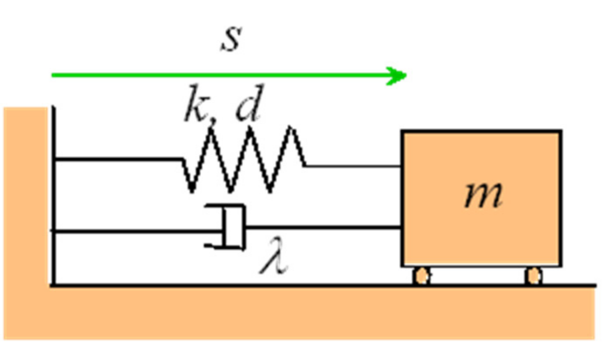
Some basic mechanical systems

Spring-mass system

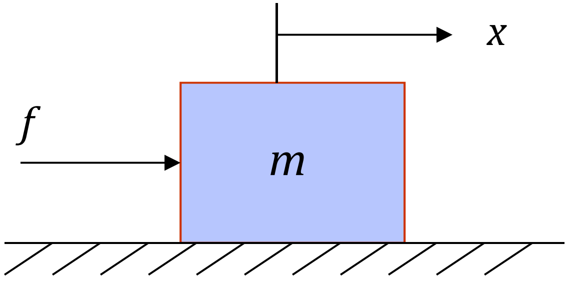


Animation courtesy of Dr. Dan Russell, Kettering University

Mass-spring-damper system



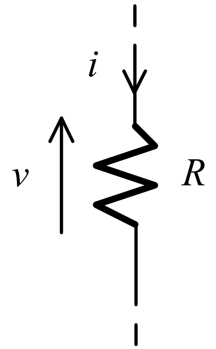
Newton's law of motion



$$f = ma = m\ddot{x}$$

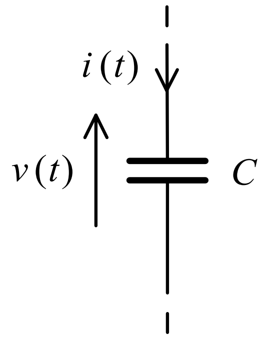
Basic laws for electrical systems

resistor



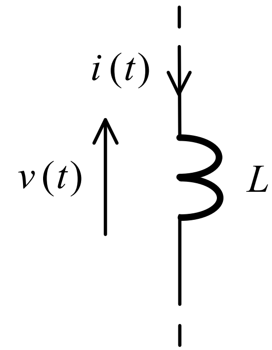
$$v = i\,R$$

capacitor



$$i = C\frac{dv}{dt}$$

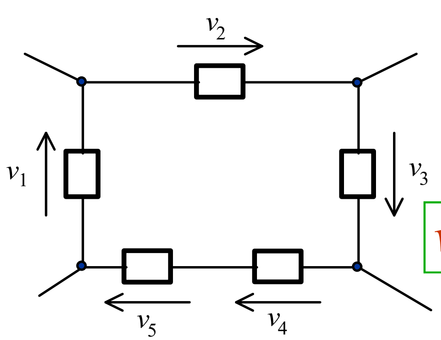
inductor



$$v = L\frac{di}{dt}$$

Kirchhoff's Voltage Law (KVL):

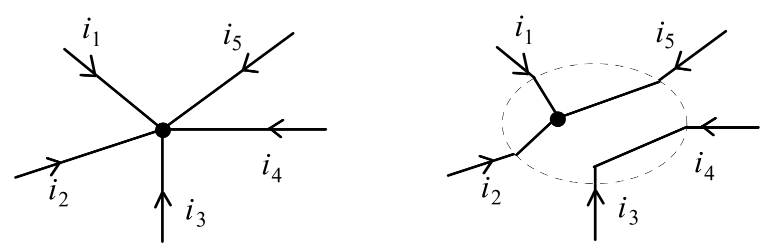
The sum of voltage drops around any close loop in a circuit is 0.



$$v_1+v_2+v_3+v_4+v_5 = 0$$

Kirchhoff's Current Law (KCL):

The sum of currents entering/ leaving a node/closed surface is 0.



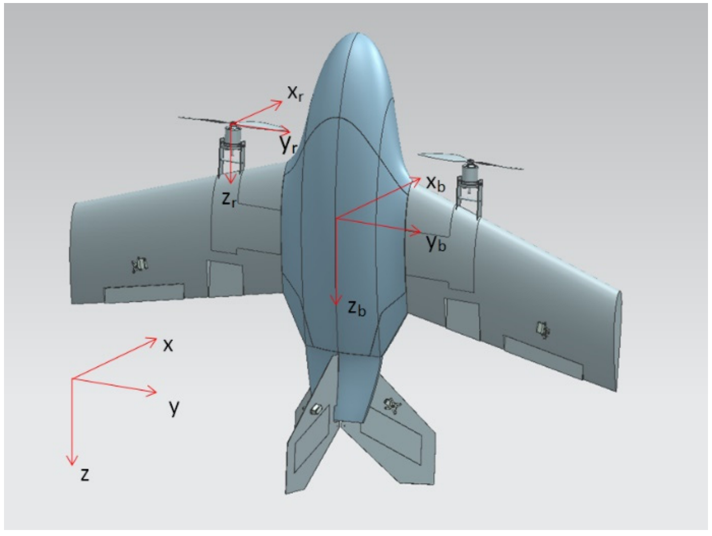
$$i_1+ i_2 + i_3 + i_4 + i_5 = 0$$

A real example for ODEs (just for information only)…

The transition dynamics of the aircraft at the bottom:

$$\begin{aligned}
\dot{u} &= \frac{1}{\mathrm{m}} F_{\mathrm{a,x}}(\boldsymbol{x}) - g\sin(\theta) - qw + \frac{1}{m} T_{\mathrm{u}} + \delta_{\mathrm{u}}(t), \\
\dot{w} &= \frac{1}{m} F_{\mathrm{a,z}}(\boldsymbol{x}) + g\cos(\theta) + qu - \frac{1}{m} T_{\mathrm{w}} + \frac{1}{m} T_{\mathrm{f}} + \delta_{\mathrm{w}}(t), \\
\dot{q} &= \frac{1}{I_{\mathrm{y}}} M_{\mathrm{a}}(\boldsymbol{x}) + \frac{l_{\mathrm{m}}}{I_{\mathrm{y}}} T_{\mathrm{w}} + \frac{l_{\mathrm{t}}}{I_{\mathrm{y}}} T_{\mathrm{f}} + \delta_{\mathrm{q}}(t), \\
\dot{\theta} &= q,
\end{aligned} \tag{1}$$

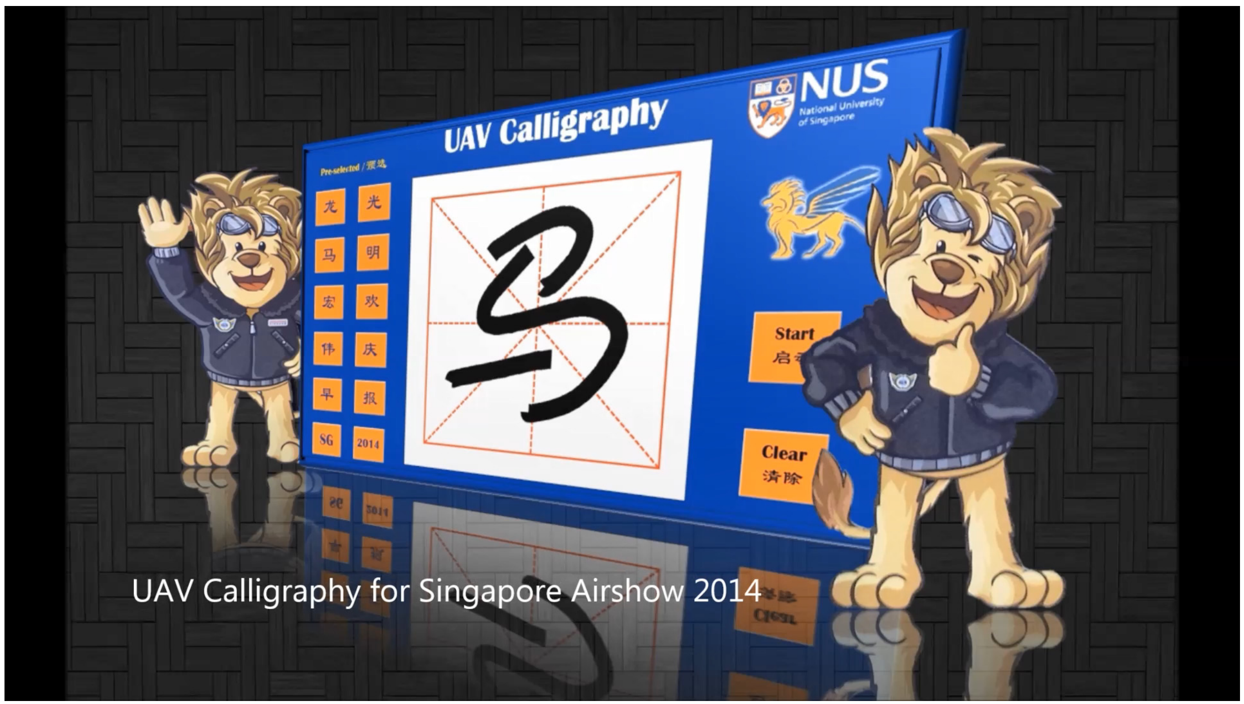
The aircraft…



Quadrotor dynamics model

The ordinary differential equations of a drone model...

$$
\begin{aligned}
\dot{u} &= -g\theta \\
\dot{v} &= g\phi \\
\dot{w} &= \frac{1}{m}u_1 \\
\dot{p} &= J_{\mathrm{XX}}^{-1}u_2 \\
\dot{q} &= J_{\mathrm{YY}}^{-1}u_3 \\
\dot{r} &= J_{\mathrm{ZZ}}^{-1}u_4 \\
\dot{\phi} &= p \\
\dot{\theta} &= q \\
\dot{\psi} &= r
\end{aligned}
$$



- If a differential equation only contains ordinary derivatives with respect to a single independent variable, it is called an **ordinary differential equation** (ODE).

- Let $y(x)$ be a function depending on a single variable x. Then an ODE is typically of the form

$$\boxed{F(x, y, y', \ldots, y^{(n)}) = 0,} \qquad (1.1)$$

where $y = y(x), y' = \frac{dy}{dx}, \ldots, y^{(n)} = \frac{d^n y}{dx^n}$.

Classifications of ODE's

- If the ODE (1.1) can be written in the form

$$\boxed{a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_0(x)y = g(x),} \tag{1.3}$$

  it is called a **linear** ODE. Otherwise, it is called a **non-linear** ODE.

- Let $\mathcal{L}[y] = a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_0(x)y$. Then the ODE (1.3) is called linear because

$$\mathcal{L}[\alpha_1 y_1 + \alpha_2 y_2] = \alpha_1 \mathcal{L}[y_1] + \alpha_2 \mathcal{L}[y_2]$$

  for any constants α_1 and α_2.

- If in addition that $g(x) = 0$ in (1.3), then the ODE is called **homogeneous**. Otherwise, it is called **non-homogeneous**.

Order of an ODE…

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_0(x)y = g(x),$$

- In (1.3), the highest derivative of y is n, and we call n the **order** of the ODE. (1.3) is called an n-th order ODE. For example,

  $$y'' + y = 2\sin x$$

  is a second order ODE.

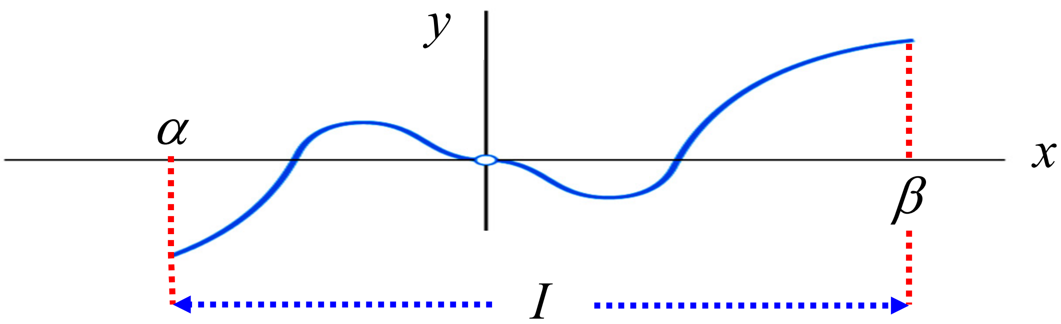
- If $a_n(x), a_{n-1}(x), \ldots, a_0(x)$ are constants in (1.3), then the ODE is called **constant coefficients linear differential equations**.

- A **solution** of the ODE (1.1) over an interval $I = (\alpha, \beta)$ is a function $y(x)$ defined on I such that

  (a) y and its derivatives $y', y'', \ldots, y^{(n)}$ exist on I; and

  (b) $F(x, y, y', \ldots, y^{(n)}) = 0$ for any $x \in I$.

The whole topic is on how to find solutions!!!



$$F(x, y, y', \ldots, y^{(n)}) = 0 \qquad (1.1)$$

Example (1a)

- The equation $y' - xy = 1 - x^2$ is a first order linear non-homogeneous ODE.
- $y(x) = 4e^{\frac{x^2}{2}} + x$ is a solution over $(-\infty, \infty)$ because

$$y' - xy = \frac{d}{dx}(4e^{\frac{x^2}{2}} + x) - x(4e^{\frac{x^2}{2}} + x) = (4xe^{\frac{x^2}{2}} + 1) - (4xe^{\frac{x^2}{2}} + x^2) = 1 - x^2.$$

Example (1b)

- The equation $y'' - 3y' + 2y = 0$ is a second order linear homogeneous ODE with constant coefficients.
- For any values of constants c_1 and c_2, $y(x) = c_1 e^{2x} + c_2 e^{x}$ is a solution over $(-\infty, \infty)$ of the ODE because

$$\begin{aligned} y'' - 3y' + 2 &= \frac{d}{dx}(2c_1 e^{2x} + c_2 e^{x}) - 3(2c_1 e^{2x} + c_2 e^{x}) + 2(c_1 e^{2x} + c_2 e^{x}) \\ &= (4c_1 e^{2x} + c_2 e^{x}) - (6c_1 e^{2x} + 3c_2 e^{x}) + (2c_1 e^{2x} + 2c_2 e^{x}) \\ &= 0. \end{aligned}$$

Example (1c)

- The equation $y \sin y' + (y''')^2 = 0$ is a third order non-linear ODE.
- Clearly, $y(x) = 0$ is a solution over $(-\infty, \infty)$. It is called the **trivial solution**.

Initial Value Problem

IVP is to make solution to ODE unique…

- Consider a first order linear ODE

$$\frac{dy}{dx} = 1 + x^2. \qquad (1.4)$$

- Integrating on both sides of (1.4) with respect to x gives a family of solutions of the ODE parameterized by a constant C as follows:

$$y = \int (1 + x^2)\, dx = x + \frac{x^3}{3} + C.$$

  Thus the ODE (1.4) has infinitely many solutions and C is called the **integration constant**.

- We can impose some conditions on the solution of an ODE at one or more points on the interval under consideration. Conditions specified at a single point are called initial conditions.

- Finding a solution of an ODE that also satisfies those initial conditions is called an **initial value problem** (IVP). Under certain conditions, the IVP has a unique solution.

Example (1f)

- To solve the IVP of ODE (1.4) with the initial condition $y(0) = 1$, letting $y(0) = 0 + \frac{0^3}{3} + C = 1$ gives $C = 1$. Hence, we obtain the unique solution of the problem:

$$y = x + \frac{x^3}{3} + 1.$$

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Forms of 1st Order ODEs

- First order differential equations are of the form

$$
\begin{aligned}
y' &= Q(x,y) && \text{(explicit form)}\\
Q(x,y,y') &= 0 && \text{(implicit form)}\\
P(x,y)\,\mathrm{d}x + Q(x,y)\,\mathrm{d}y &= 0 && \text{(differential form)}
\end{aligned}
$$

In principle, the three forms are somewhat equivalent†, but one should revert to the *explicit* or *differential* forms as much as possible.

Solving such equations involves two main techniques

- Reduction to *separable* form
- Reduction to *exact differential* form

Integrating Factor Method

- Consider a general first order linear ODE

$$\boxed{y' + p(x)y = q(x),} \tag{2.1}$$

where both $p(x)$ and $q(x)$ are continuous functions of x on some interval. For example, the first order linear ODE (1.4), i.e., ($y' = \frac{dy}{dx}$)

$$y' = 1 + x^2,$$

is a special case of (2.1) where $p(x) = 0$. This ODE can be solved simply by integrating both sides with respect to x.

- Another special case is where $q(x) = 0$, that is the homogeneous case. The equation can be rewritten as follows:

$$\frac{dy}{y} + p(x)\,dx = 0, \tag{2.2}$$

assuming that y is nonzero on the x interval of interest.

- Integrating the equation (2.2) gives

$$\ln|y| = -\int p(x)\,dx + C,$$

where C is an arbitrary constant. Thus

$$|y(x)| = e^{-\int p(x)\,dx + C} = e^{C} e^{-\int p(x)\,dx}$$

and, as a result,

$$\boxed{y(x) = Ae^{-\int p(x)\,dx}.} \tag{2.3}$$

for some constant A.

- However, the general case is more complex. One of the methods that can solve the general case is known as the **integrating factor method** and is derived as follows.

→ Integrating factor method

- First, multiply both sides of the ODE (2.1) by the

$$\text{integrating factor} = e^{\int p(x)\,dx} \qquad (2.4)$$

to obtain

$$e^{\int p(x)\,dx} y' + e^{\int p(x)\,dx} p(x) y = e^{\int p(x)\,dx} q(x). \qquad (2.5)$$

product rule

$$\left(e^{\int p(x)dx} y\right)' = \left(e^{\int p(x)dx}\right)' y + e^{\int p(x)dx} y' = e^{\int p(x)dx} p(x) y + e^{\int p(x)dx} y'$$

$$y' + p(x)y = q(x), \qquad (2.1)$$

>>>

- Using the product rule and the **Fundamental Theorem of Calculus**, rewrite the equation (2.5) as follows:

$$\left(e^{\int p(x)\,dx}y\right)' = e^{\int p(x)\,dx}q(x).$$

- Integrating both sides of the above equation with respect to x gives

$$e^{\int p(x)\,dx}y = \int \left(e^{\int p(x)\,dx}q(x)\right)dx + C,$$

where C is an arbitrary constant.

- Thus the solution of the first order linear ODE (2.1)

$$y = e^{-\int p(x)\,dx}\left[\int \left(e^{\int p(x)\,dx} q(x)\right) dx + C\right]. \tag{2.6}$$

$$y' + p(x)y = q(x)$$

- **Remark:** Since both $p(x)$ and $q(x)$ are continuous, the integrations in (2.6) exist.

Example (2a)

$$y = e^{-\int p(x)\,dx}\left[\int \left(e^{\int p(x)\,dx} q(x)\right) dx + C\right]$$

- Solve the first order linear ODE:

$$y' + 2y = 0.$$

- Since $p(x) = 2$, by (2.4), the integrating factor is

$$e^{\int p(x)\,dx} = e^{\int 2\,dx} = e^{2x}.$$

- So multiplying both sides of the ODE by the integrating factor gives

$$e^{2x}y' + 2e^{2x}y = 0.$$

- Then

$$(e^{2x}y)' = 0.$$

- Hence, the general solution is

$$y = Ce^{-2x}.$$

Example (2b)

$$y = e^{-\int p(x)\,dx}\left[\int \left(e^{\int p(x)\,dx} q(x)\right) dx + C\right]$$

- Solve the first order linear ODE:

$$y' + 3y = x.$$

- The integrating factor is $e^{\int p(x)\,dx} = e^{\int 3\,dx} = e^{3x}$. Then

$$(e^{3x}y)' = e^{3x}x.$$

- Hence, using integration by parts, the general solution is

$$\begin{aligned} y = e^{-3x}\left[\int (e^{3x}x)\,dx + C\right] &= e^{-3x}\left[\frac{1}{3}\int x\,d(e^{3x}) + C\right] \\ &= e^{-3x}\left[\frac{1}{3}\left(xe^{3x} - \int e^{3x}\,dx\right) + C\right] \\ &= e^{-3x}\left[\frac{1}{3}\left(xe^{3x} - \frac{e^{3x}}{3}\right) + C\right] \\ &= \frac{x}{3} - \frac{1}{9} + Ce^{-3x}. \end{aligned}$$

Integration by part…

$$\int u\,dv = uv - \int v\,du$$

General Solutions and Particular Solutions

$$y' + p(x)y = q(x),$$

- The right-hand side of equation (2.6)

$$y = e^{-\int p(x)\,dx}\left[\int\left(e^{\int p(x)\,dx}q(x)\right)dx + C\right].$$

contains an arbitrary constant C and is called the **general solution**.

$$y = e^{-\int p(x)\,dx}\left[\int \left(e^{\int p(x)\,dx} q(x)\right) dx + C\right] \qquad (2.6)$$

- If we replace the indefinite integral in (2.6) by a definite integral with lower limit a and upper limit x, we obtain the following expression:

$$\boxed{y = e^{-\int_a^x p(\xi)\,d\xi}\left[\int_a^x \left(e^{\int_a^\xi p(\zeta)\,d\zeta} q(\xi)\right) d\xi + C\right].} \qquad (2.7)$$

It can be seen that the difference between (2.6) and (2.7) is only a constant, thus (2.7) is another form of the general solution of (2.1). If we substitute (2.7) into (2.1), we will still lead to an identity. This is an alternative way to show that (2.7) is another form of the general solution of (2.1).

$$y = e^{-\int_a^x p(\xi)\, d\xi} \left[\int_a^x \left(e^{\int_a^\xi p(\zeta)\, d\zeta} q(\xi) \right) d\xi + C \right]. \tag{2.7}$$

- The constant C can be determined by requiring the solution to satisfy an initial condition. For example, imposing $y(a) = b$ on (2.7) shows $C = b$. Thus, we have

$$y = e^{-\int_a^x p(\xi)\, d\xi} \left[\int_a^x \left(e^{\int_a^\xi p(\zeta)\, d\zeta} q(\xi) \right) d\xi + b \right]. \tag{2.8}$$

We call (2.8) a **particular solution** of (2.1) passing through the point $(x, y) = (a, b)$. It is also the unique solution of the initial value problem of (2.1) with initial condition $y(a) = b$.

Example (2c)

$$y = e^{-\int p(x)\,dx}\left[\int \left(e^{\int p(x)\,dx} q(x)\right) dx + C\right]$$

- Solve the IVP for $x > 0$:

$$xy' + 2y = 4x^2, \quad y(1) = 2.$$

- The equation can be rewritten as

$$y' + \frac{2}{x}y = 4x.$$

$$y' + p(x)y = q(x)$$

Thus, the integrating factor is $e^{\int p(x)\,dx} = e^{\int \frac{2}{x}\,dx} = e^{2\ln|x|} = x^2$.

- Using (2.6) gives the general solution as follows:

$$y = x^{-2}\left(\int (4x^3)dx + C\right) = x^{-2}\left(\int dx^4 + C\right) = x^{-2}\left(x^4 + C\right) = x^2 + \frac{C}{x^2}$$

- Letting $y(1) = 2$ gives $C = 1$. Hence, the solution for the IVP is

$$y(1) = 1^2 + \frac{C}{1^2} = 1 + C = 2 \quad \Rightarrow \quad C = 1 \quad \Rightarrow \quad y = x^2 + \frac{1}{x^2}$$

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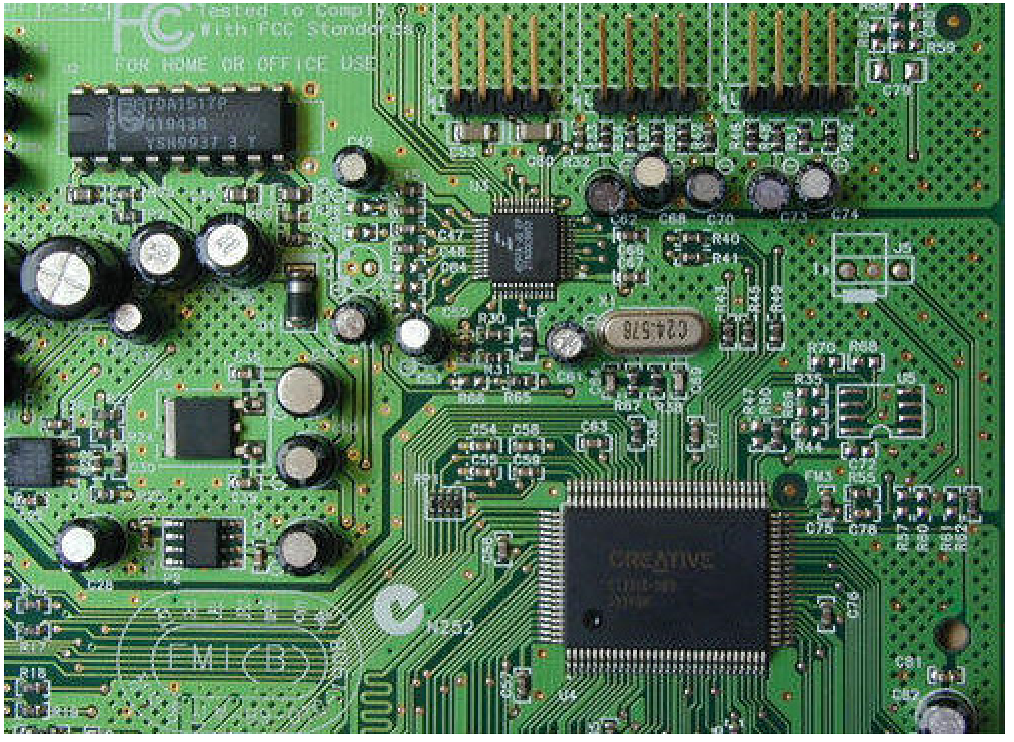
Electric Circuits

- One of the applications for ODEs is to Electrical Engineering.
- An **electric circuit** is a network consisting of some circuit elements such as resistors, inductors or capacitors.

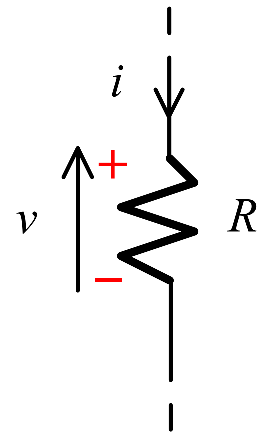


(a) **Resistor**: the voltage drop $V(t)$ across it, where t is the time, is proportional to the current $i(t)$ passing through it, i.e.,

$$\boxed{V(t) = Ri(t).} \tag{3.1}$$

Here $V(t)$ is measured in volts(V), time is measured in seconds(s), $i(t)$ is measured in amperes(A) and R is a constant called the resistance, measured in ohms(Ω). The equation (3.1) is called the **Ohm's law**.

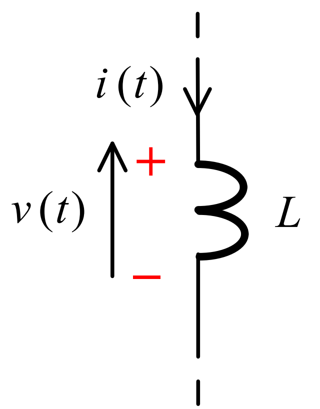
$$\boxed{v(t) = i(t)\, R}$$



(b) **Inductor**: the voltage drop $V(t)$ across it is proportional to the time rate of change of the current passing through it, i.e.,

$$\boxed{V(t) = L\frac{di(t)}{dt}.} \tag{3.2}$$

Here L is a constant called the inductance and is measured in henries(H). Physically, most inductors are coils of wire, hence the electronic symbol of an inductor looks like a coil.



$$\boxed{v(t) = L\frac{di(t)}{dt}}$$

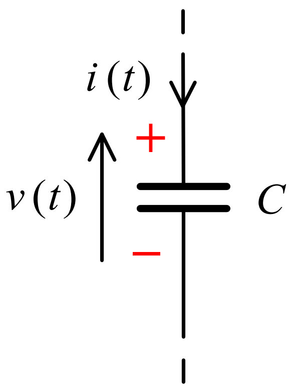
(c) **Capacitor**: the voltage drop across it is proportional to the relative charge $Q(t)$ on the capacitor, i.e.,

$$V(t) = \frac{Q(t)}{C}. \tag{3.3}$$

Here C is a constant called the capacitance and is measured in farads(F). The current flowing through a capacitor equals the time rate of change of the relative charge on the capacitor, i.e.,

$$i(t) = \frac{dQ(t)}{dt}. \tag{3.4}$$

From equations (3.3) and (3.4), it follows that the desired voltage/current relation for a capacitor is

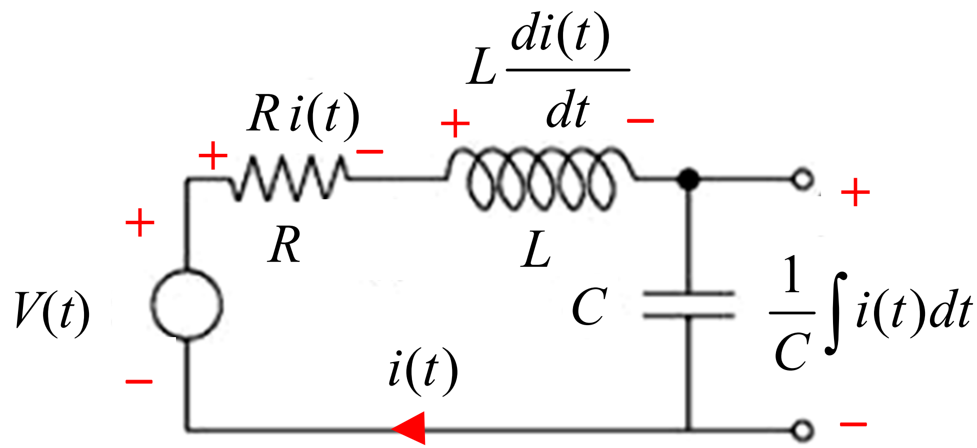


$$i(t) = C\frac{dv(t)}{dt}$$

$$V(t) = \frac{1}{C}\int i(t)\,dt. \tag{3.5}$$

RLC Series Circuits

- Consider an **RLC series circuit**, which is an electric circuit consisting of a resistor, an inductor and a capacitor, connected in series.



- From **Kirchoff's Voltage Law** (KVL), which states that the algebraic sum of the voltage drops around any closed loop of a circuit is zero, we have

$$V(t) - Ri(t) - L\frac{di(t)}{dt} - \frac{1}{C}\int i(t)\,dt = 0.$$

$$V(t) - Ri(t) - L\frac{di(t)}{dt} - \frac{1}{C}\int i(t)\,dt = 0.$$

- Differentiating the above equation with respect to t gives

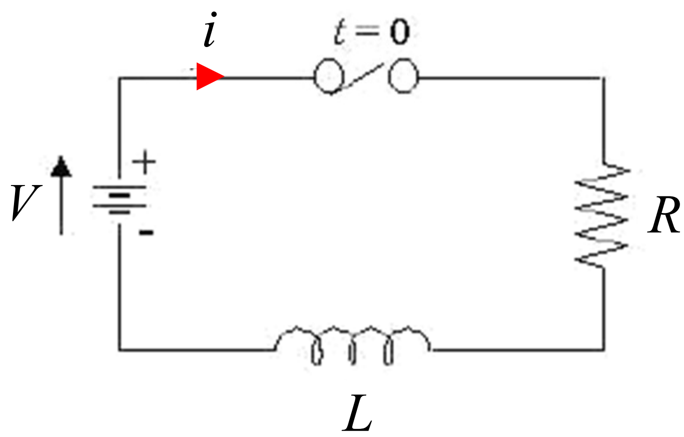
$$\boxed{L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{1}{C}i = \frac{dV}{dt},} \tag{3.6}$$

where $V = V(t)$ and $i = i(t)$. Hence, we obtain a second order linear ODE.

Gustav Kirchhoff
(1824–1887)
German Physicist

RL Circuits

- If we omit the capacitor from the RLC series circuit, the circuit becomes an **RL circuit**.



- Assume $i(0) = 0$ and a constant voltage V_0 is applied when the switch is closed, then equation (3.6) becomes

$$\boxed{L\frac{di}{dt} + Ri = V_0,} \qquad (3.7)$$

which is a first order linear ODE.

$$\boxed{y' + p(x)y = q(x)}$$

$$L\frac{di}{dt} + Ri = V_0 \quad \Rightarrow \quad \frac{di}{dt} + \frac{R}{L}i = \frac{V_0}{L}$$

- Equation (3.7) can be solved by the integrating factor method. Now, the integrating factor is $e^{\int \frac{R}{L}\,dt} = e^{\frac{Rt}{L}}$ and equation (3.7) becomes

$$\left(e^{\frac{Rt}{L}} i\right)' = e^{\frac{Rt}{L}} \frac{V_0}{L}.$$

- Thus, we have

$$i(t) = e^{-\frac{Rt}{L}} \left[\int \left(e^{\frac{Rt}{L}} \frac{V_0}{L} \right) dt + \tilde{C} \right] = e^{-\frac{Rt}{L}} \left(e^{\frac{Rt}{L}} \frac{V_0}{R} + \tilde{C} \right),$$

where $\tilde{C}$ is the integration constant.

$$y = e^{-\int p(x)\,dx} \left[\int \left(e^{\int p(x)\,dx} q(x) \right) dx + C \right] \qquad (2.6)$$

$$i(t) = e^{-\frac{Rt}{L}} \left[\int \left(e^{\frac{Rt}{L}} \frac{V_0}{L} \right) dt + \tilde{C} \right] = e^{-\frac{Rt}{L}} \left(e^{\frac{Rt}{L}} \frac{V_0}{R} + \tilde{C} \right),$$

- Since $i(0) = 0$, then $\tilde{C} = -\dfrac{V_0}{R}$. Hence, we obtain the solution of the RL circuit, i.e.,

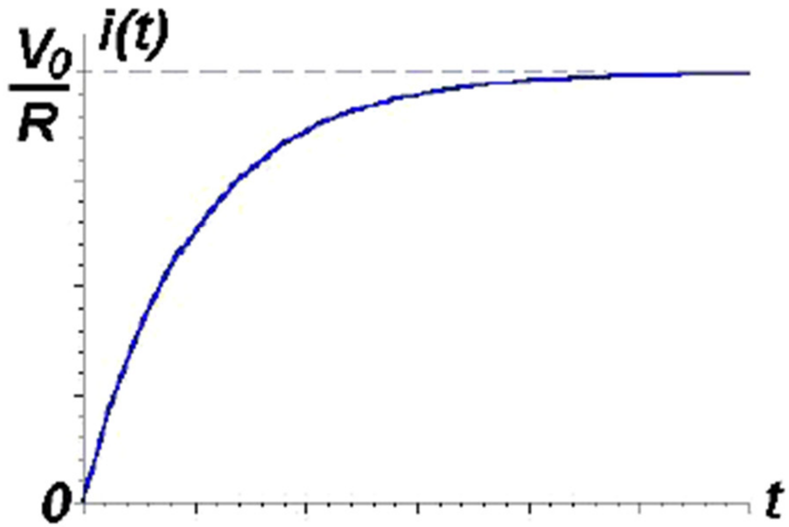
$$\boxed{i(t) = \frac{V_0}{R} \left(1 - e^{-\frac{Rt}{L}} \right).} \tag{3.8}$$

- From the solution (3.8), we have

$$i(t) \to \frac{V_0}{R} \text{ as } t \to \infty.$$

- Thus, we call the $\dfrac{V_0}{R}$ term in solution (3.8) the **steady-state** solution and the $-\dfrac{V_0}{R} e^{-\frac{Rt}{L}}$ term the **transient** part of the solution.

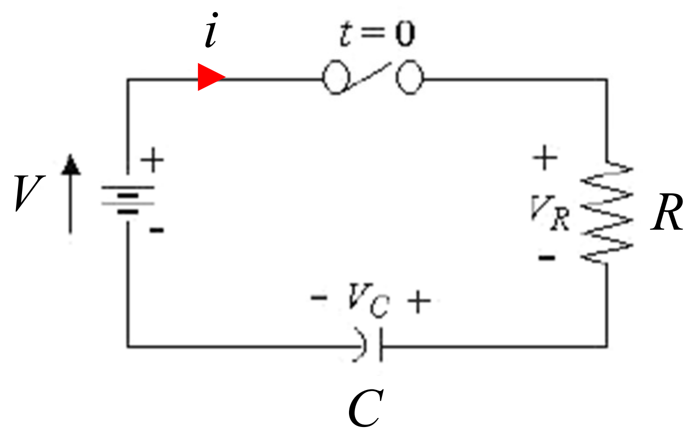
- The steady-state solution is shown as below:



$$i(t) = \frac{V_0}{R}\left(1 - e^{-\frac{Rt}{L}}\right).$$

RC Circuits

- If we omit the inductor from the RLC series circuit, the circuit becomes an **RC circuit**.



- Assume $i(0) = i_0$ and a constant voltage V_0 is applied when the switch is closed, then equation (3.6) becomes

$$\boxed{R\frac{di}{dt} + \frac{1}{C}i = 0,} \tag{3.9}$$

which is also a first order linear ODE.

- Solving equation (3.9) yields the solution

$$\boxed{i(t) = i_0 e^{-\frac{t}{RC}}.} \tag{3.10}$$

$$y = e^{-\int p(x)\,dx}\left[\int \left(e^{\int p(x)\,dx} q(x)\right) dx + C\right] \tag{2.6}$$

Ordinary Differential Equations – 4…

1 Introduction
2 Integrating Factor Method
3 Applications of First Order Linear ODEs
4 **Separable Equations**
5 Exact Equations
6 Linear Independence and Wronskian Determinants
7 Homogeneous Equations: General Theory
8 Homogeneous Equations: Constant Coefficients
9 Nonhomogeneous Equations: Method of Undetermined Coefficients
10 Euler-Cauchy Equation
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$$y' = \frac{dy}{dx}$$

Separable Equations

- Consider a special type of first order non-linear ODEs of the following form:

$$y' = f(x)g(y), \tag{4.1}$$

where both $f(x)$ and $g(y)$ are continuous functions over some intervals of the respective independent variables and $g(y) \neq 0$. Equation (4.1) is called a **separable equation** because it can be rewritten in the **differential form** as follows:

$$\frac{dy}{g(y)} = f(x)\,dx. \tag{4.2}$$

- Integrating equation (4.2) gives

$$\int \frac{dy}{g(y)} = \int f(x)\,dx + C. \tag{4.3}$$

This is the **general solution** of the separable equation (4.1).

Example (4a)

- Solve the IVP for $x > -1$:

$$y' = -y^2, \quad y(0) = 1.$$

- This equation is separable and can be rewritten in the form

$$\frac{dy}{y^2} = -dx.$$

- Integrating on both sides gives $\displaystyle\int \frac{dy}{y^2} = -\int dx + C.$
- Thus, we have

$$-\frac{1}{y} = -x + C.$$

- Using the initial condition $y(0) = 1$ gives $C = -1$. Hence, the solution is

$$y = \frac{1}{1+x}.$$

Example (4b)

- Solve the IVP:

$$y' = \frac{4x}{1+2e^y}, \quad y(0) = 1.$$

- Separating the variables and integrating on both sides gives

$$\int (1+2e^y)\, dy = \int (4x)\, dx + C.$$

- Thus, we have

$$y + 2e^y = 2x^2 + C.$$

- Using the initial condition $y(0) = 1$ gives $C = 1 + 2e$. Hence, the solution for the IVP is

$$y + 2e^y = 2x^2 + 1 + 2e.$$

- **Remark:** The solution in Example (4b) is called an **implicit solution**, i.e., the solution cannot be written in the explicit form $y = h(x)$ for some function h using elementary functions.

Example (4c)

- Find the general solution of the first order ODE:

$$y' = \frac{y(y-2)}{x(y-1)}.$$

- Separating the variables gives

$$\int \frac{y-1}{y(y-2)}\, dy = \int \frac{1}{x}\, dx + C.$$

- By partial fraction expansion,

$$\frac{y-1}{y(y-2)} = \frac{\frac{1}{2}}{y} + \frac{\frac{1}{2}}{y-2}.$$

- Thus, we have

$$\frac{1}{2}\ln|y| + \frac{1}{2}\ln|y-2| = \ln|x| + C.$$

Some manipulation gives the simplified form of the general solution (in implicit form):

$$y(y-2) = \tilde{C}x^2 \text{ for some constant } \tilde{C}.$$

Ordinary Differential Equations – 5…

1 Introduction

2 Integrating Factor Method

3 Applications of First Order Linear ODEs

4 Separable Equations

5 **Exact Equations**

6 Linear Independence and Wronskian Determinants

7 Homogeneous Equations: General Theory

8 Homogeneous Equations: Constant Coefficients

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Exact Equations

- A differential equation of the form

$$\frac{dy}{dx} = -\frac{M(x,y)}{N(x,y)} \tag{5.1}$$

can be put in the differential form

$$\boxed{M(x,y)\,dx + N(x,y)\,dy = 0.} \tag{5.2}$$

- If there exists some function $F(x,y)$ such that

$$dF(x,y) = M(x,y)\,dx + N(x,y)\,dy, \tag{5.3}$$

i.e., there exists $F(x,y)$ such that

$$\boxed{\begin{cases} \dfrac{\partial F}{\partial x} = M \\ \dfrac{\partial F}{\partial y} = N, \end{cases}} \tag{5.4}$$

then, (5.2) gives

$$dF(x,y) = 0. \tag{5.5}$$

- Now, (5.5) can be integrated to give the general solution

$$\boxed{F(x, y) = C,} \tag{5.6}$$

where C is an arbitrary constant.

- If condition (5.3) is satisfied, then $M(x, y)\,dx + N(x, y)\,dy$ is called an **exact differential**, and (5.2) is called an **exact differential equation**. The general solution of an exact differential equation is readily given, in implicit form, by $F(x, y) = C$.

$$\boxed{dF(x, y) = M(x, y)\,dx + N(x, y)\,dy,} \tag{5.3}$$

Theorem (5.1 Test for Exactness)

Let $M(x,y)$, $N(x,y)$, $\frac{\partial M}{\partial y}$ *and* $\frac{\partial N}{\partial x}$ *be continuous within a rectangle* R *in the* x-y *plane. Then* $M(x,y)\,dx + N(x,y)\,dy$ *is an exact differential in* R *if and only if*

$$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}} \tag{5.7}$$

everywhere in R.

We illustrate this result through an example…

Example (5a)

- The first order ODE: $\sin y\, dx + (x\cos y - 2y)\, dy = 0$ has $M(x,y) = \sin y$ and $N(x,y) = x\cos y - 2y$. The condition

$$\frac{\partial M}{\partial y} = \cos y = \frac{\partial N}{\partial x}$$

is satisfied. Thus the ODE is an exact equation.

- The next step is to solve the system

$$\begin{cases} \dfrac{\partial F}{\partial x} = M = \sin y \\ \dfrac{\partial F}{\partial y} = N = x\cos y - 2y. \end{cases}$$

- Integrating the equation $\frac{\partial F}{\partial x} = \sin y$ with respect to x on both sides with y being fixed gives

$$F(x,y) = \int \sin y\, dx = x\sin y + u(y) \text{ for some function } u(y).$$

- Differentiating the equation $F(x, y) = x \sin y + u(y)$ with respect to y gives

$$\frac{\partial F}{\partial y} = x \cos y + u'(y).$$

- Comparing with $\frac{\partial F}{\partial y} = x \cos y - 2y$ gives

$$x \cos y - 2y = x \cos y + u'(y).$$

  Or, after simplification,

$$u'(y) = -2y.$$

- Integrating both sides with respect to y gives

$$u(y) = -\int 2y\, dy = -y^2 + C.$$

- Finally, we have $F(x, y) = x \sin y + u(y) = x \sin y - y^2 + C$. Hence, the general solution of the original ODE is

$$x \sin y - y^2 = \tilde{C}.$$

Homework Assignment No: 3

Due Date: 6:00pm, 31 October 2019

Please place your assignment to Assignment Box 3 outside PC Lab (ERB 218)

Ordinary Differential Equations – 6…

ODE 6-11: Second Order and High Order ODEs

6 **Linear Independence and Wronskian Determinants**

7 Homogeneous Equations: General Theory

8 Homogeneous Equations: Constant Coefficients

9 Nonhomogeneous Equations: Method of Undetermined Coefficients

10 Euler-Cauchy Equation

11 Modeling: Free and Forced Oscillations

Linear Independence

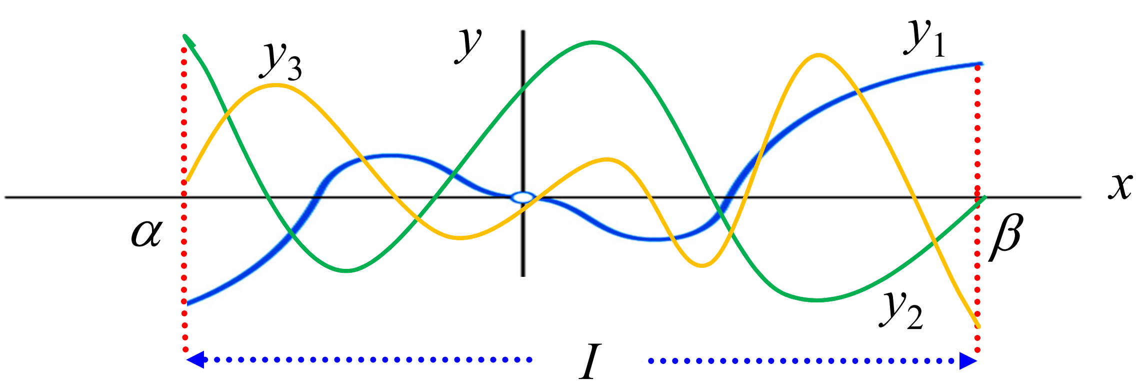
- Let I be an interval of $\mathbb{R}$. Consider the vector space V of all continuous functions defined on I, i.e.,

$$V = \{y(x) \mid y : I \to \mathbb{R},\ y(x) \text{ is continuous}\},$$

where the vector addition and the scalar multiplication are defined as usual:

$$(y_1 + y_2)(x) = y_1(x) + y_2(x) \text{ for any } y_1, y_2 \in V, x \in I;$$

$$(cy)(x) = cy(x) \text{ for any } c \in \mathbb{R}, y \in V, x \in I.$$



V is a collection of all these continuous functions…

- Suppose $y_1(x), y_2(x), \ldots, y_n(x) \in V$. They are said to be **linearly dependent** (LD) on I if there exist scalars $\alpha_j \in \mathbb{R}$, not all zero, such that

$$\boxed{\alpha_1 y_1(x) + \alpha_2 y_2(x) + \cdots + \alpha_n y_n(x) = 0 \text{ for any } x \in I.}$$

  Otherwise, $y_1(x), y_2(x), \ldots, y_n(x)$ are said to be **linearly independent** (LI) on I.

- If $y_1(x), y_2(x), \ldots, y_n(x)$ are linearly dependent on I, then at least one of them can be expressed as a linear combination of the others on I.

  For example, if $\alpha_1 \neq 0$, then

$$y_1(x) = -\frac{\alpha_2}{\alpha_1} y_2(x) - \cdots - \frac{\alpha_n}{\alpha_1} y_n(x)$$

- Otherwise, if $y_1(x), y_2(x), \ldots, y_n(x)$ are linearly independent on I, then all of them cannot be expressed as a linear combination of the others on I.

Example (6a)

- The functions $y_1(x) = \sin^2 x$ and $y_2(x) = -1 + \cos^2 x$ are LD over $\mathbb{R}$ because

$$y_1(x) + y_2(x) = \sin^2 x - 1 + \cos^2 x = 0 \text{ for any } x \in \mathbb{R}.$$

- Alternatively, since $y_1(x) = -y_2(x)$, $y_1(x)$ and $y_2(x)$ are LD over $\mathbb{R}$.

Example (6b)

- Let $\omega \neq 0$ be a real constant. Show that the functions $y_1(x) = \cos \omega x$ and $y_2(x) = \sin \omega x$ are LI over $\mathbb{R}$.
- Suppose there exist real scalars $\alpha_1, \alpha_2 \in \mathbb{R}$ such that

$$\alpha_1 \cos \omega x + \alpha_2 \sin \omega x = 0 \text{ for any } x \in \mathbb{R}. \cdots\cdots (*)$$

Differentiating $(*)$ with respect to x gives

$$-\alpha_1 \omega \sin \omega x + \alpha_2 \omega \cos \omega x = 0 \text{ for any } x \in \mathbb{R}. \cdots\cdots (**)$$

By $(*)$ and $(**)$, we have

$$\begin{pmatrix} \cos \omega x & \sin \omega x \\ -\omega \sin \omega x & \omega \cos \omega x \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Since $\begin{vmatrix} \cos \omega x & \sin \omega x \\ -\omega \sin \omega x & \omega \cos \omega x \end{vmatrix} = \omega \neq 0$, the above system has the unique solution $\alpha_1 = \alpha_2 = 0$. Hence, $y_1(x)$ and $y_2(x)$ must be LI over $\mathbb{R}$.

Wronskian Determinant

Let $y_1, y_2, \ldots, y_n \in V$ and assume $y_1, y_2, \ldots, y_n$ are $n-1$ times differentiable on I. Differentiating both sides of the equation

$$\alpha_1 y_1(x) + \alpha_2 y_2(x) + \cdots + \alpha_n y_n(x) = 0$$

with respect to x for j times, where $j = 0, 1, \ldots, n-1$, gives n equations

$$\left\{ \begin{aligned} \alpha_1 y_1(x) + \alpha_2 y_2(x) + \cdots + \alpha_n y_n(x) &= 0 \\ \alpha_1 y_1'(x) + \alpha_2 y_2'(x) + \cdots + \alpha_n y_n'(x) &= 0 \\ &\vdots \\ \alpha_1 y_1^{(n-1)}(x) + \alpha_2 y_2^{(n-1)}(x) + \cdots + \alpha_n y_n^{(n-1)}(x) &= 0. \end{aligned} \right. \tag{6.2}$$

$$\begin{cases} \alpha_1 y_1(x) + \alpha_2 y_2(x) + \cdots + \alpha_n y_n(x) = 0 \\ \alpha_1 y_1'(x) + \alpha_2 y_2'(x) + \cdots + \alpha_n y_n'(x) = 0 \\ \vdots \\ \alpha_1 y_1^{(n-1)}(x) + \alpha_2 y_2^{(n-1)}(x) + \cdots + \alpha_n y_n^{(n-1)}(x) = 0. \end{cases} \qquad (6.2)$$

$$\Rightarrow \begin{pmatrix} y_1(x) & y_2(x) & \cdots & y_n(x) \\ y_1'(x) & y_2'(x) & \cdots & y_n'(x) \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)}(x) & y_2^{(n-1)}(x) & \cdots & y_n^{(n-1)}(x) \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} = 0$$

$$\Rightarrow \text{ if } \begin{vmatrix} y_1(x) & y_2(x) & \cdots & y_n(x) \\ y_1'(x) & y_2'(x) & \cdots & y_n'(x) \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)}(x) & y_2^{(n-1)}(x) & \cdots & y_n^{(n-1)}(x) \end{vmatrix} \neq 0 \text{ for some } x = x_0 \quad \Rightarrow \quad \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} = 0$$

$\Rightarrow$ $y_1(x), y_2(x), \cdots, y_n(x)$ are Linearly Independent.

- Denote the determinant of the coefficients as:

$$W(y_1, y_2, \ldots, y_n)(x) = \begin{vmatrix} y_1(x) & y_2(x) & \cdots & y_n(x) \\ y_1'(x) & y_2'(x) & \cdots & y_n'(x) \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)}(x) & y_2^{(n-1)}(x) & \cdots & y_n^{(n-1)}(x) \end{vmatrix},$$

which is called the **Wronskian determinant** of $y_1, y_2, \ldots, y_n$.

Józef Maria Hoene-Wroński
(1776–1853)
Polish Mathematician

Theorem (6.1 Wronskian Condition for LI)

Let

$$y_1, y_2, \cdots, y_n \in V$$

and they are $n-1$ times differentiable on *I*. If the corresponding Wronskian determinant is not identically zeros on *I*, i.e.,

$$W(y_1, y_2, \cdots, y_n)(x) = \begin{vmatrix} y_1(x) & y_2(x) & \cdots & y_n(x) \\ y_1'(x) & y_2'(x) & \cdots & y_n'(x) \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)}(x) & y_2^{(n-1)}(x) & \cdots & y_n^{(n-1)}(x) \end{vmatrix} \neq 0 \quad \text{for some } x = x_0,$$

then $y_1, y_2, \cdots, y_n$ are linearly independent on *I*.

Theorem (6.2 A Necessary and Sufficient Condition for LD)

If $y_1, y_2, \ldots, y_n$ are solutions of an n-th order linear homogeneous ODE

$$y^{(n)}(x) + p_1(x)y^{(n-1)}(x) + \cdots + p_n(x)y(x) = 0,$$

where the coefficients $p_j(x)$'s are continuous on an interval I, then $W(y_1, y_2, \ldots, y_n)(x) \equiv 0$ on I is both **necessary and sufficient** *for the linear dependence of the set $\{y_1, y_2, \ldots, y_n\}$ on I.*

$$\begin{vmatrix} y_1(x) & y_2(x) & \cdots & y_n(x) \\ y_1'(x) & y_2'(x) & \cdots & y_n'(x) \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)}(x) & y_2^{(n-1)}(x) & \cdots & y_n^{(n-1)}(x) \end{vmatrix} \equiv 0 \quad \text{iff} \quad y_1,\ y_2, \cdots, y_n \quad \text{are LD.}$$

Example (7c)

- Show that $\{1, e^x, e^{-x}\}$ is a set of linearly independent solutions of the ODE $y''' - y' = 0$ over $\mathbb{R}$.
- Let $y_1(x) = 1, y_2(x) = e^x$ and $y_3(x) = e^{-x}$. Then,

$$y_1''' - y_1' = 0 - 0 = 0,$$

$$y_2''' - y_2' = e^x - e^x = 0,$$

and

$$y_3''' - y_3' = -e^{-x} - (-e^{-x}) = 0.$$

Now, direct computation gives

$$W(y_1, y_2, y_3)(x) = \begin{vmatrix} 1 & e^x & e^{-x} \\ 0 & e^x & -e^{-x} \\ 0 & e^x & e^{-x} \end{vmatrix} = 2 \neq 0.$$

Hence, $\{1, e^x, e^{-x}\}$ is a set of linearly independent solutions.

Ordinary Differential Equations – 7…

6 Linear Independence and Wronskian Determinants

7 **Homogeneous Equations: General Theory**

8 Homogeneous Equations: Constant Coefficients

9 Nonhomogeneous Equations: Method of Undetermined Coefficients

10 Euler-Cauchy Equation

11 Modeling: Free and Forced Oscillations

Principle of Superposition

- Consider an n-th order **linear homogeneous** ODE

$$y^{(n)}(x) + p_1(x)y^{(n-1)}(x) + \cdots + p_n(x)y(x) = 0. \qquad (7.1)$$

- If we write

$$\mathcal{L} = \frac{d^n}{dx^n} + p_1(x)\frac{d^{n-1}}{dx^{n-1}} + \cdots + p_n(x),$$

then $\mathcal{L}$ is an n-th order **linear differential operator**, and

$$\mathcal{L}(y) = \frac{d^n}{dx^n}y(x) + p_1(x)\frac{d^{n-1}}{dx^{n-1}}y(x) + \cdots + p_n(x)y(x)$$

$$= y^{(n)}(x) + p_1(x)y^{(n-1)}(x) + \cdots + p_n(x)y(x)$$

$$= \text{ Left-hand size of Equation (7.1)}$$

We note $\mathcal{L}$ is a notation (or defined operator) for the ease of presentation.

$$y^{(n)}(x) + p_1(x)y^{(n-1)}(x) + \cdots + p_n(x)y(x) = 0. \qquad (7.1)$$

⬇

$$\mathcal{L}[y] = 0$$

⬇

$$\boxed{\mathcal{L}[C_1y_1 + C_2y_2 + \cdots + C_ny_n] = C_1\mathcal{L}[y_1] + C_2\mathcal{L}[y_2] + \cdots + C_n\mathcal{L}[y_n]} \qquad (7.2)$$

for any functions $y_1, y_2, \ldots, y_n$ and any constants $C_1, C_2, \ldots, C_n$.

For $n = 2$, (7.2) can be verified as follows:

$$\mathcal{L}(C_1 y_1 + C_2 y_2) = \frac{d^2}{dx^2}\big(C_1 y_1(x) + C_2 y_2(x)\big) + p_1(x)\frac{d}{dx}\big(C_1 y_1(x) + C_2 y_2(x)\big) + p_2(x)\big(C_1 y_1(x) + C_2 y_2(x)\big)$$

$$= \frac{d^2}{dx^2} C_1 y_1(x) + p_1(x)\frac{d}{dx} C_1 y_1(x) + p_2(x) C_1 y_1(x) + \frac{d^2}{dx^2} C_2 y_2(x) + p_1(x)\frac{d}{dx} C_2 y_2(x) + p_2(x) C_2 y_2(x)$$

$$= C_1\left[\frac{d^2}{dx^2} y_1(x) + p_1(x)\frac{d}{dx} y_1(x) + p_2(x) y_1(x)\right] + C_2\left[\frac{d^2}{dx^2} y_2(x) + p_1(x)\frac{d}{dx} y_2(x) + p_2(x) y_2(x)\right]$$

$$= C_1 \mathcal{L}(y_1) + C_2 \mathcal{L}(y_2)$$

- The **Principle of Superposition** states that if $y_1, y_2, \ldots, y_n$ are solutions of ODE (7.1) on an interval I, then

$$\boxed{y = C_1 y_1 + C_2 y_2 + \cdots + C_n y_n}$$

is also a solution of that ODE on the same interval I for arbitrary constants $C_1, C_2, \ldots, C_n$.

- **Reason:** If $y_1, y_2, \ldots, y_n$ are solutions of the same ODE (7.1), then $\mathcal{L}[y_j] = 0$ for $j = 1, 2, \ldots, n$. It follows from (7.2) that

$$\begin{aligned}\mathcal{L}[C_1 y_1 + C_2 y_2 + \cdots + C_n y_n] &= C_1 \mathcal{L}[y_1] + C_2 \mathcal{L}[y_2] + \cdots + C_n \mathcal{L}[y_n] \\ &= C_1(0) + C_2(0) + \cdots + C_n(0) \\ &= 0,\end{aligned}$$

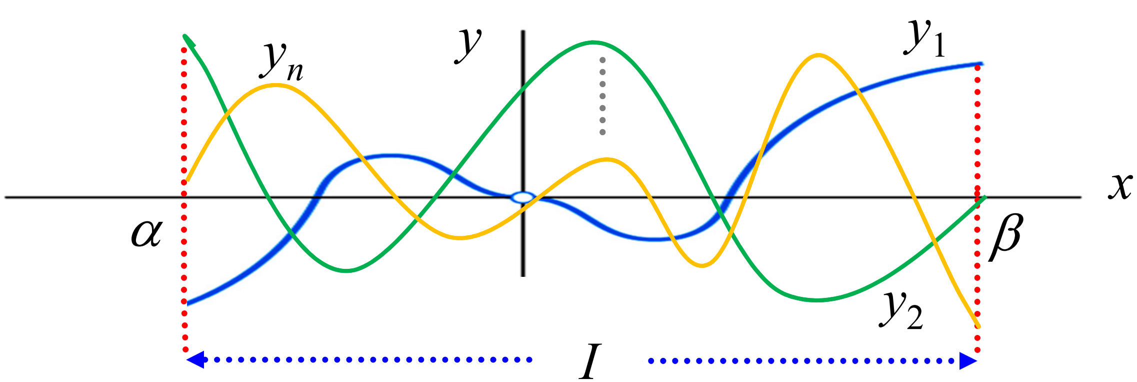
so $y = C_1 y_1 + C_2 y_2 + \cdots + C_n y_n$ is also a solution.

Theorem (7.1 General Solution of ODE (7.1))

Let the coefficients $p_j(x)$'s of ODE (7.1) are continuous on an open interval I. Then the n-th order linear homogeneous ODE (7.1) admits exactly n LI solutions. Let $y_1, y_2, \ldots, y_n$ be any n LI solutions on I, then for any solution $y(x)$ of ODE (7.1) on the interval I, there exist n constants $C_1, C_2, \ldots, C_n$ such that

$$y(x) = C_1 y_1(x) + C_2 y_2(x) + \cdots + C_n y_n(x).$$

$$y^{(n)}(x) + p_1(x) y^{(n-1)}(x) + \cdots + p_n(x) y(x) = 0. \qquad (7.1)$$



All the solutions to ODE (7.1) are linear combination of the solutions, $y_1, y_2, \ldots, y_n$

- The set $\{y_1, y_2, \ldots, y_n\}$ where $y_1, y_2, \ldots, y_n$ be any n LI solutions of ODE (7.1) on I forms a **basis**, and is called a **fundamental set** of solutions of ODE (7.1).

- The expression $C_1 y_1(x) + C_2 y_2(x) + \cdots + C_n y_n(x)$ is called a linear combination of the fundamental set of solutions and is also called a **general solution** of the n-th order linear homogeneous ODE (7.1) on the interval I.

$$y(x) = C_1 y_1(x) + C_2 y_2(x) + \cdots + C_n y_n(x)$$

A **particular solution** of ODE (7.1) is any solution of (7.1) on the interval I.

$$y^{(n)}(x) + p_1(x) y^{(n-1)}(x) + \cdots + p_n(x) y(x) = 0. \qquad (7.1)$$

Example (7a)

- It is readily verified

$$y_1(x) = 1, \quad y_2(x) = e^x, \quad y_3(x) = e^{-x}$$

are 3 LI solutions of the ODE

$$y''' - y' = 0$$

and thus $\{y_1, y_2, y_3\}$ forms a fundamental set of solutions of the equation.

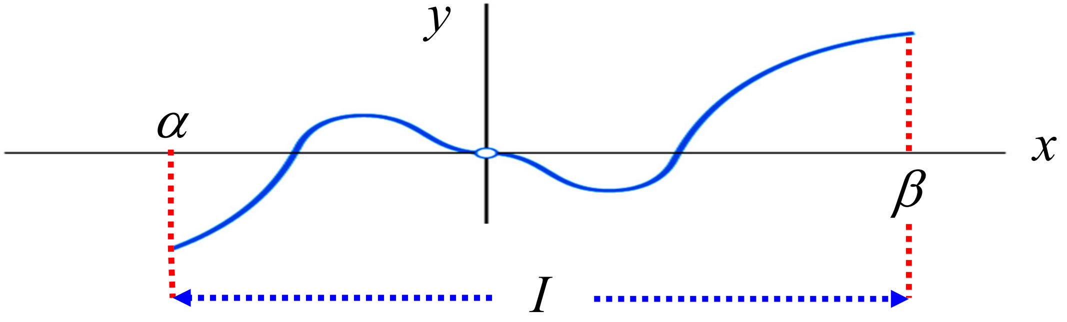
- By Theorem 7.1, the general solution of the ODE is

$$\begin{aligned} y(x) &= C_1 y_1(x) + C_2 y_2(x) + C_3 y_3(x) \\ &= C_1 + C_2 e^x + C_3 e^{-x} \end{aligned}$$

where C_1, C_2, C_3 are arbitrary constants.

Boundary Value Problem

- In contrast with IVP, we can impose conditions at two points of a solution for an ODE. Such conditions are called the boundary conditions. Normally the two points are selected at both ends of the interval I. The problem of solving an ODE that satisfies some boundary conditions is called a **boundary value problem** (BVP).



- Unlike an IVP, which has a unique solution as indicated by Theorem 7.1, a BVP may have no solution, a unique solution, or infinitely many solutions.

Example (7b)

- Consider an ODE

$$y'' + y = 0,$$

which can be readily verified that it admits a general solution

$$y(x) = C_1 \cos x + C_2 \sin x.$$

Three different sets of boundary conditions are given as follows:

- **Case 1:** $y(0) = 2, y(\pi) = 1$. Then

$$y(0) = 2 = C_1 + 0, \quad y(\pi) = 1 = -C_1 + 0,$$

which has no solution for C_1 and C_2, so the BVP has no solution for $y(x)$.

$$y(x) = C_1 \cos x + C_2 \sin x.$$

- **Case 2:** $y(0) = 2, y(\frac{\pi}{2}) = 3$. Then

$$y(0) = 2 = C_1 + 0, \quad y(\frac{\pi}{2}) = 3 = 0 + C_2,$$

so $C_1 = 2, C_2 = 3$, and the BVP has the unique solution $y(x) = 2\cos x + 3\sin x$.

- **Case 3:** $y(0) = 2, y(\pi) = -2$. Then

$$y(0) = 2 = C_1 + 0, \quad y(\pi) = -2 = -C_1 + 0,$$

so $C_1 = 2$ and C_2 is arbitrary, so the BVP has infinitely many solutions $y(x) = 2\cos x + C_2 \sin x$ parametrized by an arbitrary constant C_2.

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Second Order ODEs: Constant Coefficients

- A second order linear homogeneous equation with constant coefficients is described by

$$\boxed{y''(x) + a_1 y'(x) + a_2 y(x) = 0,} \qquad (8.4)$$

where a_1 and a_2 are real constants.

We are looking for a fundamental set solutions $\{y_1, y_2\}$ such that

(a) $y_j''(x) + a_1 y_j'(x) + a_2 y_j(x) = 0, \quad j = 1, 2;$ and

(b) $W(y_1, y_2)(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} \neq 0,$

➡ the general solution of ODE (8.4) is

$$\boxed{y(x) = C_1 y_1(x) + C_2 y_2(x),} \qquad (8.6)$$

Characteristic Equation and General Solution

- To obtain a fundamental set of solutions of the ODE

$$y''(x) + a_1 y'(x) + a_2 y(x) = 0, \tag{8.7}$$

we seek a solution in the exponential function form

$$y(x) = e^{\lambda x}, \tag{8.8}$$

where λ is some complex number to be determined.

- Substituting (8.8) into (8.7) and using the property (8.2) gives

$$(\lambda^2 + a_1\lambda + a_2)e^{\lambda x} = 0.$$

Since $e^{\lambda x}$ is not equal to zero on any interval I for any choices of λ, λ must be such that

$$\boxed{\lambda^2 + a_1\lambda + a_2 = 0.} \tag{8.9}$$

This equation and its left-hand side are called **characteristic equation** and **characteristic polynomial** of (8.7), respectively.

- Using quadratic formula, the roots of (8.9) are given by

$$\lambda = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2}}{2}. \qquad (8.10)$$

We then have three different cases for the roots in (8.10)…

- **Case 1**: Two distinct real roots $\lambda_1 \neq \lambda_2$ when $a_1^2 - 4a_2 > 0$
- **Case 2**: Repeated roots $\lambda_1 = \lambda_2$ when $a_1^2 - 4a_2 = 0$
- **Case 3**: Two complex conjugated roots when $a_1^2 - 4a_2 < 0$

$$\lambda_1 = \mu + i\omega \text{ and } \lambda_2 = \mu - i\omega.$$

- **Case 1**: Two distinct real roots $\lambda_1 \neq \lambda_2$ when $a_1^2 - 4a_2 > 0$

$$\boxed{y(x) = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x},} \tag{8.11}$$

(Optional) Reason for Case 1: When $a_1^2 - 4a_2 \neq 0$, denote the two distinct roots of (8.10) by λ_1 and λ_2. Then it can be verified that $y_1(x) = e^{\lambda_1 x}$ and $y_2(x) = e^{\lambda_2 x}$ form a solution set of ODE (8.4) and

$$\begin{aligned} W(y_1, y_2)(x) &= \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} \\ &= \begin{vmatrix} e^{\lambda_1 x} & e^{\lambda_2 x} \\ \lambda_1 e^{\lambda_1 x} & \lambda_2 e^{\lambda_2 x} \end{vmatrix} \\ &= (\lambda_2 - \lambda_1) e^{(\lambda_1 + \lambda_2) x} \neq 0. \end{aligned}$$

Thus, the general solution of ODE (8.4) is given by, according to (8.6),

$$y(x) = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}.$$

- **Case 2**: Repeated roots $\lambda_1 = \lambda_2$ when $a_1^2 - 4a_2 = 0$:

$$\boxed{y(x) = C_1 e^{\lambda_1 x} + C_2 x e^{\lambda_1 x},} \tag{8.12}$$

(Optional) Reason for Case 2: When $a_1^2 - 4a_2 = 0$, $\lambda_1 = \lambda_2 = \frac{-a_1}{2}$ which is real. $y_1(x) = e^{\lambda_1 x}$ and $y_2(x) = e^{\lambda_2 x}$ are the same thing and thus they cannot constitute a fundamental set of solutions of ODE (8.4). We need to find another solution of ODE (8.4) which is LI of $y_1(x) = e^{\lambda_1 x}$ on the interval I. In fact, assuming $y_2(x) = xe^{\lambda_1 x}$ and substituting $y_2(x)$ into the left-hand side of ODE (8.4) gives

$$y_2''(x) + a_1 y_2'(x) + a_2 y_2(x) = (2\lambda_1 + a_1)e^{\lambda_1 x} + (\lambda_1^2 + a_1\lambda_1 + a_2)xe^{\lambda_1 x}, \tag{8.14}$$

which is identically equal to zero since λ_1 is a root of $\lambda^2 + a_1\lambda + a_2 = 0$ and $\lambda_1 = \frac{-a_1}{2}$. Thus, $y_2(x) = xe^{\lambda_1 x}$ is indeed a solution of ODE (8.4) and is obviously LI of $y_1(x) = e^{\lambda_1 x}$. Thus, the general solution of ODE (8.4) for this case is

$$y(x) = C_1 e^{\lambda_1 x} + C_2 x e^{\lambda_1 x}.$$

- **Case 3**: Two complex roots When $a_1^2 - 4a_2 < 0$. $\lambda_1 = \mu + i\omega$ and $\lambda_2 = \mu - i\omega$.

$$\boxed{y(x) = (C_1 \cos \omega x + C_2 \sin \omega x)e^{\mu x},} \tag{8.13}$$

where C_1 and C_2 are arbitrary real constants. An advantage of this form of the general solution is that the right-hand side is a real function.

- **(Optional) Reason for Case 3:**

$$y_1(x) = e^{\lambda_1 x} = e^{(\mu + i\omega)x} = e^{\mu x}(\cos \omega x + i \sin \omega x).$$

and

$$y_2(x) = e^{\lambda_2 x} = e^{(\mu - i\omega)x} = e^{\mu x}(\cos \omega x - i \sin \omega x).$$

Thus, another fundamental set of solutions can be formed by

$$\frac{y_1(x) + y_2(x)}{2} = e^{\mu x} \cos \omega x \quad \text{and} \quad \frac{y_1(x) - y_2(x)}{2i} = e^{\mu x} \sin \omega x.$$

Solutions for Second Order Linear Homogeneous ODEs

- Given a second order linear homogeneous ODE

$$y''(x) + a_1 y'(x) + a_2 y(x) = 0.$$

Then the characteristic equation is

$$\lambda^2 + a_1\lambda + a_2 = 0$$

and the general solution is summarized in three different cases:

Case 1: distinct real roots where $\lambda_1 \neq \lambda_2$:

$$\boxed{y(x) = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}.}$$

Case 2: repeated roots where $\lambda_1 = \lambda_2$:

$$\boxed{y(x) = C_1 e^{\lambda_1 x} + C_2 x e^{\lambda_1 x}.}$$

Case 3: complex roots where $\lambda = \mu \pm i\omega$:

$$\boxed{y(x) = (C_1 \cos\omega x + C_2 \sin\omega x) e^{\mu x}.}$$

Example (8a)

$$y(x) = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}.$$

- Solve the ODE:

$$y'' - 9y = 0.$$

- The characteristic polynomial is $\lambda^2 - 9$ which has two distinct real roots $\lambda_1, \lambda_2 = \pm 3$. Hence, the general solution is

$$y(x) = C_1 e^{3x} + C_2 e^{-3x}.$$

Example (8b)

$$y(x) = C_1 e^{\lambda_1 x} + C_2 x e^{\lambda_1 x}.$$

- Solve the ODE:

$$y'' + 2y' + y = 0.$$

- The characteristic polynomial is $\lambda^2 + 2\lambda + 1$ which has repeated roots $\lambda_1 = \lambda_2 = -1$. Hence, the general solution is

$$y(x) = C_1 e^{-x} + C_2 x e^{-x}.$$

$$y(x) = (C_1 \cos \omega x + C_2 \sin \omega x) e^{\mu x}.$$

Example (8c)

- Solve the IVP: $y'' + 4y' + 7y = 0, \quad y(0) = 1, \quad y'(0) = 0.$
- The characteristic polynomial is $\lambda^2 + 4\lambda + 7$ which has a pair of complex roots $\lambda_1, \lambda_2 = \frac{-4 \pm \sqrt{4^2 - 4(1)(7)}}{2} = -2 \pm \sqrt{3}i$. Then, the general solution is

$$y(x) = (C_1 \cos \sqrt{3}x + C_2 \sin \sqrt{3}x) e^{-2x}.$$

- Now, differentiating both sides of the above equation with respect to x gives

$$y' = \left[(-\sqrt{3}C_1 - 2C_2) \sin \sqrt{3}x + (\sqrt{3}C_2 - 2C_1) \cos \sqrt{3}x \right] e^{-2x}.$$

- Using the initial conditions $y(0) = 1$ and $y'(0) = 0$ gives

$$C_1 = 1 \text{ and } \sqrt{3}C_2 - 2C_1 = 0,$$

which yields $C_1 = 1$ and $C_2 = \frac{2}{\sqrt{3}}$. Hence, the solution is

$$y(x) = \left(\cos \sqrt{3}x + \frac{2}{\sqrt{3}} \sin \sqrt{3}x \right) e^{-2x}.$$

Higher Order ODEs with Constant Coefficients

- Consider an n-th order linear homogeneous equation with constant coefficients, i.e.,

$$\boxed{y^{(n)}(x) + a_1 y^{(n-1)}(x) + \cdots + a_n y(x) = 0,} \tag{8.15}$$

where a_j's are real constants.

- If $\{y_1, y_2, \ldots, y_n\}$ is a fundamental set of solutions of ODE (9.15), then by the Principle of Superposition, the general solution is

$$\boxed{y(x) = C_1 y_1(x) + C_2 y_2(x) + \cdots + C_n y_n(x),} \tag{8.16}$$

where C_j's are arbitrary constants.

- A fundamental set of solutions $\{y_1, y_2, \ldots, y_n\}$ of ODE (8.15) is found by considering

$$\boxed{\lambda^n + a_1\lambda^{n-1} + \cdots + a_n = 0.} \tag{8.17}$$

This equation and its left-hand side are called **characteristic equation** and **characteristic polynomial** of (8.15), respectively.

- A root λ_j of the characteristics equation (8.17) has **multiplicity** k if $(\lambda - \lambda_j)^k$ divides the characteristic polynomial but $(\lambda - \lambda_j)^{k+1}$ does not. In other words, if a root λ_j has multiplicity k, then

$$\lambda^n + a_1\lambda^{n-1} + \cdots + a_n = (\lambda - \lambda_j)^k p(\lambda)$$

for some polynomial function $p(\lambda)$ for which $p(\lambda)$ is not divisible by $\lambda - \lambda_j$.

A fundamental set of solutions of ODE (8.15) can be formed as

(a) If the roots $\lambda_1, \lambda_2, \ldots, \lambda_k$ are real and mutually distinct, then the following k functions

$$\boxed{e^{\lambda_1 x}, e^{\lambda_2 x}, \ldots, e^{\lambda_k x}} \tag{8.18}$$

constitute k LI solutions of ODE (8.15).

(b) If λ_1 is a real root with multiplicity k, then the following k functions

$$\boxed{e^{\lambda_1 x}, x e^{\lambda_1 x}, \ldots, x^{k-1} e^{\lambda_1 x}} \tag{8.19}$$

constitute k LI solutions of ODE (8.15).

(c) If $\lambda = \mu + i\omega$ is a complex root with multiplicity k, then so does its complex conjugate $\lambda = \mu - i\omega$, and the following $2k$ functions

$$\boxed{\begin{array}{l}(\cos\omega x)e^{\mu x}, x(\cos\omega x)e^{\mu x}, \ldots, x^{k-1}(\cos\omega x)e^{\mu x}, \\ \qquad\qquad (\sin\omega x)e^{\mu x}, x(\sin\omega x)e^{\mu x}, \ldots, x^{k-1}(\sin\omega x)e^{\mu x}\end{array}} \tag{8.20}$$

constitute $2k$ LI solutions of ODE (8.15).

Example (8d)

- Solve the ODE:

$$y'''' - 3y'' + 2y = 0.$$

- The characteristic polynomial is $\lambda^4 - 3\lambda^2 + 2 = (\lambda^2 - 1)(\lambda^2 - 2)$ whose roots are $\lambda_1 = 1, \lambda_2 = -1, \lambda_3 = \sqrt{2}$ and $\lambda_4 = -\sqrt{2}$.
- Hence, the general solution is

$$y(x) = C_1 e^{x} + C_2 e^{-x} + C_3 e^{\sqrt{2}x} + C_4 e^{-\sqrt{2}x}.$$

$$\boxed{e^{\lambda_1 x}, e^{\lambda_2 x}, \ldots, e^{\lambda_k x}} \qquad (8.18)$$

Example (8e)

- Solve the ODE:

$$y'''' - y'' = 0.$$

- The characteristic polynomial is $\lambda^4 - \lambda^2 = \lambda^2(\lambda^2 - 1)$ whose roots are $\lambda_1 = \lambda_2 = 0, \lambda_3 = 1$ and $\lambda_4 = -1$.
- Hence, the general solution is

$$y(x) = C_1 + C_2 x + C_3 e^{x} + C_4 e^{-x}.$$

$$\boxed{e^{\lambda_1 x}, x e^{\lambda_1 x}, \ldots, x^{k-1} e^{\lambda_1 x}} \qquad (8.19)$$

Example (8f)

- Solve the ODE:

$$y'''' + 2y'' + y = 0.$$

- The characteristic polynomial is $\lambda^4 + 2\lambda^2 + 1 = (\lambda^2 + 1)^2$ whose roots are $\lambda_1 = \lambda_2 = i$ and $\lambda_3 = \lambda_4 = -i$.
- Hence, the general solution is

$$y(x) = C_1 \cos x + C_2 \sin x + C_3 x \cos x + C_4 x \sin x.$$

$$\boxed{\begin{array}{l}(\cos\omega x)e^{\mu x}, x(\cos\omega x)e^{\mu x}, \ldots, x^{k-1}(\cos\omega x)e^{\mu x}, \\ \qquad (\sin\omega x)e^{\mu x}, x(\sin\omega x)e^{\mu x}, \ldots, x^{k-1}(\sin\omega x)e^{\mu x}\end{array}} \quad (8.20)$$

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Nonhomogeneous ODEs

- We now study 2^{nd} order non-homogeneous linear ODEs. Let us consider the simplest case where the ODE is of constant-coefficient type described as follows:

$$y''(x) + a_1 y'(x) + a_2 y(x) = f(x), \tag{9.1}$$

where a_1 and a_2 are real constants and $f(x)$ is nonzero.

- The corresponding **homogeneous equation** is

$$y''(x) + a_1 y'(x) + a_2 y(x) = 0. \tag{9.2}$$

- Let $y_h(x)$ denote the general solution of (9.2) , i.e.,

$$y_h = C_1 y_1 + C_2 y_2,$$

where $\{y_1, y_2\}$ is a fundamental set of solutions of the homogeneous equation. Then $y_h(x)$ is called the **homogeneous solution** of (9.1) .

$$\boxed{y''(x) + a_1 y'(x) + a_2 y(x) = f(x),}$$

- Let $y_p(x)$ be any solution of the non-homogeneous equation (9.1), i.e., y_p satisfies

$$y_p''(x) + a_1 y_p'(x) + a_2 y_p(x) = f(x).$$

Then $y_p(x)$ is called a **particular solution** of (9.1).

Example (9a)

- Consider a nonhomogeneous linear ODE with constant coefficient

$$y'' - 9y = 4 + 5\sinh 3x.$$

- The associated homogeneous equation is $y'' - 9y = 0$ and thus the corresponding characteristic equation is given by

$$\lambda^2 - 9 = 0 \quad \Rightarrow \quad \lambda_{1,2} = \pm 3$$

and the homogenous solution is given by

$$y_h(x) = C_1 e^{3x} + C_2 e^{-3x}$$

Example (9a) (cont.)

- A particular solution of the ODE is $y_p(x) = -\frac{4}{9} + \frac{5}{6}x\cosh 3x$ because

$$\begin{aligned} y_p'' - 9y_p &= \frac{d}{dx}\left(\frac{5}{6}\cosh 3x + \frac{5}{6}(x)(3\sinh 3x)\right) - 9\left(-\frac{4}{9} + \frac{5}{6}x\cosh 3x\right) \\ &= \frac{5}{6}(3\sinh 3x) + \frac{5}{2}(\sinh 3x + 3x\cosh 3x) + \left(4 - \frac{15}{2}x\cosh 3x\right) \\ &= 4 + 5\sinh 3x. \end{aligned}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\sinh x = \cosh x$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\cosh x = \sinh x$$

Theorem (9.1 General Solution of Eq.(9.1))

If $y_h(x)$ and $y_p(x)$ are homogeneous and particular solutions of (9.1), respectively, on an open interval I, then a general solution of (9.1) on I is

$$\boxed{y(x) = y_h(x) + y_p(x).} \tag{9.3}$$

Proof.

$$\begin{aligned} y'' + a_1 y' + a_2 y &= \left(y_h + y_p\right)'' + a_1\left(y_h + y_p\right)' + a_2\left(y_h + y_p\right) \\ &= \left(y_h'' + a_1 y_h' + a_2 y_h\right) + \left(y_p'' + a_1 y_p' + a_2 y_p\right) \\ &= 0 + f(x) \\ &= f(x) \end{aligned}$$

This implies that $y(x)$ indeed a solution to the ODE in (9.1).

$$y''(x) + a_1 y'(x) + a_2 y(x) = f(x) \tag{9.1}$$

- If $f(x)$ is a linear combination of several functions $f_1, \ldots, f_k$, then,

$$\boxed{y''(x) + a_1 y'(x) + a_2 y(x) = A_1 f_1 + \ldots + A_k f_k} \qquad (9.4)$$

Theorem (9.2 General Solution of Eq.(9.4))

If $y_h(x)$ is a homogeneous solution of (9.4) on an open interval I, and $y_{p_1}(x), \ldots, y_{p_k}(x)$ are particular solutions of

$$y''(x) + a_1 y'(x) + a_2 y(x) = f_1,$$

$$\vdots$$

$$y''(x) + a_1 y'(x) + a_2 y(x) = f_k$$

respectively, then a general solution of (9.4) on I, where $A_1, \ldots, A_k$ are some real numbers, is

$$\boxed{y(x) = y_h(x) + A_1 y_{p_1}(x) + \cdots + A_k y_{p_k}(x).} \qquad (9.5)$$

This result is a **Superposition Principle**.

Example (9b)

- It is readily verified in Example (9a) that

$$y_h(x) = C_1 e^{3x} + C_2 e^{-3x}$$

and

$$y_p(x) = -\frac{4}{9} + \frac{5}{6} x \cosh 3x$$

are homogeneous and particular solutions of the ODE

$$y'' - 9y = 4 + 5 \sinh 3x,$$

respectively.

This can be done by showing that $y_{p_1}(x) = -\dfrac{1}{9}$ is a particular solution to

$$y'' - 9y = f_1 = 1$$

and $y_{p_2}(x) = \frac{1}{6} x \cosh 3x$ is a particular solution to

$$y'' - 9y = f_2 = \sinh 3x$$

$$\frac{\mathrm{d}}{\mathrm{d}x} \sinh x = \cosh x$$
$$\frac{\mathrm{d}}{\mathrm{d}x} \cosh x = \sinh x$$

$$y''_{p_2} - 9y_{p_2} = \left(\frac{1}{6} \cosh 3x + \frac{1}{2} x \sinh 3x \right)' - \frac{3}{2} x \cosh 3x$$
$$= \frac{1}{2} \sinh 3x + \frac{1}{2} \sinh 3x + \frac{3}{2} x \cosh 3x - \frac{3}{2} x \cosh 3x$$
$$= \sinh 3x = f_2$$

- Hence, by the Superposition Principle in Theorem 9.2, the general solution is

$$y(x) = y_h(x) + 4y_{p_1}(x) + 5y_{p_2}(x) = C_1 e^{3x} + C_2 e^{-3x} - \frac{4}{9} + \frac{5}{6} x \cosh 3x$$

Method of Undetermined Coefficients

- Let $f(x)$ be a smooth function defined on some interval I. Define a set F of functions as follows:

$$\boxed{F = \{f(x), f'(x), f''(x), f^{(3)}(x), \dots\},} \qquad (9.6)$$

that is, F consists of $f(x)$ and its successive derivatives. If F contains only a finite number of LI terms, then it is called the **family generated by $f(x)$**. In this case, there exists an integer m such that $f^{(j)}(x), j = 0, 1, \dots, m-1$, are LI, and, for any integer $k \geq m$, $f^{(k)}(x)$ is a linear combination of $f^{(j)}(x), j = 0, 1, \dots, m-1$. We call

$$f^{(j)}(x), \quad j = 0, 1, \dots, m-1,$$

a **basis** of F and m the **dimension** of F.

- The **method of undetermined coefficients** can be used to find a particular solution y_p of Eq.(9.1) under the following two conditions:

  (a) Eq.(9.1) is of constant-coefficient type.

  (b) $f(x)$ will generate a family F, i.e., the set (9.6), with a finite number of LI terms.

- Example (1): if $f(x) = e^{\mu x}$, then

$$F = \{e^{\mu x}, \mu e^{\mu x}, \mu^2 e^{\mu x}, \ldots\}.$$

We see that the set $\{e^{\mu x}\}$ is a basis of F with dimension 1 and thus $f(x)$ satisfies condition (b).

$$y''(x) + a_1 y'(x) + a_2 y(x) = f(x) \qquad (9.1)$$

- Example (2): if $f(x) = x^k$ (k is a non-negative integer), then

$$F = \{x^k, kx^{k-1}, \ldots, k!x, k!, 0, 0, 0, \ldots\}.$$

We see that the set $\{x^k, x^{k-1}, \ldots, x, 1\}$ is a basis of F with dimension $(k+1)$ and thus $f(x)$ satisfies condition (b).

- Example (3), if $f(x) = \cos \omega x$, then

$$F = \{\cos \omega x, -\omega \sin \omega x, -\omega^2 \cos \omega x, \ldots\}.$$

We see that $\{\cos \omega x, \sin \omega x\}$ is a basis of F with dimension 2 and thus $f(x)$ satisfies condition (b).

- Example (4), if $f(x) = \frac{1}{x}$, then

$$F = \left\{ \frac{1}{x}, -\frac{1}{x^2}, \frac{2}{x^3}, \dots \right\}.$$

We see that the sequence contains an infinite number of LI terms

$$\left\{ \frac{1}{x}, \frac{1}{x^2}, \frac{1}{x^3}, \dots \right\}.$$

Hence $f(x)$ does not satisfy condition (b).

- Case 1 where $f(x) = e^{\mu x}$ where μ is some real number and μ is not a root of the characteristic polynomial of (9.1):

$$\boxed{y_p = Ce^{\mu x},} \tag{9.7}$$

- Case 2 where $f(x) = x^k$ where $k = 0, 1, \ldots$ and 0 is not a root of the characteristic polynomial of (9.1):

$$\boxed{y_p = C_k x^k + C_{k-1} x^{k-1} + \cdots + C_0,} \tag{9.8}$$

where C_j's are undetermined coefficients.

Recall the characteristic polynomial of (9.1) (actually should be (9.2))…

$$\lambda^2 + a_1 \lambda + a_2 = 0$$

- Case 3 where $f(x) = \cos \omega x$ or $f(x) = \sin \omega x$ where ω is some positive number and $i\omega$ is not a root of the characteristic polynomial of (9.1):

$$\boxed{y_p = C_1 \cos \omega x + C_2 \sin \omega x,} \tag{9.9}$$

where C_1 and C_2 are undetermined coefficients.

- For the previous three cases, the form of the particular solution $y_p(x)$ is a linear combination of the basis of the set F which is the family generated by the individual $f(x)$.

$$y_p = Ce^{\mu x},$$

$$y_p = C_k x^k + C_{k-1} x^{k-1} + \cdots + C_0,$$

$$y_p = C_1 \cos \omega x + C_2 \sin \omega x,$$

We note that $y_p(x)$ is a particular solution to the given ODE, i.e.,

$$y_p''(x) + a_1 y_p'(x) + a_2 y_p(x) = f(x)$$

The unknown constant coefficients can be determined through this equation.

Choices for the Particular Solution

Types of $f(x)$	Choices for y_p
$e^{\mu x}$	$y_p = Ce^{\mu x}$
$x^k (k = 0, 1, \ldots)$	$y_p = C_k x^k + C_{k-1} x^{k-1} + \cdots + C_0$
$\cos \omega x$ $\sin \omega x$	$\Big\} \; y_p = C_1 \cos \omega x + C_2 \sin \omega x$

where neither μ, nor 0, nor $i\omega$, is a root of the characteristic polynomial of (9.1) and C_j's are coefficients to be determined.

- In case where $f(x)$ is a linear combination of some functions, say,

$$f(x) = A_1 e^{\mu x} + A_2 x^k + A_3 \cos \omega x + A_4 \sin \omega x,$$

we have to use the Superposition Principle in Theorem 9.2.

Example (9c)

- Solve the ODE:

$$y'' + 4y = 8x^2.$$

- The characteristic polynomial is $\lambda^2 + 4$ which has two roots $\lambda = \pm 2i$. Thus the homogeneous solution is

$$y_h = A\cos 2x + B\sin 2x,$$

where A and B are arbitrary constants.

- Since 0 is not a root of the characteristic polynomial $\lambda^2 + 4$, we can assume $y_p = C_2x^2 + C_1x + C_0$. Substituting it to the original ODE gives

$$\left(C_2x^2 + C_1x + C_0\right)'' + 4\left(C_2x^2 + C_1x + C_0\right) = 8x^2$$

$$\Downarrow$$

$$2C_2 + 4(C_2x^2 + C_1x + C_0) = 8x^2.$$

Comparing the coefficients gives $4C_2 = 8, 4C_1 = 0$ and $2C_2 + 4C_0 = 0$. Therefore, $C_2 = 2, C_1 = 0$ and $C_0 = -1$.

A particular solution is $y_p = 2x^2 - 1$. Hence, the general solution is

$$y(x) = y_h + y_p = A\cos 2x + B\sin 2x + 2x^2 - 1.$$

To verify that $y(x)$ is indeed a solution to the given ODE, we check…

$$\left(A\cos 2x + B\sin 2x + 2x^2 - 1\right)'' + 4\left(A\cos 2x + B\sin 2x + 2x^2 - 1\right)$$

$$= \left(-2A\sin 2x + 2B\cos 2x + 4x\right)' + 4A\cos 2x + 4B\sin 2x + 8x^2 - 4$$

$$= -4A\cos 2x - 4B\sin 2x + 4 + 4A\cos 2x + 4B\sin 2x + 8x^2 - 4$$

$$= 8x^2$$

Indeed, $y(x)$ is a solution to the ODE,

$$y'' + 4y = 8x^2$$

Example (9d)

- Solve the ODE:

$$y'' - 3y' - 4y = 2\sin x.$$

- The characteristic polynomial is $\lambda^2 - 3\lambda - 4$ which has two roots $\lambda_1 = 4$ and $\lambda_2 = -1$. Thus the homogeneous solution is

$$y_h = Ae^{4x} + Be^{-x}.$$

- Since i is not a root of the characteristic polynomial $\lambda^2 - 3\lambda - 4$, we can assume $y_p = C_1 \cos x + C_2 \sin x$. Substituting it to the original ODE gives

$$(C_1 \cos x + C_2 \sin x)'' - 3(C_1 \cos x + C_2 \sin x)' - 4(C_1 \cos x + C_2 \sin x) = 2\sin x$$

$$\Downarrow$$

$$(-5C_1 - 3C_2)\cos x + (3C_1 - 5C_2)\sin x = 2\sin x.$$

Comparing coefficients gives $-5C_1 - 3C_2 = 0$ and $3C_1 - 5C_2 = 2$ which yields $C_1 = \frac{3}{17}$ and $C_2 = -\frac{5}{17}$.

We thus obtain a particular solution

$$y_p(x) = \frac{3}{17}\cos x - \frac{5}{17}\sin x$$

and the general solution

$$y(x) = y_h(x) + y_p(x) = Ae^{4x} + Be^{-x} + \frac{3}{17}\cos x - \frac{5}{17}\sin x$$

to the given ODE

$$y'' - 3y' - 4y = 2\sin x$$

Exercise: Verify it!

Modified Form of the Particular Solution

- **Non-duplication condition** means that none of the members of the basis of F is proportional to any member of the basis of the fundamental set of solutions of the homogeneous equation (9.2).

- Under this condition, the undetermined coefficients can be determined by solving some linear algebraic equations. However, if this condition is violated, then these linear algebraic equations do not have a solution as shown by the next example.

Example (9e)

- Consider the ODE

$$y'' - y' = e^x. \tag{9.10}$$

The characteristic polynomial of ODE (9.10) is $\lambda^2 - \lambda$ which has two roots $\lambda_1 = 0$ and $\lambda_2 = 1$ and thus a basis of the homogeneous solution is $\{1, e^x\}$.

- On the other hand, $\mu = 1$ is a root of the characteristic polynomial. Also, the basis of $F = \{e^x, e^x, \ldots\}$ is $\{e^x\}$ which duplicates the term e^x in the basis of the homogeneous solution.

- If we still let $y_p = Ce^x$ and substitute $y_p = Ce^x$ into ODE (9.10), we obtain

$$Ce^x - Ce^x = e^x,$$

which yields $0 = e^x$, a contradiction. Therefore, this form of y_p cannot be a particular solution of ODE (9.10).

- Nevertheless, even in this case, by modifying y_p, we can still obtain a particular solution of the non-homogeneous equation (9.10). In fact, let

$$y_p = Cxe^x$$

and substitute it into ODE (9.10), we obtain

$$C(2e^x + xe^x) - C(e^x + xe^x) = e^x,$$

which yields $C = 1$. Hence, $y_p = xe^x$ is a particular solution of the equation.

- **Remark:** Notice that the function xe^x is not proportional to any member of the basis of the homogeneous solution.

- From Example (9e), we see that even if $y_p = Ce^x$ is not a particular solution, we can still obtain a particular solution by modifying y_p. In what follows, we will modify the expressions of the particular solutions for the previous three cases where $\mu, 0$, or $i\omega$ is one of the roots of the characteristic polynomial of the ODE.

- Case 1 where $f(x) = e^{\mu x}$ with μ some real number and μ is a root of the characteristic polynomial of (9.2) of multiplicity m:

$$\boxed{y_p = Cx^m e^{\mu x},} \tag{9.11}$$

Recall the characteristic polynomial of (9.2)...

$$\lambda^2 + a_1\lambda + a_2 = 0$$

- Case 2 where $f(x) = x^k$ where $k = 0, 1, \ldots$ and 0 is a root of the characteristic polynomial of (9.2) of multiplicity m. Then,

$$\boxed{y_p = x^m(C_k x^k + C_{k-1}x^{k-1} + \cdots + C_0),} \tag{9.12}$$

- Case 3 where $f(x) = \cos\omega x$ or $f(x) = \sin\omega x$ where ω is some positive number and $i\omega$ is a root of the characteristic polynomial of (9.2) of multiplicity m. Then,

$$\boxed{y_p = x^m(C_1\cos\omega x + C_2\sin\omega x),} \tag{9.13}$$

where C_1 and C_2 are the undetermined coefficients.

Recall the characteristic polynomial of (9.2)…

$$\lambda^2 + a_1\lambda + a_2 = 0$$

Example (9f)

- Solve the ODE:

$$y'' - 3y' + 2y = e^x - e^{-x}.$$

- The characteristic polynomial is $\lambda^2 - 3\lambda + 2$ which has two roots $\lambda_1 = 1$ and $\lambda_2 = 2$. Thus the homogeneous solution is

$$y_h = Ae^x + Be^{2x}.$$

- Let $f(x) = f_1(x) + f_2(x)$ where $f_1(x) = e^x$ and $f_2(x) = -e^{-x}$. Then, by the Superposition Principle,

$$y_p = y_{p_1} + y_{p_2},$$

  where $y_{p_i}, i = 1, 2$, are particular solutions of $y'' - 3y' + 2y = f_i(x), i = 1, 2$, respectively.

- To obtain y_{p_1}, note that $\mu = 1$ is a simple root of the characteristic polynomial, we can assume $y_{p_1} = C_1 x e^x$. Determining the unknown

$$y''_{p_1} - 3y'_{p_1} + 2y_{p_1} = \left(C_1 x e^x\right)'' - 3\left(C_1 x e^x\right)' + 2\left(C_1 x e^x\right)$$

$$= C_1\left(e^x + xe^x\right)' - 3C_1\left(e^x + xe^x\right) + 2C_1 xe^x = C_1\left(2e^x + xe^x\right) - 3C_1 e^x - C_1 xe^x$$

$$= -C_1 e^x = f_1(x) = e^x \quad \Rightarrow \quad C_1 = -1 \quad \Rightarrow \quad y_{p_1} = -xe^x$$

- To obtain y_{p_2}, note that $\mu = -1$ is not a root of the characteristic polynomial, we can assume $y_{p_2} = C_2 e^{-x}$. Determining the unknown coefficient C_2 gives $C_2 = -\frac{1}{6}$. Thus

$$y_{p_2} = -\frac{1}{6} e^{-x}.$$

- Finally, a general solution of the original ODE is

$$\begin{aligned} y(x) &= y_h + y_{p_1} + y_{p_2} \\ &= Ae^x + Be^{2x} - xe^x - \frac{1}{6} e^{-x}. \end{aligned}$$

Ordinary Differential Equations – 10…

6 Linear Independence and Wronskian Determinants

7 Homogeneous Equations: General Theory

8 Homogeneous Equations: Constant Coefficients

9 Nonhomogeneous Equations: Method of Undetermined Coefficients

10 Euler-Cauchy Equation

11 Modeling: Free and Forced Oscillations

Euler-Cauchy Equation

The following ODE is called Euler-Cauchy Equation:

$$x^2 y'' + axy' + by = 0$$

where a and b are constants. We try a solution

$$y = x^m$$

Substituting this and its derivatives into the Euler-Cauchy Equation, we obtain

$$x^2 m(m-1)x^{m-2} + axmx^{m-1} + bx^m = 0$$

$$\Rightarrow \quad m(m-1)x^m + amx^m + bx^m = 0 \quad \Rightarrow \quad m(m-1) + am + b = 0$$

$$\Rightarrow \quad m^2 + (a-1)m + b = 0 \quad \Rightarrow \quad m_{1,2} = \frac{1 - a \pm \sqrt{(a-1)^2 - 4b}}{2}$$

Case 1: If

$$m_{1,2} = \frac{1-a \pm \sqrt{(a-1)^2 - 4b}}{2}$$

has two distinct real roots, we obtain a general solution to the Euler-Cauchy equation as

$$y = c_1 x^{m_1} + c_2 x^{m_2}$$

with c_1 and c_2 being free constants.

We check...

$$\underline{x^2 y'' + axy' + by} = x^2 (c_1 x^{m_1} + c_2 x^{m_2})'' + ax(c_1 x^{m_1} + c_2 x^{m_2})' + b(c_1 x^{m_1} + c_2 x^{m_2})$$

$$= x^2 c_1 m_1 (m_1 - 1) x^{m_1 - 2} + x^2 c_2 m_2 (m_2 - 1) x^{m_2 - 2} + axc_1 m_1 x^{m_1 - 1} + axc_2 m_2 x^{m_2 - 1} + bc_1 x^{m_1} + bc_2 x^{m_2}$$

$$= \left(c_1 m_1^2 - c_1 m_1 + ac_1 m_1 + bc_1 \right) x^{m_1} + \left(c_2 m_2^2 - c_2 m_2 + ac_2 m_2 + bc_2 \right) x^{m_2}$$

$$= c_1 \left(m_1^2 + (a-1) m_1 + b \right) x^{m_1} + c_2 \left(m_2^2 + (a-1) m_2 + b \right) x^{m_2}$$

$$= c_1 \cdot 0 \cdot x^{m_1} + c_1 \cdot 0 \cdot x^{m_2} \underline{= 0} \quad \checkmark$$

Example 11.1: Solve the following Euler-Cauchy equation

$$x^2 y'' - 2.5xy' - 2.0y = 0$$

Solution: For $a = -2.5$ and $b = -2.0$, we obtain

$$m_{1,2} = \frac{1 - a \pm \sqrt{(a-1)^2 - 4b}}{2} = \frac{1 + 2.5 \pm \sqrt{(-2.5-1)^2 + 8}}{2}$$
$$= \frac{1 + 2.5 \pm 4.5}{2} = -0.5,\ 4$$

and thus the general solution to the ODE

$$y = c_1 x^{-0.5} + c_2 x^4 = \frac{c_1}{\sqrt{x}} + c_2 x^4$$

Case 2: If

$$m_{1,2} = \frac{1 - a \pm \sqrt{(a-1)^2 - 4b}}{2}$$

has two repeated roots, it can be showed that a general solution to the Euler-Cauchy equation is given as

$$y = \left(c_1 + c_2 \ln x\right) x^{(1-a)/2}$$

Example 11.2: Solve the following Euler-Cauchy equation

$$x^2 y'' - 3xy' + 4y = 0$$

Solution: For $a = -3$ and $b = 4$, we obtain $m_{1,2} = 2$ and thus the general solution to the ODE

$$y = \left(c_1 + c_2 \ln x\right) x^2$$

Case 3: If

$$m_{1,2} = \frac{1-a \pm \sqrt{(a-1)^2 - 4b}}{2}$$

has two repeated roots $m_1 = \mu + i\nu$ and $m_2 = \mu - i\nu$, it can be showed that a general solution to the Euler-Cauchy equation is given as

$$y = x^{\mu}\left[c_1 \cos(\nu \ln x) + c_2 \sin(\nu \ln x)\right]$$

Example 11.3: Solve the following Euler-Cauchy equation

$$x^2 y'' + 7xy' + 13y = 0$$

Solution: For $a = 7$ and $b = 13$, we obtain $m_{1,2} = -3 \pm 2i$ and thus the general solution to the ODE

$$y = x^{-3}\left[c_1 \cos(2\ln x) + c_2 \sin(2\ln x)\right]$$

Ordinary Differential Equations – 11…

6 Linear Independence and Wronskian Determinants

7 Homogeneous Equations: General Theory

8 Homogeneous Equations: Constant Coefficients

9 Nonhomogeneous Equations: Method of Undetermined Coefficients

10 Euler-Cauchy Equation

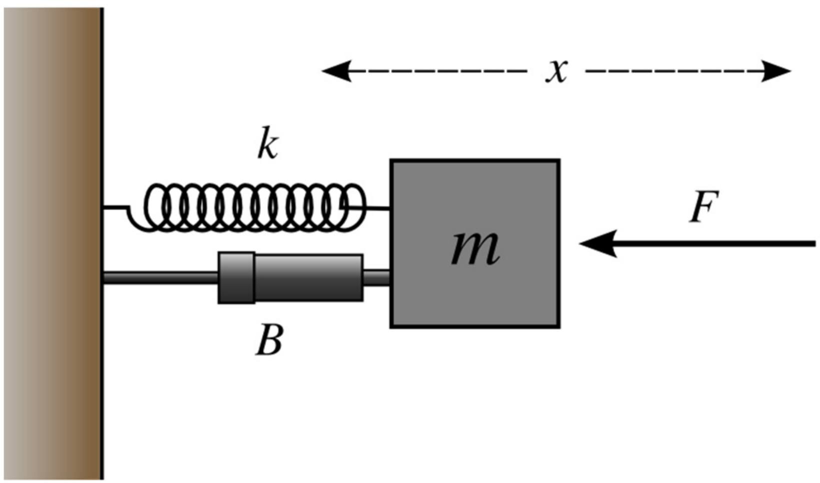
11 **Modeling: Free and Forced Oscillations**

Spring-mass-damper System

- **spring-mass-damper system**

$$\boxed{mx'' + cx' + kx = F(t),} \qquad (11.1)$$

where $c \geq 0$ is the **damping coefficient**, and $k \geq 0$ is the **spring stiffness**.



Free Oscillations

- $F(t) = 0.$ the motion equation is simplified as

$$\boxed{mx'' + cx' + kx = 0.} \tag{11.2}$$

- The characteristic equation

$$\boxed{m\lambda^2 + c\lambda + k = 0,} \tag{11.3}$$

whose roots are

$$\boxed{\lambda_1, \lambda_2 = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m}.} \tag{11.4}$$

- **Case 1**: **Undamped harmonic oscillation** $c = 0.$

$$\boxed{mx'' + kx = 0.} \qquad (11.5)$$

The two roots are $\lambda_1, \lambda_2 = \pm i\omega$

$\omega = \sqrt{\frac{k}{m}}$ is called the **natural frequency** of the system.

The general solution of system (11.5) is

$$\boxed{x(t) = A\cos\omega t + B\sin\omega t.} \qquad (11.6)$$

The **frequency** of the oscillation in Hertz and the **period** are

$$f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \quad \text{and} \quad T = \frac{1}{f},$$

- The general solution in another form

$$\boxed{x(t) = C\sin(\omega t + \phi),} \qquad (11.7)$$

C and ϕ are called the **amplitude** and the **phase angle**,

$$C = \sqrt{A^2 + B^2} \quad \text{and} \quad \phi = \tan^{-1}\frac{A}{B}.$$

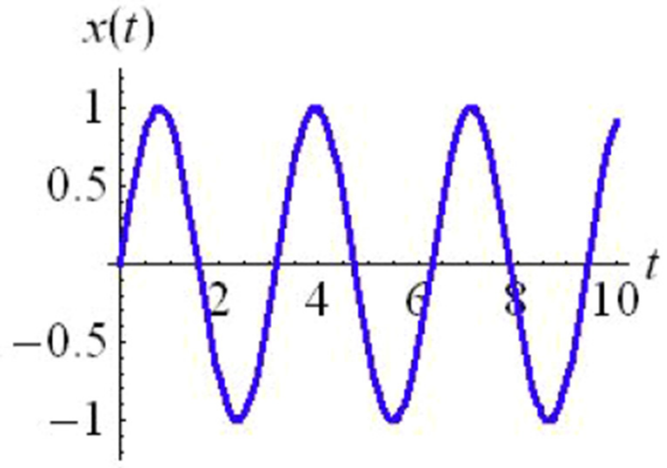


Fig. 11.2. an undamped harmonic oscillation:
$x'' + 4x = 0, x(0) = 0, x'(0) = 2.$

- **Case 2: Nonzero damping case** when $c > 0$.

Let $c_{critical} = \sqrt{4mk}$ **critical damping**.

⇨ consider three subcases depending on whether

$$c < c_{critical}\,, \; c = c_{critical} \text{ or } c > c_{critical}\,.$$

$$\boxed{\lambda_1, \lambda_2 = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m}.}$$

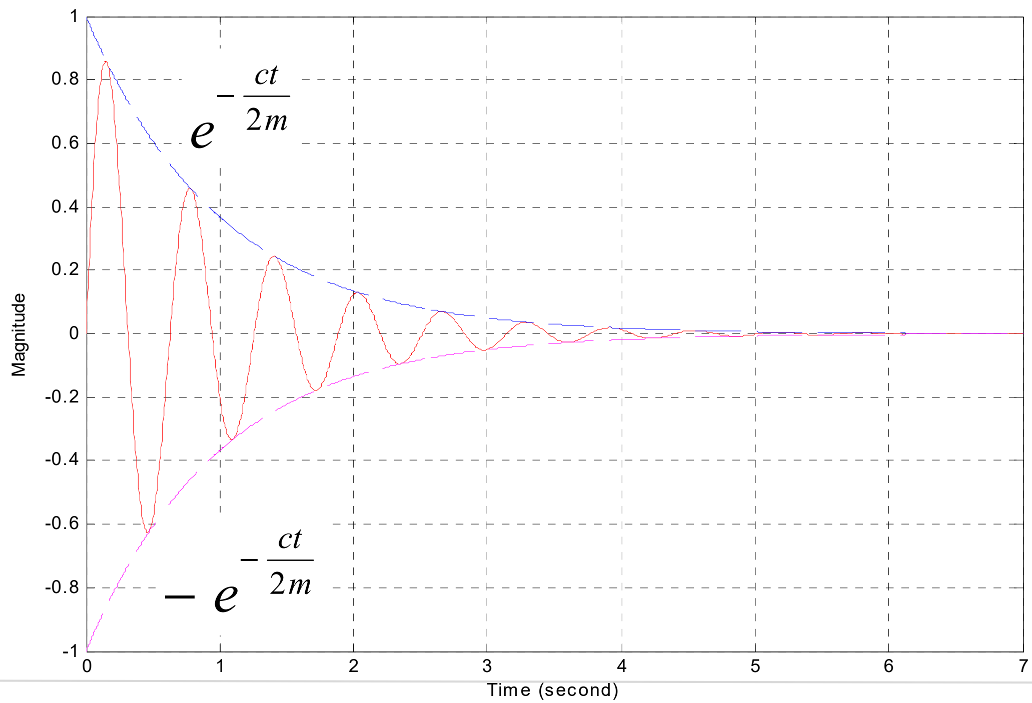
(a) When $c < c_{critical}$, ⇨ **underdamped free oscillation**

The general solution is

$$x(t) = (A\cos\eta t + B\sin\eta t)e^{-\frac{ct}{2m}}, \quad (11.8)$$

where $\eta = \sqrt{\omega^2 - \frac{c^2}{4m^2}}$. The asymptotic behavior of the general solution is

$x(t) \to 0$ as $t \to \infty$.



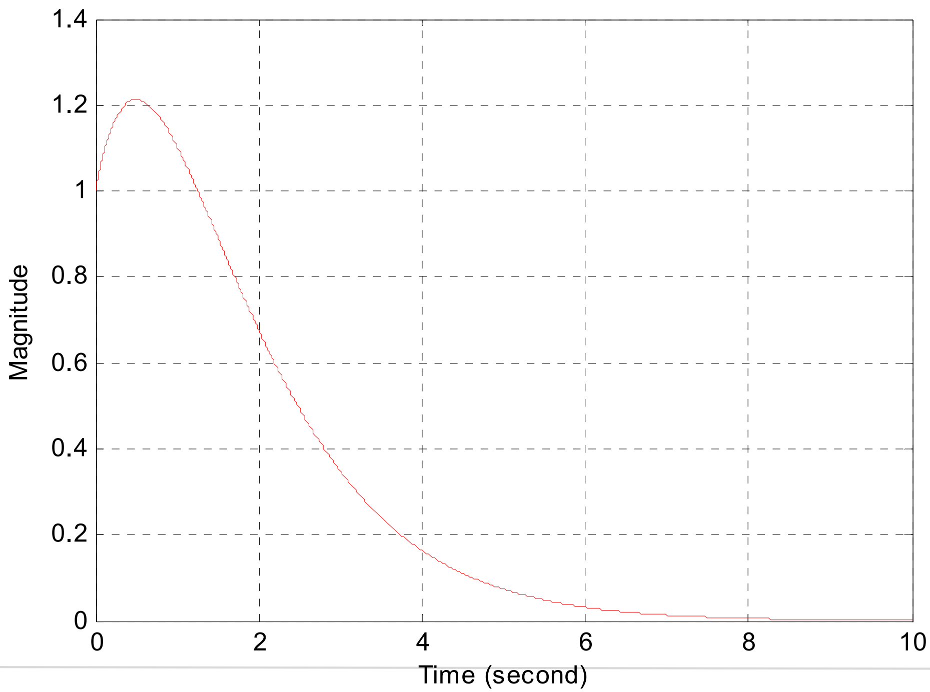
(b) When $c = c_{critical}$, ⇨ **critically damped free oscillation**

The general solution is

$$\boxed{x(t) = (A + Bt)e^{-\frac{ct}{2m}},} \qquad (11.9)$$

Note also that the asymptotic behavior of the general solution is

$$x(t) \to 0 \text{ as } t \to \infty.$$



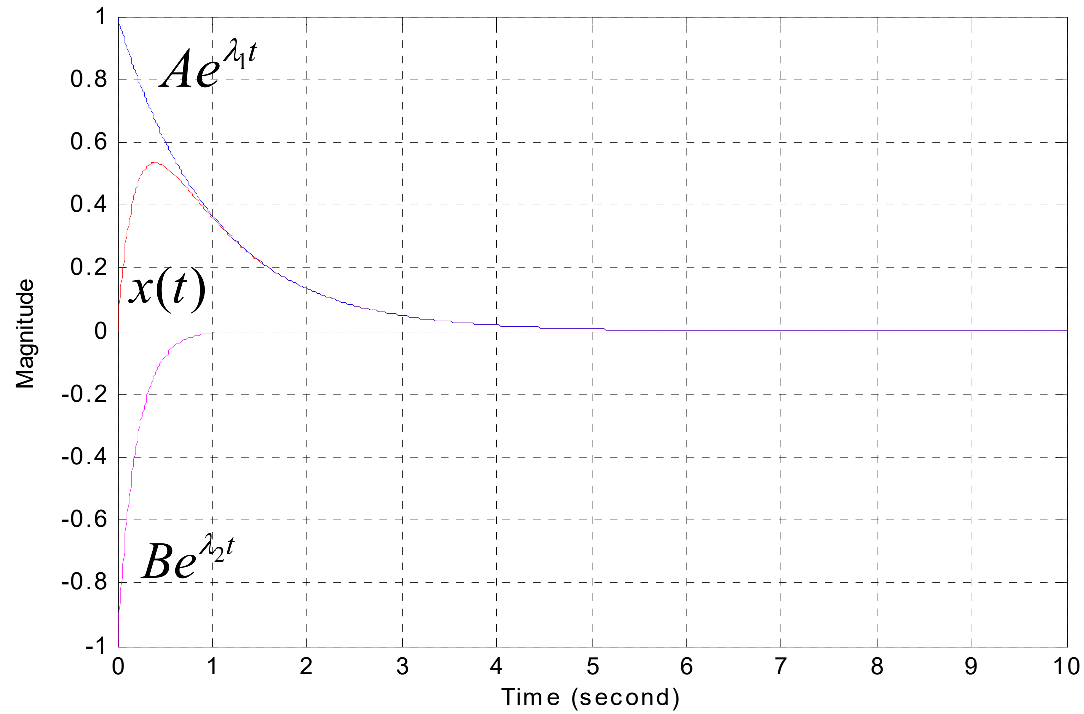
(c) When $c > c_{critical}$, ⇒ **overdamped free oscillation**

The general solution is

$$\boxed{x(t) = Ae^{\lambda_1 t} + Be^{\lambda_2 t},} \qquad (11.10)$$

where $\lambda_1, \lambda_2 = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m}$.

the asymptotic behavior of the general solution is $x(t) \to 0$ as $t \to \infty$.



Forced Oscillations

$$\boxed{mx'' + cx' + kx = F_0 \cos \Omega t.} \qquad (11.11)$$

- **Case 1:** When $c = 0$, i.e., there is no damping, equation (11.11) is reduced to

$$\boxed{mx'' + kx = F_0 \cos \Omega t.} \qquad (11.12)$$

This is a second order linear non-homogeneous equation with constant coefficients, which can be solved by the method of undetermined coefficients. The associated homogeneous equation is

$$mx'' + kx = 0,$$

and thus the homogeneous solution is given by

$$x_h = A \cos \omega t + B \sin \omega t,$$

where $\omega = \sqrt{\frac{k}{m}}$. Therefore, the general solution of equation (11.12) depends on whether $\Omega \neq \omega$ or $\Omega = \omega$.

(a) When $\Omega \neq \omega$, the general solution is

$$x(t) = A\cos\omega t + B\sin\omega t + \frac{F_0}{m(\omega^2 - \Omega^2)}\cos\Omega t.$$

Suppose we impose the initial conditions that $x(0) = x'(0) = 0$, then a phenomenon called **beat** occurs. The solution is

$$\boxed{x(t) = \frac{F_0}{m(\omega^2 - \Omega^2)}(\cos\Omega t - \cos\omega t).} \qquad (11.13)$$

This is the sum of two periodic functions of different periods with the same amplitude.

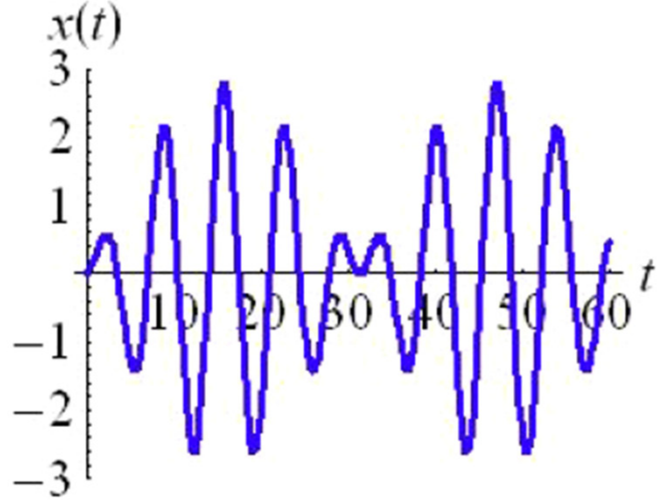


Fig. 11.6. an undamped forced oscillation (beat):
$x'' + x = 0.5\cos 0.8t, x(0) = x'(0) = 0.$

(b) When $\Omega = \omega$ and suppose we impose the same initial conditions that $x(0) = x'(0) = 0$, then a phenomenon called **resonance** occurs. The solution is

$$\boxed{x(t) = \frac{F_0}{2m\omega} t \sin \omega t.} \tag{11.14}$$

Note the oscillation becomes unbounded because

$|x(t)|$ is unbounded as $t \to \infty$.

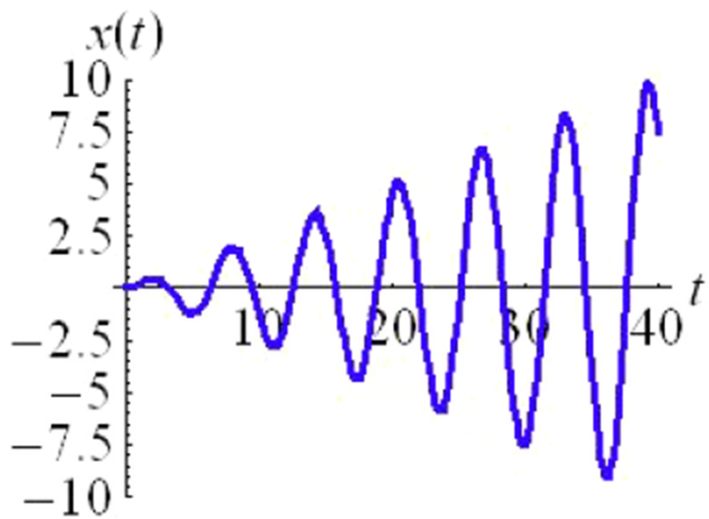


Fig. 11.7. an undamped forced oscillation (resonance):
$x'' + x = 0.5 \cos t, x(0) = x'(0) = 0.$

Homework Assignment No: 4

Due Date: 6:00pm, 14 November 2019

Please place your assignment to Assignment Box 3 outside PC Lab (ERB 218)

Summary of Ordinary Differential Equations…

General n-th order **Ordinary Differential Equation**:

$$F\left(x, y, y', \cdots, y^{(n)}\right) = 0$$

We look for all possible solutions $y(x)$ that satisfy this ODE.

Linear n-th order **O**rdinary **D**ifferential **E**quation:

$$a_n(x)y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_0(x)y = g(x)$$

It is a **homogenous** ODE if $g(x) = 0$. Otherwise, it is **non-homogenous**.

Solving the following 1st order ODE by **integrating factor method**:

$$y' + p(x)y = q(x) \quad \Rightarrow \quad y(x) = e^{-\int p(x)dx}\left[\int\left(e^{\int p(x)dx} q(x)\right)dx + C\right]$$

Summary of Ordinary Differential Equations (cont.)…

Solution to **separable 1st order ODE**:

$$y' = f(x)g(y) \quad \Rightarrow \quad \int \frac{dy}{g(y)} = \int f(x)dx + C$$

Solution to the 1st order ODE of **exact differential form**:

$$\frac{dy}{dx} = -\frac{M(x,y)}{N(x,y)} \quad \text{with} \quad \exists F(x,y) \text{ such that } \begin{cases} \dfrac{\partial F}{\partial x} = M \\ \dfrac{\partial F}{\partial y} = N \end{cases} \quad \Rightarrow \quad F(x,y) = 0$$

Summary of Ordinary Differential Equations (cont.)…

Vector space of all continuous functions defined on *I*…

$$V = \left\{ y(x) \,\middle|\, y : I \to \mathbb{R},\ y(x) \text{ is continous} \right\}$$

Given $y_1(x), y_2(x), \cdots, y_n(x) \in V$, they are said to be **linearly dependent on** *I*, if there exist real scalars α_j, not all zero, such that

$$\alpha_1 y_1(x) + \alpha_2(x)\ y_2(x) + \cdots + \alpha_n\ y_n(x) = 0 \text{ for any } x \in I$$

Otherwise, they are said to be **linearly independent** on *I*.

We are looking for a set of linearly independent solutions for linear ODE, i.e., a set of solutions with their **Wronskian determinant** being not identically zero

$$W(y_1, y_2, \cdots, y_n)(x) = \begin{vmatrix} y_1(x) & y_2(x) & \cdots & y_n(x) \\ y_1'(x) & y_2'(x) & \cdots & y_n'(x) \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)}(x) & y_2^{(n-1)}(x) & \cdots & y_n^{(n-1)}(x) \end{vmatrix} \neq 0 \quad \text{for some } x$$

The general solution to the *n*-th order linear ODE,

$$a_n(x) y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0(x) y = 0$$

can then be characterized by the linear combination of n linearly independent solutions

$$y(x) = C_1 y_1(x) + C_2 \; y_2(x) + \cdots + C_n \; y_n(x)$$

Summary of Ordinary Differential Equations (cont.)...

General solution to the 2nd order linear homogenous ODE with constant coefficients,

$$y''(x) + a_1 y'(x) + a_2 y(x) = 0$$

is depended on the roots of its **characteristic polynomial equation**:

$$\lambda^2 + a_1 \lambda + a_2 = 0$$

Case 1: distinct real roots where $\lambda_1 \neq \lambda_2$:

$$\boxed{y(x) = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}.}$$

Case 2: repeated roots where $\lambda_1 = \lambda_2$:

$$\boxed{y(x) = C_1 e^{\lambda_1 x} + C_2 x e^{\lambda_1 x}.}$$

Case 3: complex roots where $\lambda = \mu \pm i\omega$:

$$\boxed{y(x) = (C_1 \cos \omega x + C_2 \sin \omega x) e^{\mu x}.}$$

General solution to the n-th order **linear homogenous ODE with constant coefficients**,

$$y^{(n)}(x)+a_1 y^{(n-1)}(x)+\cdots+a_0 y(x)=0$$

is depended on the roots of its characteristic polynomial equation:

$$\lambda^n+a_1\lambda^{n-1}+\cdots+a_0=0$$

If $\lambda_1, \lambda_2, \cdots, \lambda_k$ are distinct roots, we have the following independent solutions to the ODE

$$e^{\lambda_1 x},\ e^{\lambda_2 x},\ \cdots,\ e^{\lambda_k x}$$

If λ_1 is a real root with multiplicity of k, the we have the following independent solutions

$$e^{\lambda_1 x},\ xe^{\lambda_1 x},\ \cdots,\ x^{k-1}e^{\lambda_1 x}$$

If $\lambda_1 = \mu + i\,\omega$ is a complex root with multiplicity of k, the we have the independent solutions

$$(\cos\omega x)e^{\mu x},\ (\cos\omega x)xe^{\mu x},\ \cdots,\ (\cos\omega x)x^{k-1}e^{\mu x}$$

$$(\sin\omega x)e^{\mu x},\ (\sin\omega x)xe^{\mu x},\ \cdots,\ (\sin\omega x)x^{k-1}e^{\mu x}$$

Solution to the 2nd order **linear non-homogenous ODE** with constant coefficients,

$$y''(x) + a_2 y'(x) + a_2 y(x) = f(x)$$

is given by

$$y(x) = y_h(x) + y_p(x)$$

with $y_h(x)$ being the general solution to its corresponding homogenous ODE

$$y''(x) + a_2 y'(x) + a_2 y(x) = 0$$

and $y_p(x)$ being any **particular solution** to the ODE, i.e.,

$$y_p''(x) + a_2 y_p'(x) + a_2 y_p(x) = f(x)$$

Types of $f(x)$	Choices for y_p
$e^{\mu x}$	$y_p = Ce^{\mu x}$
$x^k (k = 0, 1, \ldots)$	$y_p = C_k x^k + C_{k-1} x^{k-1} + \cdots + C_0$
$\cos \omega x$, $\sin \omega x$	$y_p = C_1 \cos \omega x + C_2 \sin \omega x$

If neither μ, nor 0, nor $i\omega$, is a root of the characteristic polynomial of the given ODE.

Solution to the 2nd order **linear non-homogenous ODE** with constant coefficients,

$$y''(x) + a_2 y'(x) + a_2 y(x) = f(x)$$

is given by

$$y(x) = y_h(x) + y_p(x)$$

with $y_h(x)$ being the general solution to its corresponding homogenous ODE

$$y''(x) + a_2 y'(x) + a_2 y(x) = 0$$

and $y_p(x)$ being any **particular solution** to the ODE, i.e.,

$$y_p''(x) + a_2 y_p'(x) + a_2 y_p(x) = f(x)$$

Types of $f(x)$	Choices for y_p
$e^{\mu x}$	$y_p = Cx^m e^{\mu x}$
$x^k (k = 0, 1, \ldots)$	$y_p = x^m (C_k x^k + C_{k-1} x^{k-1} + \cdots + C_0)$
$\cos \omega x$, $\sin \omega x$	$y_p = x^m (C_1 \cos \omega x + C_2 \sin \omega x)$

If μ, or 0, or $i\omega$, is a root of the characteristic polynomial with a multiplicity of m.

Summary of Ordinary Differential Equations (cont.)...

General solution to **Euler-Cauchy Equation**:

$$x^2 y'' + axy' + by = 0$$

can be done by solving the following quadratic equation

$$m^2 + (a-1)m + b = 0$$

If it has two distinct real roots,

$$y = c_1 x^{m_1} + c_2 x^{m_2}$$

two repeated roots,

$$y = \left(c_1 + c_2 \ln x\right) x^{(1-a)/2}$$

Two complex conjugated roots, $m_1 = \mu + i\nu$ and $m_2 = \mu - i\nu$,

$$y = x^{\mu}\left[c_1 \cos(\nu \ln x) + c_2 \sin(\nu \ln x)\right]$$