ENGG 2420

Complex Analysis & Differential Equations for Engineers

Part 1: Complex Analysis

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Course Outlines

1. **Complex Analysis**

   Analytic functions and Cauchy-Riemann equations; complex integration, Cauchy principal value; elementary complex valued functions; exponential functions, Euler's formula, trigonometric and hyperbolic functions, logarithm and general powers; power series, Taylor series and convergence tests.

2. **Differential Equations**

   Classification of differential equations; 1st order ordinary differential; 2nd order differential equations; partial differential equations.

textbook

Erwin Kreyszig

Advanced Engineering Mathematics

10th International Edn.

John Wiley & Sons

2011

General Announcements

1. Assessment Scheme
    - Homework Assignments, Quizzes, etc. 25%
    - Mid-term Exam (common) 25%
    - Final Exam (common) 50%
2. The mid-term exam would include questions on Complex Analysis and perhaps some part of ordinary differential equations (ODEs), e.g., 1st order ODE.
3. The mid-term exam would be arranged on 24 October 2019 (Thursday) during the tutorial session.
4. Both the mid-term and final exam are closed-book. One double-sided A4 handwritten cheat sheet and calculators are allowed.
5. Students are not allowed to switch classes.

General Announcements

6. Tutorial classes start in Week 2

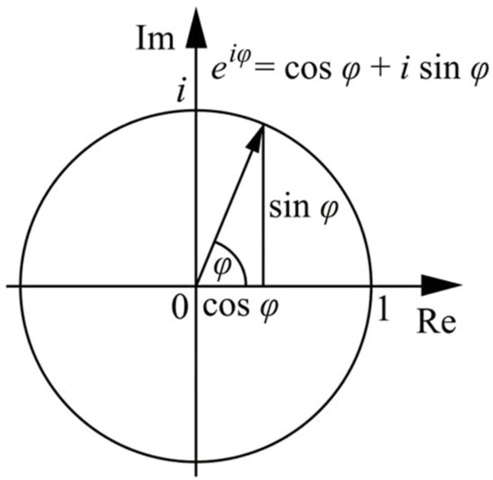
Tutorial classes will be conducted by the following tutors this semester…

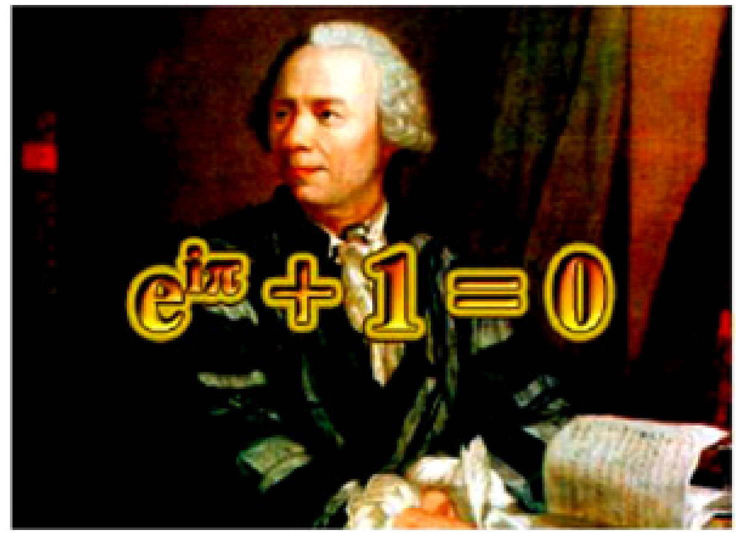
	Ext.	Tutorial Sessions in Charge
	33271	Weeks 2 & 3
	38056	Weeks 4 & 5
	34231	Weeks 6 & 7
	34229	Weeks 8 & 9
	34223	Weeks 10 & 11
	34237	Weeks 12 & 13
	34237	Coordination of exams & assignments…

Tutorial sessions will be held in YIA LT3, Thursday, 10:30–12:15pm…

Complex Analysis

Euler's formula





Leonhard Euler
(1707–1783)
Swiss Mathematician

$$e^{ix} = \cos x + i \sin x$$

Complex Analysis – 1…

1. Complex Numbers
2. Functions of One Complex Variable
3. Complex Differentiation
4. Complex Integration and Cauchy's Theorem
5. Cauchy Integral Formula
6. Complex Series, Power Series and Taylor Series

Complex Numbers

A complex number is defined (in the Cartesian form) as

$$z = x + iy$$

where

$$i = \sqrt{-1} \qquad \text{or} \qquad i^2 = -1$$

x is called the **Re**al part and y the **Im**aginary part of z, written

$$x = \text{Re } z, \qquad y = \text{Im } z$$

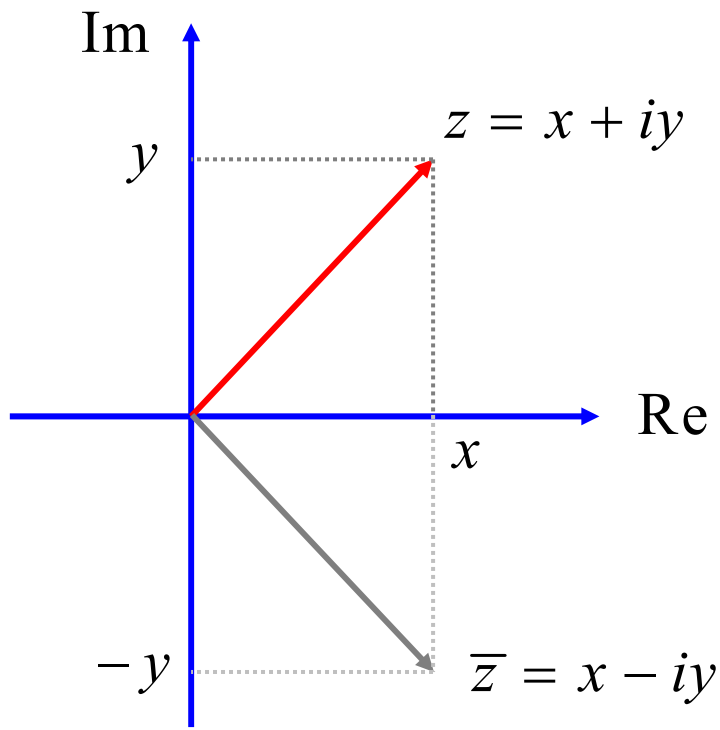
both being a real number. For example,

$$z = -4 + 2i, \quad z = -3 - 5i, \quad z = 5 - 8i$$

all are complex numbers. Occasionally, we might treat a complex number as an ordered pair (x, y) of real numbers x and y, written

$$z = (x, y)$$

Since a complex number has two parts, we can depict it on a 2D-plane, which is called a complex plane.



$\bar{z} = x - iy$ is called the conjugate of z

Additions: It is easy to do additions (subtractions) in Cartesian coordinate, i.e.,

$$(a + ib) + (v + iw) = (a + v) + i(b + w)$$

Multiplication:

$$z_1 z_2 = (x_1 + y_1 i)(x_2 + y_2 i) = x_1 x_2 + x_1 y_2 i + x_2 y_1 i + y_1 y_2 i^2$$
$$= (x_1 x_2 - y_1 y_2) + (x_1 y_2 + x_2 y_1) i.$$

Division: The **Quotient** $z = \frac{z_1}{z_2} (z_2 \neq 0)$

$$\frac{z_1}{z_2} = \frac{x_1 + y_1 i}{x_2 + y_2 i}$$
$$= \frac{x_1 + y_1 i}{x_2 + y_2 i} \frac{x_2 - y_2 i}{x_2 - y_2 i}$$
$$= \frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2} + \frac{x_2 y_1 - x_1 y_2}{x_2^2 + y_2^2} \cdot i$$

Example (1a)

- $$\frac{2+i}{3-4i} = \frac{2+i}{3-4i}\frac{3+4i}{3+4i} = \frac{2+11i}{25},$$

  which result can be checked by showing that $3-4i$ times $\frac{2+11i}{25}$ gives $2+i$.
- The real part and the imaginary part are $\frac{2}{25}$ and $\frac{11}{25}$, respectively.
- Also, the modulus of this complex number is

$$\left|\frac{2+i}{3-4i}\right| = \left|\frac{2+11i}{25}\right| = \frac{\sqrt{2^2+11^2}}{25} = \frac{1}{\sqrt{5}}.$$

- The **complex conjugate** of $z = x + yi$ is defined as

$$\boxed{\bar{z} = x - yi.} \tag{1.7}$$

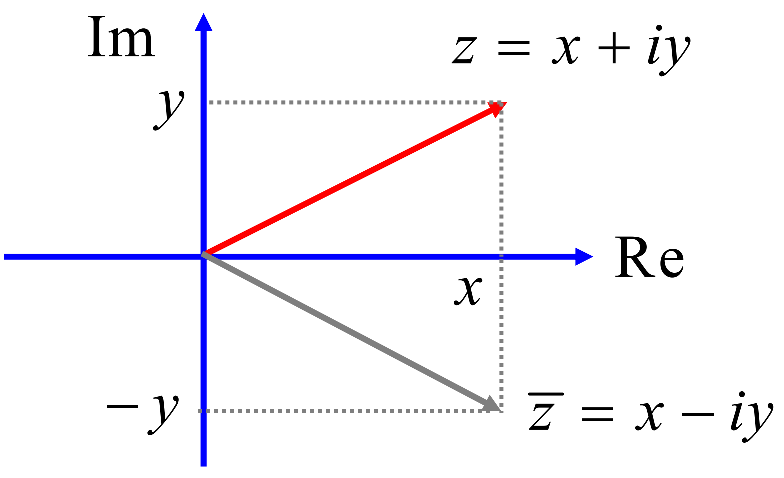
It can be shown that

$$\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2, \quad \overline{z_1 z_2} = \bar{z}_1 \bar{z}_2,$$

and

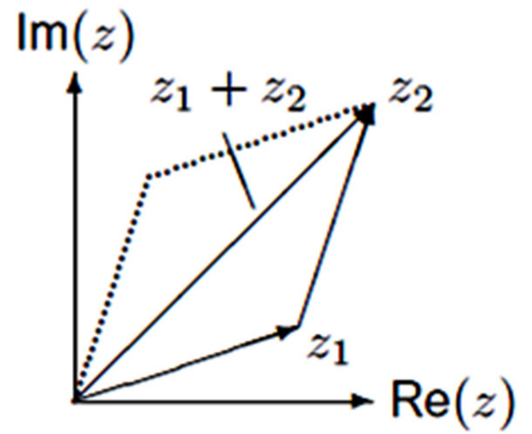
$$\boxed{|z|^2 = z\bar{z}.} \tag{1.8}$$

Geometrically, $\bar{z}$ is the reflection of z along the real axis.



- We have the **triangle inequality**

$$|z_1 + z_2| \le |z_1| + |z_2|,$$



and the **reverse triangle inequality**

$$|z_1 - z_2| \ge \big|\, |z_1| - |z_2| \,\big|$$

Proof… $|z_1| = |(z_1 - z_2) + z_2| \le |z_1 - z_2| + |z_2| \quad \Rightarrow \quad |z_1 - z_2| \ge |z_1| - |z_2|$

$|z_2| = |(z_2 - z_1) + z_1| \le |z_2 - z_1| + |z_1| \quad \Rightarrow \quad |z_1 - z_2| \ge |z_2| - |z_1|$

⇩

$$|z_1 - z_2| \ge \big|\, |z_1| - |z_2| \,\big| = \big|\, |z_2| - |z_1| \,\big|$$

Polar Representation of Complex Numbers

There is another way (polar representation) to represent a complex scalar

$$z = re^{i\theta}$$

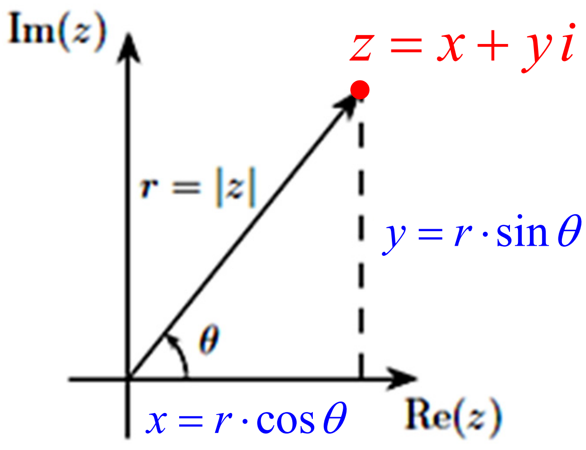
Using the Euler's formula

$$e^{i\theta} = \cos\theta + i\sin\theta$$

we obtain (Polar to Cartesian representations)

$$z = re^{i\theta} = r(\cos\theta + i\sin\theta) = (r\cos\theta) + (r\sin\theta)i = x + yi$$

Conversely,



$$r^2 = |z| = x^2 + y^2$$

$$\tan\theta = \frac{y}{x}$$

$$\Rightarrow \quad \theta = \tan^{-1}\frac{y}{x}$$

Cartesian to Polar Representation

$$x + yi = re^{i\theta}$$

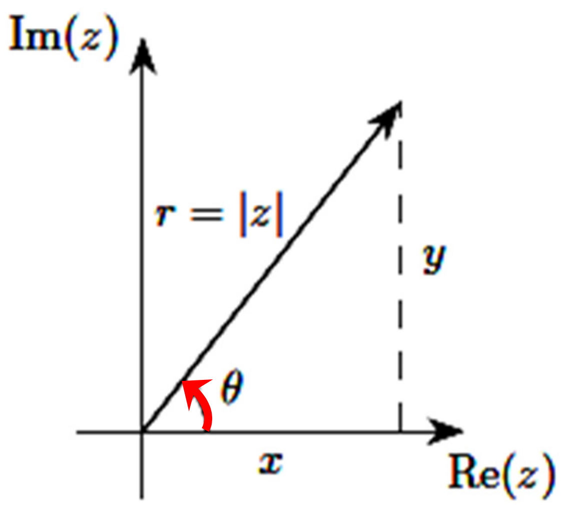
- Note that r is the **absolute value** or **modulus** of z, i.e.,

$$\boxed{r = |z| = \sqrt{x^2 + y^2}.} \tag{1.11}$$

The angle θ, called the **argument** of z, is denoted by $\theta = \arg(z)$, which can be determined from the formula

$$\boxed{\theta = \arg(z) = \tan^{-1}\left(\frac{y}{x}\right)} \tag{1.12}$$

for $z \neq 0$; for $z = 0$ the angle θ is undefined.



All angles (arguments) are measured in radians and **positive** in counter-clockwise sense.

- Given any point $z \neq 0$, the angle θ can be determined only to within an arbitrary integer multiple of 2π. Now, it is sometimes convenient to choose a particular value of θ. The value of θ satisfying $-\pi < \theta \leq \pi$ is called the **principal argument** of z, denoted by $\theta_0 = \text{Arg}(z)$. Then, we have

$$\theta = \arg(z) = \text{Arg}(z) + 2k\pi = \theta_0 + 2k\pi$$

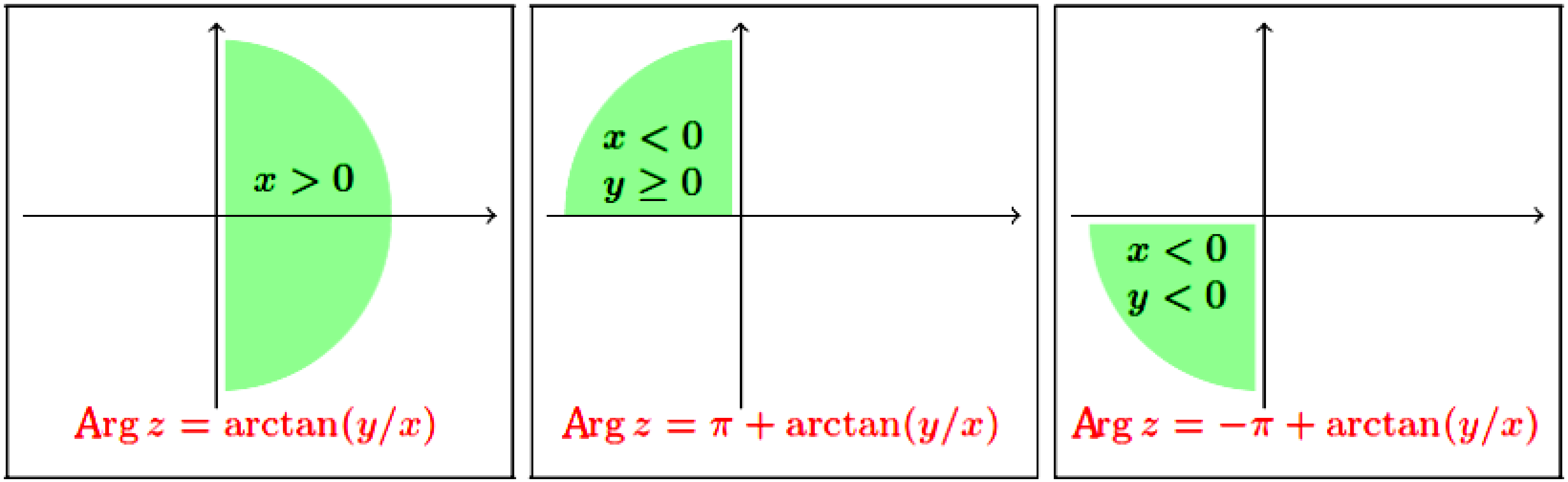
for $k = 0, \pm 1, \pm 2, \ldots$, as is evident in Fig. 1.2.

$$\boxed{\theta = \arg(z) = \tan^{-1}\left(\frac{y}{x}\right)}$$

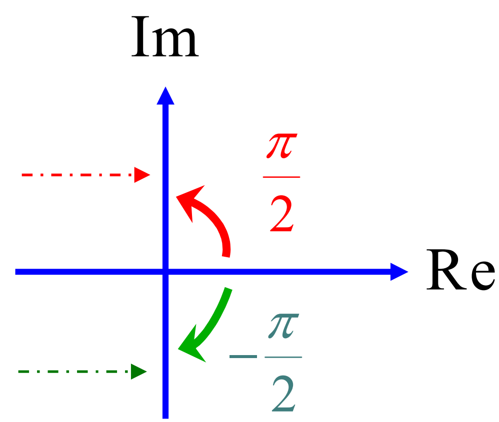
For example,

$$\theta = tan^{-1}\left(\frac{y}{x}\right) = tan^{-1}(1) \Rightarrow \theta = \frac{\pi}{4} + 2k\pi, \ldots$$

- Explicit expression of Arg(z): depends on the location of $z = x + iy$.



$$\text{and, otherwise, Arg } z = \begin{cases} \pi/2, & \text{if } x = 0 \text{ and } y > 0 \\ \text{undefined}, & \text{if } x = 0 \text{ and } y = 0 \\ -\pi/2, & \text{if } x = 0 \text{ and } y < 0 \end{cases}$$



- The polar form of complex numbers is especially convenient for their multiplication and division. For example, let $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$. Then

$$z = z_1 z_2 = (r_1 e^{i\theta_1})(r_2 e^{i\theta_2}) = r_1 r_2 e^{i(\theta_1+\theta_2)} = re^{i\theta}$$

or

$$\left| z_1 z_2 \right| = \left| r_1 r_2 [\cos(\theta_1 + \theta_2) + i(\sin(\theta_1 + \theta_2))] \right| = r_1 \cdot r_2$$

and

$$\frac{z_1}{z_2} = \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}.$$

or

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i(\sin(\theta_1 - \theta_2))].$$

- **Question:** $|z_1 z_2| = ?$ $\quad \arg(z_1 z_2) = ?$

Euler's formula $\boxed{e^{i\theta} = \cos\theta + i\sin\theta,} \Rightarrow \left|e^{i\theta}\right| = \left|\cos\theta + i\sin\theta\right| = \sqrt{\cos^2\theta + \sin^2\theta} = 1$

- In particular, the integer power of z can be computed easily by

$$z^n = \left(re^{i\theta}\right)^n = r^n e^{in\theta} = r^n\left(\cos n\theta + i\sin n\theta\right) \qquad (1.13)$$

where n is any integer. This is known as the **de Moivre's formula**.

- The de Moivre's formula (1.13) gives a way to compute the fractional power $z^{\frac{1}{n}}$. We call $z^{\frac{1}{n}}$ the **n-th root** of z, then

$$\boxed{z^{\frac{1}{n}} = r^{\frac{1}{n}} e^{\frac{i(\theta_0 + 2k\pi)}{n}}} \qquad (1.14)$$

for $k = 0, 1, \ldots, n-1$. Note that $z^{\frac{1}{n}}$ is a **multi-valued** function.

$\boxed{z = re^{i\theta}.}$ **Euler's formula** $\boxed{e^{i\theta} = \cos\theta + i\sin\theta,}$

Example (1b)

- evaluate the values of $(1+i)^{\frac{1}{3}}$.

$$r = \sqrt{1^2 + 1^2} = \sqrt{2}$$

$$\theta_\circ = \tan^{-1} 1 = \frac{\pi}{4}.$$

principal argument

$$\Rightarrow \quad 1 + i = \sqrt{2} e^{\frac{i\pi}{4}}$$

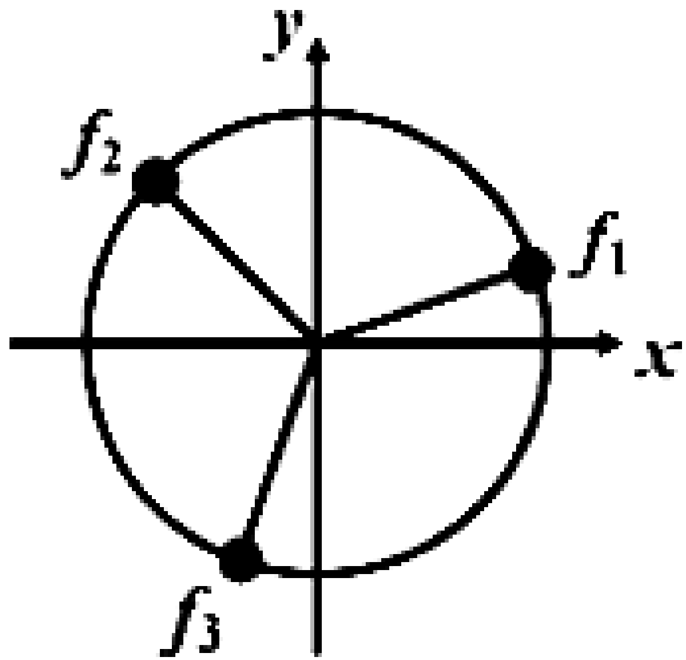
$$z^{\frac{1}{n}} = r^{\frac{1}{n}} e^{\frac{i(\theta_0 + 2k\pi)}{n}}$$

$$\Rightarrow (1+i)^{\frac{1}{3}} = 2^{\frac{1}{6}} e^{i\left(\frac{\pi}{12} + \frac{2k\pi}{3}\right)}, \quad k = 0, 1, -1$$

$$= 2^{\frac{1}{6}} e^{\frac{\pi i}{12}}, 2^{\frac{1}{6}} e^{\frac{3\pi i}{4}} \text{ and } 2^{\frac{1}{6}} e^{-\frac{7\pi i}{12}}$$

Geometrical description of these three roots

$$f_1 = 2^{\frac{1}{6}} e^{\frac{\pi i}{12}}, \; f_2 = 2^{\frac{1}{6}} e^{\frac{3\pi i}{4}} \text{ and } f_3 = 2^{\frac{1}{6}} e^{-\frac{7\pi i}{12}}$$



Complex Analysis – 2…

1 Complex Numbers

2 **Functions of One Complex Variable**

3 Complex Differentiation

4 Complex Integration and Cauchy's Theorem

5 Cauchy Integral Formula

6 Complex Series, Power Series and Taylor Series

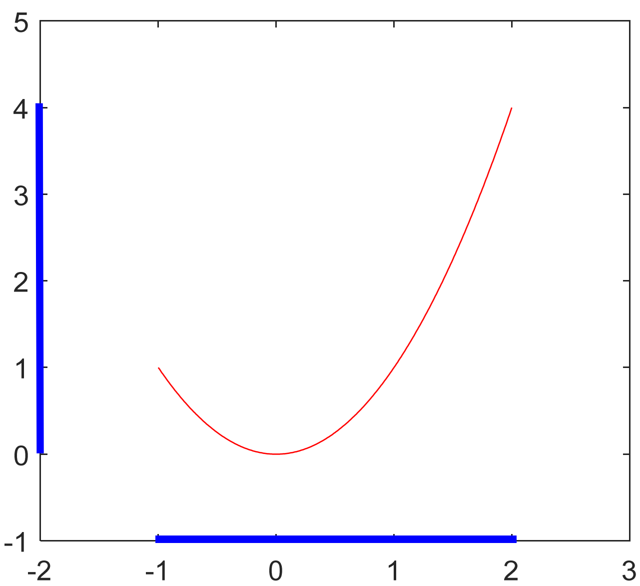
Revisit: Real functions of a real variable

$$y = f(x)$$

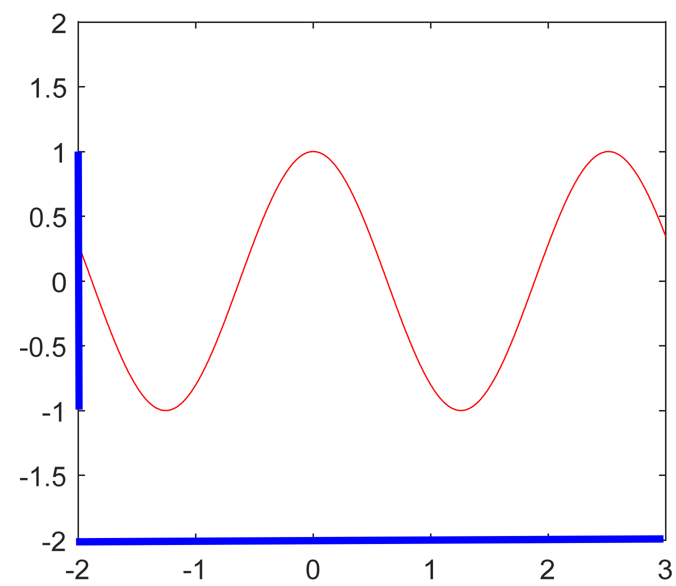
which is a mapping from a set of real scalar x to another set of real scalar y.

Examples:

$$y = x^2, \quad -1 \le x \le 2$$



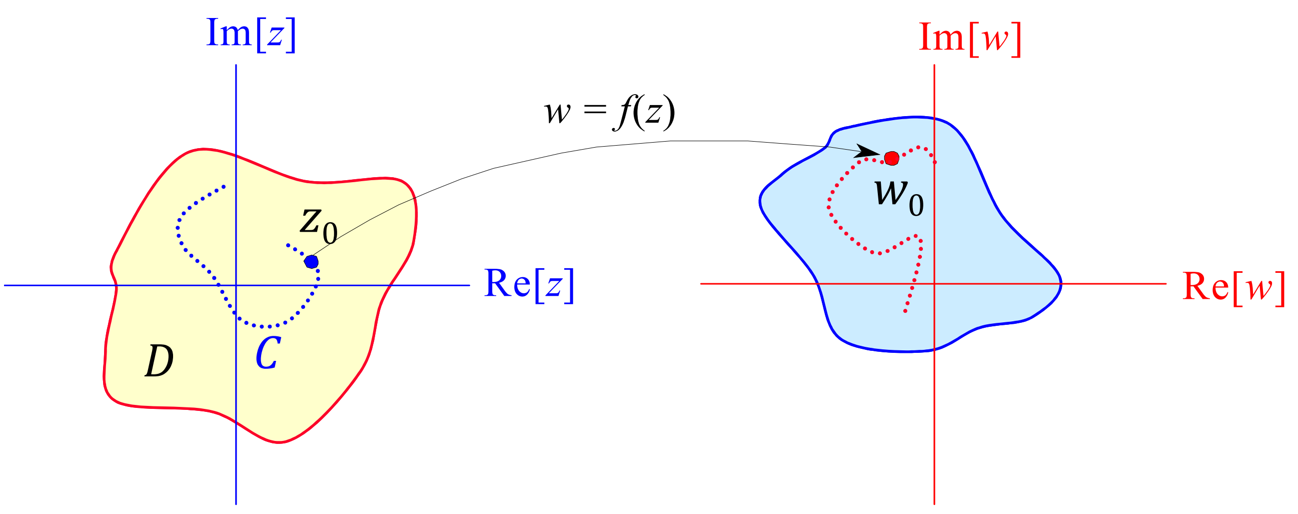
$$y = \cos 2.5x, \quad -2 \le x \le 3$$



Basics: Complex functions of a complex variable…

- Functions of one complex variable z are mapping from a complex plane to a complex plane. For convenience we label the former as the z-**plane** and the later as the w-**plane**.
- Functions of one complex variable are usually denoted by

$$\boxed{w = f(z) = f(x + yi).} \qquad (2.1)$$



The complex function

$$\boxed{w = f(z) = f(x + yi)} \quad (2.1)$$

can be expressed as follows…

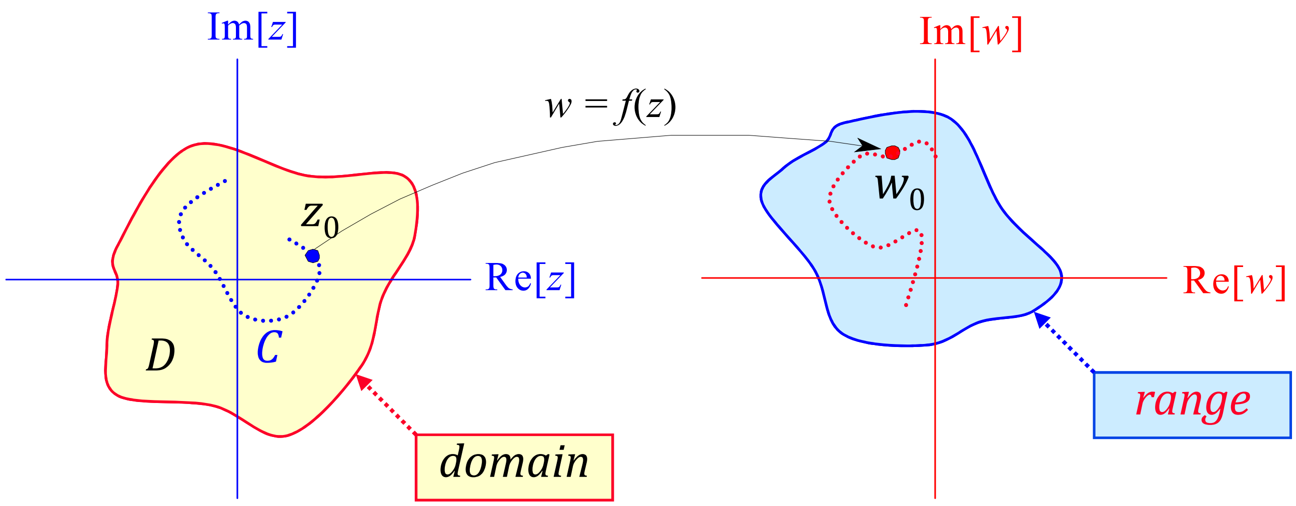
$$\boxed{f(z) = u(x, y) + i\, v(x, y),} \quad (2.2)$$

a function of one complex variable z can be regarded as a function of two real variables x and y.

$u(x, y)$ **real part** of f

$v(x, y)$ **imaginary part** of f.

- As usual, we define the set on which f is defined as the **domain** of f and the set of all values of $f(z)$ (z is in the domain of f) as the **range** of f.



Example (2a)

- Consider the complex function $w = f(z) = z^2$ defined on the first quadrant of the z-plane: $0 < x < \infty, 0 < y < \infty$. Then

$$f(z) = f(x + yi) = (x + yi)^2 = (x^2 - y^2) + 2xyi.$$

Thus, $u(x, y) = x^2 - y^2$ and $v(x, y) = 2xy$.

- Now, $-\infty < u(x, y) < \infty$ and $0 < v(x, y) < \infty$. Therefore, the range of f is the entire upper half plane, i.e., $\text{Im}(z) > 0$.

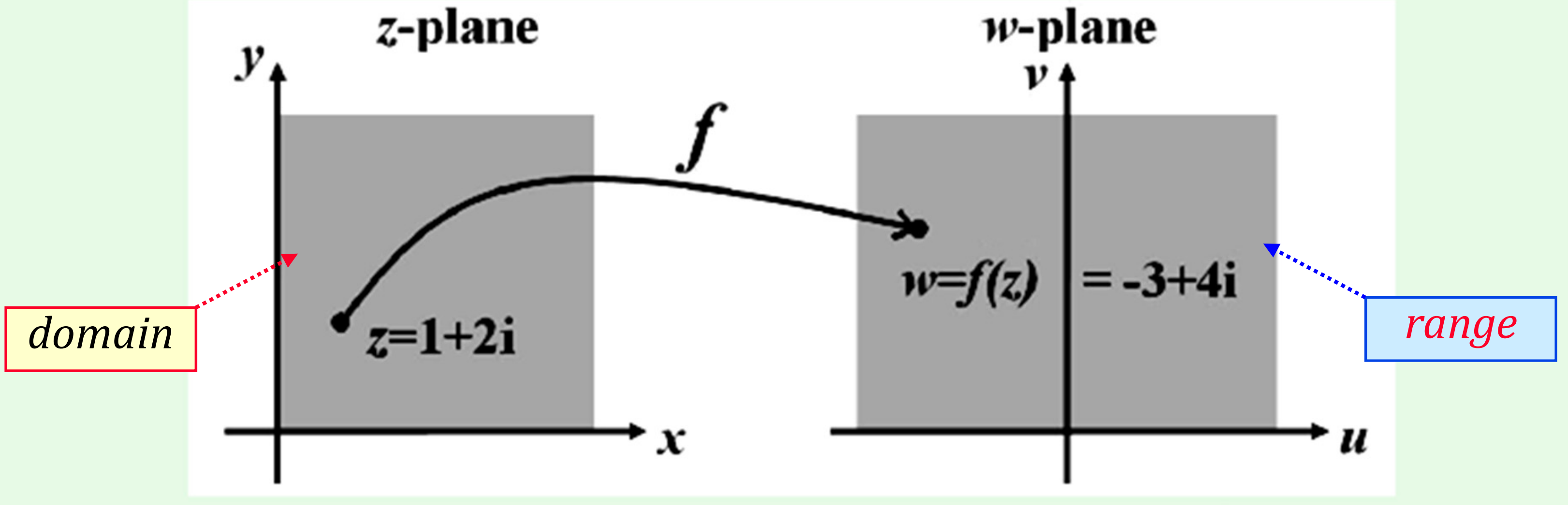
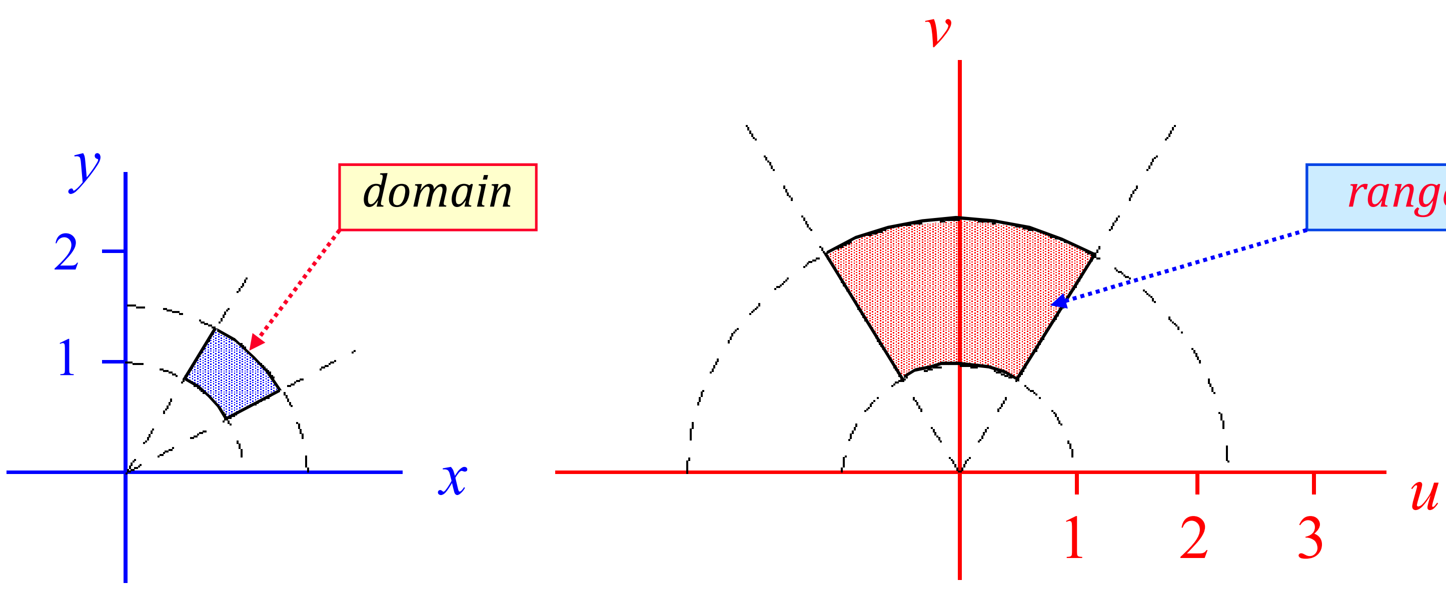


Fig. 2.1. Mapping defined by $w = f(z) = z^2$.

Example: z^2 (cont.)

In polar coordinate: $w = f(z) = R\,e^{i\theta} = z^2 = (r\,e^{i\phi})^2 = r^2\,e^{i2\phi}$

For example, the set of the region $1 \le r \le 3/2,\ \pi/6 \le \phi \le \pi/3$ under the mapping $w = z^2$ is $1 \le R \le 9/4,\ \pi/3 \le \theta \le 2\pi/3$



elementary functions.

The first function is **exponential** of z

$$e^z$$

Exponential Functions

- $$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots . \qquad (2.3)$$

Writing $z = x + yi$ and using Euler's formula

$$\Rightarrow \quad e^z = e^x e^{iy}$$

$$= e^x(\cos y + i \sin y).$$

Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta,$

- Note that

$$|e^{iy}| = |\cos y + i \sin y| = \sqrt{\cos^2 y + \sin^2 y} = 1 \quad \text{for all } y$$

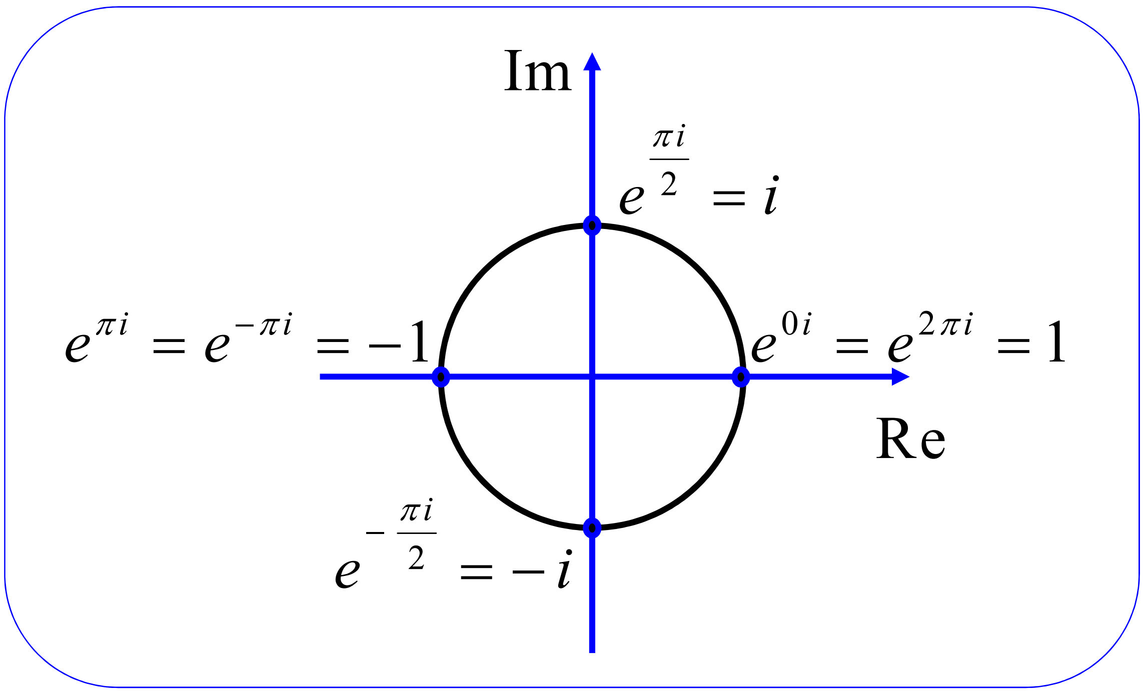
⇨ $$|e^z| = |e^x(\cos y + i \sin y)| = |e^x||\cos y + i \sin y| = |e^x| \quad \text{for all } z.$$

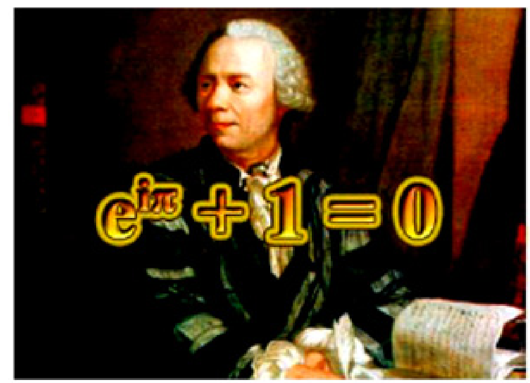
Therefore, $e^z \neq 0$ in the entire z-plane (entire: see section 3).

- From the **Euler's formula** $e^{i\theta} = \cos\theta + i\sin\theta,$

$$\Rightarrow \quad e^{2\pi i} = 1$$

$$e^{\frac{\pi i}{2}} = i, \quad e^{\pi i} = -1, \quad e^{\frac{-\pi i}{2}} = -i, \quad e^{-\pi i} = -1.$$





- Periodicity of e^x with period $2\pi i$,

$$e^{z+2\pi i} = e^z \, e^{2\pi i} = e^z \cdot 1 = e^z$$

Example: e^z (cont.)

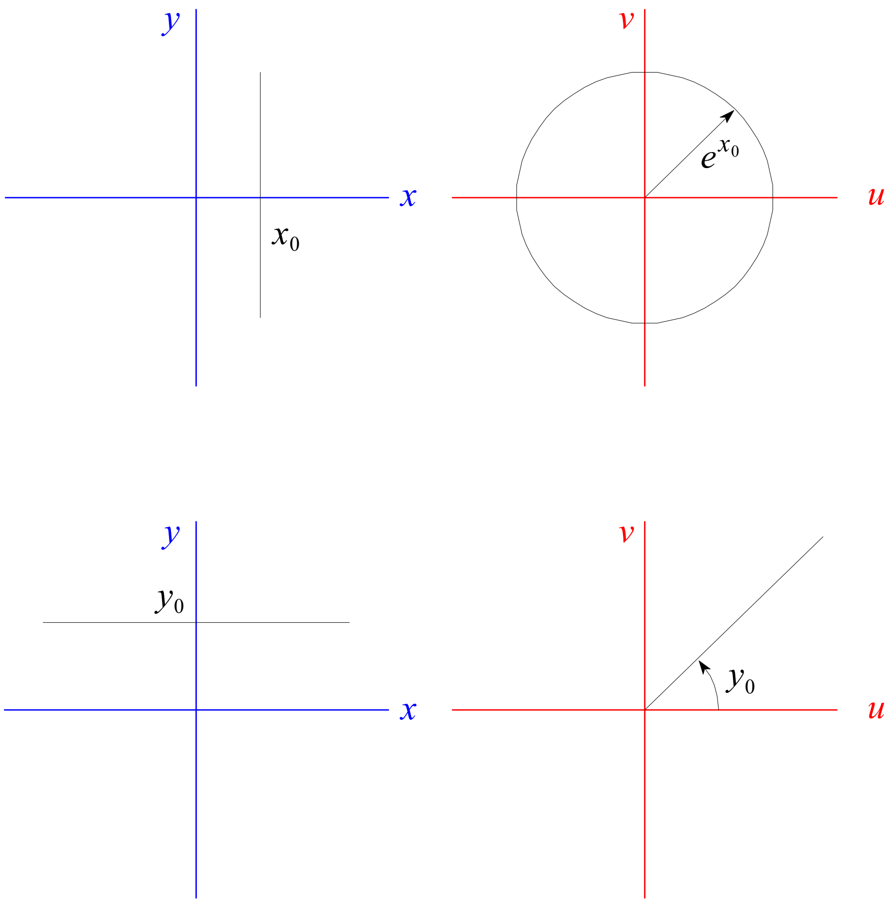
$$w = e^z = e^{x+iy} = e^x e^{iy} = Re^{i\theta}$$

For $w = e^z$, consider the images of:

1. Straight lines $x = x_0 = \text{const}$ and $y = y_0 = \text{const}$

   From $R = e^x$, $\theta = y$, we see that $x = x_0$ is mapped onto the circle $|w| = e^{x_0}$ and $y = y_0$ is mapped onto the ray $\arg(w) = y_0$

$$R = e^x, \quad \theta = y$$



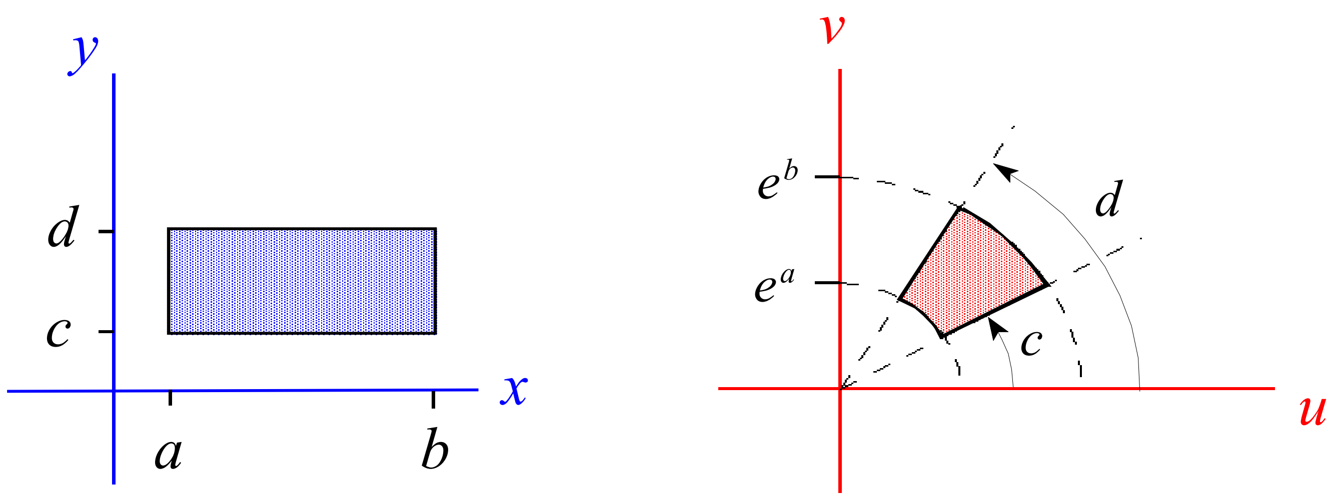
Example: e^z (cont.)

$$R = e^x, \quad \theta = y$$

2. Rectangle $D = \{ z = x + iy \mid a \le x \le b,\ c \le y \le d \}$:

   From 1, we can conclude that any rectangle with side parallel to the coordinate axes is mapped onto a region bounded by portions of rays and circles. Therefore the range of D is

$$D' = \{ w = Re^{i\theta} \mid e^a \le R \le e^b, c \le \theta \le d \}$$



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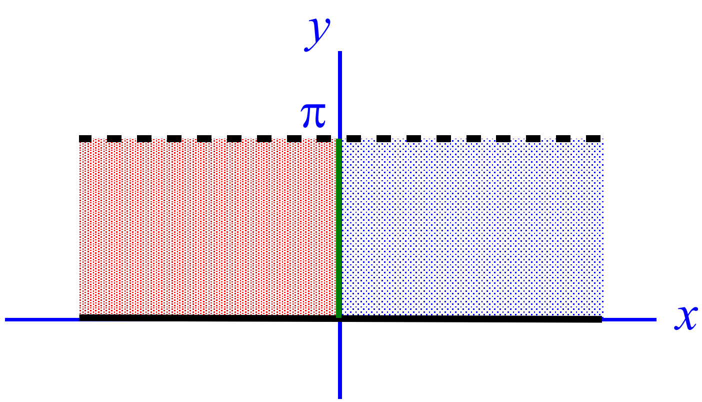
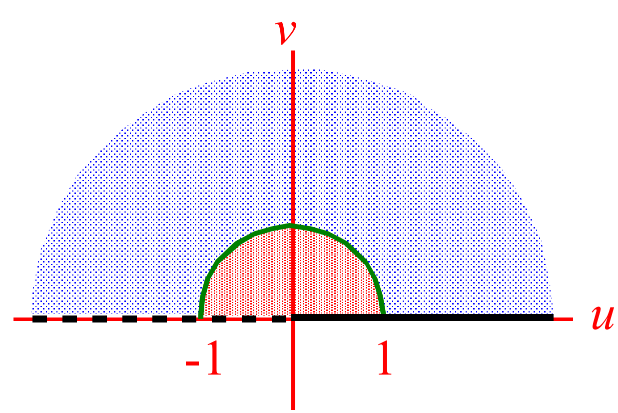
Example: e^z (cont.)

3. The fundamental region $-\pi \le y \le \pi$:

The fundamental region is mapped onto the entire *w*-plane, excluding the origin. The strip $0 \le y \le \pi$ is mapped onto the upper half-plane





More generally, every horizontal strip $c \le y \le c + 2\pi$ is mapped onto the full *w*-plane excluding the origin.

$$R = e^x, \quad \theta = y$$

Trigonometric and Hyperbolic Functions

- Changing θ to y and $-y$ in Euler's formula (1.9), then we get

$$e^{iy} = \cos y + i \sin y, \quad e^{-iy} = \cos y - i \sin y. \tag{2.5}$$

- **Note**: Euler's formula is valid in complex.

  Solving (2.5) for $\cos y$ and $\sin y$, we obtain

$$\cos y = \frac{e^{iy} + e^{-iy}}{2}, \quad \sin y = \frac{e^{iy} - e^{-iy}}{2i}.$$

$$\boxed{\cos z = \frac{e^{iz} + e^{-iz}}{2}, \quad \sin z = \frac{e^{iz} - e^{-iz}}{2i}.}$$

- As in calculus, we define

$$\tan z = \frac{\sin z}{\cos z}, \quad \cot z = \frac{\cos z}{\sin z}.$$

$$\sec z = \frac{1}{\cos z}, \quad \csc z = \frac{1}{\sin z}.$$

- define the **hyperbolic cosine** and **hyperbolic sine functions**

$$\cosh z = \frac{e^z + e^{-z}}{2},$$
$$\sinh z = \frac{e^z - e^{-z}}{2}. \qquad (2.7)$$

$$\cos z = \frac{e^{iz} + e^{-iz}}{2},$$
$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}.$$

Their derivatives are

$$(\cosh z)' = \sinh z, \quad (\sinh z)' = \cosh z.$$

- The other hyperbolic functions are defined

$$\tanh z = \frac{\sinh z}{\cosh z}, \quad \coth z = \frac{\cosh z}{\sinh z}.$$

and

$$\operatorname{sech} z = \frac{1}{\cosh z}, \quad \operatorname{csch} z = \frac{1}{\sinh z}.$$

- **Complex trigonometric and hyperbolic functions are related:**

  the connections between trigonometric and hyperbolic functions are

$$\cos(iz) = \cosh z, \quad \sin(iz) = i \sinh z,$$
$$\cosh(iz) = \cos z, \quad \sinh(iz) = i \sin z.$$

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}, \quad \sin z = \frac{e^{iz} - e^{-iz}}{2i}.$$

$$\cosh z = \frac{e^{z} + e^{-z}}{2}, \quad \sinh z = \frac{e^{z} - e^{-z}}{2}.$$

- Based on definitions of e^z, $\cos z$, $\sin z$, $\cosh z$ and $\sinh z$ most familiar formulas for real exponentials, trigonometric and hyperbolic functions still apply.

Example

$$
\begin{aligned}
\sin^2 z + \cos^2 z &= -\frac{1}{4}(e^{iz} - e^{-iz})^2 + \frac{1}{4}(e^{iz} + e^{-iz})^2 \\
&= \frac{1}{4}(\underline{-e^{2iz}} + 2 - \underline{e^{-2iz}} + \underline{e^{2iz}} + 2 + \underline{e^{-2iz}}) \\
&= 1.
\end{aligned}
$$

Logarithmic Functions

- define **logarithmic function** $\ln z$ (sometimes also by $\log z$).

  for $z \neq 0$, express z in polar form and write

$$\begin{aligned}\ln z &= \ln(re^{i\theta}) = \ln r + \ln(e^{i\theta}) \\ &= \ln r + i\theta, \qquad (r = |z| > 0, \theta = \arg z)\end{aligned}$$

or

$$\boxed{\ln z = \ln r + i(\theta_0 + 2k\pi)}$$

for $\theta = \arg z = \theta_0 + 2k\pi, -\pi < \theta_0 \leq \pi, k = 0, \pm 1, \pm 2, \ldots$.

- Since the argument of z is determined only up to integer multiples of 2π, the complex natural logarithm $\ln z (z \neq 0)$ is **infinitely many-valued.**

$$\theta = \arg z = \theta_0 + 2k\pi, -\pi < \theta_0 \leq \pi, k = 0, \pm 1, \pm 2, \ldots.$$

$$\Rightarrow \quad \ln z = \ln r + i\theta,$$

Example $\quad \ln 1 = 0 + 2k\pi i = 0, \pm 2\pi i, \pm 4\pi i, \cdots\cdots$

- The value of $\ln z$ corresponding to the principal value Arg$z$ is denoted by Lnz (Ln with capital L) and is called the **Principal value** of $\ln z (z \neq 0)$.

$$\boxed{\text{Ln}z = \ln|z| + i\text{Arg}z} \tag{2.9}$$

- The uniqueness of Arg$z$ for given $z(\neq 0)$ implies that Lnz is single-valued.

- Since the other values of arg$z$ differ by integer multiples of 2π, the other values of ln$z$ are given by

$$\ln z = \text{Ln} z + 2k\pi i \qquad (k = \pm 1, \pm 2, \ldots),$$

- **Note**: All have the same real part, and imaginary parts differ by integer multiples of 2π.
- If z is positive real, then Arg$z = 0$, and Lnz becomes identical with the real natural logarithm ; If z is negative real, then Arg$z = \pi$ and

$$\text{Ln} z = \ln |z| + \pi i, \qquad \text{(z negative real)}$$

Examples

$$\text{Ln}(-1) = \ln|-1| + \pi i = \pi i$$

$$\text{Ln}(i) = \ln|i| + \frac{\pi}{2} i = \frac{\pi}{2} i$$

$$\text{Ln}(-i) = \ln|-i| - \frac{\pi}{2} i = -\frac{\pi}{2} i$$

Example (2b)

- Since $1+i=\sqrt{2}e^{\frac{i\pi}{4}}$,

$$\ln(1+i)=\frac{\ln 2}{2}+i\left(\frac{\pi}{4}+2k\pi\right), \quad k=0,\pm1,\pm2,\ldots.$$

$$\mathrm{Ln}(1+i)=\ln\sqrt{2}+\frac{\pi}{4}i$$

<u>Just for fun...</u> Compute

$$i^i=\left(e^{i\pi/2}\right)^i=e^{i^2\pi/2}=e^{-\pi/2}=0.20787957635076\cdots\cdots$$

$$\mathrm{Ln}\left(i^i\right)=\ln e^{-\pi/2}=-\frac{\pi}{2}$$

General Powers of z

- Suppose z and c are both complex numbers, we have

$$z^c = e^{\ln z^c} = e^{c \ln z}$$

- Since $\ln z$ is infinitely many-valued, z^c will be multivalued, and

$$z^c = e^{c\left[\ln r + i(\theta_0 + 2k\pi)\right]} \qquad \text{for } k = 0, \pm 1, \pm 2, \ldots.$$

c-**power** of z

The particular value $(k = 0)$

$$z^c = e^{c \operatorname{Ln} z}$$

is called the **principal value** of z^c.

- *c*-**power** of z as

$$z^c = e^{c\left[\ln r + i(\theta_0 + 2k\pi)\right]}.$$

In particular, when c is real, then

$$z^c = \left(re^{i\theta}\right)^c = r^c e^{i(\theta_0+2k\pi)c}. \qquad (2.11)$$

Example (2c)

- Since $\ln i = \ln 1 + i(\frac{\pi}{2} + 2k\pi) = i(\frac{\pi}{2} + 2k\pi)$,

$$i^i = e^{i \ln i} = e^{-(\frac{\pi}{2}+2k\pi)}, \qquad k = 0, \pm 1, \pm 2, \ldots.$$

$$\ln z = \ln r + i(\theta_0 + 2k\pi)$$

$$z^c = e^{c \ln z}$$

Complex Analysis – 3…

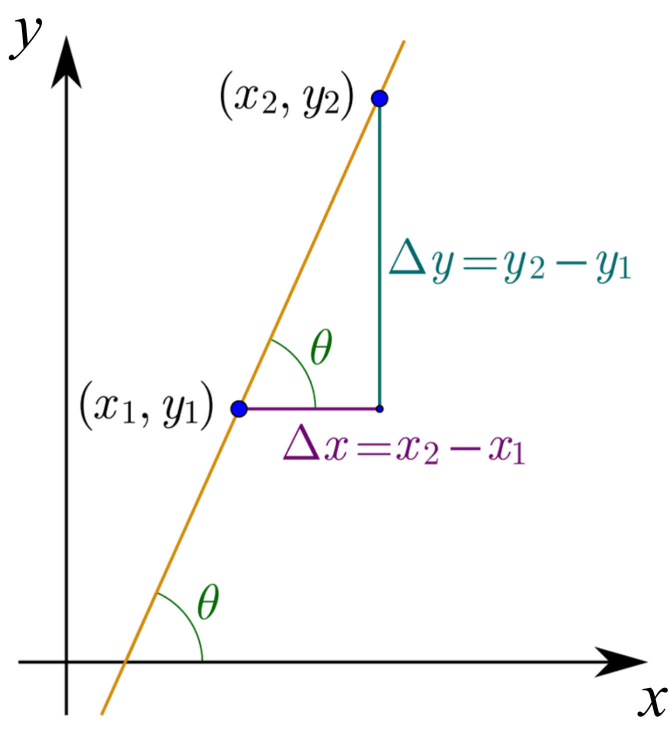
1 Complex Numbers
2 Functions of One Complex Variable
3 **Complex Differentiation**
4 Complex Integration and Cauchy's Theorem
5 Cauchy Integral Formula
6 Complex Series, Power Series and Taylor Series

Revisit: The derivative of a real function

$$y = f(x)$$

is a measure of the rate at which the value y of the function changes with respect to the change of x.

Example: Consider the function (a straight line) plotted in the figure on the right. The derivative of the function (or the rate of changes) of the function is its slope.



Note that the derivative cannot be defined on a single point. We need an interval of x, i.e., Δx, to define a derivative.

Similarly, we need a 2D region in a complex function domain to define a complex derivative as a complex function is actually a mapping from a 2D plane to another 2D plane.

Terminologies

- **neighborhood** of a point z_0 with radius r

$$N(z_0; r) = \{z \in \mathbb{C} : |z - z_0| < r\}.$$

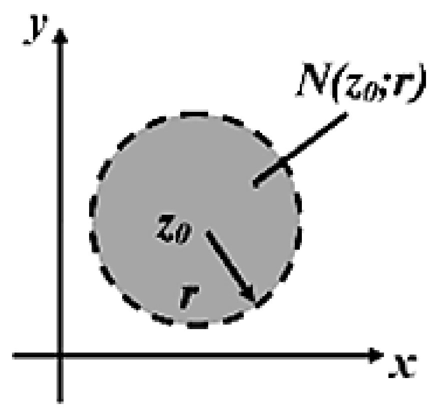
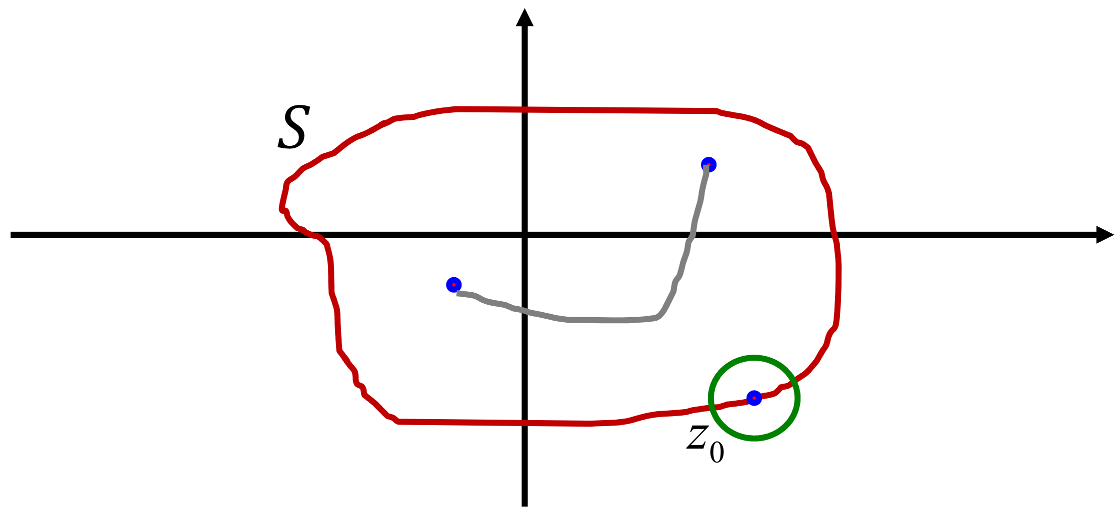


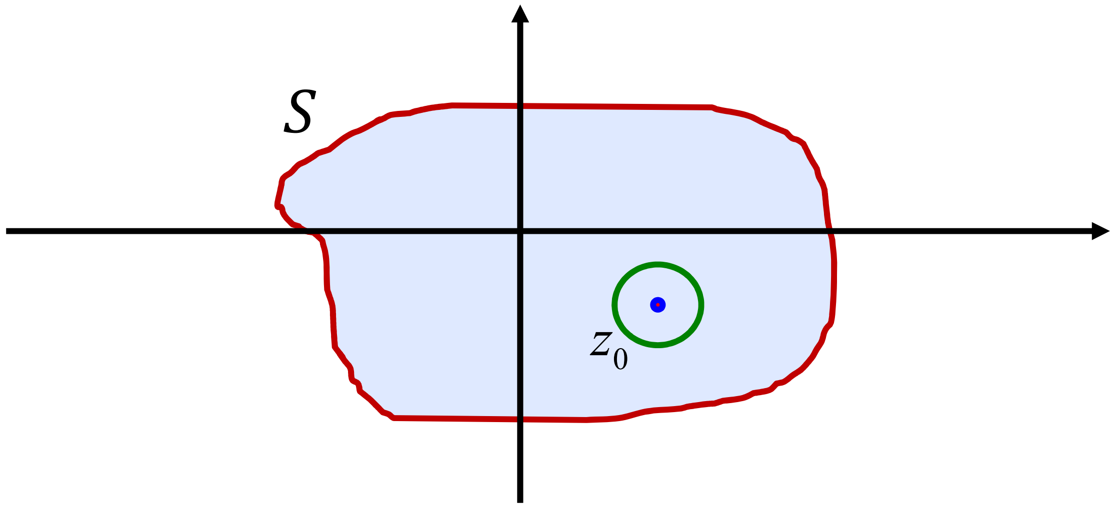
Fig. 1.4. Neighborhood of a point z_0 with radius r.

- A set S is called **connected** if every point in S can be joined by an unbroken line entirely within S.



- A point z_0 is called a **boundary point** of S if every neighborhood of z_0 contains a point in S and a point not in S. The set of all boundary points of S is called the **boundary** of S.

- A point z_0 is called an **interior point** of S if there exists a neighborhood $N(z_0;\epsilon)$ of z_0 lying entirely within S. The set of all interior points of S is called the **interior** of S.



- A connected set is called a **region**.
- S is called an **open region** or a **domain** if it contains none of its boundary points. S is called a **closed region** if it contains all of its boundary points.

- A **simple closed path** is a closed path that does not intersect or touch itself as shown in Fig. 1.5

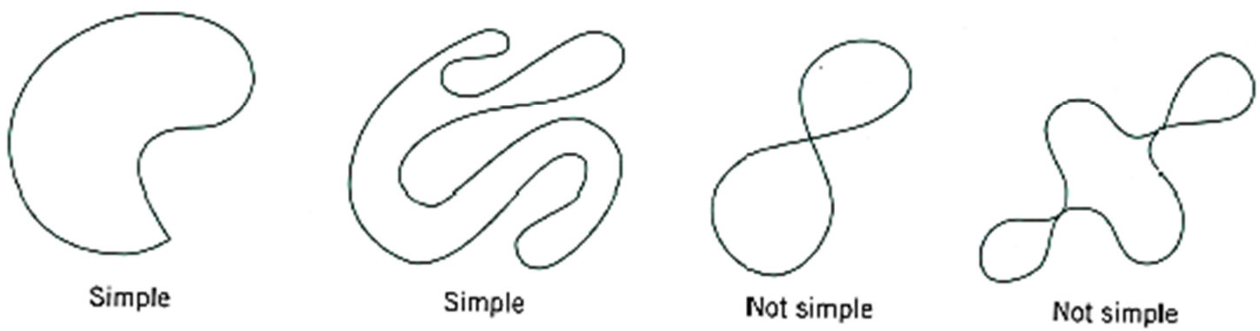


Fig. 1.5. Closed paths

- A **simply connected domain** D in the complex plane is a domain such that every simple closed path in D encloses only points of D. A domain that is not simply connected is called **multiply connected**

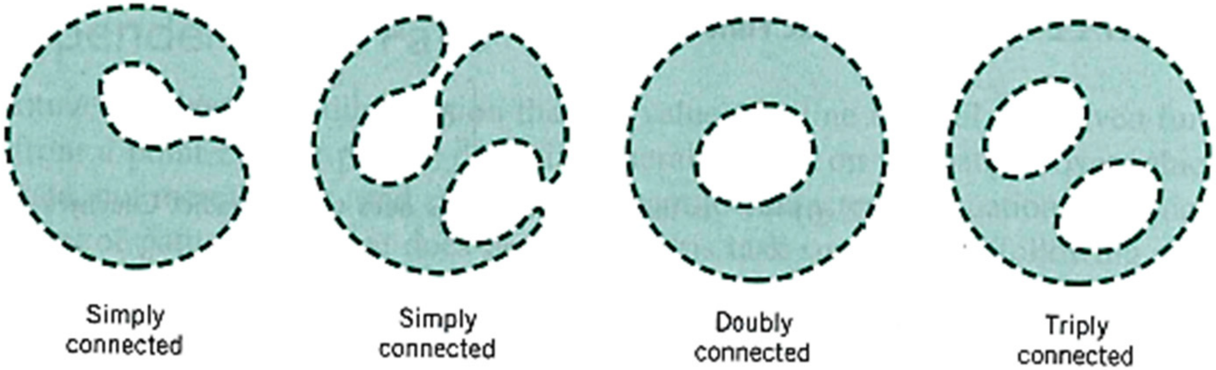
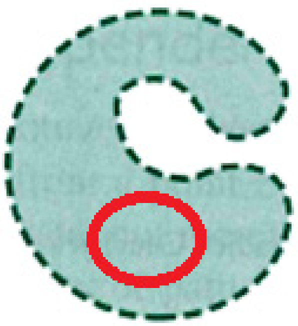
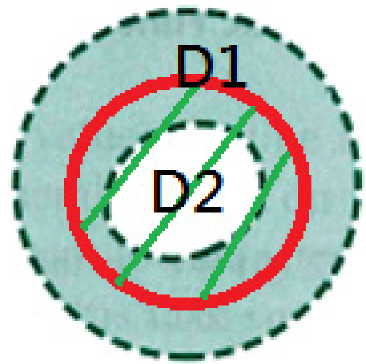


Fig. 1.6. Simply and multiply connected domains





Example (1c)

- Consider the set $S' = \{z \in \mathbb{C} : |z - (1+i)| \leq 1\}$.

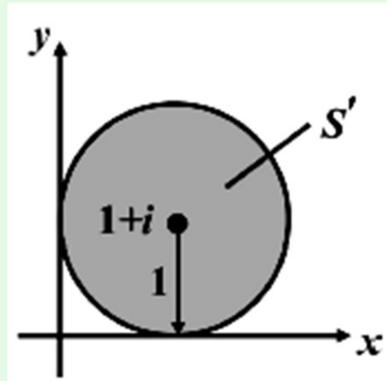


Fig. 1.7. The sketch of S'.

- Then,
  (a) S' is connected, and also simply connected.
  (b) The boundary of S' is $|z - (1+i)| = 1$. The interior of S' is $|z - (1+i)| < 1$.
  (c) The union of boundary of S' and interior of S' is S'.
  (d) The intersection of boundary of S' and interior of S' is $\varnothing$, i.e., the empty set.

Limit and Continuity

- Let z_0 be an interior point in the domain of $f(z)$. We say that the **limit** of $f(z)$, as z approaches a point z_0, is l, i.e.,

$$\boxed{\lim_{z \to z_0} f(z) = l,} \tag{3.1}$$

if for each $\epsilon > 0$ there exists a $\delta > 0$ such that

$$|f(z) - l| < \epsilon \text{ whenever } 0 < |z - z_0| < \delta.$$

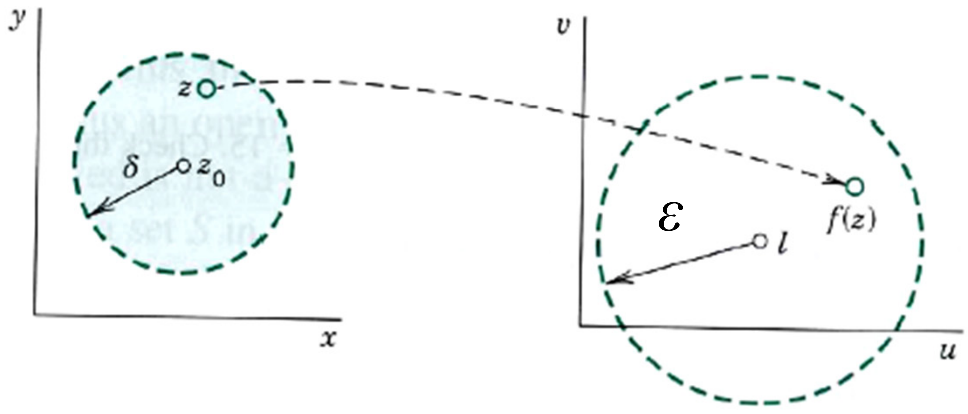
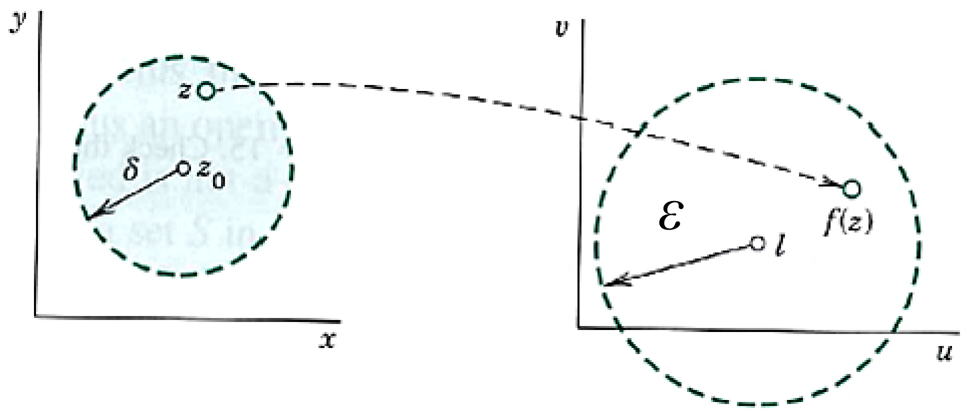


Fig. 3.1. Limit.

- $f(z)$ is **continuous** at $z = z_0$ if

$$\boxed{\lim_{z \to z_0} f(z) = f(z_0).} \qquad (3.2)$$



- In many cases, we can manipulate complex limits like real limits. example,

$$\lim_{z \to i}(z^2 + iz) = i^2 + i^2 = -2.$$

Complex Differentiation and Analytic Functions

- Let z_0 be an interior point in the domain of $f(z)$. We define the **derivative** of f at $z = z_0$ as

$$\boxed{f'(z_0) = \lim_{\triangle z \to 0} \frac{f(z_0 + \triangle z) - f(z_0)}{\triangle z},} \qquad (3.3)$$

provided that the limit exists.

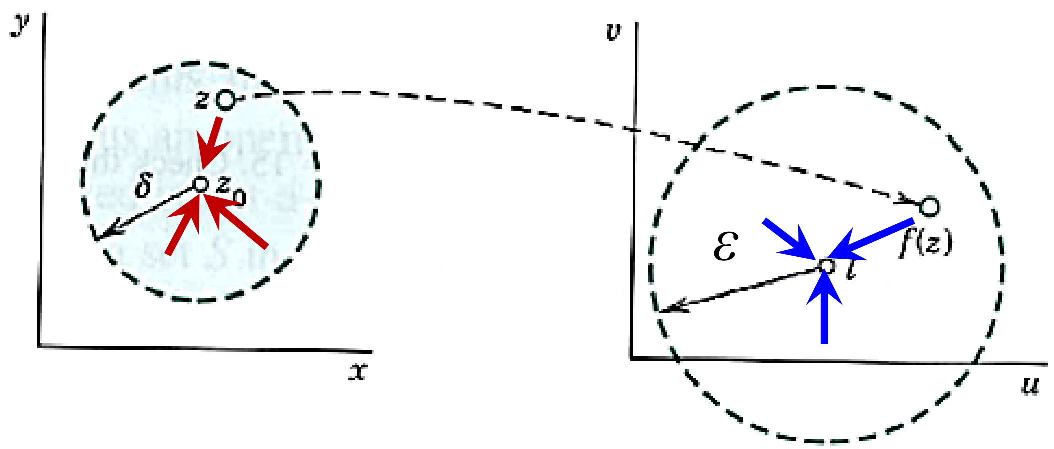
(3.3) can be rewritten as

$$\boxed{f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}.} \qquad (3.4)$$

If $f'(z_0)$ exists, we say that f is **differentiable** at $z = z_0$.

- Note: By the definition of limit, $f(z)$ is defined in a neighborhood of z_0 and z in (3.4) may approach z_0 from any direction in the complex plane. Hence, differentiability at z_0 means that, along whatever path z approaches z_0, the quotient in (3.4) always approaches a certain value and all these values are equal.

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}. \quad (3.4)$$



Example (3a)

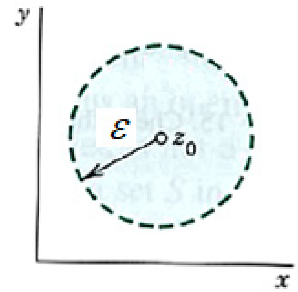
- The function $f(z) = z^2$ is differentiable for all z and has the derivative $f'(z) = 2z$ because

$$f'(z_0) = \lim_{\triangle z \to 0} \frac{(z + \triangle z)^2 - z^2}{\triangle z} = \lim_{\triangle z \to 0} \frac{z^2 + 2z\triangle z + (\triangle z)^2 - z^2}{\triangle z}$$

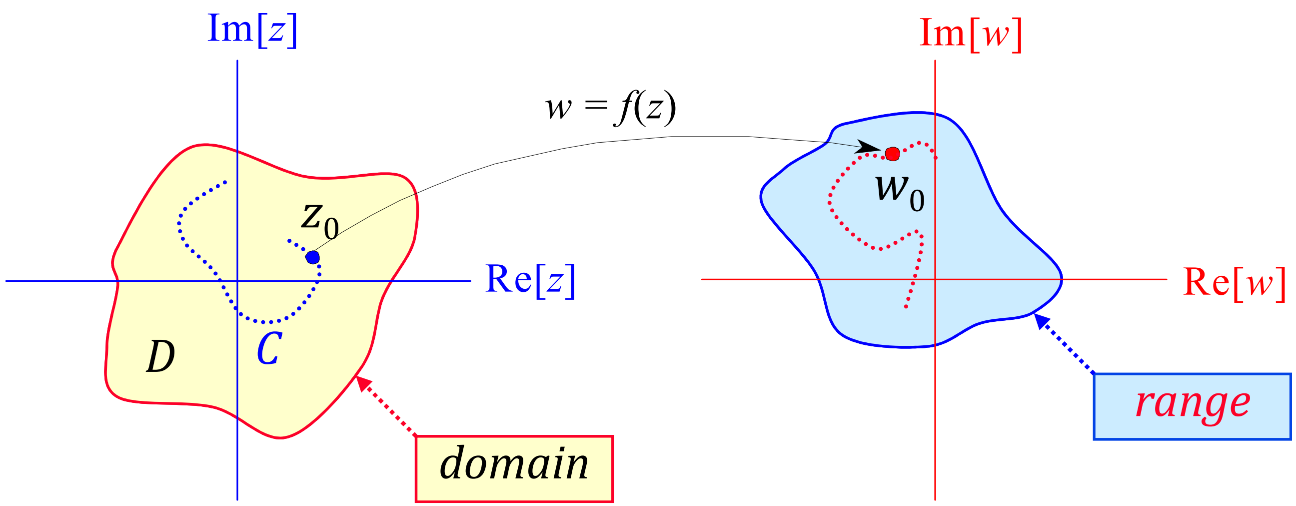
$$= \lim_{\triangle z \to 0} (2z + \triangle z) = 2z$$

Analytic Functions

- A complex function $f(z)$ is **analytic** at $z = z_0$ if there exists a neighborhood $N(z_0; \epsilon)$ of z_0 such that f is differentiable at every point in $N(z_0; \epsilon)$. If it is not analytic at z_0, it is **singular** there.



- It is called an **analytic function** at a domain $D \subset \mathbb{C}$ if it is analytic at every point in D. Functions that are analytic everywhere in the z-plane are called **entire functions**.

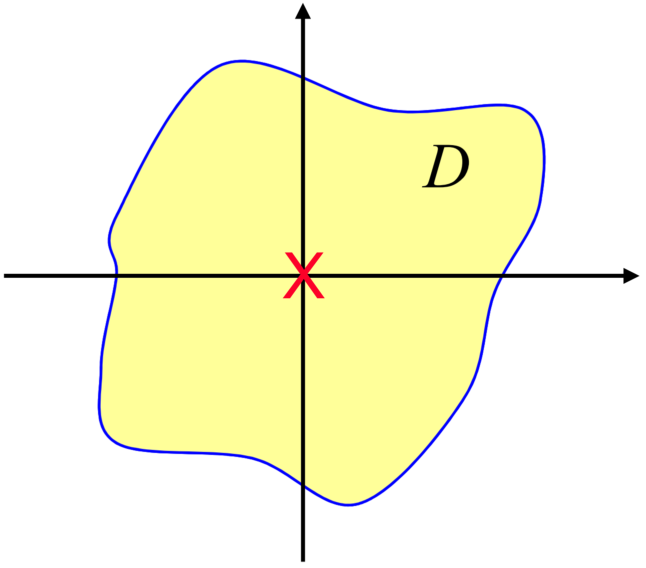


Singularities

Points where a function is not analytic are called **singular points** or **singularities** or **poles** sometimes.

Example:

$f(z) = \frac{1}{z}$ is analytic everywhere in D except $z = 0$, which is thus the singular point or pole of the function.



Note that a function is either analytic or singular at any given point…

Rules of Complex Differentiation

- The familiar rules of real differentiation carry over to the complex case. example,

  the product rule $$[f(z)g(z)]' = f'(z)g(z) + f(z)g'(z)$$

  and the chain rule $$\frac{d}{dz} f(g(z)) = f'(g(z))g'(z).$$

the complex differentiability of f at a point $z = z_0$ implies the continuity of f at that point.

- useful results of complex differentiation:

$$(z^n)' = nz^{n-1}, \quad (e^z)' = e^z, \quad (\sin z)' = \cos z,$$

$$(\cos z)' = -\sin z, \quad (\sinh z)' = \cosh z, \quad (\cosh z)' = \sinh z$$

$$\left(\frac{1}{z}\right)' = -\frac{1}{z^2}, \quad \frac{d}{dz}\mathrm{Ln}(z) = \frac{1}{z}.$$

- Remark: Exponential, trigonometric and hyperbolic functions are entire functions, while z^{-1} is analytic on $\mathbb{C}\backslash\{0\}$.

Cauchy-Riemann Equations

- A necessary condition for the differentiability of a complex function $f(z) = u(x, y) + iv(x, y)$ is to satisfy the relation

$$\boxed{\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y}, \\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x}. \end{aligned}} \qquad (3.5)$$

Or simply written as

$$\boxed{u_x = v_y, \quad u_y = -v_x.} \qquad (3.6)$$

They are known as the **Cauchy-Riemann equations** (C-R equations).

Augustin-Louis Cauchy
(1789–1857)
French Mathematician

Bernhard Riemann
(1826–1866)
German Mathematician

Theorem (3.1 Cauchy-Riemann Equations)

Let $f(z) = u(x, y) + iv(x, y)$ be defined and continuous in some neighborhood of a point $z = x + iy$ and differentiable at z itself. Then, at that point, the first-order partial derivatives of u and v exist and satisfy the Cauchy-Riemann equations (3.6).

Hence, if $f(z)$ is analytic in a domain D, those partial derivatives exist and satisfy (3.6) at all points of D.

- If f is differentiable, then f' is given by any of these four equivalent expressions:

$$\boxed{f' = u_x + iv_x = v_y - iu_y = u_x - iu_y = v_y + iv_x.} \qquad (3.7)$$

- **Remark:** The four equivalent expressions are obtained by simply applying the Cauchy-Riemann equations (3.6).

$$\boxed{u_x = v_y, \quad u_y = -v_x.}$$

Proof (optional)...

In spite of these similarities, there is a fundamental difference between differentiation for functions of real variables and differentiation for functions of a complex variable. Let $z = (x, y)$ and suppose that h is real. Then

$$f'(z) = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h} = \frac{\partial f}{\partial x}(z) = f_x(z).$$

But if $h = ik$ is purely imaginary, then

$$f'(z) = \lim_{k \to 0} \frac{f(x, y+k) - f(x, y)}{ik} = \frac{1}{i}\frac{\partial f}{\partial y}(z) = -if_y(z).$$

Thus, the existence of a complex derivative forces the function to satisfy the partial differential equation

$$f'(z) = f_x = -if_y .$$

Writing $f(z) = u(z) + iv(z)$, where u and v are real-valued functions of a complex variable, and equating the real parts and imaginary parts of

$$u_x + iv_x = f_x = -if_y = v_y - iu_y ,$$

we obtain the **Cauchy-Riemann** differential equations

$$u_x = v_y , \qquad v_x = -u_y .$$

Example (3b)

$$f(z) = \bar{z} = x - iy = u + iv$$

$$\frac{\partial u}{\partial x} = 1 \quad , \quad \frac{\partial v}{\partial y} = -1 \quad , \quad \frac{\partial u}{\partial y} = 0 = \frac{\partial v}{\partial x}$$

$\Rightarrow$ $f(z)$ is not analytic anywhere

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\boxed{u_x = v_y, \quad u_y = -v_x.}$$

Theorem (3.2 Cauchy-Riemann Equations)

If two real-valued continuous functions $u(x, y)$ and $v(x, y)$ of two real variables x and y have continuous first partial derivatives that satisfy the Cauchy-Riemann equations in some domain D, then the complex function $f(z) = u(x, y) + iv(x, y)$ is analytic in D.

Example (3c)

Is $f(z) = u(x, y) + iv(x, y) = e^x(\cos y + i \sin y)$ analytic?

e^z

Solution:

we have $u = e^x \cos y, v = e^x \sin y$ and by differentiation

$$u_x = e^x \cos y, \quad v_y = e^x \cos y, \quad u_y = -e^x \sin y, \quad v_x = e^x \sin y$$

The Cauchy-Riemann equations are satisfied and conclude that $f(z)$ is analytic for all z.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\boxed{u_x = v_y, \quad u_y = -v_x.}$$

Example:

$$f(z) = z^2 = x^2 - y^2 + i\,2xy = u + iv$$

$$\frac{\partial u}{\partial x} = 2x = \frac{\partial v}{\partial y} \quad , \quad \frac{\partial u}{\partial y} = -2y = -\frac{\partial v}{\partial x}$$

and the partial derivatives are continuous $\forall\, z$.

Consequently, $f(z)$ is analytic $\forall\, z$.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\boxed{u_x = v_y, \quad u_y = -v_x.}$$

Example:

$$f(z)=\frac{\bar{z}}{|z|^2}=\frac{x}{x^2+y^2}-i\frac{y}{x^2+y^2}=u+iv$$

$$\frac{\partial u}{\partial x}=\frac{y^2-x^2}{(x^2+y^2)^2}=\frac{\partial v}{\partial y}\quad,\quad \frac{\partial u}{\partial y}=\frac{-2xy}{(x^2+y^2)^2}=-\frac{\partial v}{\partial x}$$

$\Rightarrow$ $f(z)$ is analytic everywhere, except where $x^2+y^2=0$ i.e. at the origin.

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$$

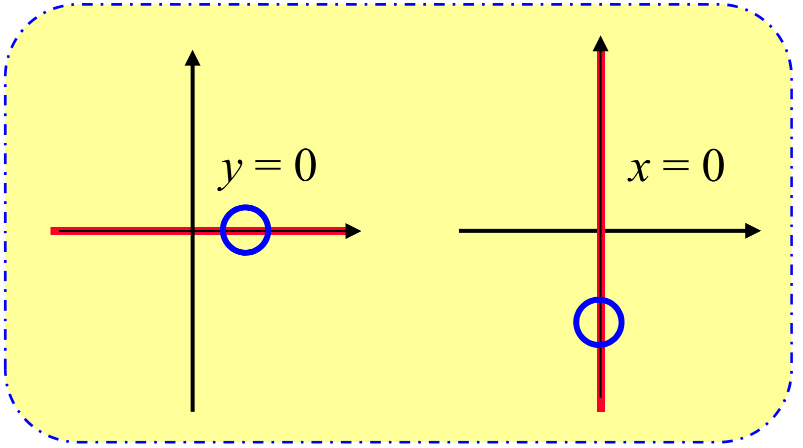
$$\boxed{u_x = v_y, \quad u_y = -v_x.}$$

Example:

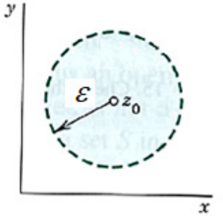
$$f(z) = x^2 y^2 + i\,2x^2 y^2 = u + iv$$

$$\frac{\partial u}{\partial x} = 2xy^2 \quad , \quad \frac{\partial v}{\partial y} = 4x^2 y \quad , \quad \frac{\partial u}{\partial y} = 2x^2 y \quad , \quad \frac{\partial v}{\partial x} = 4xy^2$$

The Cauchy-Riemann equations only hold for $x = 0$ and/or $y = 0$. Since the function is not analytic in a neighbourhood of $x = 0$ or $y = 0$, $f(z)$ is not analytic anywhere.



- A complex function $f(z)$ is **analytic** at $z = z_0$ if there exists a neighborhood $N(z_0; \epsilon)$ of z_0 such that f is differentiable at every point in $N(z_0; \epsilon)$. If it is not analytic at z_0, it is **singular** there.



Observations

1. The sum or product of analytic functions is analytic.
2. All polynomials are analytic.
3. A rational function (the quotient of two polynomials) is analytic, except at zeroes of the denominator.
4. An analytic function of an analytic function is analytic.
5. Functions e^z, $\sin z$, $\cos z$, $\sinh z$, $\cosh z$ are analytic everywhere.

Theorem (3.3 Laplace's Equations)

If $f(z) = u(x, y) + iv(x, y)$ is analytic in a domain D, then both u and v satisfy Laplace's equations

$$\boxed{\begin{aligned} \nabla^2 u = u_{xx} + u_{yy} = 0, \\ \nabla^2 v = v_{xx} + v_{yy} = 0 \end{aligned}} \qquad (3.11)$$

in D and have continuous second partial derivatives in D.

- **Remark:** The above theorem follows from Cauchy-Riemann equations (3.6):

$$u_{xx} + u_{yy} = (u_x)_x + (u_y)_y = (v_y)_x + (-v_x)_y = 0,$$

$$v_{xx} + v_{yy} = (v_x)_x + (v_y)_y = (-u_y)_x + (u_x)_y = 0,$$

assuming that u and v are C^2.

$$\boxed{u_x = v_y, \quad u_y = -v_x.}$$

Harmonic Functions

A function $h(x, y)$ is harmonic if it is a twice continuously differentiable that satisfies Laplace's equation: $h_{xx} + h_{yy} = 0.$

Note that if $f(z) = u(x, y) + i\, v(x, y)$ is analytic, then the harmonic function u and v are a related pair.

We refer to them as conjugate harmonic functions…

$$\nabla^2 u = u_{xx} + u_{yy} = 0$$
$$\nabla^2 v = v_{xx} + v_{yy} = 0$$

Pierre-Simon Laplace
(1749–1827)
French Mathematician

Homework Assignment No: 1

Due Date: 6:00pm, 3 October 2019

Please place your assignment to Assignment Box 3 outside PC Lab (ERB 218)

Complex Analysis – 4…

1 Complex Numbers

2 Functions of One Complex Variable

3 Complex Differentiation

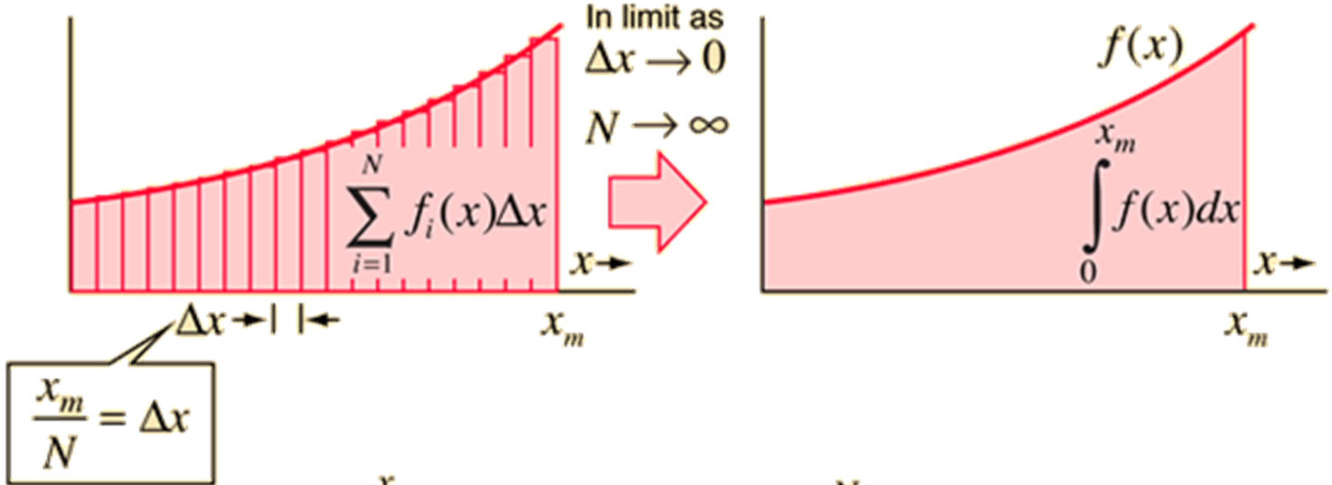
4 **Complex Integration and Cauchy's Theorem**

5 Cauchy Integral Formula

6 Complex Series, Power Series and Taylor Series

Revisit: Real Integration…

Sum becomes Integral



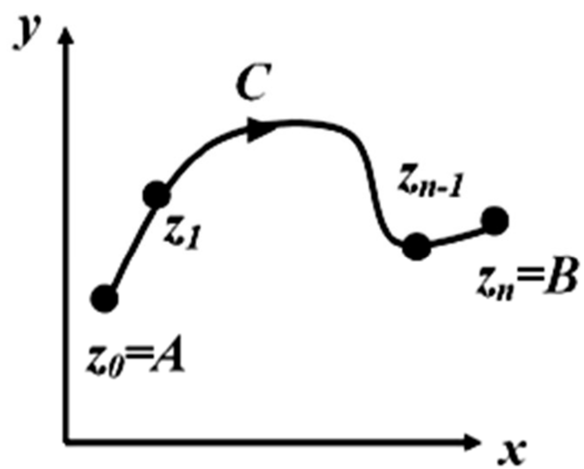
$$Area = \int_0^{x_m} f(x)dx = \lim_{\Delta x \to 0} \sum_{i=1}^{N} f_i(x)\Delta x$$

Line Integral in the Complex Plane

- Complex definite integrals are called (complex) **line integrals**. They are written

$$\boxed{I = \int_C f(z)\, dz} \qquad (4.1)$$

Here the integrand $f(z)$ is integrated over a given curve C in the complex z-plane, called the **path of integration**.



- Such a curve C can be represented by a parametric representation

$$\boxed{z(t) = x(t) + iy(t)} \tag{4.2}$$

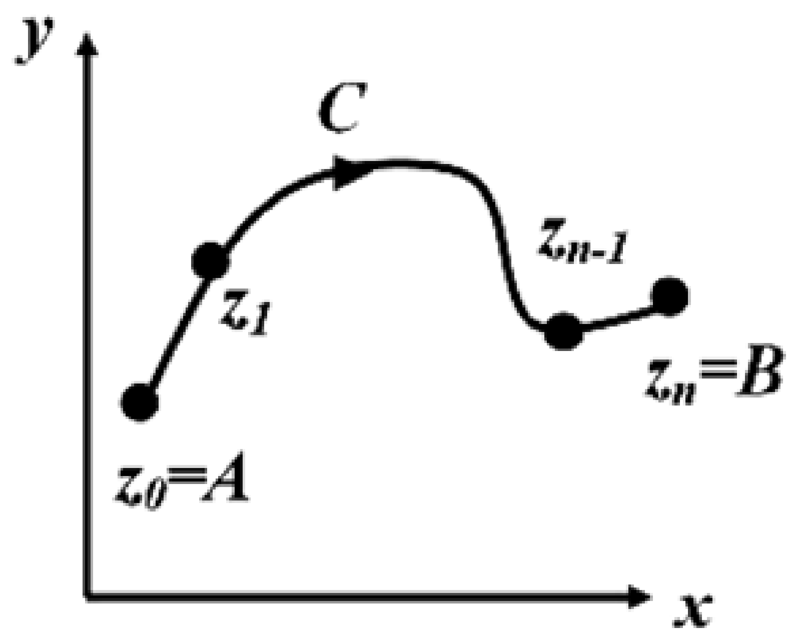


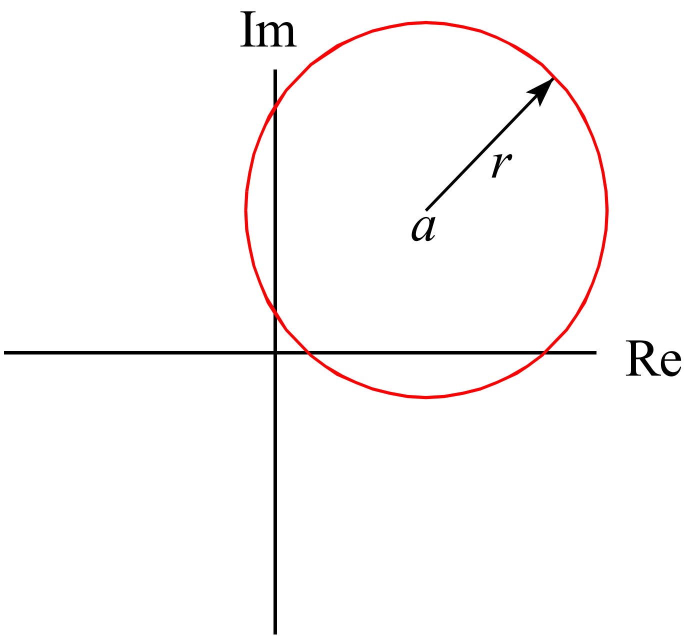
Fig. 4.1

Special Curve: Circle

- **Circle**

  The parametric description for a **circle** centred at complex point a and with a radius r is

$$z(t) = a + re^{it}, \quad t \in [0, 2\pi]$$

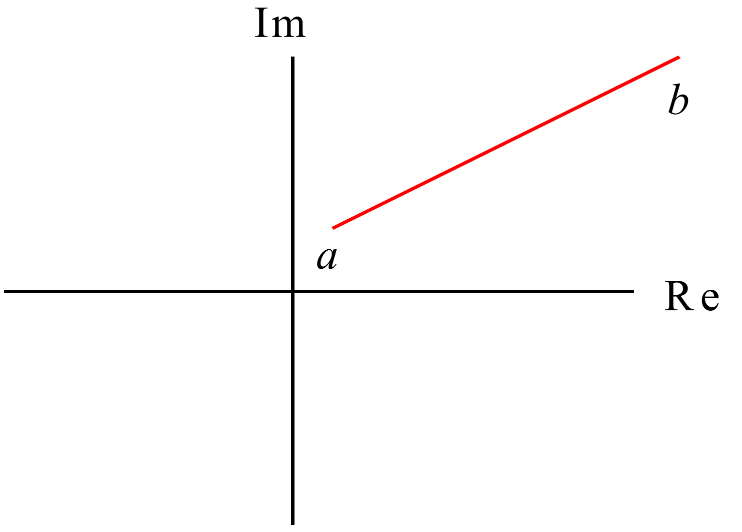


Special Curve: Straight Line

- **Straight Line**

  The parametric description of a **straight line** segment with starting point a and endpoint b is

$$z(t) = (b - a)t + a, \quad t \in [0,1]$$



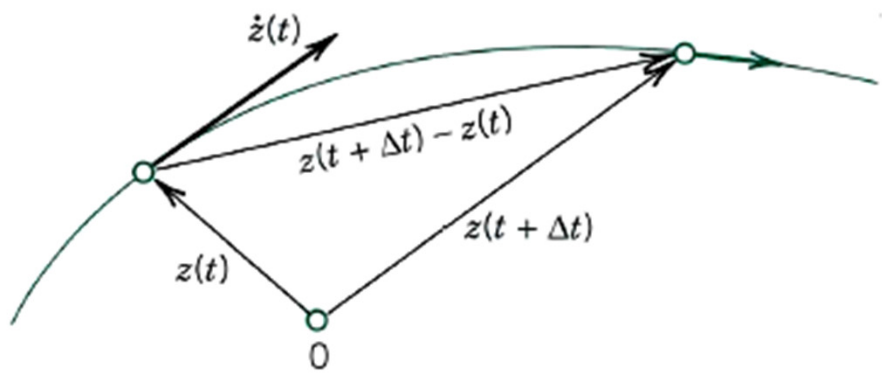
- We assume C to be a **smooth curve**, i.e., C has a continuous and nonzero derivative

$$\dot{z}(t) = \frac{dz}{dt} = \dot{x}(t) + i\dot{y}(t)$$

at each point.

- Geometrically, this means that C has a continuously turning tangent everywhere.

$$\dot{z}(t) = \lim_{\Delta t \to 0} \frac{z(t+\Delta t) - z(t)}{\Delta t}$$



- Recall a curve C is simple if it does not intersect itself and it is called a **closed path** if $A = B$ in Fig.4.1. In such case,

$$\boxed{I = \int_C f(z)\,dz} \quad \Rightarrow \quad I = \oint_C f(z)\,dz.$$

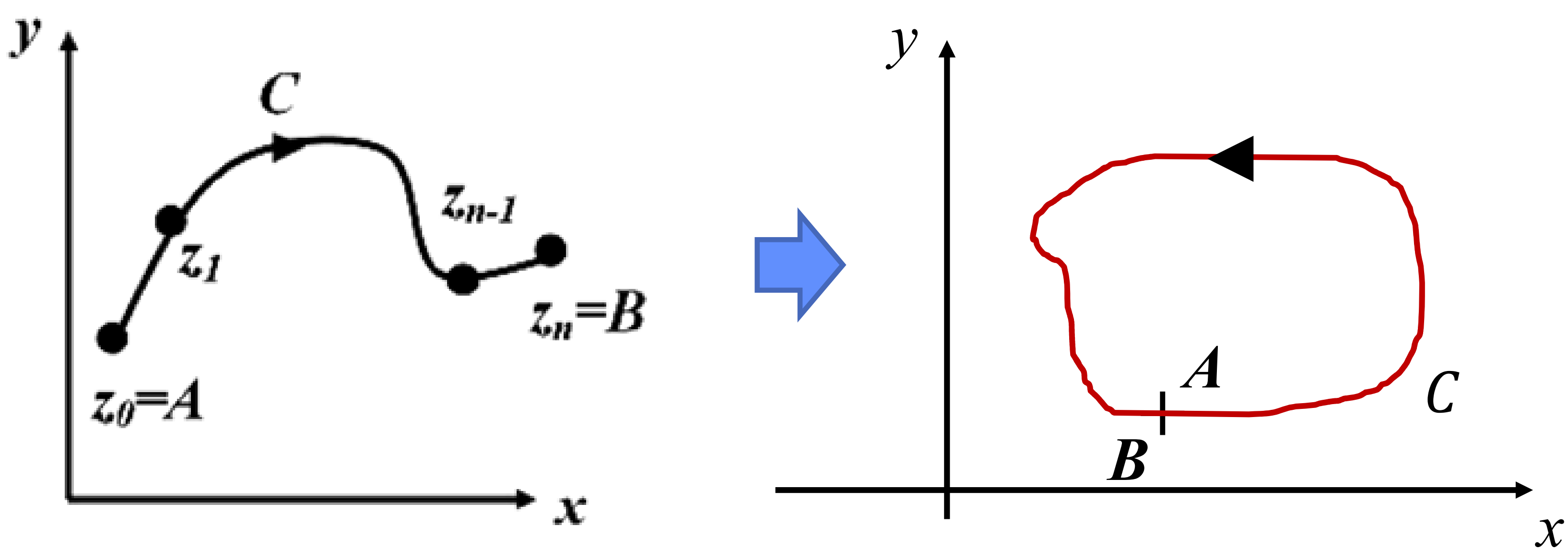


Fig. 4.1

Definition of Complex Line Integral

- Consider a smooth curve C in the complex plane given by

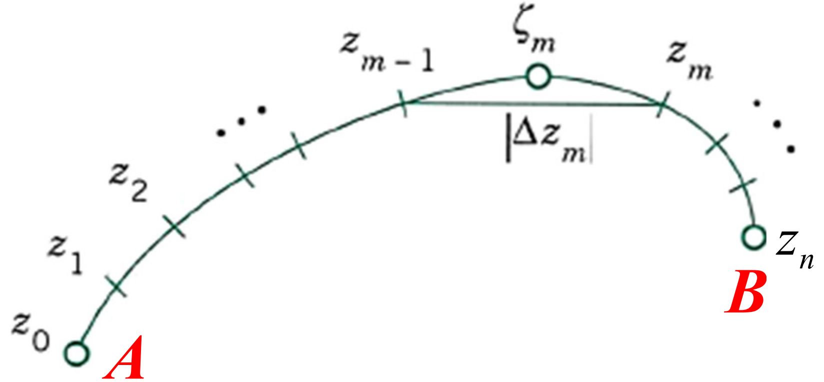
$$z(t) = x(t) + iy(t), \quad a \le t \le b$$

Subdivide the interval $a \le t \le b$ by points

$$a = t_0,\ t_1,\ t_2,\ \ldots,\ t_{n-1},\ t_n = b$$

- Suppose that C has initial point and end points at $z = A$ and $z = B$, respectively, the Corresponding to points on C will be,

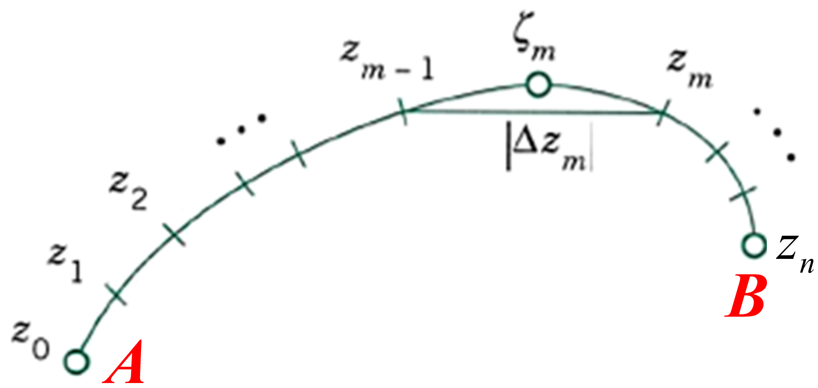
$$A = z_0,\ z_1,\ z_2,\ \ldots,\ z_{n-1},\ z_n = B \qquad z_i = z(t_i)$$



- Form the sum

$$S_n = \sum_{m=1}^{n} f(\zeta_m)(z_m - z_{m-1}) \tag{4.3}$$

where ζ_m is some point between the arc from z_{m-1} to z_m. The choice of the z_m's and ζ_m's defines a **partition** of C, and we call the largest $|\Delta z_m| = |z_m - z_{m-1}|$ the **norm** of the partition.



- The partition is chosen such that the norm of the n-th partition tends to zero as $n \to \infty$. If the corresponding sequence of the sums $S_1, S_2, \ldots$ converges to a limit, we call that limit the complex integral $\int_C f(z)\,dz$ and say that the integral exists, i.e.,

$$\boxed{\int_C f(z)\,dz = \lim_{n\to\infty} S_n} \tag{4.4}$$

Properties of Complex Integrals

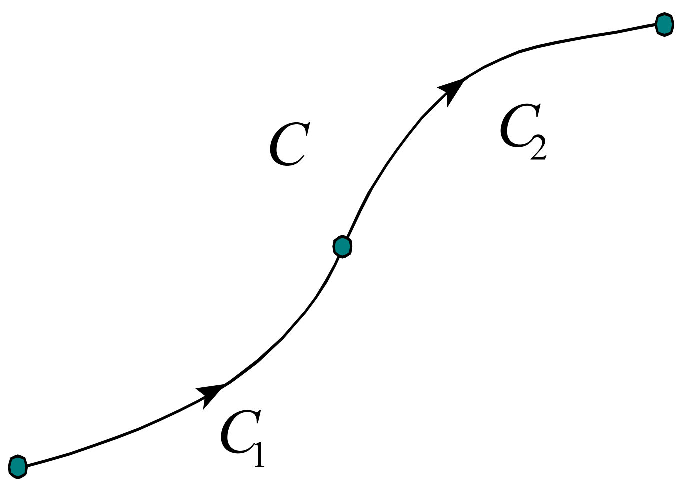
As for real integrals, the following rules apply:

1. $$\int_C [f(z)+g(z)]\,dz = \int_C f(z)\,dz + \int_C g(z)\,dz$$

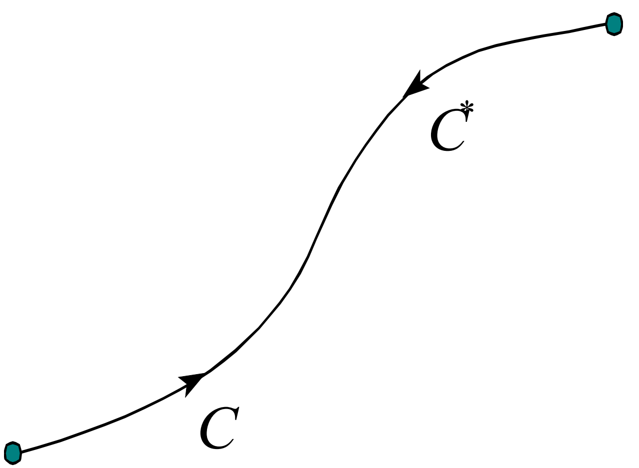
2. $$\int_C k\, f(z)\,dz = k\int_C f(z)\,dz\,,\ \ k \text{ complex}$$

Properties of Complex Integrals

3. $\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz$



4. $\int_C f(z)\,dz = -\int_{C^*} f(z)\,dz$



Estimation of a Complex Integral

- Let $f(z)$ be continuous on $C: t \to z(t),\ t \in [\alpha, \beta]$. If $|f(z)| \le M$ on C, then

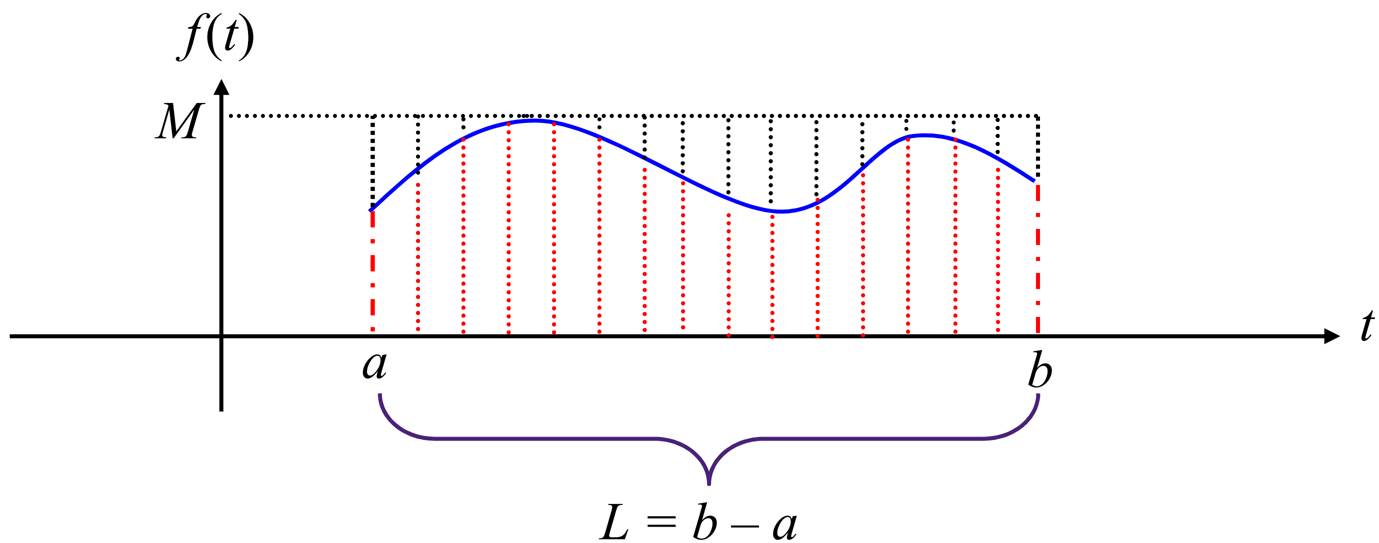
$$\left| \int_C f(z)\, dz \right| \le M L$$

where L is the length of the curve C, i.e.

$$L = \int_\alpha^\beta |z'(t)|\, dt = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Estimation of Complex Integral – An Illustration

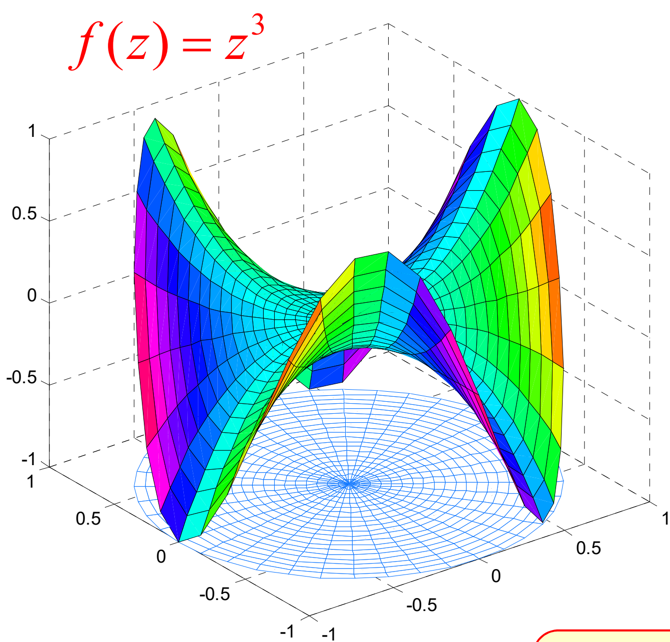
Graphically, take real integration as an example,

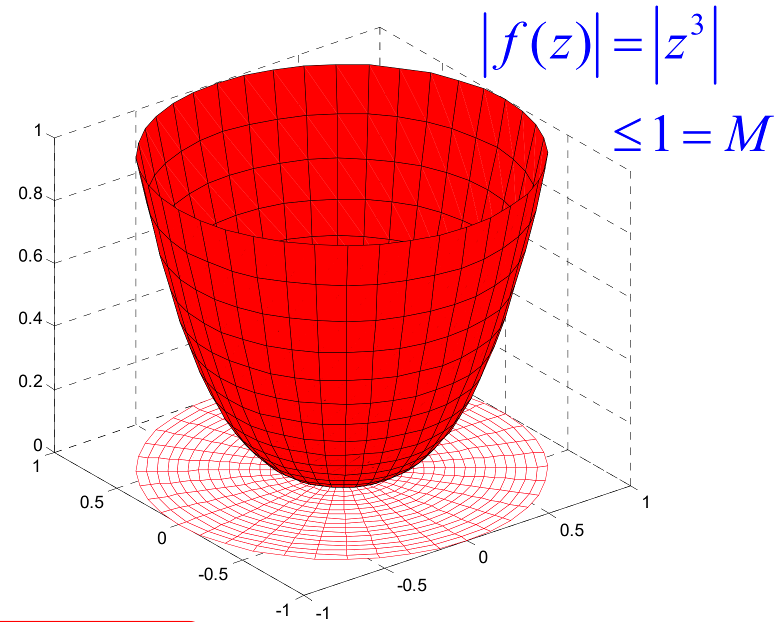


$$\left| \int_a^b f(t)dt \right| = \text{shaded area with red lines} \le M \cdot L$$

Estimation of Complex Integral – An Illustrative example

For complex cases, for example, we take $f(z) = z^3$ and C to be a unit circle





$$\left|\int_C f(z)dz\right| \leq ML = 2\pi$$

Evaluation Method: Indefinite Integration and Substitution of Limits

- An **indefinite integral** is a function whose derivative equals a given analytic function in a region.

Theorem (4.1 Indefinite Integration of Analytical Functions)

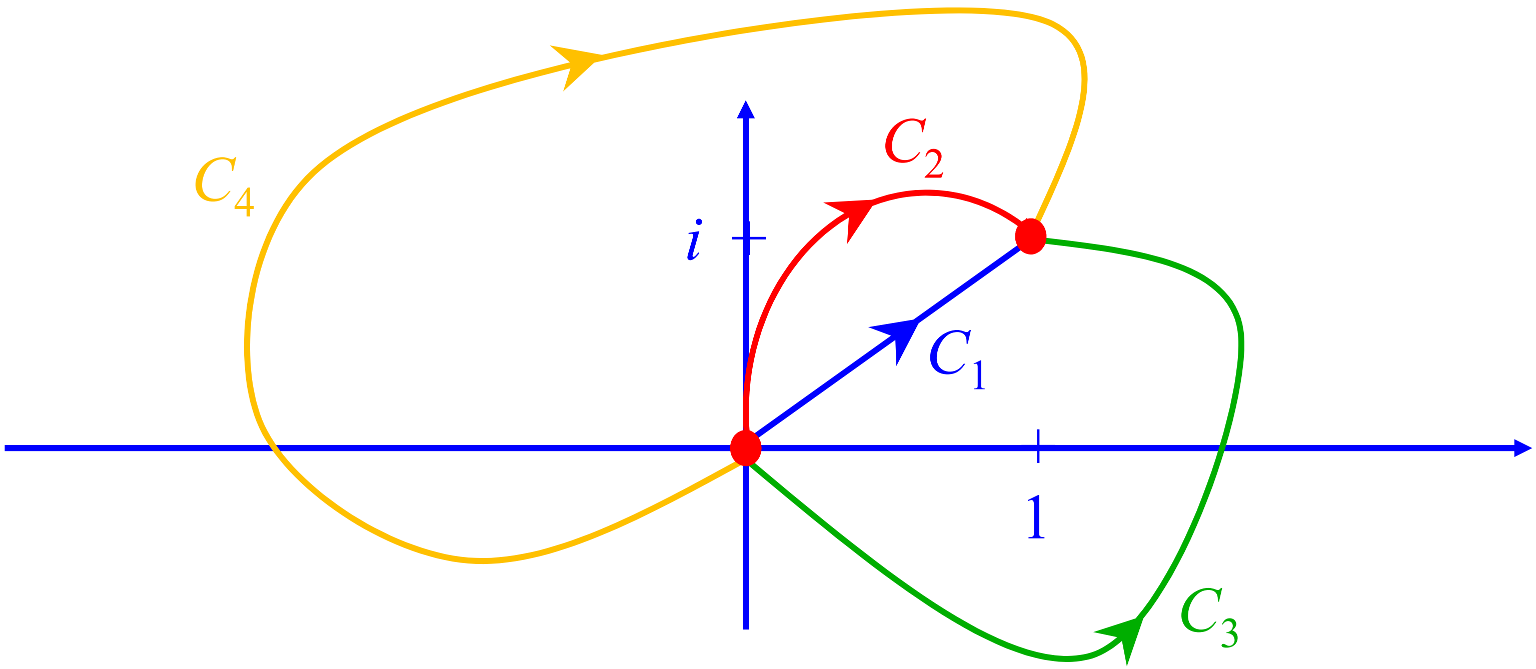
Let $f(z)$ be analytic in a simply connected domian D. Then there exists an indefinite integral of $f(z)$ in the domain D, that is, an analytic function $F(z)$ such that $F'(z) = f(z)$ in D, and for all paths in D joining two points z_0 and z_1 in D we have

$$\int_{z_0}^{z_1} f(z)\,dz = F(z_1) - F(z_0) \qquad [F'(z) = f(z)] \tag{4.5}$$

(Note that we can write z_0 and z_1 instead of C, since we get the same value for all those C from z_0 and z_1.)

Example (4a)

$$\int_0^{1+i} z^2\, dz = \frac{1}{3} z^3 \Big|_0^{1+i} = \frac{1}{3}(1+i)^3 = -\frac{2}{3} + \frac{2}{3} i$$



Evaluation Method: Use of a Representation of a Path

Theorem (4.2 Integration by the Use of the Path)

Let C be a piecewise smooth path, represented by $z = z(t)$, where $t \in [a, b]$. Let $f(z)$ be a continuous function on C. Then

$$\int_C f(z)\,dz = \int_a^b f(z(t))z'(t)\,dt. \tag{4.6}$$

According to Theorem 4.2, the integral depends on the path/contour chosen. This is generally true for non-analytic functions.

Example (4b)

- Show that by integrating $\frac{1}{z}$ counterclockwise around the unit circle (the circle of radius 1 and center 0)

$$\boxed{\oint_C \frac{dz}{z} = 2\pi i} \qquad (4.7)$$

- (a). Represent the unit circle C by

$$z(t) = \cos t + i \sin t = e^{it} \qquad 0 \leq \ t \leq 2\pi$$

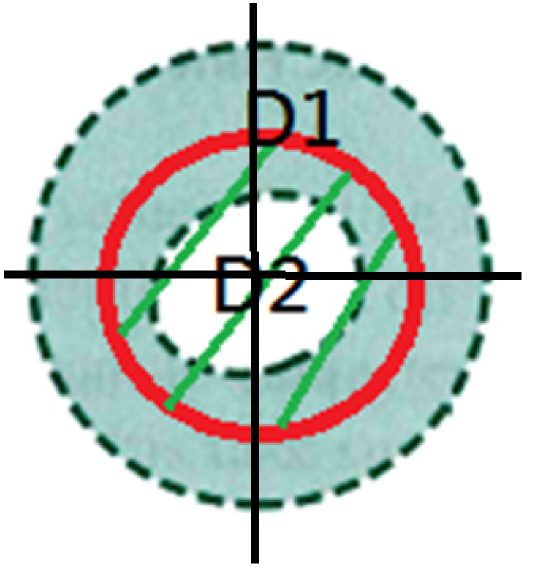
so that counterclockwise integration corresponds to an increase of $t \in [0\ 2\pi]$.

- (b). Differentiation gives $\dot{z}(t) = ie^{it}$.
- (c). By substitution, $f(z(t)) = \frac{1}{z(t)} = e^{-it}$.
- (d). From Eq.(4.6), we obtain

$$\oint_C \frac{dz}{z} = \int_0^{2\pi} e^{-it}\ ie^{it}\ dt = i \int_0^{2\pi} dt = 2\pi i.$$

(Check this result by using $z(t) = \cos t + i \sin t$)

- Note: Simple connectedness is essential in Theorem 4.1.
- Eq.(4.5) in Theorem 4.1 gives 0 for any closed path because then $z_1 = z_0$, so that $F(z_1) - F(z_0) = 0$.
- Now, $\frac{1}{z}$ is not analytic at $z = 0$. But any simply connected domain containing the unit circle must contain $z = 0$, so that Theorem 4.1 does not apply.
- It is not enough that $\frac{1}{z}$ is analytic in an annulus, say, $\frac{1}{2} < |z| < \frac{3}{2}$, because an annulus is not simply connected!



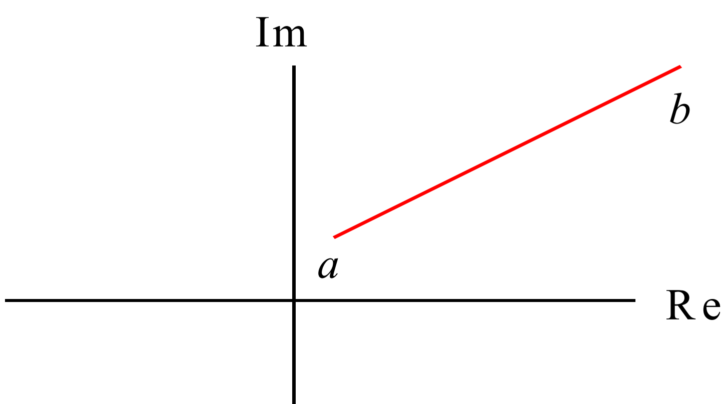
Example (4c)

- Suppose we want to evaluate the complex integral

$$\int_C |z|^2 \, dz,$$

where C is a straight line from $z = 0$ to $z = 1 + i$.

- Parametrize C as $z(t) = t + ti, t \in [0, 1]$, then $z'(t) = 1 + i$. Hence, by property (4.6),



$$z(t)=(b-a)t+a, \quad t\in[0,1]$$

$$\begin{aligned}\int_C |z|^2 \, dz &= \int_0^1 |t + ti|^2 (1+i) \, dt \\ &= \int_0^1 (2t^2)(1+i) \, dt \\ &= (1+i) \frac{2t^3}{3} \Big|_{t=0}^1 \\ &= \frac{2}{3}(1+i).\end{aligned}$$

Example (4d)

- Evaluate

$$I = \oint_C (z-a)^n \, dz,$$

where a is a given complex number, n is any integer and C is a circle of radius R, centered at a and oriented in an anticlockwise direction as follows.

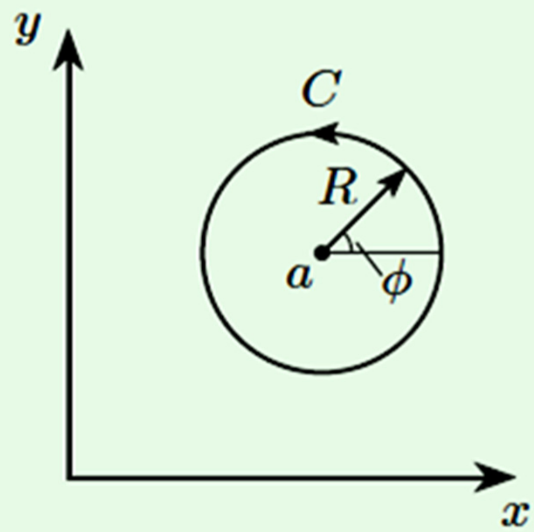
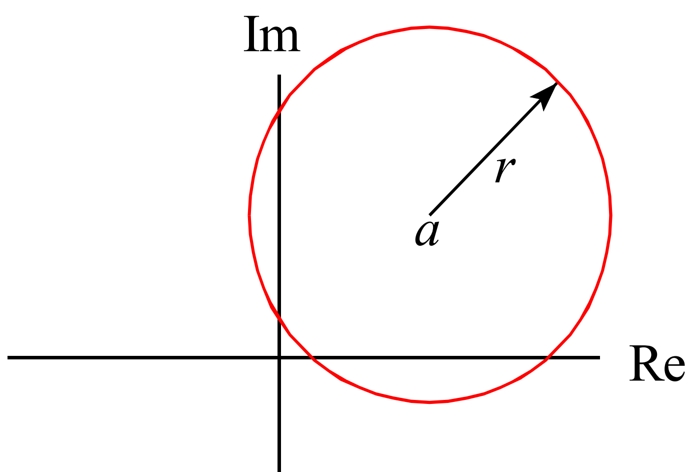


Fig. 4.3. The closed contour C.



$$z(t) = a + re^{it}, \quad t \in [0, 2\pi]$$

- Parametrize C as

$$z = a + Re^{i\phi}, \quad 0 \le \phi \le 2\pi.$$

Then, by property (4.6),

$$I = \int_0^{2\pi} (Re^{i\phi})^n (Rie^{i\phi})\, d\phi$$
$$= iR^{n+1} \int_0^{2\pi} e^{i(n+1)\phi}\, d\phi$$
$$= \frac{R^{n+1}}{n+1} e^{i(n+1)\phi} \Big|_{\phi=0}^{2\pi} = 0,$$

provided that $n \neq -1$.

- If $n = -1$, then $I = iR^0 \int_0^{2\pi} e^{i0\phi}\, d\phi = i \int_0^{2\pi} d\phi = 2\pi i$. Hence,

$$I = \oint_C (z-a)^n\, dz = \begin{cases} 2\pi i, & \text{if } n = -1, \\ 0, & \text{if } n \neq -1. \end{cases}$$

Different Paths different Values

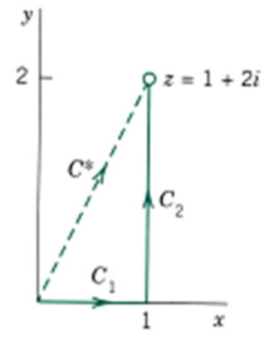
- In general, a complex line integral depends not only on the end points of the path but also on the path itself.

Example (4e)

- Integrate $f(z) = Re(z) = x$ from 0 to $1 + 2i$ (a) along C^*; (b) along C consisting of C_1 and C_2.
- (a). C^* can be represented by $z(t) = t + 2it \quad 0 \le t \le 1$.

$$\frac{dz(t)}{dt} = 1 + 2i \quad and \quad f[z(t)] = x(t) = t \quad on \quad C^*.$$

$$I^* = \int_{C^*} Rez dz = \int_0^1 t(1+2i)dt = \frac{1}{2}(1+2i) = \frac{1}{2} + i.$$



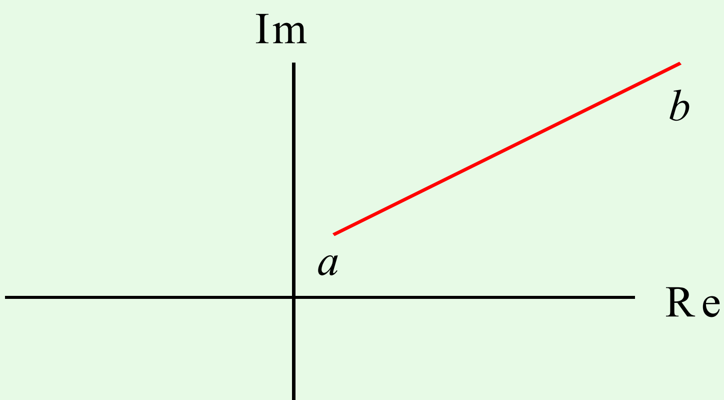
- (b).

$$C_1 : z(t) = t, \quad \dot{z}(t) = 1, \quad f(z(t)) = x(t) = t \quad 0 \le t \le 1$$

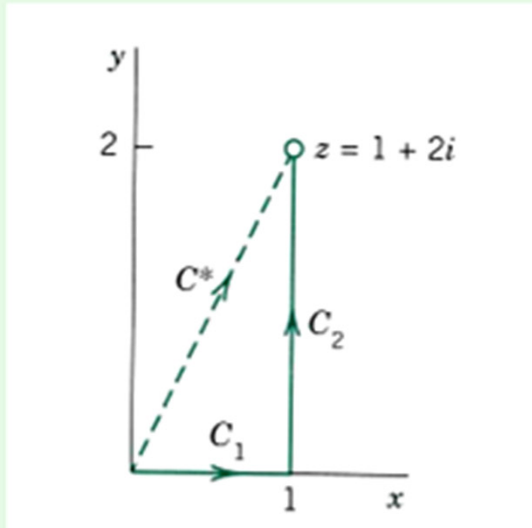
$$C_2 : \quad z(t) = 2it + 1, \quad \dot{z}(t) = 2i, \quad f(z(t)) = x(t) = 1, \quad 0 \le t < 1$$

$$I = \int_C \mathrm{Re}\, z dz = \int_{C_1} \mathrm{Re}\, z dz + \int_{C_2} \mathrm{Re}\, z dz = \int_0^1 t dt + \int_0^1 2i dt = \frac{1}{2} + 2i.$$

- Note that this result differs from the result in (a).



$z(t)=(b-a)t+a, \quad t\in[0,1]$

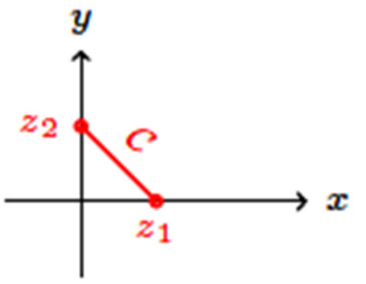


Path Dependent

Complex Integration: Path dependent

We want to compute the integral $\int_C \bar{z}\,\mathrm{d}z$ where C is the

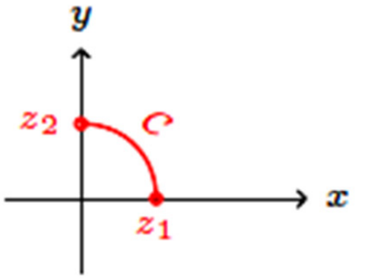
- *line between* $z_1 = 1$ *and* $z_2 = i$



$$z(t) = 1 + t(i-1),\ t \in [0,1]$$

Then $$\int_C \bar{z}\,\mathrm{d}z = \int_0^1 \big(1 + t(-i-1)\big)(i-1)\,\mathrm{d}t$$
$$= (i-1)\left(t - \frac{t^2}{2}(i+1)\right)\Big|_0^1 = (i-1)\frac{1-i}{2} = i$$

- *arc of unit circle between* z_1 *and* $z_2 = i$



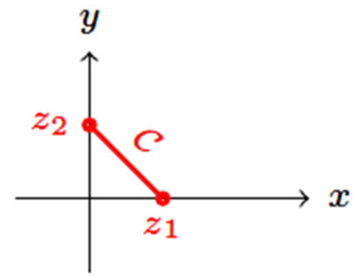
$$z(t) = \mathrm{e}^{it},\ t \in [0, \pi/2]$$

Then $$\int_C \bar{z}\,\mathrm{d}z = \int_0^{\pi/2} \mathrm{e}^{-it} i \mathrm{e}^{it}\,\mathrm{d}t$$
$$= it\Big|_0^{\pi/2} = i\pi/2$$

NOTE: The result of the integration is *path-dependent*.

We want to compute the integral $\int_C z\,\mathrm{d}z$ where C is the

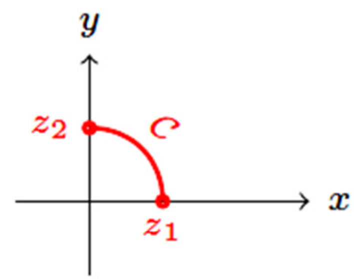
- *line between* $z_1 = 1$ *and* $z_2 = i$



$$z(t) = 1 + t(i-1),\ t \in [0,1]$$

Then $$\int_C z\,\mathrm{d}z = \int_0^1 \big(1 + t(i-1)\big)(i-1)\,\mathrm{d}t$$
$$= (i-1)\left(t + \frac{t^2}{2}(i-1)\right)\Big|_0^1 = (i-1)\frac{1+i}{2} = -1$$

- *arc of unit circle between* z_1 *and* $z_2 = i$



$$z(t) = \mathrm{e}^{it},\ t \in [0, \pi/2]$$

Then $$\int_C z\,\mathrm{d}z = \int_0^{\pi/2} \mathrm{e}^{it} i \mathrm{e}^{it}\,\mathrm{d}t$$
$$= \mathrm{e}^{2it}/2\Big|_0^{\pi/2} = \frac{e^{\pi i} - 1}{2} = \frac{\cos\pi - \sin\pi - 1}{2} = -1$$

NOTE: The result of the integration is the same for the two contours. Is it that the integral is *path-independent* and if so, why?

Path Independent

$$\int_C z\,dz$$ **Path Independent**

why?

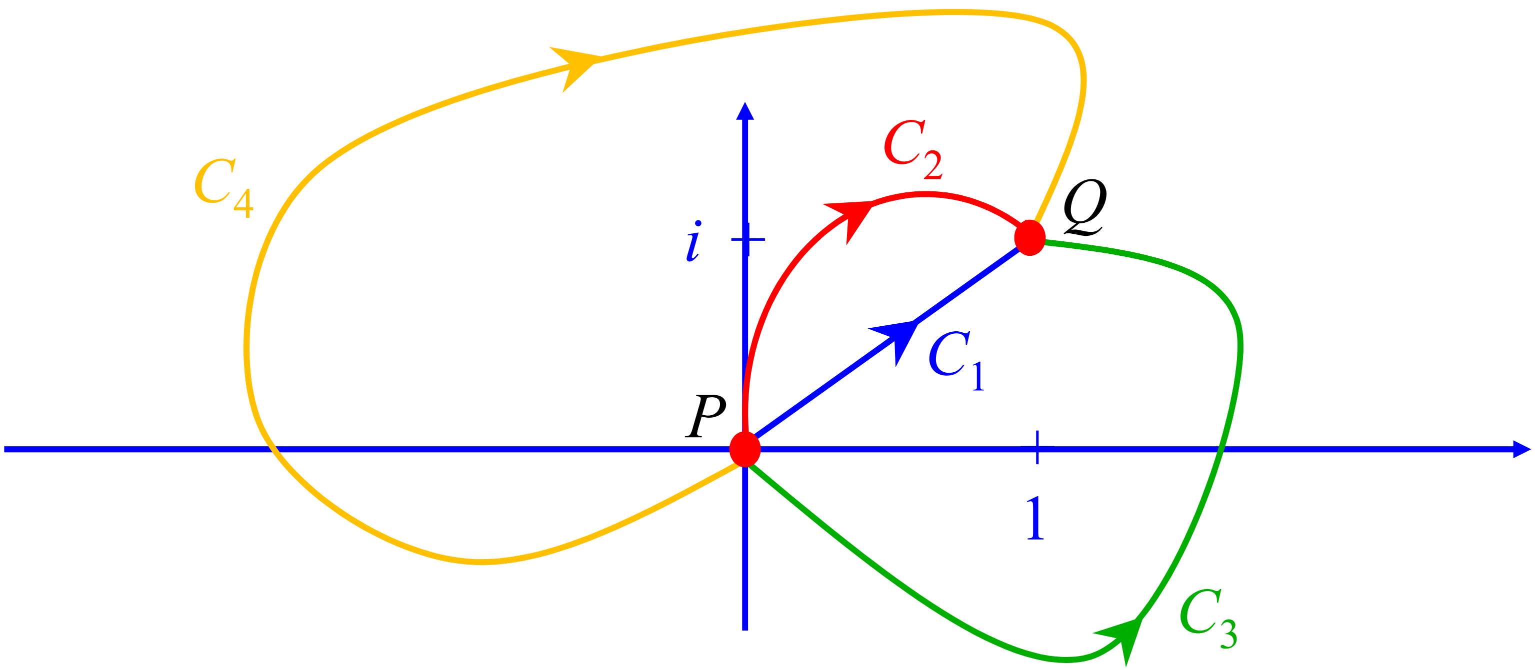
$$\int_C \bar{z}\,dz$$ **Path Dependent**

⇨ z is analytic while $\overline{z}$ is not!

Path Independence Theorem

Theorem (Path Independence)

If $f(z)$ is analytic in a simply connected domain D, then $\int_C f(z)\,dz$ is independent of path in D. That is, given any initial point P in D and for any final point Q in D, the value of $\int_C f(z)\,dz$ is the same for every piecewise smooth path C, lying entirely within D, from P to Q.



Cauchy's Integral Theorem

Theorem (4.3 Cauchy's Integral Theorem)

If $f(z)$ is analytic in a simply connected domain D, then

$$\oint_C f(z)\,dz = 0 \tag{4.9}$$

for every simple closed path C in D.

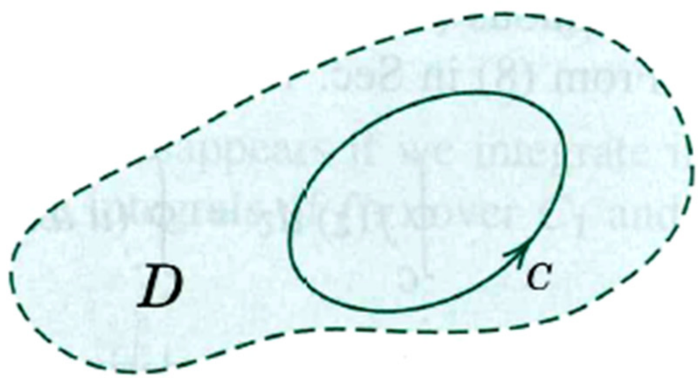


Fig. 4.4. Cauchy's integral theorem.

$$\int_{z_0}^{z_1} f(z)\,dz = F(z_1) - F(z_0) \qquad [F'(z) = f(z)] \tag{4.5}$$

Example (4g)

- Entire functions

$$\oint_C e^z dz = 0, \quad \oint_C cos z dz = 0, \quad \oint_C z^n dz = 0, \quad (n = 0, 1, ...)$$

for any closed path, since these functions are entire (analytic for all z).

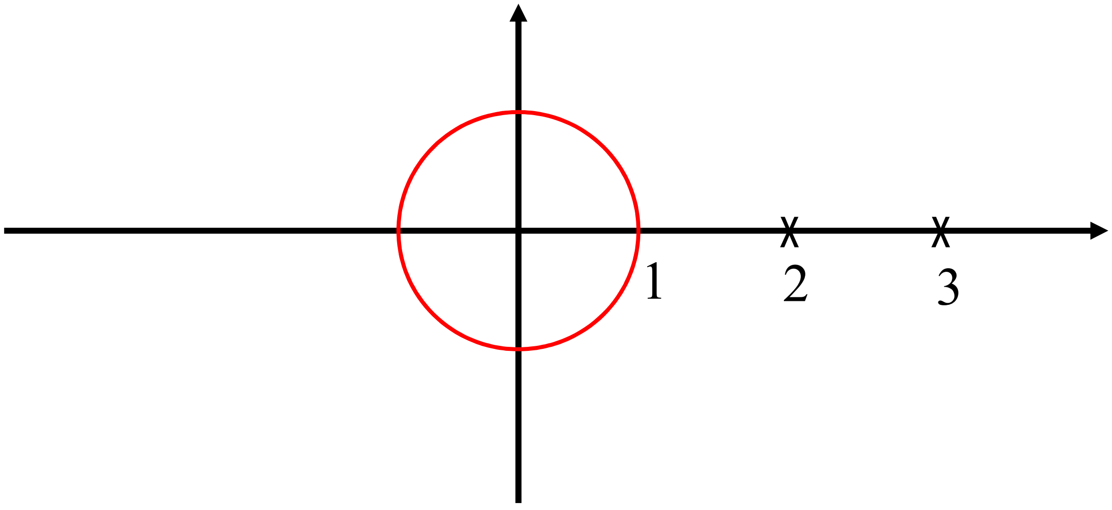
Example (4h)

- Consider

$$I = \oint_C \frac{dz}{z^2 - 5z + 6} = \oint_C \frac{dz}{(z-2)(z-3)},$$

where C is the unit circle $|z| = 1$ oriented in an anticlockwise direction.

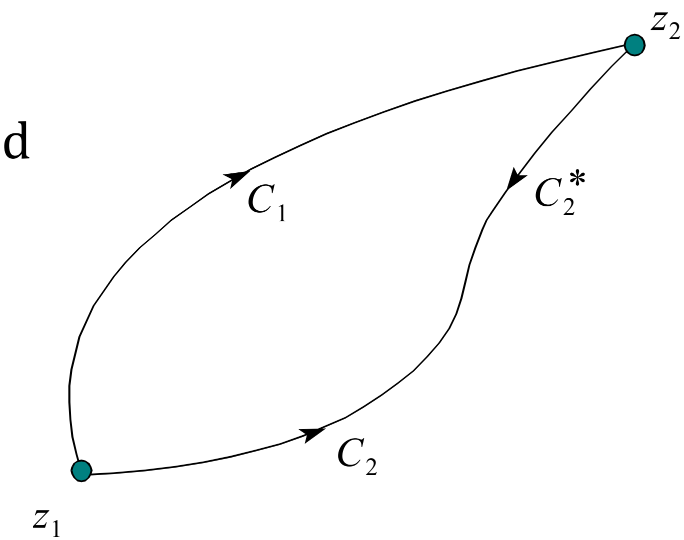
- Now, the integrand $f(z) = \frac{1}{(z-2)(z-3)}$ is analytic everywhere except at $z = 2$ and $z = 3$. Since the curve C does not enclose these two points, $I = 0$ by Cauchy's theorem (4.9).



Applications of Cauchy's Theorem

Applications:

1. If $f(z)$ is analytic in a simply connected domain D, then the integral of $f(z)$ is independent of path in D.



$$\int_{C_1} f(z)\,dz + \int_{C_2^*} f(z)\,dz = 0$$

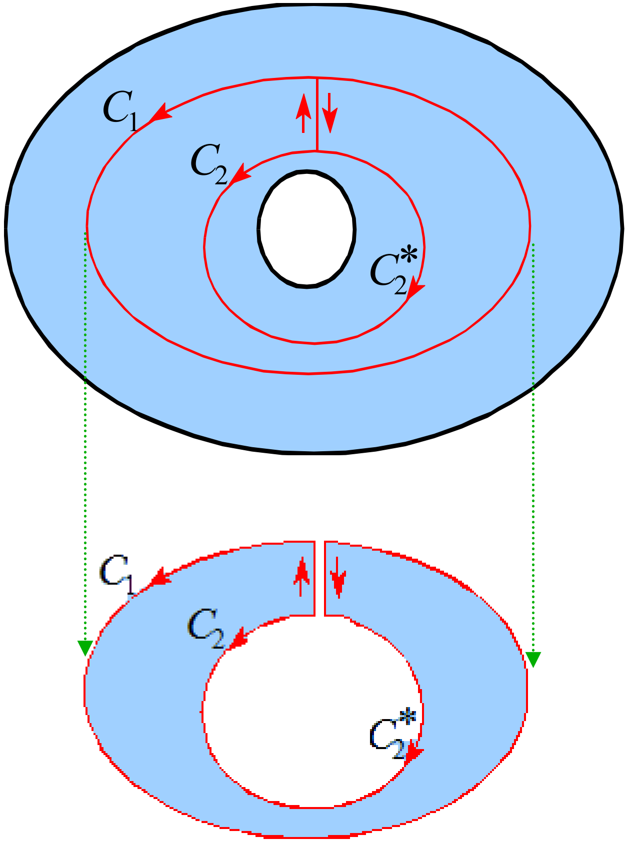
$$\Rightarrow \quad \int_{C_1} f(z)\,dz = -\int_{C_2^*} f(z)\,dz = \int_{C_2} f(z)\,dz$$

Applications (cont.)

2. Consider a doubly connected domain D. If the function $f(z)$ is analytic in D, then the integral of $f(z)$ is the same around any closed path that encircles the opening.

$$\int_{C_1} f(z)\,dz + \int_{C_2^*} f(z)\,dz = 0$$

$$\Rightarrow \quad \int_{C_1} f(z)\,dz = -\int_{C_2^*} f(z)\,dz = \int_{C_2} f(z)\,dz$$



Note that as such we can choose C_2 to be a circle…

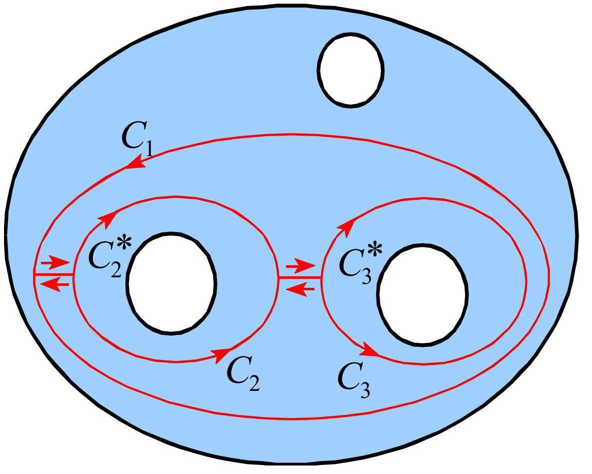
Applications (cont.)

3. The integral along a closed path C_1 of the function $f(z)$ which is analytic in the multiply connected domain D, is given by the sum of the integrals around paths which encircle all openings within the region bounded by C_1, e.g.

$$\int_{C_1} f(z)\,dz + \int_{C_2^*} f(z)\,dz + \int_{C_3^*} f(z)\,dz = 0$$

Thus

$$\int_{C_1} f(z)\,dz = -\int_{C_2^*} f(z)\,dz - \int_{C_3^*} f(z)\,dz = \int_{C_2} f(z)\,dz + \int_{C_3} f(z)\,dz$$



Note that as such we can choose both C_2 and C_3 to be a circle…

Complex Analysis – 5…

1 Complex Numbers

2 Functions of One Complex Variable

3 Complex Differentiation

4 Complex Integration and Cauchy's Theorem

5 **Cauchy Integral Formula**

6 Complex Series, Power Series and Taylor Series

Recap...

Cauchy's Integral Theorem

Theorem (4.3 Cauchy's Integral Theorem)

If $f(z)$ is analytic in a simply connected domain D, then

$$\boxed{\oint_C f(z)\,dz = 0} \tag{4.9}$$

for every simple closed path C in D.

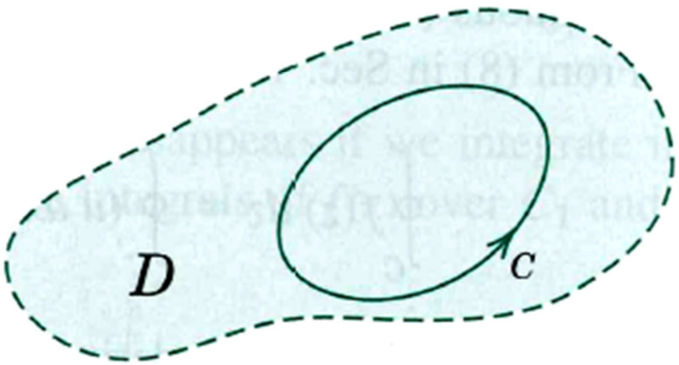


Fig. 4.4. Cauchy's integral theorem.

Cauchy Integral Formula

Theorem (5.1 Cauchy Integral Formula)

Let $f(z)$ be analytic in a simply connected domain D. Then for any point z_0 in D and any simple closed path C in D that encloses z_0 (Fig.5.1)

$$\oint_C \frac{f(z)}{z - z_0} dz = 2\pi i\, f(z_0). \tag{5.1}$$

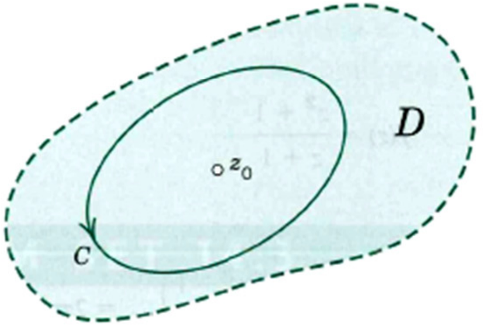
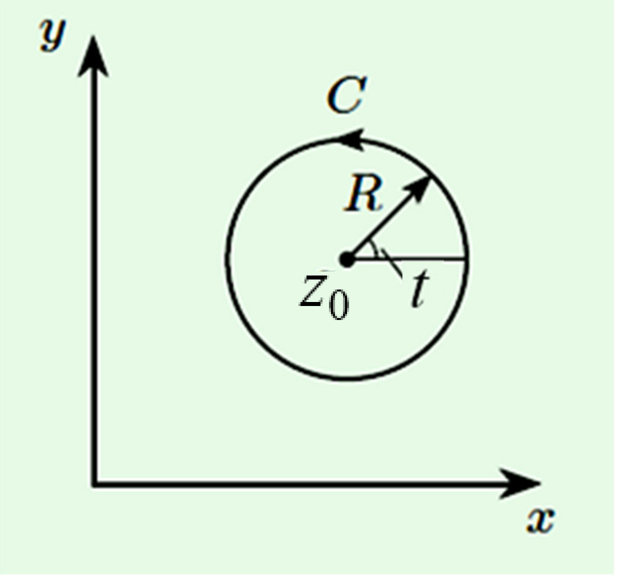


Fig. 5.1. Cauchy's integral formula.

Consider a circle $z = z_0 + Re^{it}$, $t \in [0, 2\pi]$ with centre z_0 . Then

$$\oint_C \frac{f(z)}{z-z_0}\,dz = \int_0^{2\pi} \frac{f(z_0+Re^{it})}{Re^{it}}\, iRe^{it}dt = i\int_0^{2\pi} f(z_0+Re^{it})\,dt$$

Since $f(z)$ is continuous and the integral will have the same value for all values of R, it follows that



$$\oint_C \frac{f(z)}{z-z_0}\,dz = \lim_{R\to 0}\; i\int_0^{2\pi} f(z_0+Re^{it})\,dt$$

$$= i\int_0^{2\pi} f(z_0)\,dt$$

$$= i f(z_0)\int_0^{2\pi} dt$$

$$= 2\pi i f(z_0)$$

Example (5a)

- Evaluate

$$I = \oint_C \frac{e^z}{(z-2)(z+4)}\, dz,$$

where C is a counterclockwise circle of radius 3, centered at the origin.

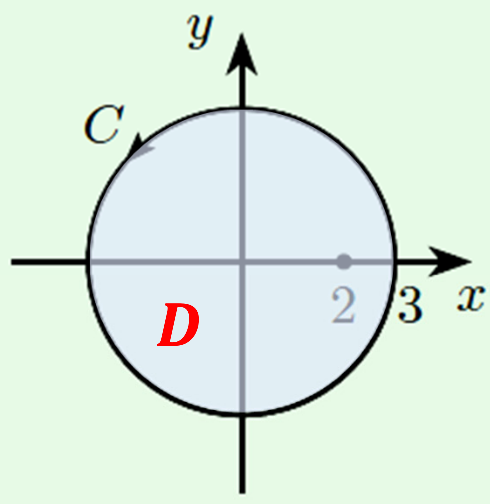


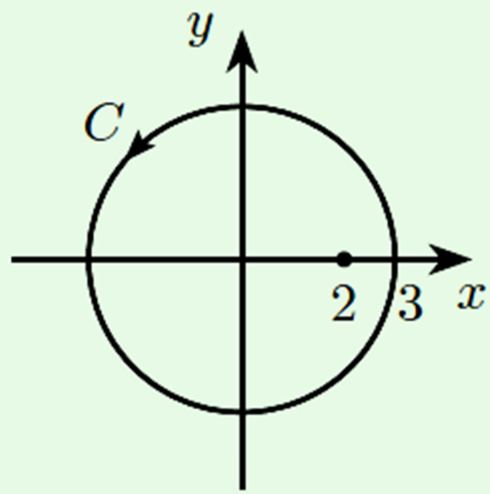
Fig. 5.3. The contour C.

Example (5a)

- Evaluate

$$I = \oint_C \frac{e^z}{(z-2)(z+4)}\, dz,$$

where C is a counterclockwise circle of radius 3, centered at the origin.



- Let $f(z) = \dfrac{e^z}{z+4}$ and $z_0 = 2$, then $f(z)$ is analytic inside C. Hence, by the Cauchy integral formula (5.1),

$$I = \oint_C \frac{f(z)}{z-z_0}\, dz = 2\pi i\, f(z_0) = 2\pi i \left(\frac{e^2}{6}\right) = \frac{\pi e^2 i}{3}.$$

- The Cauchy integral formula enables us to evaluate any integral where the integrand has a "first order singularity" at some point $z = z_0$ within the contour C. If the singularity is second order or higher, then we have the **generalized Cauchy integral formula**

$$\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz = \frac{2\pi i}{n!} f^{(n)}(z_0), \tag{5.2}$$

where $n = 0, 1, 2, \ldots$ if we have the same assumption as in Theorem 5.1.

$$\oint_C \frac{f(z)}{z-z_0}\,dz = 2\pi i\, f(z_0).$$

From the generalized Cauchy integral formula...

$$\boxed{\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz = \frac{2\pi i}{n!} f^{(n)}(z_0),} \qquad (5.2)$$

- **Remark:** Observe that having assumed only that $f(z)$ is analytic (once differentiable), one finds with no further assumption that $f(z)$ possesses derivatives of all orders:

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz.$$

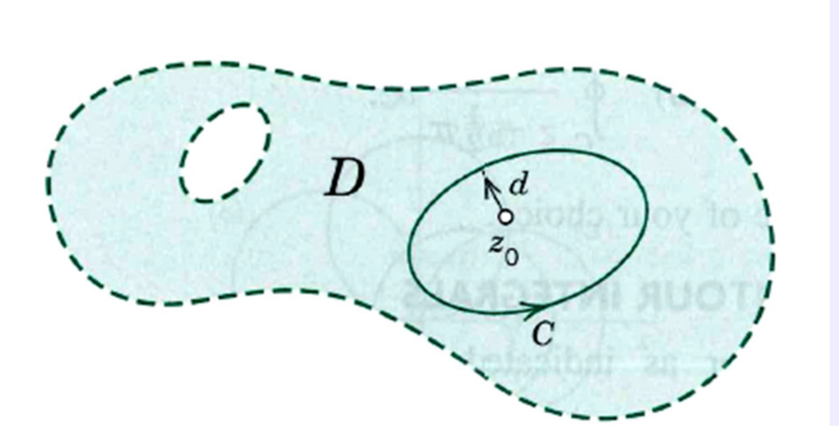
Derivatives of Analytic Functions

Theorem (5.2 Derivatives of Analytic Function)

If $f(z)$ is analytic in a domain D, then it has derivatives of all orders in D, which are then also analytic functions in D. The values of these derivatives at a point z_0 in D are given by the formulas

$$f'(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^2}\, dz.$$

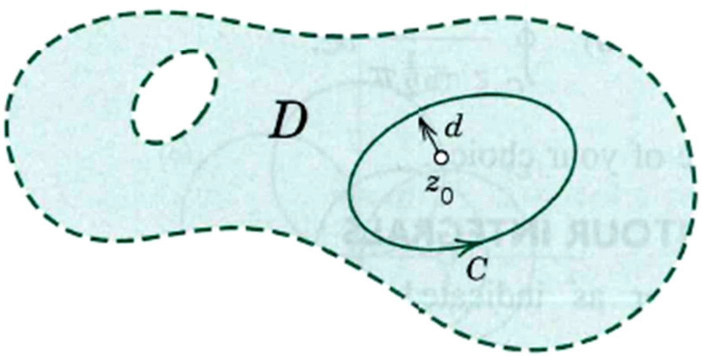
$$f''(z_0) = \frac{2!}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^3}\, dz.$$



or in general

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}}\, dz. \qquad (5.3)$$

Here C is any simple closed path in D that encloses z_0 and whose full interior belongs to D; and we integrate counterclockwise around C.



Example (5a)

- Evaluate

$$I = \oint_C \frac{e^z}{z^3}\, dz,$$

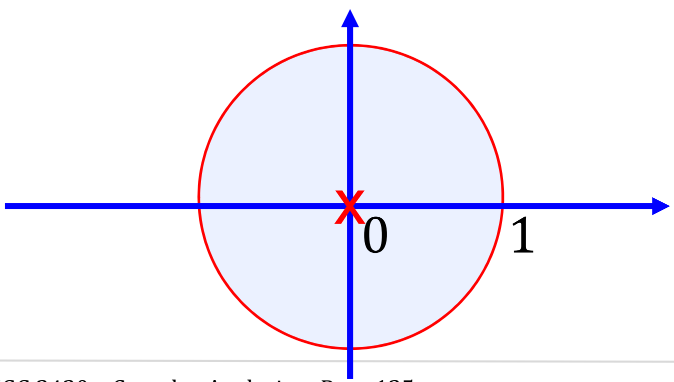
where C is the unit circle $|z| = 1$ oriented in an anticlockwise direction.

- Rewrite I as follows

$$I = \oint_C \frac{e^z}{(z-0)^3}\, dz$$

for comparison with the generalized Cauchy integral formula (5.2). It can be seen that $n = 2$, $z_0 = 0$, and $f(z) = e^z$ so (5.2) gives

$$I = \frac{2\pi i}{2!}\left(\frac{d^2}{dz^2}e^z\right)\Big|_{z=0} = \pi i.$$



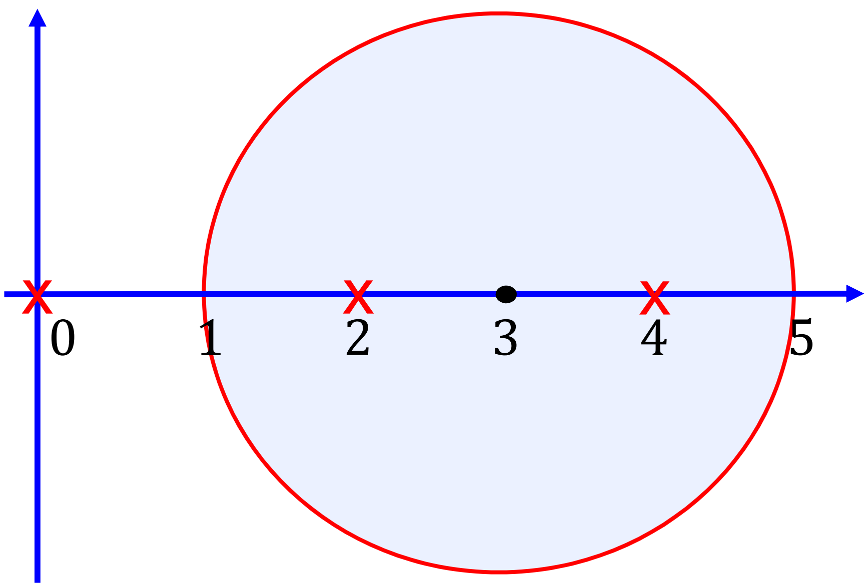
$$\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\, dz = \frac{2\pi i}{n!} f^{(n)}(z_0), \qquad (5.2)$$

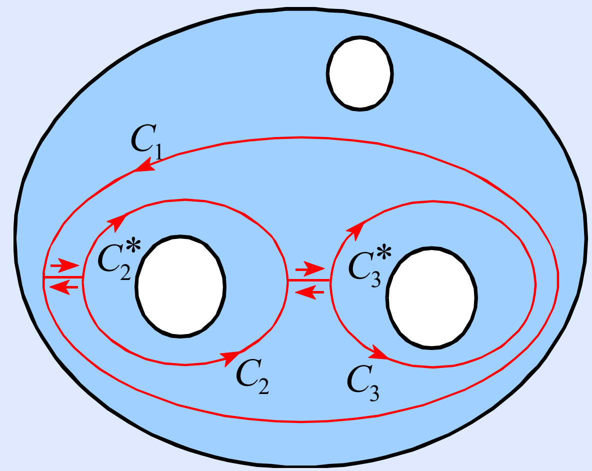
Example (5b)

- Evaluate

$$I = \oint_C \frac{z+1}{z(z-2)(z-4)^3}\, dz,$$

where C is the circle $|z-3| = 2$ oriented in an anticlockwise direction.



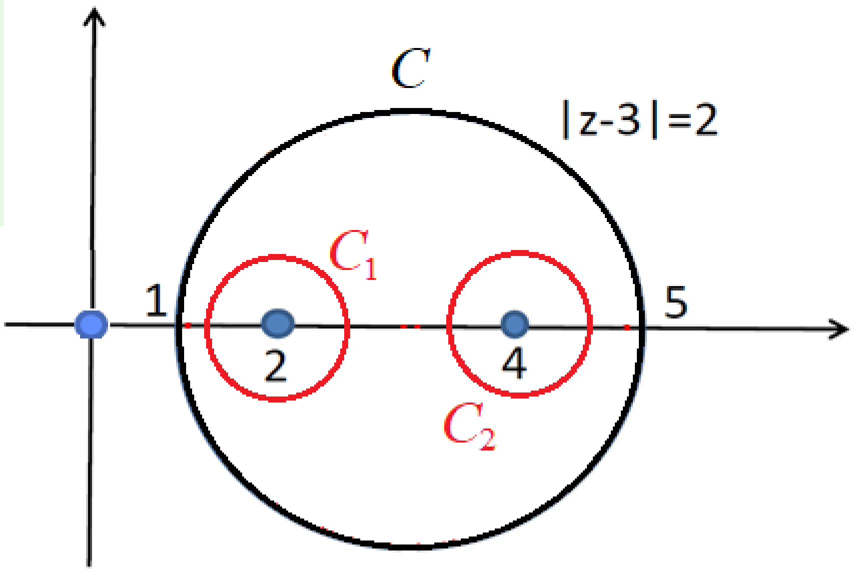


$$\oint_{C_1} f(z)dz = \oint_{C_2} f(z)dz + \oint_{C_3} f(z)dz$$

- The integrand has singularities at $z = 0, 2$ and 4, of which 2 and 4 fall within the contour C. If we deform C into two closed contours C_1 and C_2 so that 2 lies only within C_1 and 4 lies only within C_2, then the generalized Cauchy integral formula (5.2) gives

$$\begin{aligned} I &= \oint_{C_1} \left[\frac{z+1}{z(z-4)^3}\right] \frac{dz}{z-2} + \oint_{C_2} \left[\frac{z+1}{z(z-2)}\right] \frac{dz}{(z-4)^3} \\ &= 2\pi i \left[\frac{z+1}{z(z-4)^3}\right]\Bigg|_{z=2} + \frac{2\pi i}{2!} \frac{d^2}{dz^2} \left[\frac{z+1}{z(z-2)}\right]\Bigg|_{z=4} \\ &= -\frac{3\pi i}{8} + \frac{23\pi i}{64} \\ &= -\frac{\pi i}{64}. \end{aligned}$$

$$\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\, dz = \frac{2\pi i}{n!} f^{(n)}(z_0),$$



Complex Analysis – 6…

1 Complex Numbers
2 Functions of One Complex Variable
3 Complex Differentiation
4 Complex Integration and Cauchy's Theorem
5 Cauchy Integral Formula
6 **Complex Series, Power Series and Taylor Series**

Sequences

- A **sequence** is obtained by assigning to each positive integer n a number z_n, called a **term** of the sequence, and is written

$$z_1, z_2, \ldots, \quad or \quad \{z_1, z_2, \ldots\} \quad or \quad \{z_n\}$$

- A **real sequence** is one whose terms are real.

Examples…

$$1, 2, 3, \ldots, n, \ldots\ldots$$

and

$$2+i, (2+i)^2, (2+i)^3, \ldots, (2+i)^n, \ldots$$

- A **convergent sequence** $z_1, z_2, \ldots$ is one that has a limit c, written

$$\lim_{n\to\infty} z_n = c \quad or\ simply \quad z_n \to c.$$

By definition of **limit**, this means that for every $\epsilon > 0$ we can find an N such that $|z_n - c| < \epsilon \quad for\ all\ n > N$. Geometrically, all term z_n with $n > N$ lie in the open disk of radius ϵ and center c, and only finitely many terms do not lie in that disk.

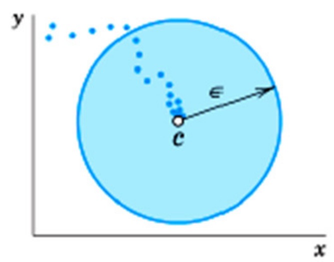


Fig. 6.1. Convergent complex sequence.

- A **divergent sequence** is one that does not converge.

Example (6a)

- The sequence

$$\{\frac{i^n}{n}\} = \{i, -\frac{1}{2}, -\frac{i}{3}, \frac{1}{4}, \ldots\}$$

  is convergent with limit 0.

- The sequence

$$\{z_n\} = \{(1+i)^n\}$$

  is divergent.

Theorem (6.1 Sequences of the Real and the Imaginary Parts)

A sequence $z_1, z_2, \ldots, z_n, \ldots$ of complex numbers $z_n = x_n + iy_n$ (where $n = 1, 2, \ldots$) converges to $c = a + ib$ if and only if the sequence of the real parts $x_1, x_2, \ldots$ converges to a and the sequence of the imaginary parts $y_1, y_2, \ldots$ converges to b.

Series

- Given a sequence $z_1, z_2, \ldots, z_m, \ldots$, we may form the sequence of the sums

$$s_1 = z_1, \quad s_2 = z_1 + z_2, \quad s_3 = z_1 + z_2 + z_3, \ldots$$

and in general

$$s_n = z_1 + z_2 + \ldots + z_n \qquad (n = 1, 2, \ldots).$$

s_n is called the **nth partial sum** of the **series**

$$\sum_{m=1}^{\infty} z_m = z_1 + z_2 + \ldots.$$

The $z_1, z_2, \ldots$ are called the **terms** of the series.

- A **convergent series** is one whose sequence of partial sums converges, i.e.,

$$\lim_{n\to\infty} s_n = s. \quad \textit{Then we write} \quad s = \sum_{m=1}^{\infty} z_m = z_1 + z_2 + \ldots .$$

  and call s the **sum** of the series. A series that is not convergent is called **divergent series**.

Theorem (6.2 Real and the Imaginary Parts)

A series $\sum_{m=1}^{\infty} z_m = z_1 + z_2 + \ldots .$ *with* $z_m = x_m + iy_m$ *converges and has the sum* $s = u + iv$ *if and only if* $x_1 + x_2 + \ldots$ *converges and has the sum* u *and* $y_1 + y_2 + \ldots$ *converges and has the sum* v.

Theorem (6.3 Divergence)

If a series $z_1 + z_2 + \ldots.$ *converges, then* $\lim_{m\to\infty} z_m = 0$. *Hence if this does not hold, the series diverges.*

- For a simple test, the series $z_1 + z_2 + \ldots. + z_n + \ldots$ converges only if $z_n \to 0$ as $n \to \infty$. On the other hand, if a complex series does not converge, it **diverges**.

Example (6b)

- Determine the convergence or divergence of the series

$$\sum_{n=0}^{\infty} \left(\frac{3+n}{4+n}\right)^{100}.$$

- Since $\left(\frac{3+n}{4+n}\right)^{100} = \left(\frac{\frac{3}{n}+1}{\frac{4}{n}+1}\right)^{100} \to 1$ as $n \to \infty$, the series diverges.

Theorem (6.4 Cauchy's Convergence Principle for Series)

A series $z_1 + z_2 + \dots .$ is convergent if and only if for every given $\epsilon > 0$ (no matter how small) we can find an N (which depends on ϵ, in general) such that

$$|z_{n+1} + z_{n+2} + \dots + z_{n+p}| < \epsilon \qquad for\ every\ n > N\ and\ p = 1, 2, \dots$$

$$z_1 + z_2 + \cdots + z_N \;\; + z_{N+1} + z_{N+2} + \cdots$$

$$\left| z_{N+1} + z_{N+2} + \cdots \right| < \varepsilon$$

← the tail of the series

- **Absolute Convergence**: A series $z_1 + z_2 + \ldots.$ is called **Absolute Convergent** if the series of the absolute values of the terms

$$\sum_{m=1}^{\infty} |z_m| = |z_1| + |z_2| + \ldots$$

is convergent.

If $z_1 + z_2 + \ldots.$ converges but $|z_1| + |z_2| + \ldots$ diverges, then the series $z_1 + z_2 + \ldots.$ is called **Conditionally Convergent**.

Example (6c) (A conditionally Convergent Series)

The series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + - \ldots \qquad = \ln 2$$

converges, but only conditionally since the harmonic series diverges.

Harmonic series… $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots.$

Theorem (6.5 Comparison Test)

If a series $z_1 + z_2 + \ldots.$ is given and we can find a convergent series $b_1 + b_2 + \ldots.$ which nonnegative real terms such that $|z_1| \leq b_1, |z_2| \leq b_2, \ldots,$ then the given series converges, even absolutely.

- A good comparison series is the geometric series, which behaves as follows.

Theorem (6.6 Geometric Series)

The geometric series

$$\sum_{m=0}^{\infty} q^m = 1 + q + q^2 + \ldots$$

converges with the sum $\frac{1}{1-q}$ *if* $|q| < 1$ *and diverges if* $|q| \geq 1$.

Example (6d)

- Determine the convergence or divergence of the series

$$\sum_{n=0}^{\infty} \frac{i^n}{n!} = 1 + i + \frac{i^2}{2!} + \frac{i^3}{3!} + \cdots .$$

- Now, $\left|\frac{i^n}{n!}\right| = \frac{1}{n!} < \frac{1}{2^n}$ for all $n \geq 4$, and $\sum_{n=0}^{\infty} \frac{1}{2^n} = \frac{1}{1-\frac{1}{2}} = 2$ is a convergent geometric series. By comparison test, the original series converges.

$$n! = \underbrace{1\cdot 2\cdot 3\cdot 4}_{24}\cdot 5\cdots n > \underbrace{2\cdot 2\cdot 2\cdot 2}_{16}\cdot 2\cdots 2 = 2^n \quad \Rightarrow \quad \frac{1}{n!} < \frac{1}{2^n}, \quad n \geq 4$$

The geometric series

$$\sum_{m=0}^{\infty} q^m = 1 + q + q^2 + \ldots$$

converges with the sum $\frac{1}{1-q}$ *if* $|q| < 1$ *and diverges if* $|q| \geq 1$.

Theorem (6.7 Ratio Test)

If a series $z_1 + z_2 + \dots$ *with* $z_n \neq 0 \quad (n = 1, 2, \dots)$ *has the property that for every* n *greater than some* N,

$$\left| \frac{z_{n+1}}{z_n} \right| \leq q < 1 \qquad (n > N) \tag{6.1}$$

(where $q < 1$ *is fixed), this series converges absolutely. If for every* $n > N$,

$$\left| \frac{z_{n+1}}{z_n} \right| \geq 1 \qquad (n > N) \tag{6.2}$$

the series diverges.

- The inequality Eq.(6.1) implies $\left|\frac{z_{n+1}}{z_n}\right| < 1$, but this does **not** imply convergence, as we see from the harmonic series, which satisfies $\frac{z_{n+1}}{z_n} = \frac{n}{n+1} < 1$ for all n but diverges.

Theorem (6.8 Ratio Test)

If a series $z_1 + z_2 + \ldots$. *with* $z_n \neq 0 \quad (n = 1, 2, \ldots)$ *is such that* $\lim_{n\to\infty} \left| \frac{z_{n+1}}{z_n} \right| = L$, *then the series converges absolutely if* $L < 1$ *and diverges if* $L > 1$. *No information is obtained if* $L = 1$ *or if the limit does not exist.*

Example (6e)

- Determine the convergence or divergence of the series

$$\sum_{n=0}^{\infty} \frac{(1+i)^n}{n!}.$$

- Since

$$\lim_{n\to\infty} \left| \frac{z_{n+1}}{z_n} \right| = \lim_{n\to\infty} \left| \frac{\frac{(1+i)^{n+1}}{(n+1)!}}{\frac{(1+i)^n}{n!}} \right| = \lim_{n\to\infty} \left| \frac{1+i}{n+1} \right| = \sqrt{2} \lim_{n\to\infty} \left| \frac{1}{n+1} \right| = 0,$$

$L = 0$. By ratio test, the series converges.

Power Series

- Generally, the terms in a series may be some functions of z, then the series becomes

$$\sum_{n=0}^{\infty} f_n(z) = f_0(z) + f_1(z) + \cdots .$$

The set of all points in the z-plane for which the series converges is called the **region of convergence** of the series.

- If we let $f_n(z) = a_n(z - z_0)^n$, where a_n's and z_0 are some complex (or real) constants in general, then the resulting series

$$\boxed{\sum_{n=0}^{\infty} a_n(z - z_0)^n = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots ,} \qquad (6.5)$$

is called a **power series** (about the point $z = z_0$). a_n's are called coefficients of the power series (6.5).

Convergence Behavior of Power Series

- Power series have variable terms (functions of z), but if we fix z, then all the concepts for series with constant terms in the last section apply.
- A series with variable terms will converge for some z and diverge for others.
- For a power series, e.g., (6.5), it may converge in a disk with center z_0 or in the whole z-plane or only at z_0.

Example (6g)

The geometric series

$$\sum_{n=0}^{\infty} z^n = 1 + z + z^2 + \ldots$$

converges absolutely if $|z| < 1$ and diverges if $|z| \geq 1$.

Example (6h)

The power series

$$\sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots$$

is absolutely convergent for every z. (Check by using the ratio test)

$$\lim_{n\to\infty}\left|\frac{z_{n+1}}{z_n}\right| = \lim_{n\to\infty}\left|\frac{\dfrac{z^{n+1}}{(n+1)!}}{\dfrac{z^n}{n!}}\right| = \lim_{n\to\infty}\left|\frac{z}{n+1}\right| = 0$$

Theorem (6.11 Convergence of a Power Series)

(a) *Every power series (6.5) converges at the center z_0.*

(b) *If (6.5) converges at a point $z = z_1 \neq z_0$, it converges absolutely for every z closer to z_0 than z_1, that is, $|z - z_0| < |z_1 - z_0|$.*

(c) *If (6.5) diverges at $z = z_2$, it diverges for every z fartheraway from z_0 than z_2.*

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots \qquad (6.5)$$

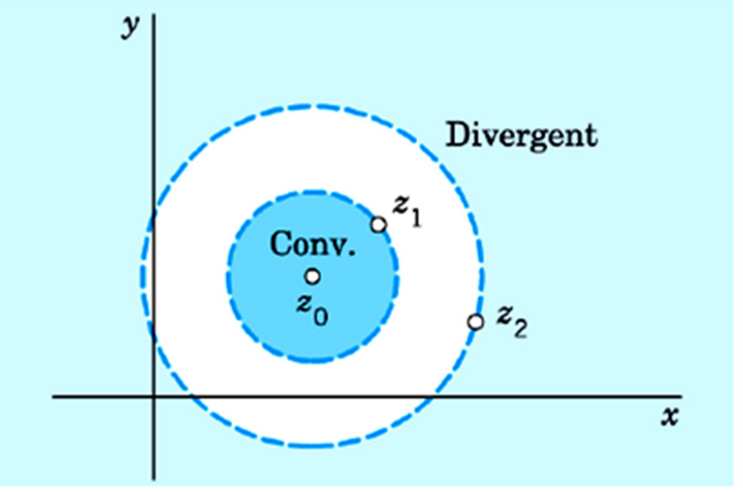


Fig. 6.2. Theorem 6.11.

Radius of Convergence of a Power Series

- To determine the smallest circle with center z_0 that includes all the points at which a given power series (6.5) converges.
- Let R denote its radius, the circle,

$$|z - z_0| = R$$

is called the **circle of convergence** and its radius R the **radius of convergence** of (6.5).

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots \qquad (6.5)$$

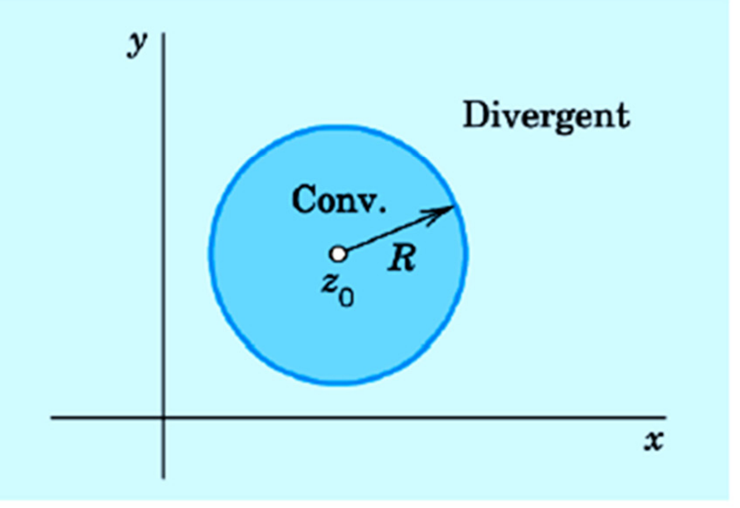


Fig. 6.3. Circle of convergence

- Theorem 6.11 implies convergence everywhere within that circle, i.e., for all z for which

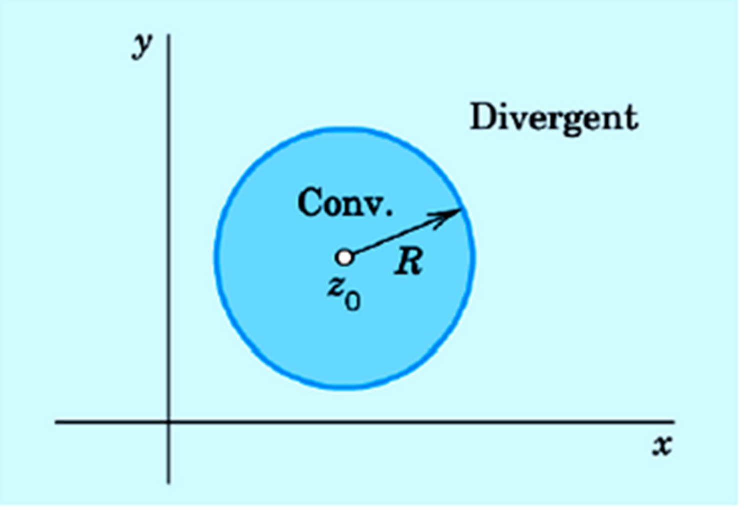
$$|z - z_0| < R \tag{6.6}$$

- Also, since R is as small as possible, the series diverges for all z for which

$$|z - z_0| > R \tag{6.7}$$

- No general statements can be made about the convergence of a power series (6.5) on the circle of convergence.

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots \tag{6.5}$$



Example (6i)

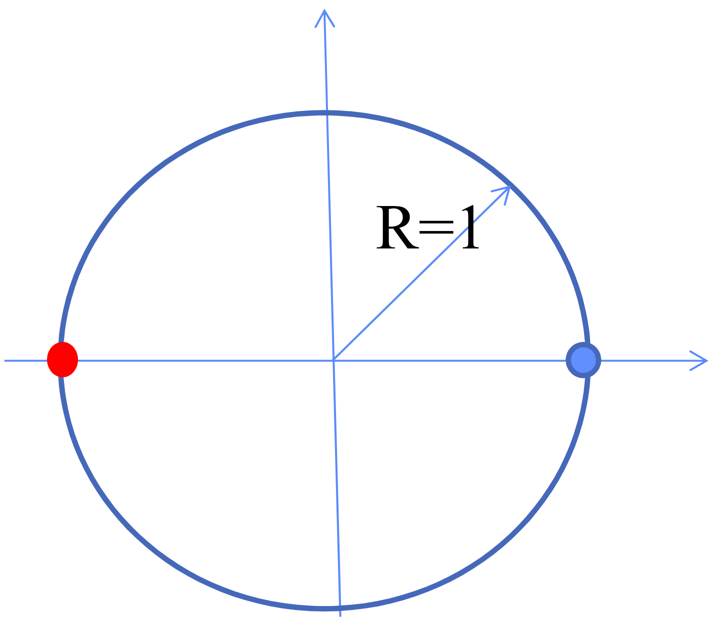
On the circle of convergence (radius $R = 1$ in all three series),

(a) $\sum \frac{z^n}{n^2}$ converges everywhere since $\sum \frac{1}{n^2}$ converges.

(b) $\sum \frac{z^n}{n}$ converges at -1 but diverges at 1.

(c) $\sum z^n$ diverges everywhere.

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$



$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + - \ldots = \ln 2$$

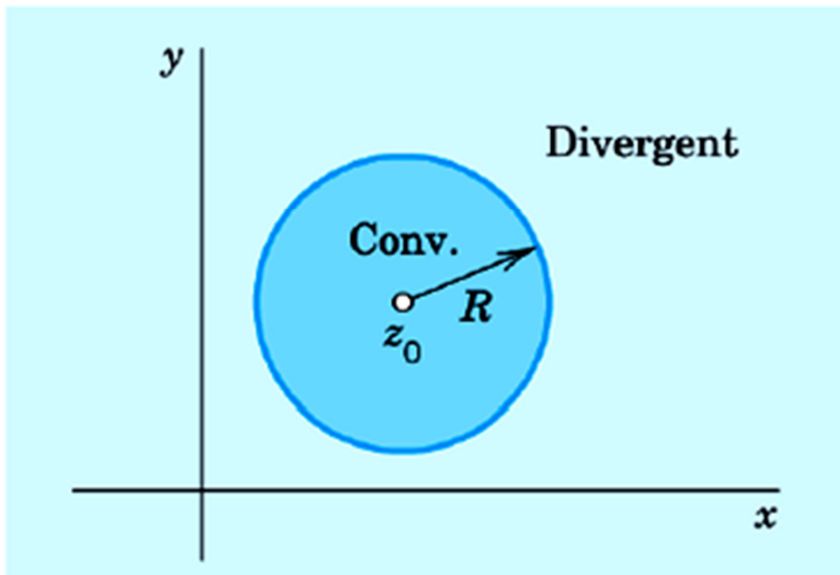
$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$$

Determination of the Radius of Convergence from the Coefficients

Notations $R = \infty$ *and* $R = 0$:

(a) $R = \infty$ if the series (6.5) converges for all z,

(b) $R = 0$ if (6.5) converges only at the center $z = z_0$.



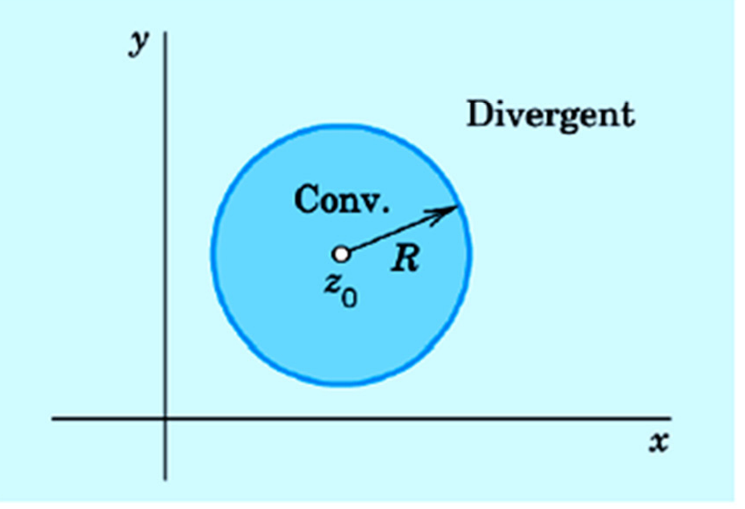
Theorem (6.12 Radius of Convergence R)

Suppose that the sequence $\left|\frac{a_{n+1}}{a_n}\right|$, $n = 1, 2, \ldots$, *converges with limit* L^*. *If* $L^* = 0$, *then* $R = \infty$; *that is, the power series (6.5) converges for all* z. *If* $L^* \neq 0$ $(hence\ L^* > 0)$, *then*

$$\boxed{R = \frac{1}{L^*} = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|} \qquad (6.8)$$

If $\left|\frac{a_{n+1}}{a_n}\right| \to \infty$, *then*, $R = 0$ *(convergence only at the center* z_0*).*

$$\sum_{n=0}^{\infty} a_n(z-z_0)^n = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots \qquad (6.5)$$



$$\sum_{n=0}^{\infty} a_n(z-z_0)^n = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots \qquad (6.5)$$

By Theorem 6.8 (ratio test), the above series converges when

$$\lim_{n\to\infty}\left|\frac{z_{n+1}}{z_n}\right| = \lim_{n\to\infty}\left|\frac{a_{n+1}(z-z_0)^{n+1}}{a_n(z-z_0)^n}\right| = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|\cdot|z-z_0| = L^*\cdot|z-z_0| < 1$$

$$\Downarrow$$

$$|z-z_0| < \frac{1}{L^*} = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = R$$

Theorem (6.8 Ratio Test)

If a series $z_1 + z_2 + \ldots.$ *with* $z_n \neq 0 \quad (n = 1, 2, \ldots)$ *is such that* $\lim_{n\to\infty}\left|\frac{z_{n+1}}{z_n}\right| = L$, *then the series converges absolutely if* $L < 1$ *and diverges if* $L > 1$. *No information is obtained if* $L = 1$ *or if the limit does not exist.*

Example (6j) (Radius of Convergence)

By Eq.(6.8), the radius of convergence of the power series $\sum_{n=0}^{\infty} \frac{(2n)!}{(n!)^2}(z-3i)^n$ is

$$R = \lim_{n\to\infty}\left[\frac{\frac{(2n!)}{(n!)^2}}{\frac{(2n+2)!}{((n+1)!)^2}}\right]$$

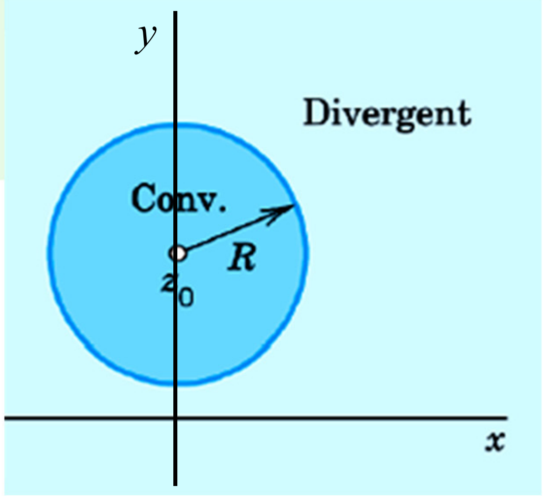
$$= \lim_{n\to\infty}\left[\frac{(2n!)}{(2n+2)!}\cdot\frac{((n+1)!)^2}{(n!)^2}\right] = \lim_{n\to\infty}\frac{(2n)!}{(2n)!(2n+1)(2n+2)}\cdot\frac{(n!)^2(n+1)^2}{(n!)^2}$$

$$= \lim_{n\to\infty}\frac{(n+1)^2}{(2n+2)(2n+1)}$$

$$= \frac{1}{4}.$$

The series converges in the open disk $|z-3i| < \frac{1}{4}$ of radius $\frac{1}{4}$ and center $3i$.

$$R = \frac{1}{L^*} = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$$



Functions Given by Power Series

- To simplify the formulas, we take $z_0 = 0$, and write Eq. (6.5) as

$$\boxed{\sum_{n=0}^{\infty} a_n z^n} \tag{6.9}$$

- If any given power series (6.9) has a nonzero radius of convergence R (thus $R > 0$), its sum is a function of z, say $f(z)$. Then we write

$$\boxed{f(z) = \sum_{n=0}^{\infty} a_n z^n = a_0 + a_1 z + a_2 z^2 + \ldots} \tag{6.10}$$

we say that $f(z)$ is <u>represented</u> by the power series or that it is <u>developed</u> in the power series.

Uniqueness of a Power Series Representation

Theorem (6.13 Uniqueness of Power Series)

*Let the power series $a_0 + a_1 z + a_2 z^2 + \ldots$ and $b_0 + b_1 z + b_2 z^2 + \ldots$ both be convergent for $|z| < R$, where R is positive, and let them both have the same sum for all these z, Then the series are identical, that is, $a_0 = b_0, a_1 = b_1, a_2 = b_2, \ldots$. Hence if a function $f(z)$ can be represented by a power series with any center z_0, this representation is **unique**.*

- If a_n, b_n are coefficients of two power series and $a_n = b_n$, then it is sure that

$$\sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} b_n z^n, \tag{6.11}$$

i.e., the two power series are the same about the point $z = 0$. $\Rightarrow$

A function $f(z)$ cannot be represented by two different power series with the same center. That is, if $f(z)$ can at all be developed in a power series with center z_0, the development is unique.

Power Series Represent Analytic Functions

Theorem (6.15 Analytic Functions. Their Derivatives)

A power series with a nonzero radius of convergence R represents an analytic function at every point interior to its circle of convergence.

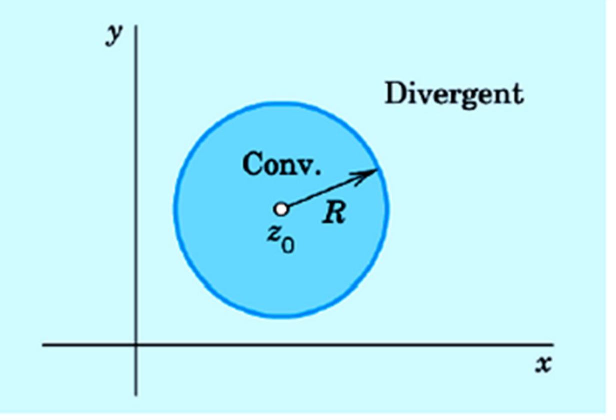
The derivatives of this function are obtained by differentiating the original series term by term.

All the series thus obtained have the same radius of convergence as the original series. **Why?**

Hence, by the first statement, each of them represents an analytic function.

$$f(z) = \sum_{n=0}^{\infty} a_n z^n = a_0 + a_1 z + a_2 z^2 + \dots$$

$$f'(z) = \left(\sum_{n=0}^{\infty} a_n z^n \right)' = 0 + a_1 + a_2 z + 2a_3 z^2 + \cdots$$



Why? We note that

$$f'(z)=\left(\sum_{n=0}^{\infty}a_n z^n\right)'=\sum_{n=0}^{\infty}a_n n z^{n-1}=\sum_{m=0}^{\infty}a_{m+1}(m+1)z^m=\sum_{n=0}^{\infty}a_{n+1}(n+1)z^n$$

$$\Rightarrow \quad R=\lim_{n\to\infty}\left|\frac{a_{n+1}(n+1)}{a_{n+2}(n+2)}\right|=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_{n+2}}\right|\cdot\lim_{n\to\infty}\left|\frac{n+1}{n+2}\right|=\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|\cdot 1=\frac{1}{L^*}$$

For a convergent series, we have...

$$f(z)=\sum_{n=0}^{\infty}a_n(z-z_0)^n=a_0+a_1(z-z_0)+a_2(z-z_0)^2+\cdots \quad \Rightarrow \quad \boxed{f(z_0)=a_0}$$

$$f'(z)=a_1+2a_2(z-z_0)+3a_3(z-z_0)^2+\cdots \quad \Rightarrow \quad \boxed{f'(z_0)=a_1}$$

$$f''(z)=2a_2+3\times 2\times a_3(z-z_0)+\cdots \quad \Rightarrow \quad f''(z_0)=2a_2$$

$$\Rightarrow \quad \boxed{a_2=\frac{f''(z_0)}{2!}} \quad \cdots\cdots \quad \boxed{a_n=\frac{f^{(n)}(z_0)}{n!}}$$

Taylor Series

- If a function $f(z)$ is analytic at $z = z_0$, then it admits derivatives of all orders there by generalized Cauchy integral formula, i.e., $f^{(n)}(z_0)$ exist for any integer $n \geq 0$. If we let $a_n = \frac{f^{(n)}(z_0)}{n!}$ in the power series (6.5), we have

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n. \qquad (6.13)$$

which is called the **Taylor series** of $f(z)$ about the point $z = z_0$.

Brook Taylor
(1685–1731)
English Mathematician

Colin Maclaurin
(1698–1746)
Scottish Mathematician

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n.$$

- or, by (5.3),

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}} dz$$

$\Longrightarrow$

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z^*)}{(z^* - z_0)^{n+1}} dz^*. \qquad (6.14)$$

- If we let $z_0 = 0$ in Taylor series (6.13), then the Taylor series about $z = 0$ is called a **Maclaurin series**, i.e.,

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n. \qquad (6.15)$$

Example (6l)

- Find the Maclaurin series of $\text{Ln}(1+z)$ and find its radius of convergence R.
- Let $f(z) = \text{Ln}(1+z)$. Since $f^{(n)}(z) = \frac{(-1)^{n+1}(n-1)!}{(1+z)^n}, n \geq 1$, we have

$$a_n = \frac{f^{(n)}(0)}{n!} = \frac{(-1)^{n+1}}{n}, \quad n \geq 1,$$

and $a_0 = \text{Ln}(1) = 0$. Now, $L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{n}{n+1} \right| = 1$. Thus $R = \frac{1}{L} = 1$ and the Maclaurin series (6.15) is

$$\text{Ln}(1+z) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{z^n}{n} = z - \frac{z^2}{2} + \frac{z^3}{3} - \cdots, \quad |z| < 1.$$

Let $z = 1 \Rightarrow$ $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + - \ldots = \ln 2$

Important Special Taylor Series

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \qquad = 1 + z + z^2 + \cdots \qquad \text{in } |z| < 1,$$

$$\text{Ln}(1+z) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{z^n}{n} \qquad = z - \frac{z^2}{2} + \frac{z^3}{3} - \cdots \qquad \text{in } |z| < 1,$$

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \qquad = 1 + z + \frac{z^2}{2!} + \cdots \qquad \text{in } |z| < \infty,$$

$$\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots \qquad \text{in } |z| < \infty,$$

$$\cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} \qquad = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots \qquad \text{in } |z| < \infty,$$

$$\sinh z = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!} \qquad = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots \qquad \text{in } |z| < \infty,$$

$$\cosh z = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!} \qquad = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots \qquad \text{in } |z| < \infty.$$

Homework Assignment No: 2

Due Date: 6:00pm, 21 October 2019

Please place your assignment to Assignment Box 3 outside PC Lab (ERB 218)

Flow Chart of Materials Covered in Complex Analysis

$z = x + iy$

Complex number manipulations
Euler's formula

$z(t) = x(t) + iy(t)$

Complex function of a real variable
~ define a ***curve*** on a complex plane

$f(z) = u + iv = u(x, y) + iv(x, y)$

Complex function of a complex variable
~ define a complex mapping from one complex plane to another

$f'(z)$

Complex Differentiation
Cauchy-Riemann equations, analytic functions, singularities

$$\int_C f(z)\,dz = \int_\alpha^\beta f[z(t)]\,z'(t)\,dt$$

Complex integral
~ integration of a complex function (of a complex variable) on a complex curve
~ *upper bound of a complex integration*

$$\oint_C f(z)\,dz = 0$$

Cauchy's Integral theorem
~ valid when $f(z)$ is ***analytic*** for every point encircled by a closed curve C

to be continued on the next page…

Flow Chart of Materials Covered in Complex Analysis (cont.)

$$\oint_C \frac{f(z)}{z-z_0}\,dz = 2\pi i\, f(z_0)$$

Cauchy's integral formula

~ C is a closed curve encloses z_0 and $f(z)$ is analytic for every point inside C

$$\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz = 2\pi i\, \frac{f^{(n)}(z_0)}{n!}$$

Another integral formula

~ $n \geq 0$, C is a closed curve encloses z_0 and $f(z)$ is analytic inside C

$$\sum_{n=0}^{\infty} z_n = z_0 + z_1 + z_2 + \cdots$$

Complex series

~ *Convergence, divergence, absolute convergence, ratio test*

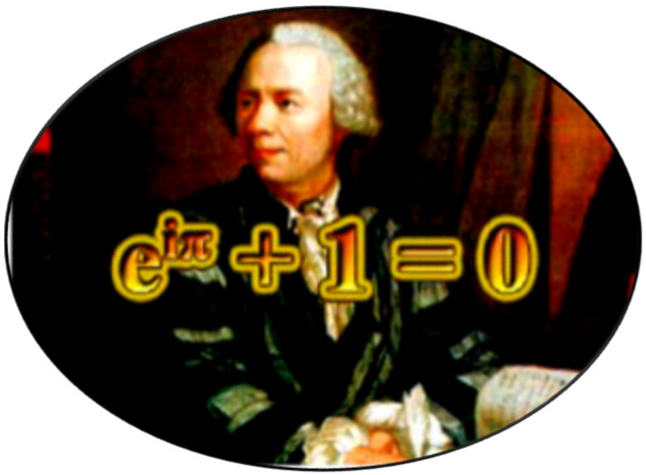
$$\sum_{n=0}^{\infty} a_n (z-z_0)^n$$

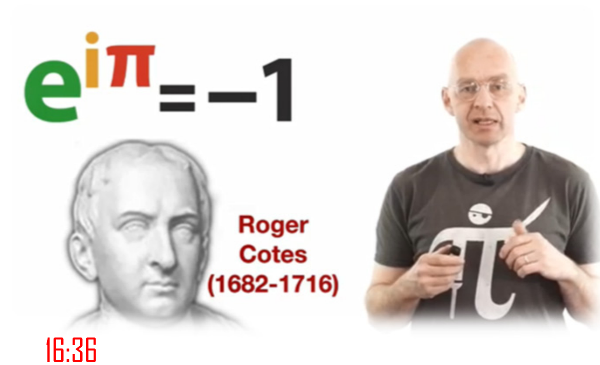
Power series

~ *Region of convergence, ratio test Taylor series*

Remarks for Complex Analysis…

In this part, we have learnt the most beautiful formula in math…





It has all the beautiful constants: e, i, π,1 and 0. We are ready to prove by video…