# EG1103 Electrical Engineering

Ben M. Chen

Associate Professor

Office: E4-6-7

Phone: 874-2289

Email: bmchen@nus.edu.sg

http://vlab.ee.nus.edu.sg/~bmchen



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# Lecture Format & Course Outlines



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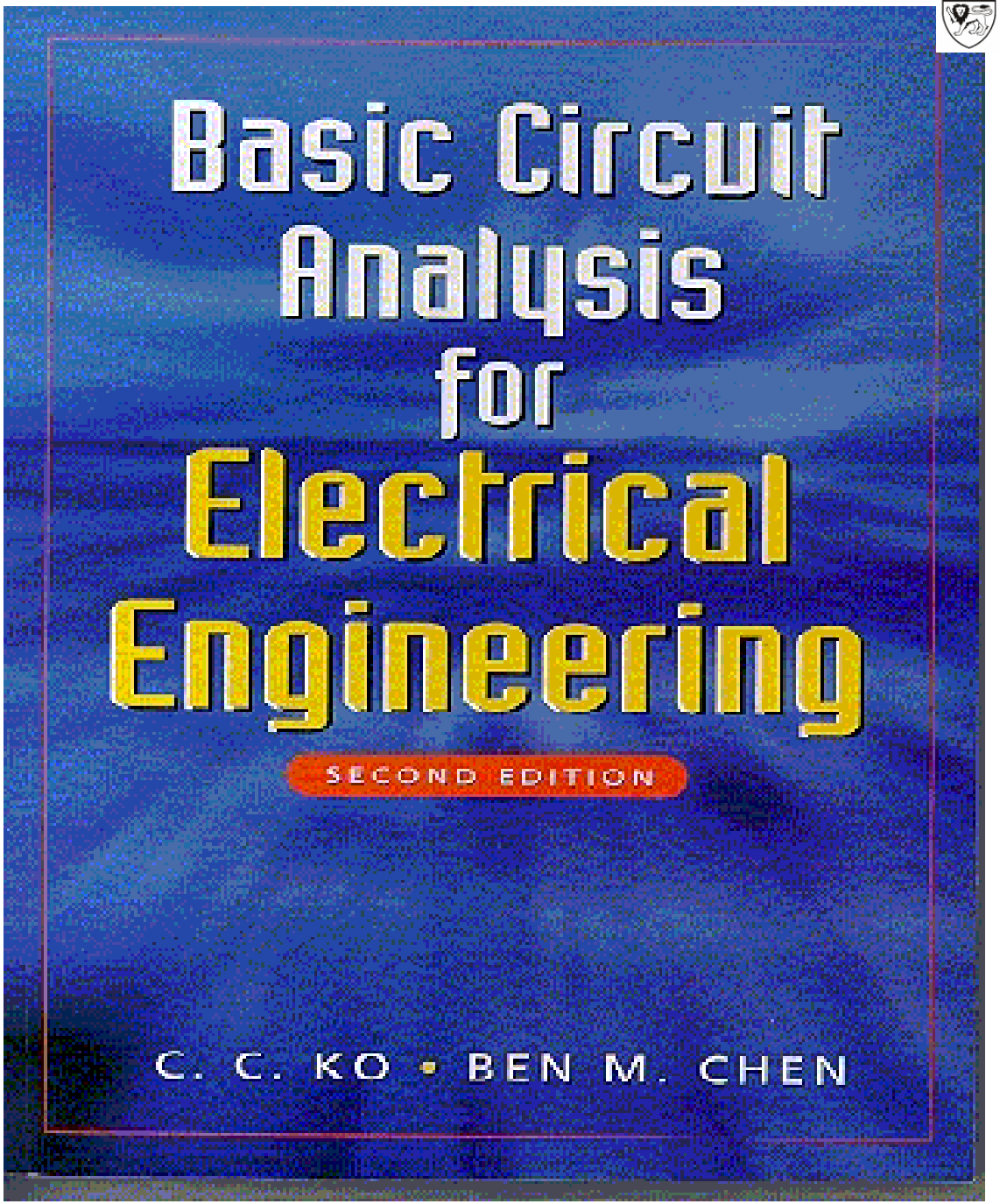
text

book

or

lecture

notes



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# Electrical Engineering

**Well known electrical engineering companies:**

- Singapore Telecom (largest in Singapore)
- Creative Technology (largest manufacturer of PC sound boards)
- Disk Drive Companies (largest producer of PC hard disk Drives)

**"Yan can Cook":**

- Ingredients + Recipe + (Funny Talk) = Good Food

**Electrical Systems:**

- Components + Method + (Funny Talk) = Good Electrical System

**EG1103 Module:**

- To introduce basic electrical components & analysis methods.



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# Reference Textbooks

- **D. E. Johnson, J. R. Johnson and J. L. Hilburn, *Electric Circuits Analysis*, 2nd Ed., Prentice Hall, 1992.**

- **S. A. Boctor, *Electric Circuits Analysis*, 2nd Ed., Prentice Hall, 1992.**



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# Lectures

Lectures will follow closely (but not 100%) the materials in the textbook.

However, certain parts of the textbook will not be covered and examined and this will be made known during the classes.

Attendance is essential.

ASK any question at any time during the lecture.



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# Tutorials

The tutorials will start on Week 4 of the semester. (Week 1 corresponds to the Orientation Week.)

Although you should make an effort to attempt each question before the tutorial, it is NOT necessary to finish all the questions.

Some of the questions are straightforward, but quite a few are difficult and meant to serve as a platform for the introduction of new concepts.

ASK your tutor any question related to the tutorials and the course.



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# Examination

The examination paper is 2-hour in duration.

You will be provided with a list of important results. This list is given under “Summary of Important Results” in the Appendix of the textbook.

To prepare for the examination, you may wish to attempt some of the questions in examinations held in previous years. These papers are actually the Additional Problems in the Appendix of the textbook (no solutions to these problems will be given out to the class).

However, note that the topics covered may be slightly different and some of the questions may not be relevant. Use your own judgement to determine the questions you should attempt.



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# Mid-term Test

There will be an one-hour (actually 50 minutes) test. It will be given some time around mid-term (most likely after the recess week). The test will consists 15% of your final grade, i.e., your final grade in this course will be computed as follows:

Your Final Grade = 15% of Your Mid-term Test Marks (max. = 100)

+ 85% of Your Examination Marks (max = 100)



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# Outline of the Course

## 1. DC Circuit Analysis

SI Units. Voltage, current, power and energy. Voltage and current sources. Resistive circuits. Kirchhoff's voltage and current laws. Nodal and mesh analysis. Ideal and practical sources. Maximum power transfer. Thevenin's and Norton's equivalent circuits. Superposition. Dependent sources. Introduction to non-linear circuit analysis.

## 2. AC Circuits

Root mean square value. Frequency and phase. Phasor. Capacitor and Inductor. Impedance. Power. Power factor. Power factor improvement. Frequency response. Tune circuit. Resonance, bandwidth and Q factor. Periodic signals. Fourier series.



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# Outline of the Course (Cont.)

## 3. Transient

First order RL and RC circuits. Steady state and transient responses. Time constant. Voltage and current continuity. Second order circuit.

## 4. Magnetic Circuit

Magnetic flux and mmf. Ampere's law. Force between surfaces. Transformers.

## 5. Electrical Measurement

Current and voltage measurement. Common instruments. Oscilloscope.



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# Chapter 1. SI Units



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## 1.1 Important Quantities and Base SI Units

| | | |
|---|---|---|
| Length | metre | m |
| Mass, $m$ | kilogram | kg |
| Time, $t$ | second | s |
| Electric current, $i$ | ampere | A |
| Thermodynamic temperature | kelvin | K |
| Plane angle | radian | rad |



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# Chapter 2. DC Circuit Analysis



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## 2.1 Voltage Source

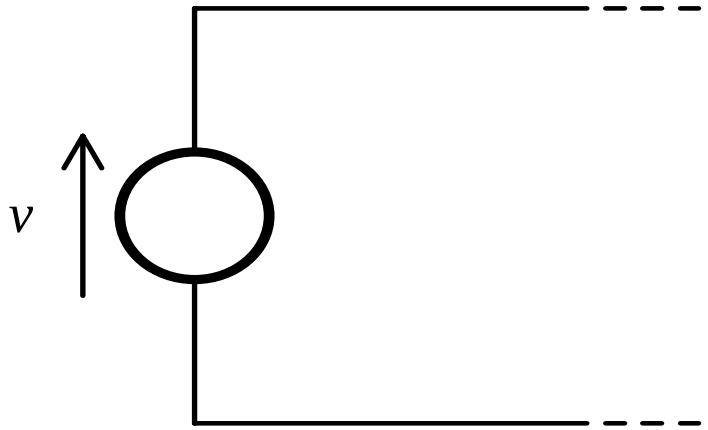
Two common dc (direct current) voltage sources are:

Dry battery (AA, D, C, etc.)

Lead acid battery in car

Regardless of the load connected and the current drawn, the above sources have the characteristic that the supply voltage will not change very much.

The definition for an ideal voltage source is thus one whose output voltage does not depend on what has been connected to it. The circuit symbol is



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Basically, the arrow and the value signifies that the top terminal has a potential of $v$ with respect to the bottom terminal regardless of what has been connected and the current being drawn.

Note that the current being drawn is not defined but depends on the load connected. For example, a battery will give no current if nothing is connected to it, but may be supplying a lot of current if a powerful motor is connected across its terminals. However, in both cases, the terminal voltages will be roughly the same.



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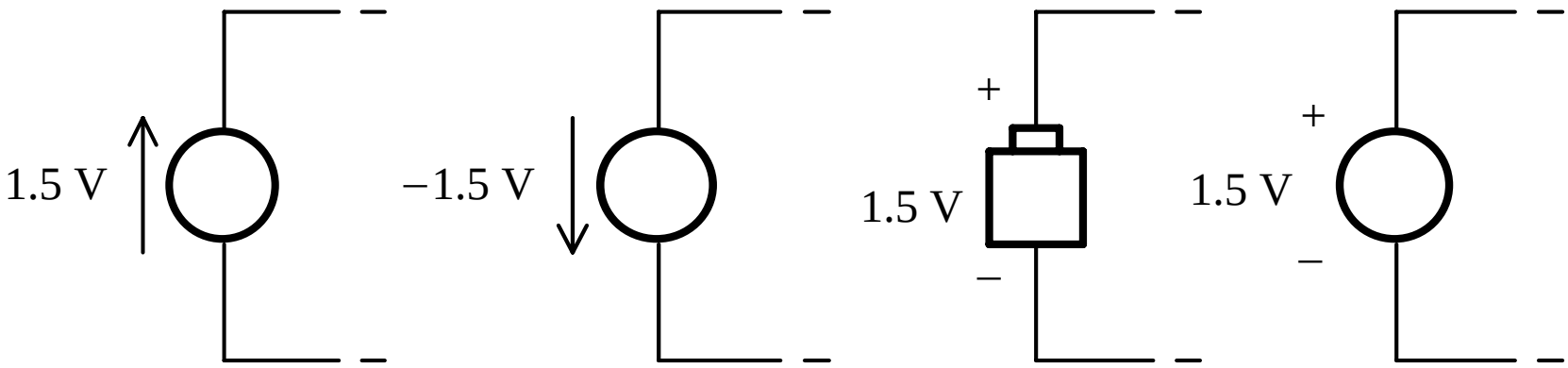
Using the above and other common circuit symbol, the following are identical:




Note that on its own, the arrow does not correspond to the positive terminal. Instead, the positive terminal depends on both the arrow and the sign of the voltage which may be negative.



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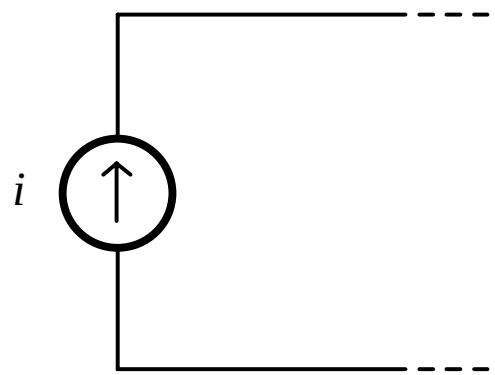
## 2.2 Current Source

In the same way that the output voltage of an ideal voltage source does not depend on the load and the current drawn, the current delivers by an ideal current source does not depend on what has been connected and the voltage across its terminals. Its circuit symbol is




Note that ideal voltage and current sources are idealisations and do not exist in practice. Many practical electrical sources, however, behave like ideal voltage and current sources.



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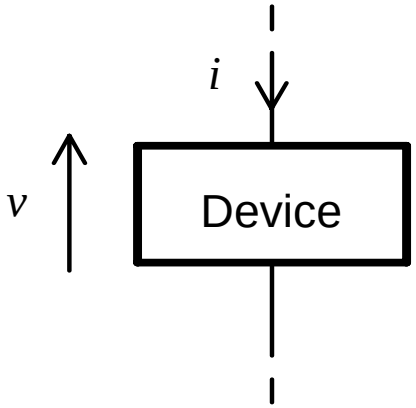
## 2.3 Power and Energy

Consider the following device,




Power Consumed by Device $p = vi$

In 1 second, there are $i$ charges passing through the device. Their electric potential will decrease by $v$ and their electric potential energy will decrease by $iv$. This energy will have been absorbed or consumed by the device.

The ***power*** or the rate of energy ***consumed*** by the device is thus $p = iv$.



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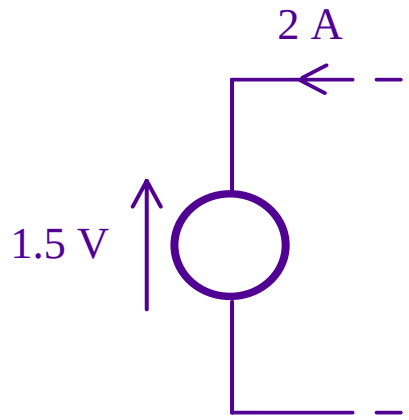
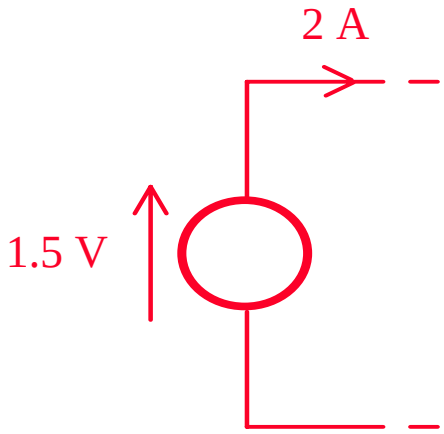
Note that $p = v\,i$ gives the power consumed by the device if the voltage and current arrows are opposite to one another. The following examples illustrate this point:




Power consumed/ absorbed by source = 3 W

Energy absorbed in 100 hr = 300 W hr = 0.3 kW hr = 0.3 unit in PUB bill

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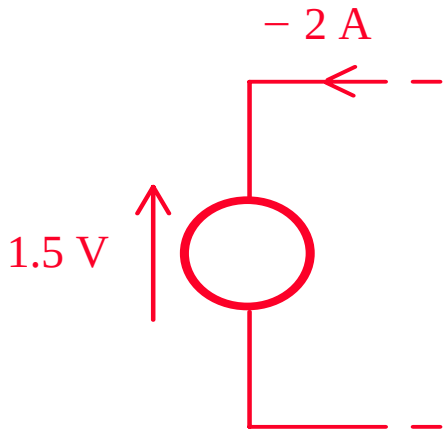


Power supplied by source = 3 W

Power absorbed by source = − 3 W



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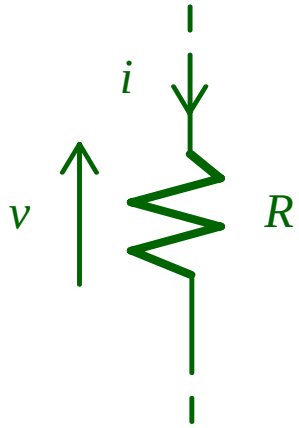
## 2.4 Resistor

The symbol for an ideal resistor is




Provided that the voltage and current arrows are in opposite directions, the voltage-current relationship follows Ohm's law:

$$v = iR$$

The power consumed is

$$p = vi = i^2 R = \frac{v^2}{R}$$

Common practical resistors are made of carbon film, wires, etc.



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## 2.5 Relative Power

Powers, voltages and currents are often measured in relative terms with respect to certain convenient reference values. Thus, taking

$$p_{ref} = 1\ \text{mW}$$

as the reference (note that reference could be any value), the power

$$p = 2\ \text{W}$$

will have a relative value of

$$\frac{p}{p_{ref}} = \frac{2\,\text{W}}{1\,\text{mW}} = \frac{2\,\text{W}}{10^{-3}\,\text{W}} = 2000$$

The log of this relative power or power ratio is usually taken and given a dimensionless unit of *bel*. The power $p = 2\ W$ is equivalent to

$$\log\left(\frac{p}{p_{ref}}\right) = \log(2000) = \log(1000) + \log(2) = 3.3\,\text{bel}$$



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As bel is a large unit, the finer sub-unit, *decibel* or dB (one-tenth of a Bel), is more commonly used. In dB, $p = 2W$ is the same as

$$10\log\left(\frac{p}{p_{ref}}\right)=10\log(2000)=33\text{dB}$$

As an example:

| Reference | Actual power | Relative | Power |
|---|---|---|---|
| $p_{ref}$ | $p$ | $p/p_{ref}$ | $10\log(p/p_{ref})$ |
| $1\text{mW}$ | $1\text{mW}$ | $1$ | $0\text{ dB}$ |
| $1\text{mW}$ | $2\text{mW}$ | $2$ | $3\text{dB}$ |
| $1\text{mW}$ | $10\text{mW}$ | $10$ | $10\text{dB}$ |
| $1\text{mW}$ | $20\text{mW}$ | $20=10\times2$ | $13\text{dB}=10\text{dB}+3\text{dB}$ |
| $1\text{mW}$ | $100\text{mW}$ | $100$ | $20\text{dB}$ |
| $1\text{mW}$ | $200\text{mW}$ | $200=100\times2$ | $23\text{dB}=20\text{dB}+3\text{dB}$ |



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Although dB measures relative power, it can also be used to measure relative voltage or current which are indirectly related to power.

For instance, taking $v_{ref} = 0.1\text{V}$

as the reference voltage (again reference voltage could be any value), the power consumed by applying $v_{ref}$ to a resistor $R$ will be

$$p_{ref} = \frac{v_{ref}^{\,2}}{R}$$

Similarly, the voltage

$$v = 1\ \text{V}$$

will lead to a power consumption of $p = \frac{v^2}{R}$



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The voltage $v$ relative to $v_{ref}$ will then give rise to a relative power of

$$\frac{p}{p_{ref}} = \frac{\frac{v^2}{R}}{\frac{v_{ref}^2}{R}} = \left(\frac{v}{v_{ref}}\right)^2 = \left(\frac{1}{0.1}\right)^2 = 100$$

or in dB:

$$10\log\left(\frac{p}{p_{ref}}\right)\text{dB} = 10\log\left(\frac{v}{v_{ref}}\right)^2\text{dB} = 20\log\left(\frac{v}{v_{ref}}\right)\text{dB} = 20\log\left(\frac{1}{0.1}\right)\text{dB} = 20\text{dB}$$

This is often used as a measure of the relative voltage $v/v_{ref}$ .

Key point: When you convert relative power to dB, you multiply its log value by 10. You should multiply its log value by 20 if you are converting relative voltage or current.



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As an example:

| Reference | Actual voltage | Relative | Voltage |
|---|---|---|---|
| $v_{ref}$ | $v$ | $v/v_{ref}$ | $20\log(v/v_{ref})$ |
| $0.1\,\mathrm{V}$ | $0.1\,\mathrm{V}$ | $1$ | 0dB |
| $0.1\,\mathrm{V}$ | $0.1\sqrt{2}\,\mathrm{V}$ | $\sqrt{2}$ | 3dB |
| $0.1\,\mathrm{V}$ | $0.2\,\mathrm{V}$ | $2=\sqrt{2}\times\sqrt{2}$ | 6dB=3dB+3dB |
| $0.1\,\mathrm{V}$ | $0.1\sqrt{10}\,\mathrm{V}$ | $\sqrt{10}$ | 10dB |
| $0.1\,\mathrm{V}$ | $0.1\sqrt{20}\,\mathrm{V}$ | $\sqrt{20}=\sqrt{10}\times\sqrt{2}$ | 13dB=10dB+3dB |
| $0.1\,\mathrm{V}$ | $1\,\mathrm{V}$ | $10=\sqrt{10}\times\sqrt{10}$ | 20dB=10dB+10dB |

The measure of relative current is the same as that of relative voltage and can be done in dB as well.



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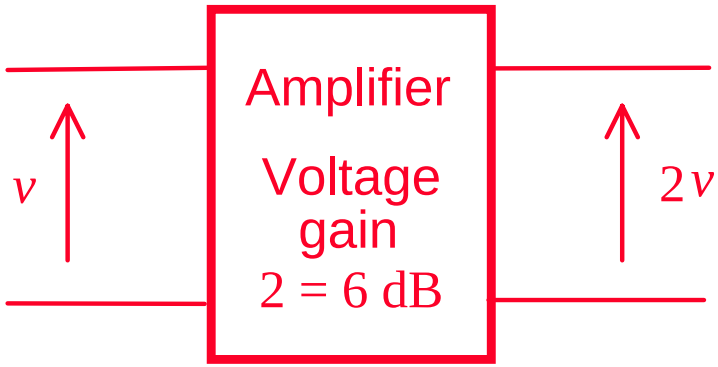
The advantage of measuring relative power, voltage and current in dB can be seen from considering the following voltage amplifier:




The voltage gain of the amplifier is given in terms of the output voltage relative to the input voltage or, more conveniently, in dB:

$$g = \frac{2v}{v} = 2 = 20\log(2)\,\text{dB} = 6\,\text{dB}$$



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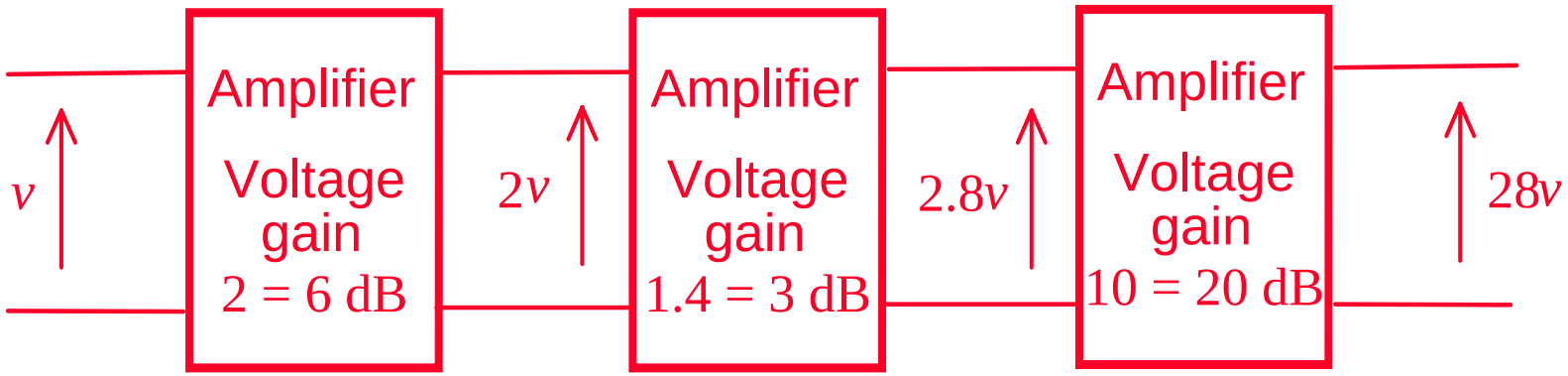
If we cascade 3 such amplifiers with different voltage gains together:




the overall voltage gain will be

$$g_{total} = 2 \times 1.4 \times 10 = 28$$

However, in dB, it is simply:

$$g_{total} = 6\text{dB} + 3\text{dB} + 20\text{dB} = 29\text{dB}$$

Under dB which is log based, multiplication's become additions.



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## Frequently Asked Questions

Q: Does the arrow associated with a voltage source always point at the + (high potential) terminal?

A: No. The arrow itself is meaningless. As re-iterated in the class, any voltage or current is actually characterized by two things: *its direction and its value*. The arrow of the voltage symbol for a voltage source could point at the - terminal (in this case, the value of the voltage will be negative) or at the + terminal (in this case, its value will be positive).

Q: What is the current of a voltage source?

A: The current of a voltage source is depended on the other part of circuit connected to it.



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## Frequently Asked Questions

Q: Does a volt source always supply power to other components in a circuit?

A: NO. A voltage source might be consuming power if it is connected to a circuit which has other more powerful sources. Thus, it is a bad idea to pre-determine whether a source is consuming power or supplying power. The best way to determine it is to follow the definition in our text and computer the power. If the value turns out to be positive, then the source will be consuming power. Otherwise, it is supplying power to the other part of the circuit.

Q: Is the current of a voltage source always flowing from + to - terminals?

A: NO. The current of a voltage source is not necessarily flowing from the positive terminal to the negative terminal.



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## Frequently Asked Questions

Q: What is the voltage cross over a current source?

A: It depends on the circuit connected to it.

Q: Is the reference power (or voltage, or current) in the definition of the relative power (or voltage, or current) unique?

A: No. The reference power (voltage or current) can be any value. Remember that whenever you deal with the relative power (voltage or current), you should keep in your mind that there are *a reference power* (voltage or current) and *an actual power* (voltage or current) associated with it.



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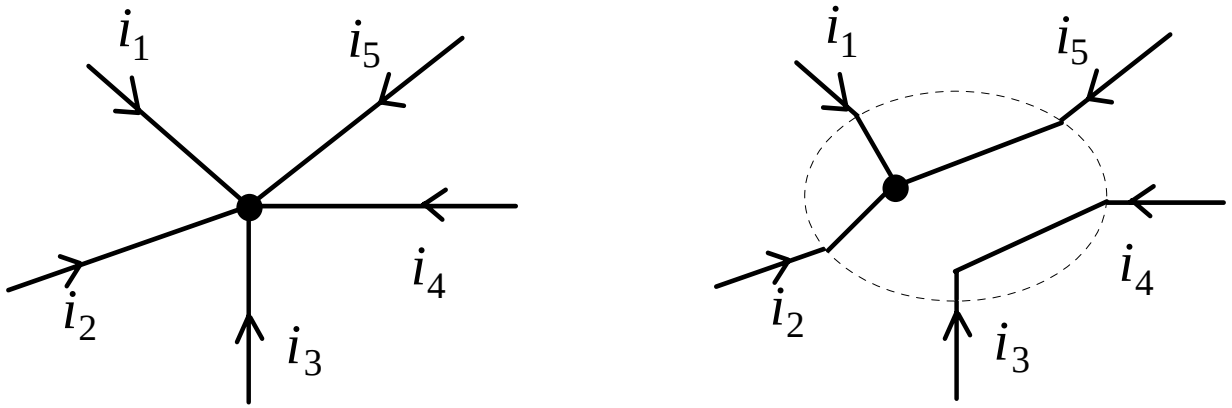
## 2.6 Kirchhoff's Current Law (KCL)

As demonstrated by the following examples, this states that the algebraic sum of the currents entering/leaving a node/closed surface is 0 or equivalently to say that the total currents flowing into a node is equal to the total currents flowing out from the node.




$i_1 + i_2 + i_3 + i_4 + i_5 = 0$ for both cases.

Since current is equal to the rate of flow of charges, KCL actually corresponds to the conservation of charges.



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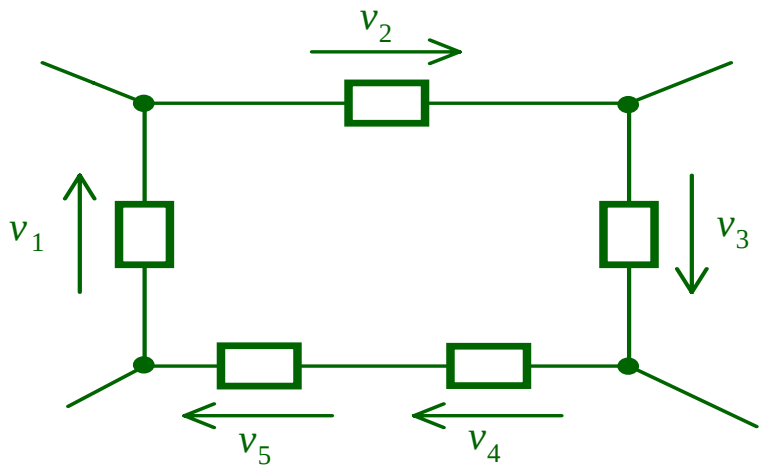
## 2.7 Kirchhoff's Voltage Law (KVL)

As illustrated below, this states that the algebraic sum of the voltage drops around any close loop in a circuit is 0.




$$v_1+v_2+v_3+v_4+v_5=0$$

**(note that all voltages are in the same direction)**

Since a charge $q$ will have its electric potential changed by $qv_1$, $qv_2$, $qv_3$, $qv_4$ , $qv_5$ as it passes through each of the components, the total energy change in one full loop is $q$ ( $v_1$ + $v_2$ + $v_3$ + $v_4$ + $v_5$ ). Thus, from the conservation of energy: $v_1 + v_2 + v_3 + v_4 + v_5 = 0$



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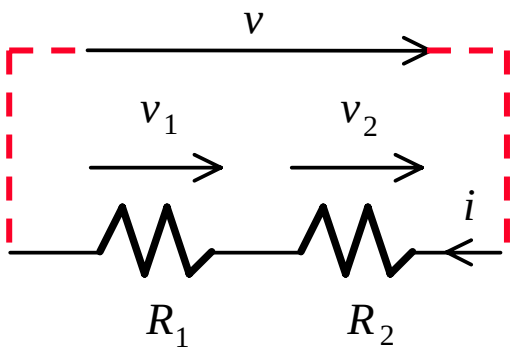
## 2.8 Series Circuit

Consider 2 resistors connected in series:



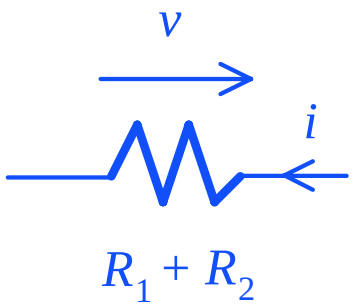

By KVL: $-v + v_1 + v_2 = 0$

$v_1 = i R_1 \qquad v_2 = i R_2$

$v = v_1 + v_2$

the voltage-current relationship is $v = i(R_1 + R_2)$

Now consider




the voltage-current relationship is $v = i(R_1 + R_2)$



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Since the voltage/current relationships are the same for both circuits, they are equivalent from an electrical point of view. In general, for $n$ resistors $R_1$, ..., $R_n$ connected in series, the equivalent resistance $R$ is

$$R = R_1 + \cdots + R_n$$

Clearly, the resistance's of resistors connected in series add (Prove it).



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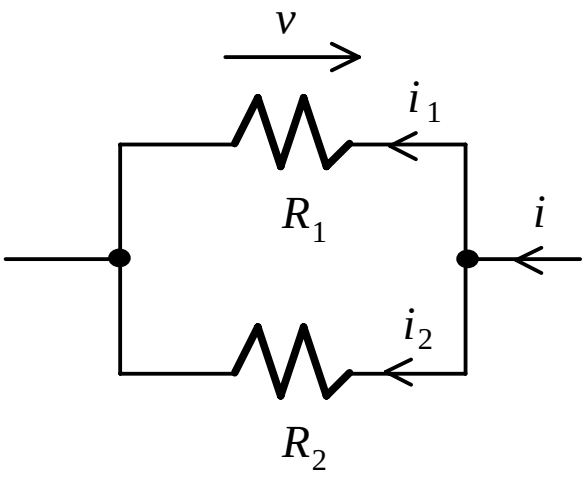
## 2.9 Parallel Circuit

Consider 2 resistors connected in parallel:




$$i_1 = \frac{v}{R_1}$$

$$i_2 = \frac{v}{R_2}$$

$$i = i_1 + i_2 = v\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$$

Clearly, the parallel circuit is equivalent to a resistor $R$ with voltage/current relationship

$$i = \frac{v}{R} \qquad \text{with} \qquad \frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$$



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In general, for $n$ resistors $R_1$, …, $R_n$, connected in parallel, the equivalent resistance $R$ is given by

$$\frac{1}{R} = \frac{1}{R_1} + \cdots + \frac{1}{R_n}$$

Note that $1/R$ is often called the ***conductance*** of the resistor $R$. Thus, the conductances of resistors connected in parallel add.



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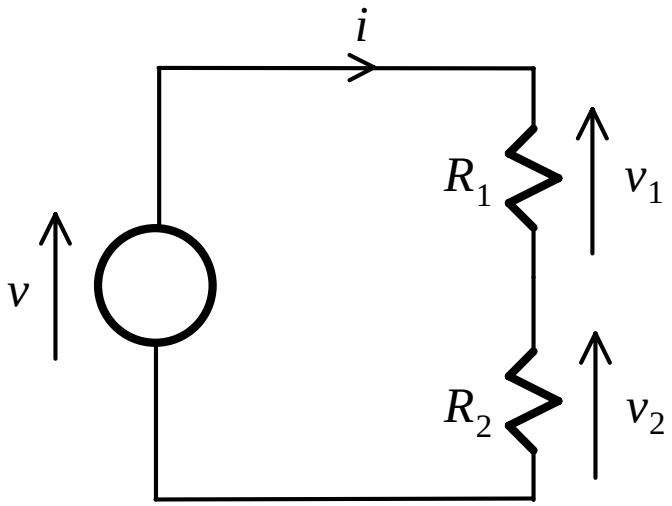
## 2.10 Voltage Division

Consider 2 resistors connected in series:




$$i = \frac{v}{R_1 + R_2}$$

$$v_1 = iR_1 = \left( \frac{R_1}{R_1 + R_2} \right) v$$

$$v_2 = iR_2 = \left( \frac{R_2}{R_1 + R_2} \right) v$$

The total resistance of the circuit is $R_1 + R_2$.

Thus,

$$\frac{v_1}{v_2} = \frac{R_1}{R_2}$$



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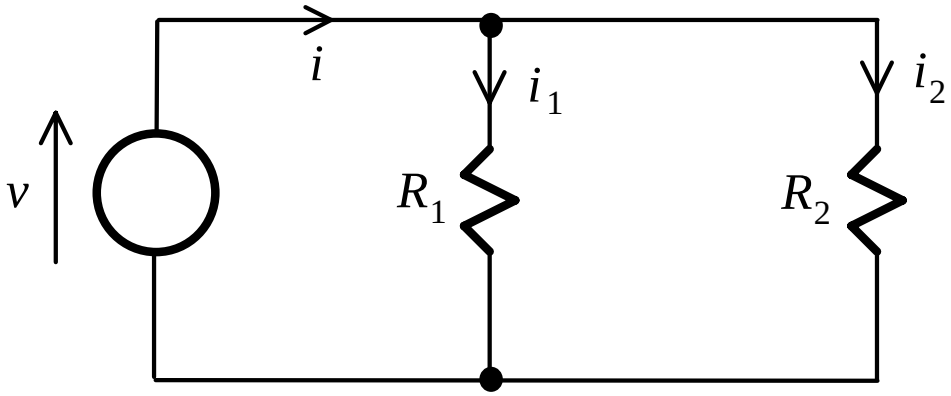
## 2.11 Current Division

Consider 2 resistors connected in parallel:




$$i_1 = \frac{v}{R_1} = \left(\frac{\frac{1}{R_1}}{\frac{1}{R_1}+\frac{1}{R_2}}\right) i = \left(\frac{R_2}{R_1+R_2}\right) i$$

$$i_2 = \frac{v}{R_2} = \left(\frac{\frac{1}{R_2}}{\frac{1}{R_1}+\frac{1}{R_2}}\right) i = \left(\frac{R_1}{R_1+R_2}\right) i$$

Thus,

$$\frac{i_1}{i_2} = \frac{R_2}{R_1}$$

The total conductance of the circuit is $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$

while the equivalent resistance is $v = i\,R = \frac{i}{\frac{1}{R_1}+\frac{1}{R_2}}$



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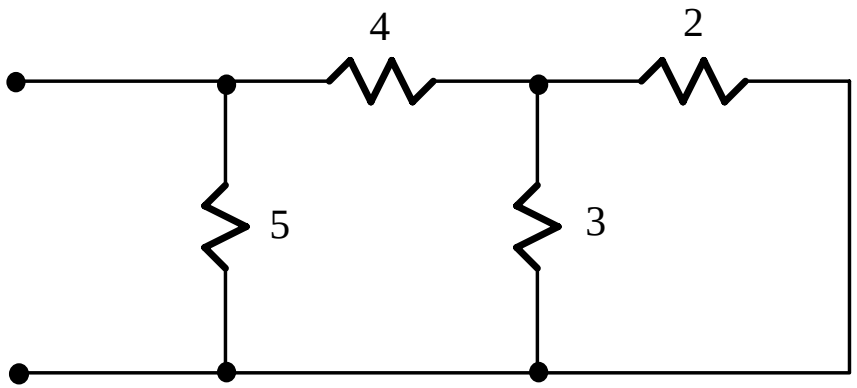
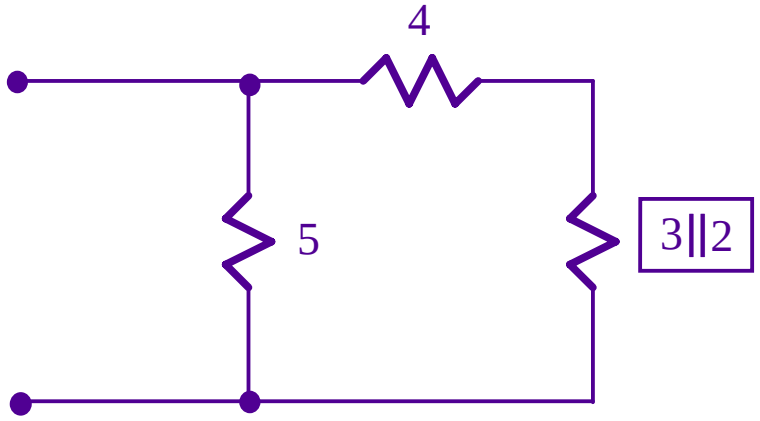
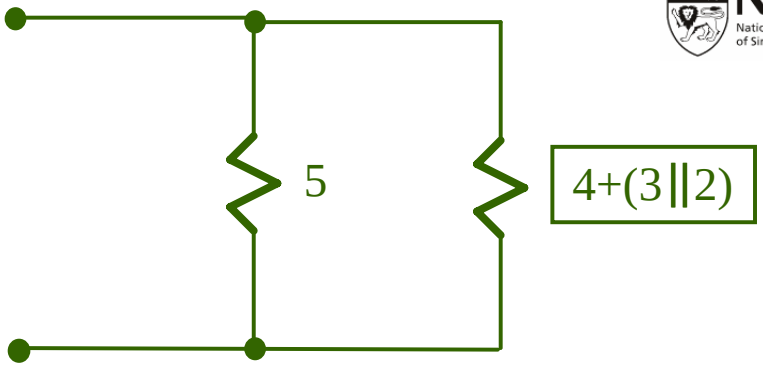
## 2.12 Ladder Circuit

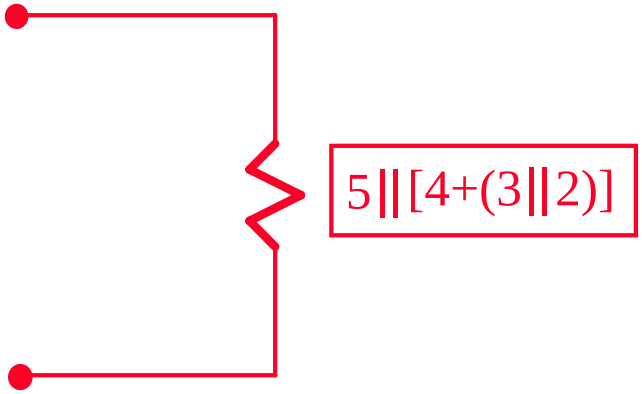
Consider the following ladder circuit:




The equivalent resistance can be determined as follows:




The network is equivalent to a resistor with resistance

$$R = 5 \,||\, [4+(3 \,||\, 2)] = \frac{1}{\frac{1}{5} + \frac{1}{4+(3 \,||\, 2)}} = \frac{1}{\frac{1}{5} + \frac{1}{4 + \frac{1}{\frac{1}{3} + \frac{1}{2}}}}$$



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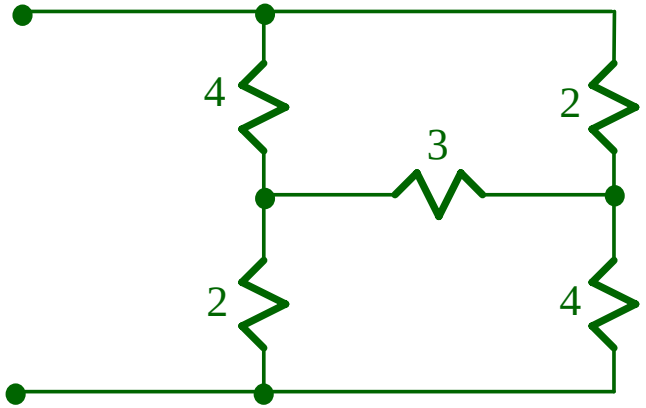
## 2.13 Branch Current Analysis

Consider the problem of determining the equivalent resistance of the following ***bridge*** circuit:




Since the components are not connected in straightforward series or parallel manner, it is not possible to use the series or parallel connection rules to simplify the circuit. However, the voltage-current relationship can be determined and this will enables the equivalent resistance to be calculated.

One method to determine the voltage-current relationship is to use the ***branch current method***.



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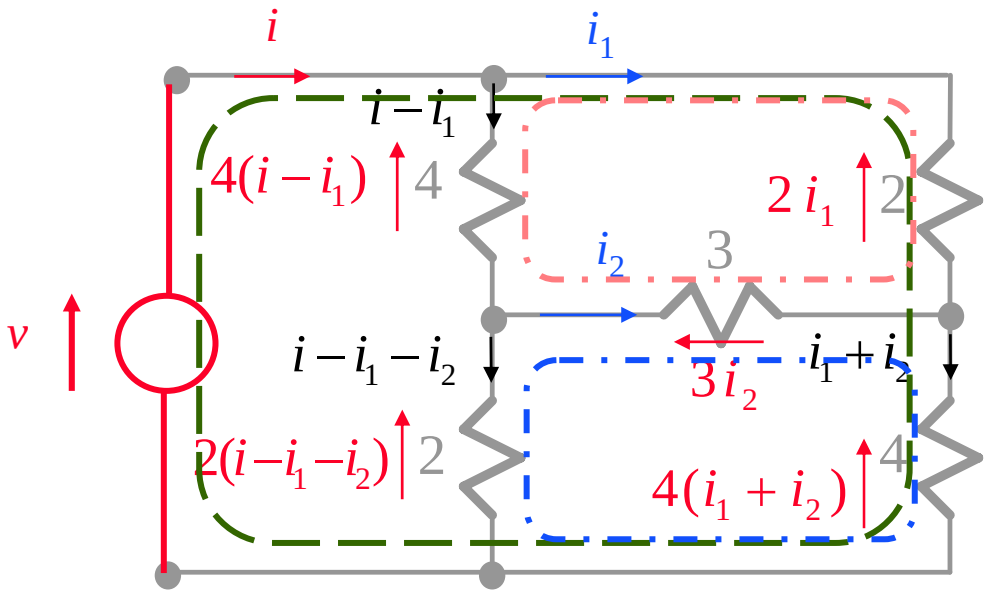
# Branch Current Analysis: Example One




$\frac{v}{i}$ = Equivalent Resistance

KVL: $v = 2i_1 + 4(i_1 + i_2) = 6i_1 + 4i_2$

KVL: $4(i - i_1) + 3i_2 = 2i_1$

KVL: $4(i_1 + i_2) + 3i_2 = 2(i - i_1 - i_2)$

**1. Assign branch currents (with any directions you prefer so that currents in other branches can be found)**

**2. Find all other branch currents (with any directions you prefer) (Use KCL to find them)**

**3. Write down branch voltages**

**4. Identify independent loops**

**Eliminate $i_1$ and $i_2$,**

$$i_1 = \frac{7i}{12} \qquad i_2 = -\frac{i}{6}$$

$$\boxed{v = \frac{17}{6} i}$$



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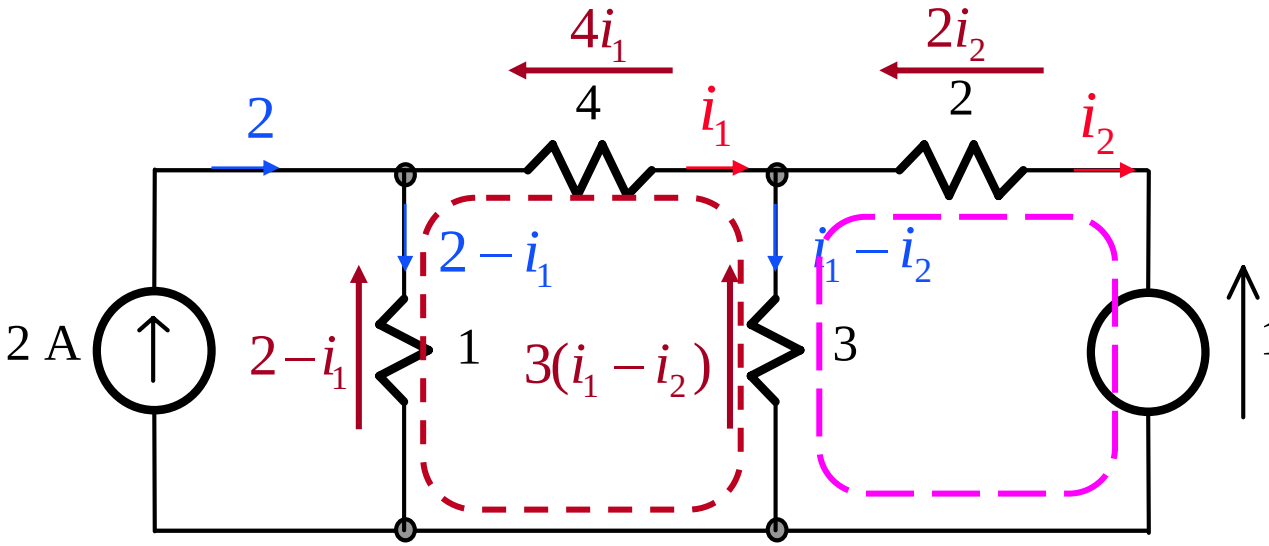
# Branch Current Analysis: Example Two




1. Assign branch currents (so that currents in other branches can be found).

2. Find all other branch currents (KCL)

3. Write down voltages across components

KVL: $2-i_1=4i_1+3(i_1-i_2)$

4. Identify independent loops (ex. 2A branch)

KVL: $1+2i_2=3(i_1-i_2)$

This implies:

$$\begin{aligned}3i_1-5i_2&=1\\8i_1-3i_2&=2\end{aligned}\Rightarrow\begin{bmatrix}3&-5\\8&-3\end{bmatrix}\begin{bmatrix}i_1\\i_2\end{bmatrix}=\begin{bmatrix}1\\2\end{bmatrix}\Rightarrow\begin{bmatrix}i_1\\i_2\end{bmatrix}=\begin{bmatrix}3&-5\\8&-3\end{bmatrix}^{-1}\begin{bmatrix}1\\2\end{bmatrix}=\frac{1}{31}\begin{bmatrix}7\\-2\end{bmatrix}$$



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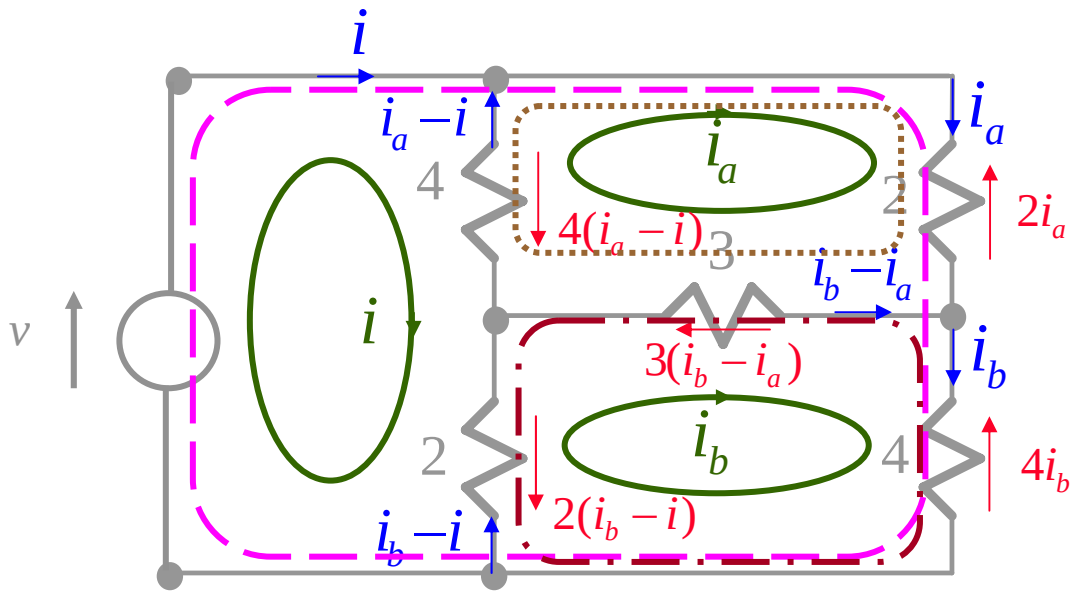
# 2.14 Mesh (Loop Current) Analysis




1. Assign fictitious loop currents.

2. Find branch currents (KCL)

3. Write down branch voltages

4. Identify independent loops

5. Simplify the equations obtained, we get

KVL: $v = 2i_a + 4i_b$

KVL: $4(i_a - i) - 3(i_b - i_a) + 2i_a = 0$

KVL: $2(i_b - i) + 4i_b + 3(i_b - i_a) = 0$

$$v = \frac{17i}{6} \quad \Rightarrow \quad R_{equivalent} = \frac{17}{6}$$



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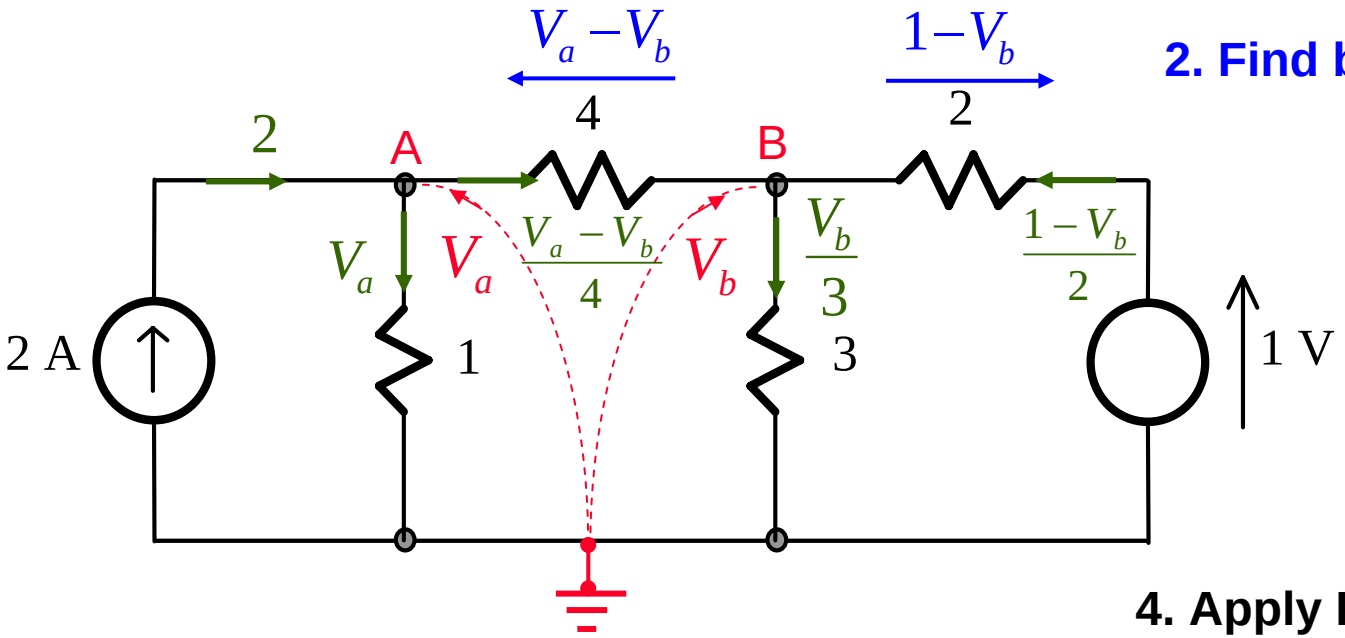
# 2.15 Nodal Analysis




**2. Find branch voltages (KVL)**

**3. Determine branch currents**

**4. Apply KCL to Nodes A & B**

**1. Assign nodal voltage w.r.t. the reference node**

$$\begin{bmatrix} 5 & -1 \\ 3 & -13 \end{bmatrix} \begin{bmatrix} V_a \\ V_b \end{bmatrix} = \begin{bmatrix} 8 \\ -6 \end{bmatrix} \Leftarrow \begin{aligned} 5V_a - V_b &= 8 \\ 3V_a - 13V_b &= -6 \end{aligned}$$

Node A:

$$\Leftarrow \quad 2 = V_a + \frac{V_a - V_b}{4}$$

Node B:

$$\frac{V_a - V_b}{4} + \frac{1 - V_b}{2} = \frac{V_b}{3}$$

$$\Rightarrow \quad V_a = \frac{55}{31} \quad V_b = \frac{27}{31}$$



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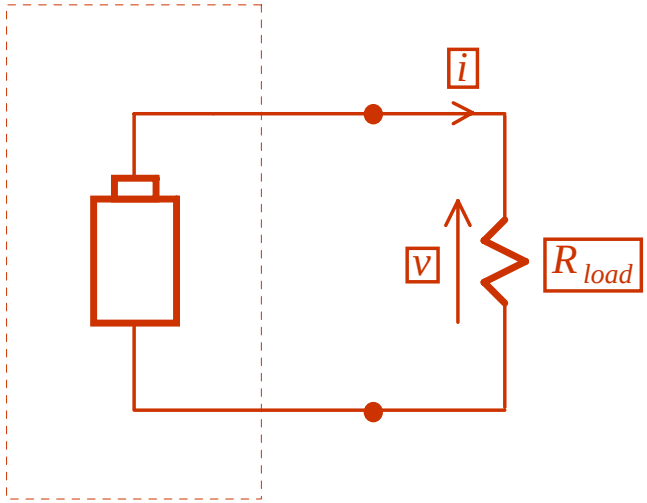
## 2.16 Practical Voltage Source

An ideal voltage source is one whose terminal voltage does not change with the current drawn. However, the terminal voltages of practical sources usually decrease slightly as the currents drawn are increased.

A commonly used model for a practical voltage source is:

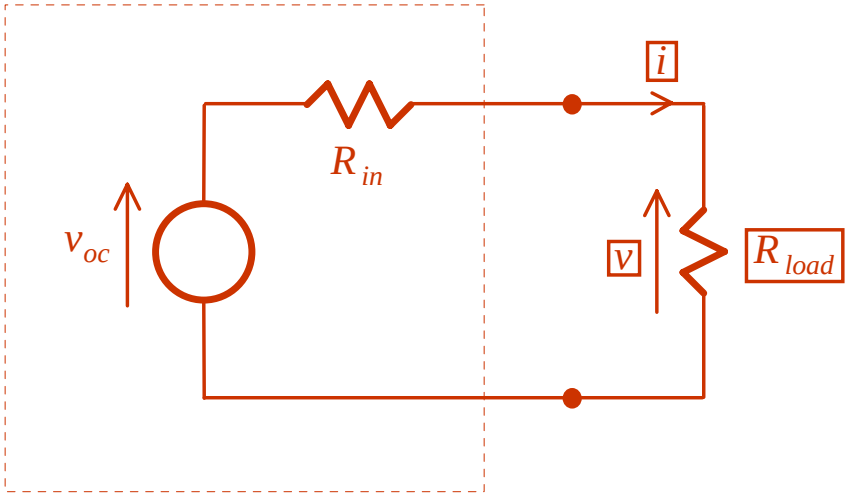
To represent it as a series of an ideal voltage source & an internal resistance.

$$v_{oc} = v + iR_{in} \Leftarrow$$




Practical voltage source

|||




Model for voltage source



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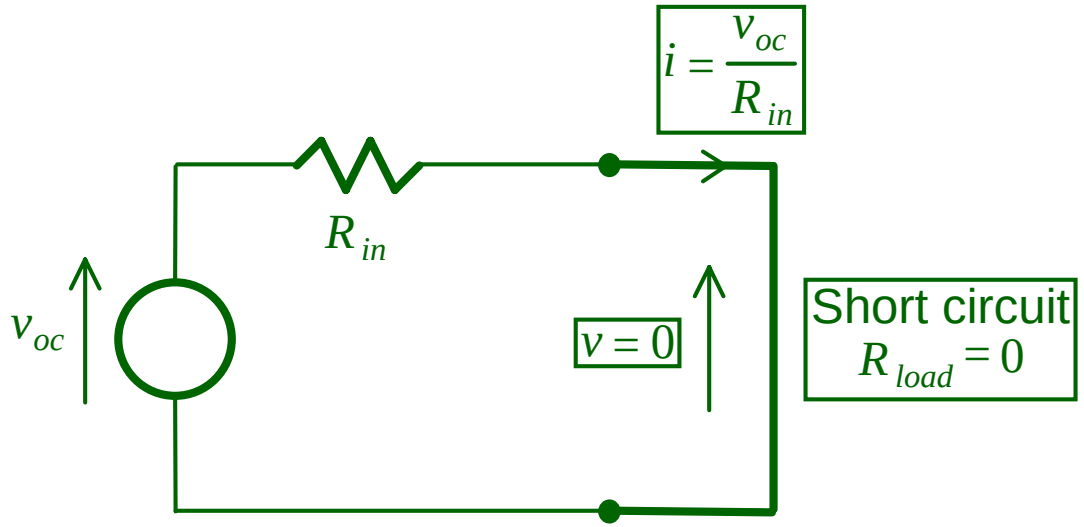
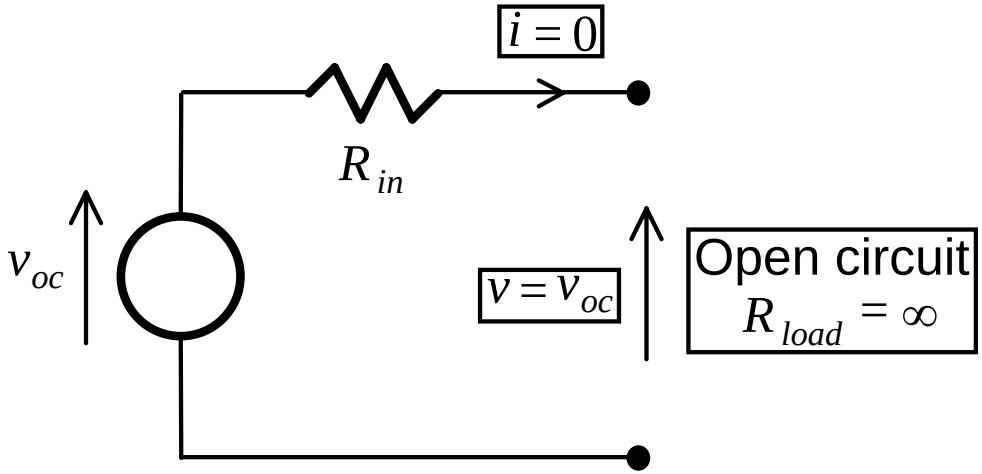
When $R_{load} = 0$ or when the source is ***short circuited*** so that $v = 0$ :




When $R_{load} = \infty$ or when the source is ***open circuited*** so that $i = 0$ :



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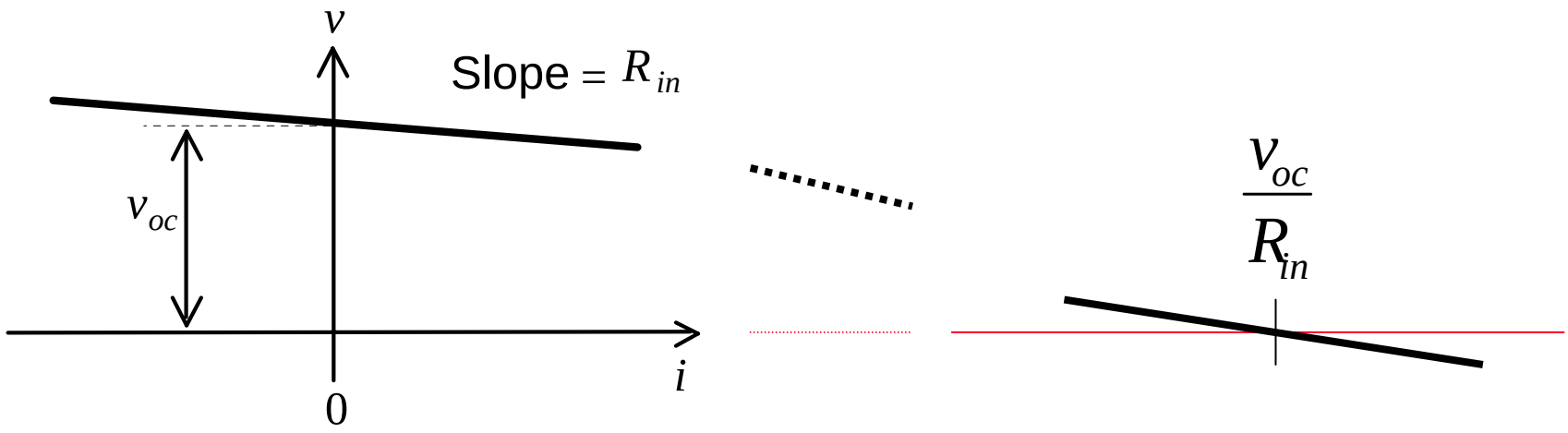
Graphically:




Good practical voltage source should therefore have small ***internal resistance***, so that its output voltage will not deviate very much from the ***open circuit voltage***, under any operating condition.

The internal resistance of an ideal voltage source is therefore zero so that $v$ does not change with $i$ .



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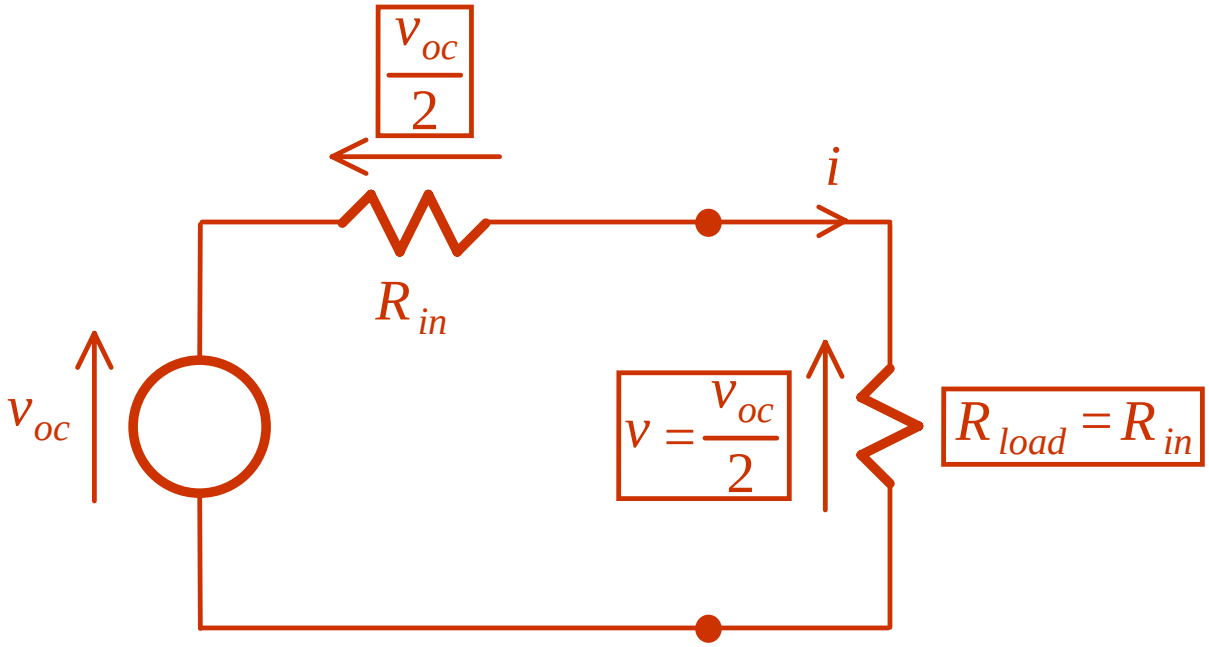
To determine the two parameters $v_{oc}$ and $R_{in}$ that characterize, say, a battery, we can measure the output voltage when the battery is open-circuited (nothing connected except the voltmeter). This will give $v_{oc}$ .

Next, we can connect a load resistor and vary the load resistor such that the voltage across it is $v_{oc}/2$ . The load resistor is then equal to $R_{in}$ :



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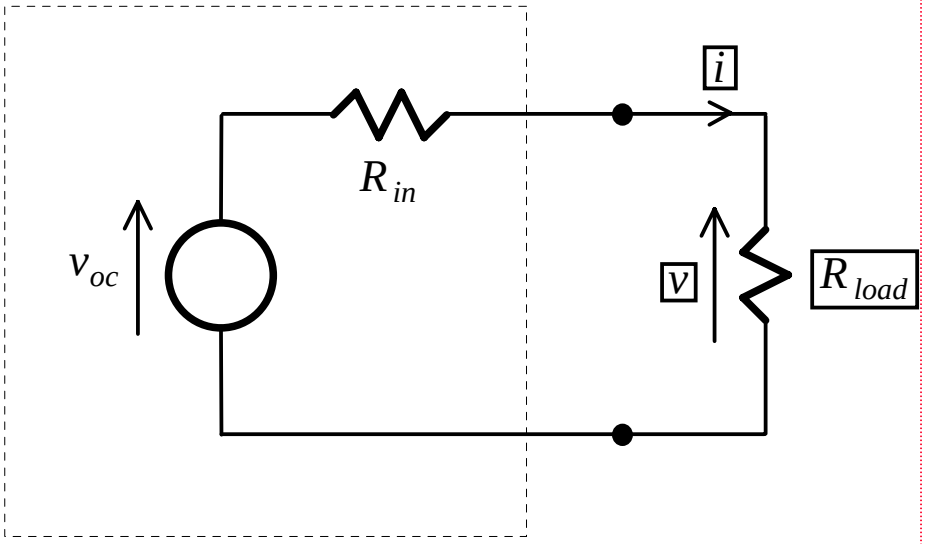
## 2.17 Maximum Power Transfer

Consider the following circuit:




Model for voltage source

The current in the load resistor is

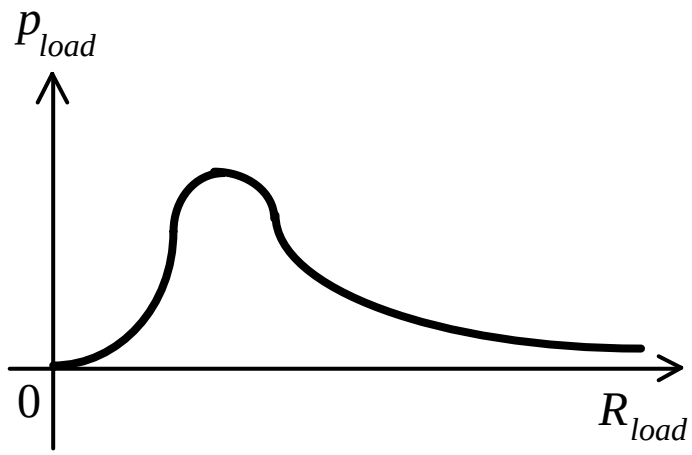
$$i = \frac{v_{oc}}{R_{in} + R_{load}}$$

The power absorbed by the load resistor is

$$p_{load} = i^2 R_{load} = \frac{v_{oc}^2 R_{load}}{(R_{in} + R_{load})^2}$$

This is always positive. However, if

$$R_{load} = 0 \quad \text{or} \quad R_{load} = \infty, \; P_{load} = 0.$$



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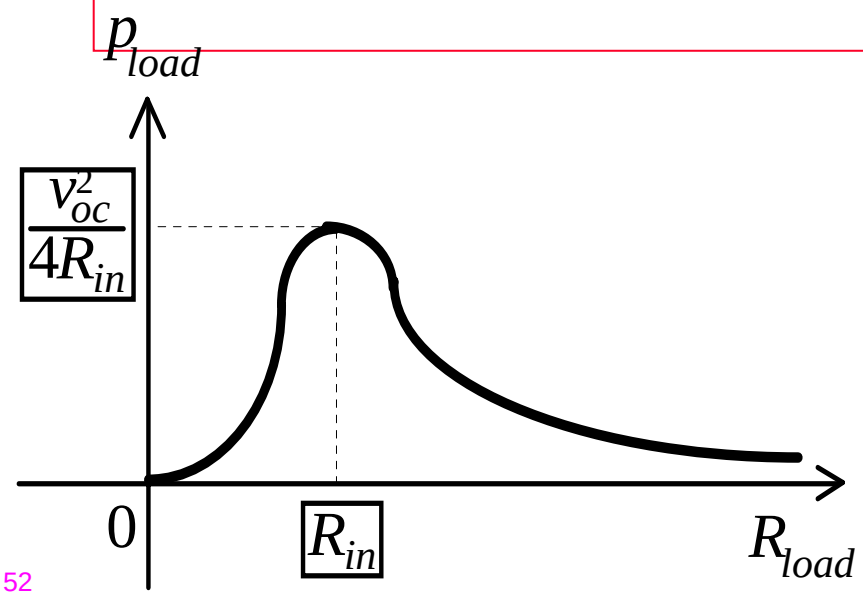
Differentiating:

$$\frac{dp_{load}}{dR_{load}} = v_{oc}^2\left[\frac{1}{(R_{in}+R_{load})^2} - \frac{2R_{load}}{(R_{in}+R_{load})^3}\right] = v_{oc}^2\left[\frac{R_{in}-R_{load}}{(R_{in}+R_{load})^3}\right]$$

The load resistor will be absorbing the maximum power or the source will be transferring the maximum power if the load and source internal resistances are matched, i.e., $R_{in}=R_{load}$. The maximum power transferred is given by

$$p_{load}=i^2R_{load}=\frac{v_{oc}^2R_{load}}{(R_{in}+R_{load})^2} \quad \Rightarrow \quad p_{\max\ load}=\frac{v_{oc}^2}{4R_{in}}$$




When the load absorbs the maximum power from the source, the overall power efficiency of 50%, which is too low for a usual electric system.



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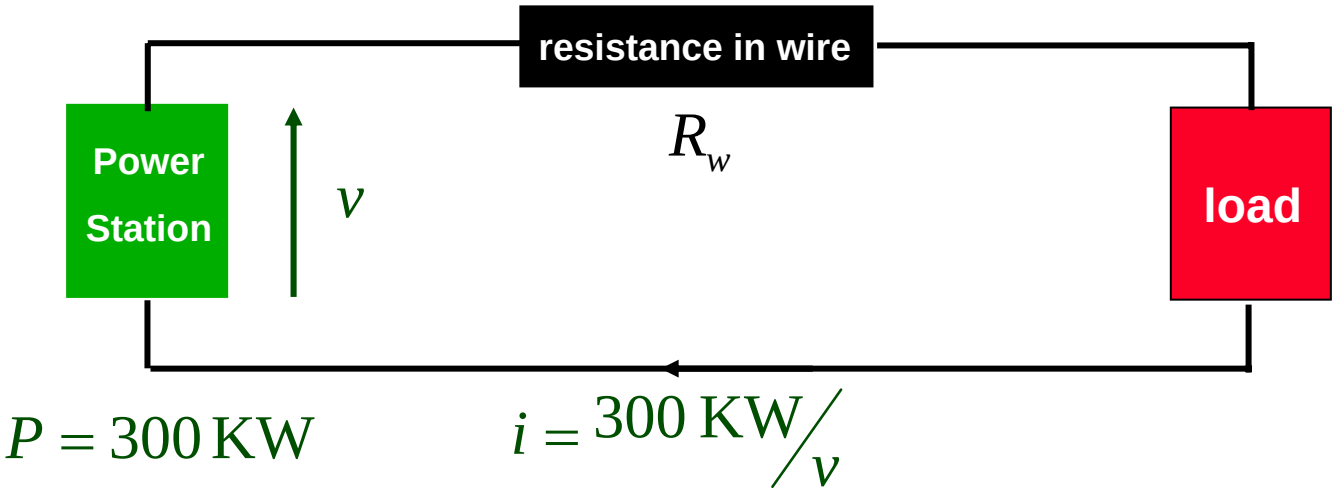
Why is the electric power transferred from power stations to local stations in high voltages?




Power loss in the transmission line:

$$P_{loss} = i^2 R_w = \frac{(300\,\mathrm{KW})^2 R_w}{v^2}$$

The higher voltage $v$ is transmitted, the less power is lost in the wire.



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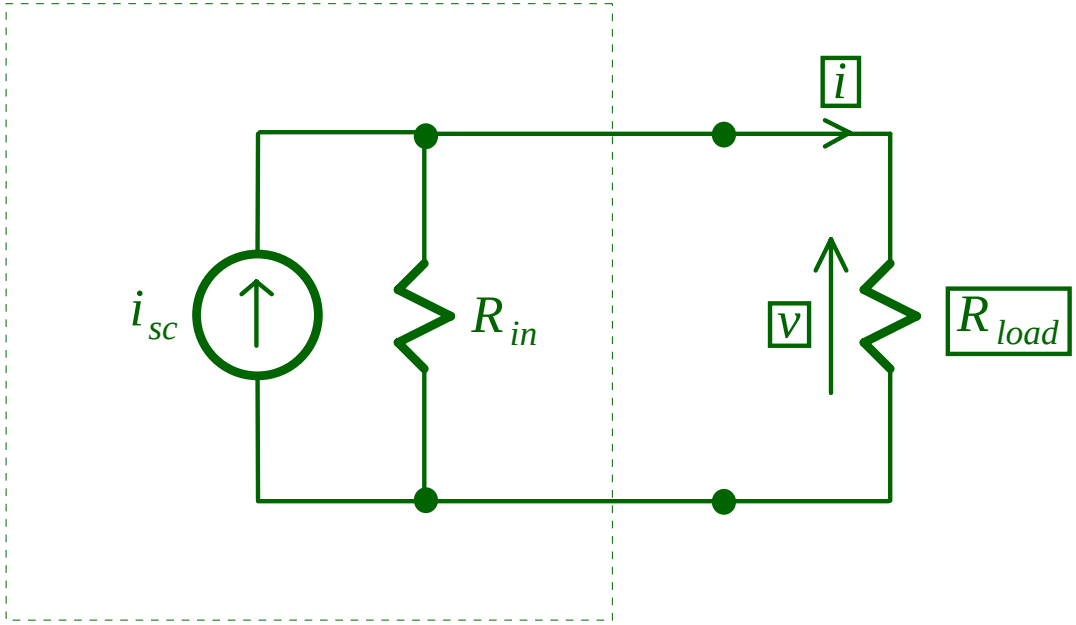
## 2.18 Practical Current Source

An ideal current source is one which delivers a constant current regardless of its terminal voltage. However, the current delivered by a practical current source usually changes slightly depending on the load and the terminal voltage.

A commonly used model for a current source is:




$$\Rightarrow \quad i_{sc} = \frac{v}{R_{in}} + i$$

Model for current source



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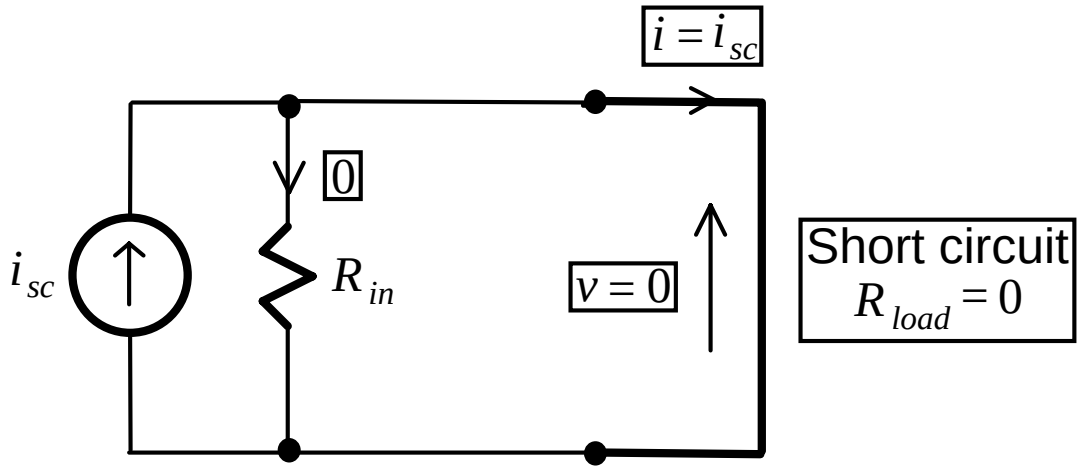
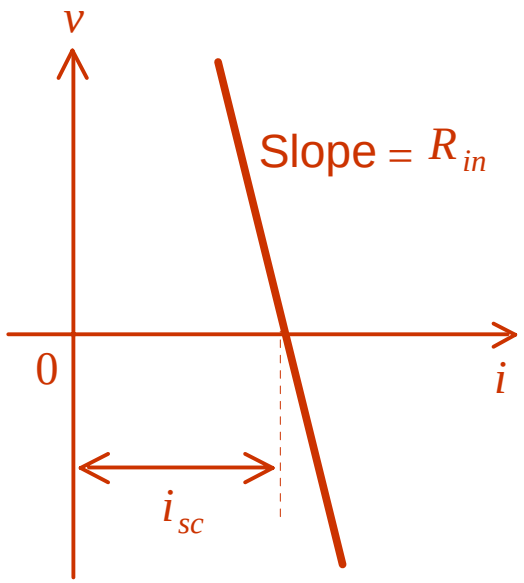
When $R_{load} = 0$ or when the source is short-circuited so that $v = 0$ :




Graphically:




Good practical current source should therefore have large internal resistance so that the current delivered does not deviate very much from the ***short circuit current*** under any operating condition.

The internal resistance of an ideal current source is therefore infinity so that $i$ does not change with $v$.



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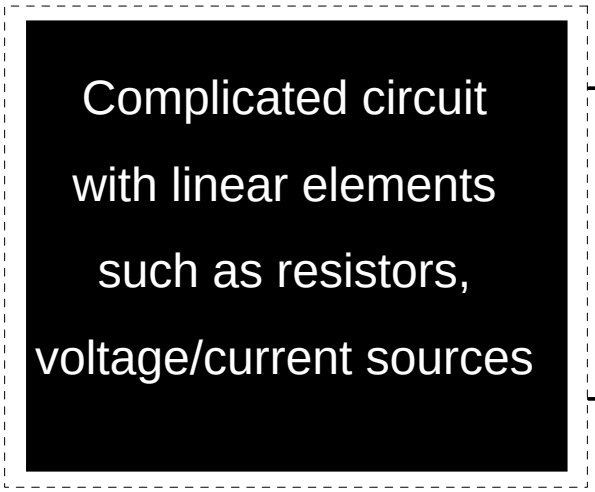
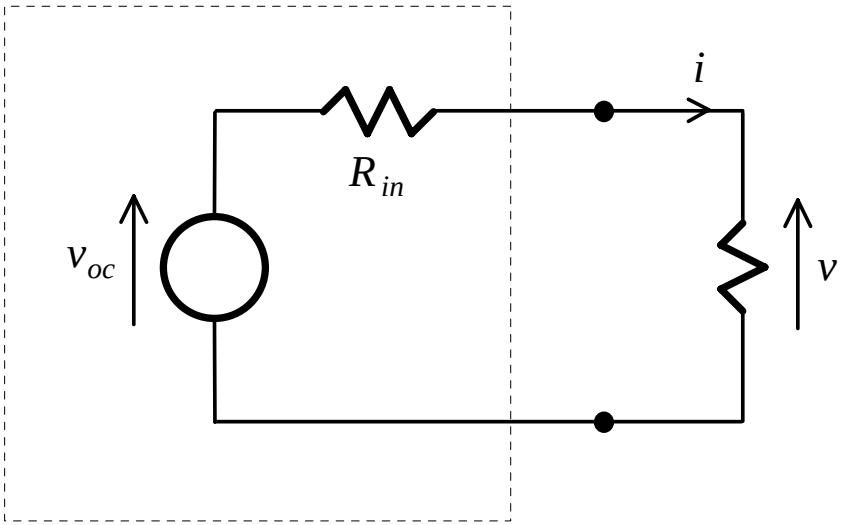
# 2.19 Thevenin's Equivalent Circuit




$$v_{oc} = v + iR_{in}$$




$$v_{oc} = v + iR_{in}$$

Thevenin's equivalent circuit

**Key points:**

1. The black box (i.e., the part of the circuit) to be simplified must be linear.

2. The black box must have two terminals connected to the rest of the circuit.



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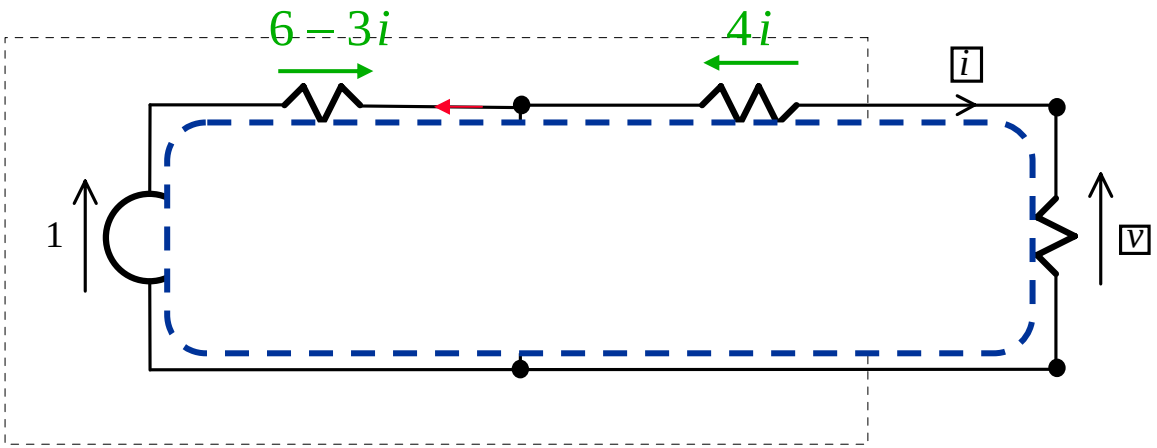
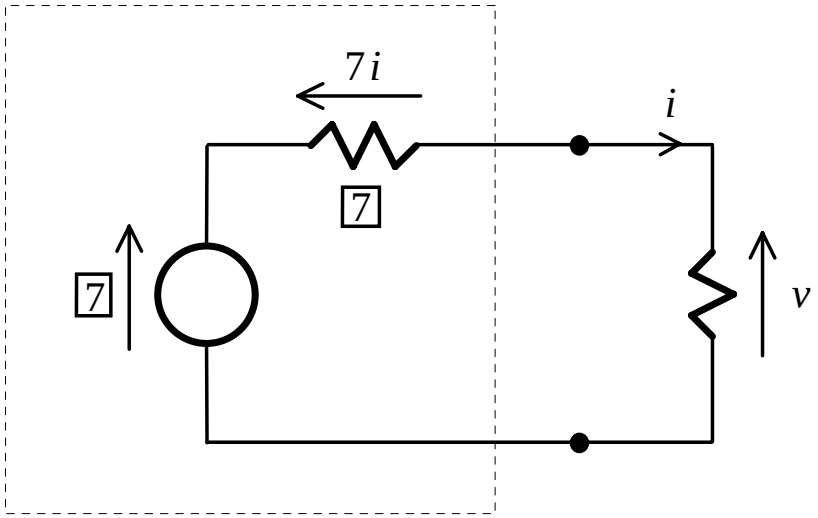
# Thevenin's Equivalent Circuit (An Example)




Applying KVL: $1 + (6 - 3i) = 4i + v \quad \Rightarrow \quad 7 = v + 7i$

The circuit is equivalent to:




$v_{oc} = 7 \quad R_{in} = 7 \quad \Leftarrow$



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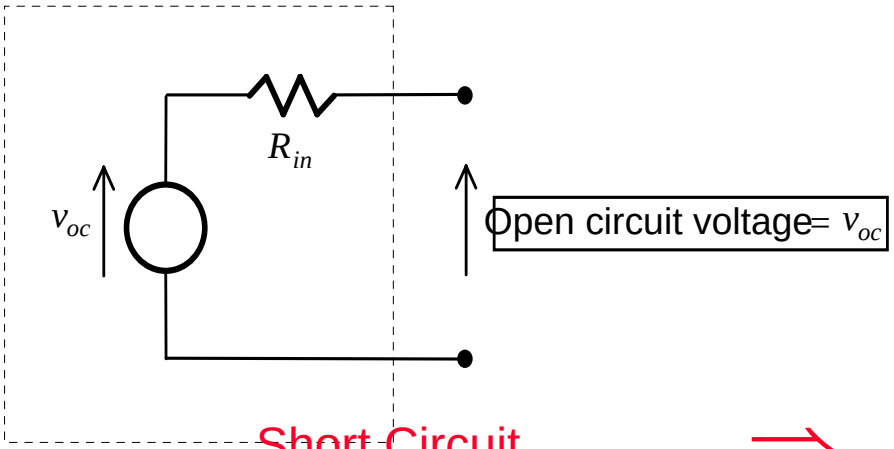
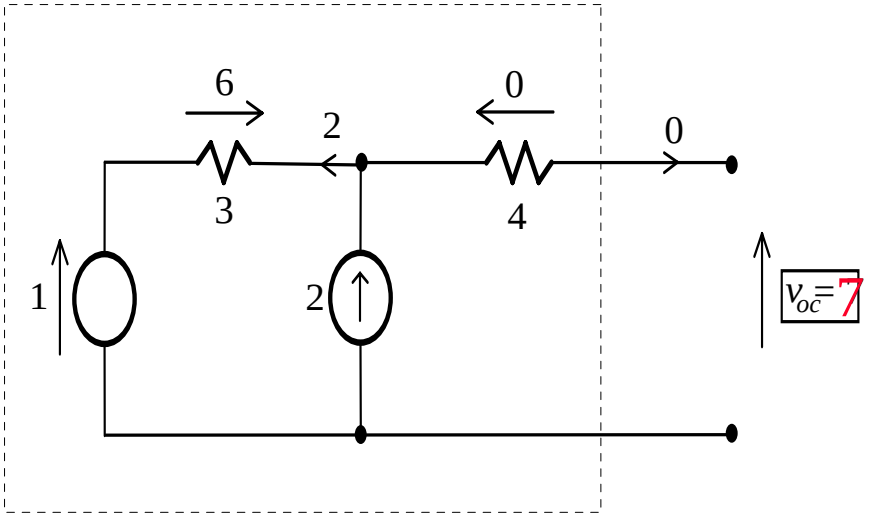
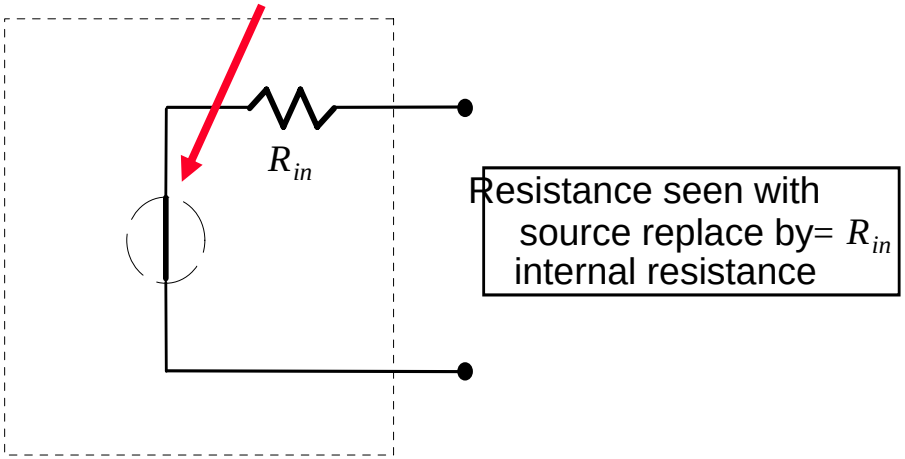
Alternatively, note that from the Thevenin's equivalent circuit:



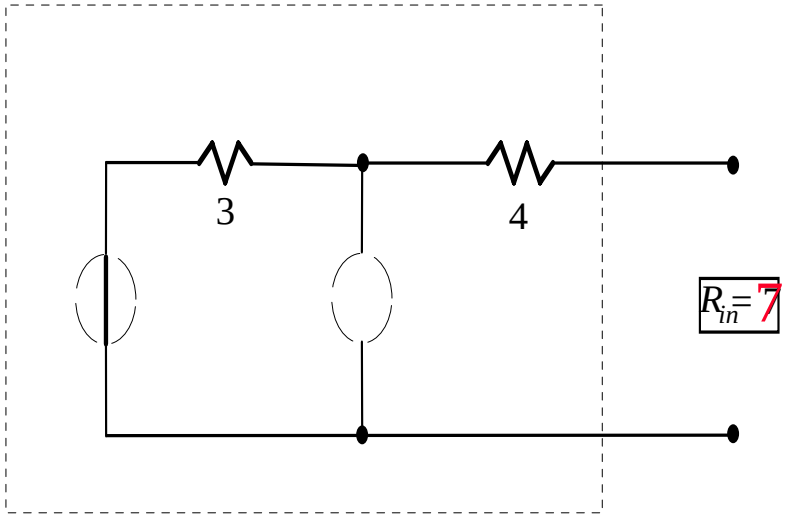

Open circuit voltage= $v_{oc}$




Short Circuit ⇒




Resistance seen with source replace by= $R_{in}$ internal resistance



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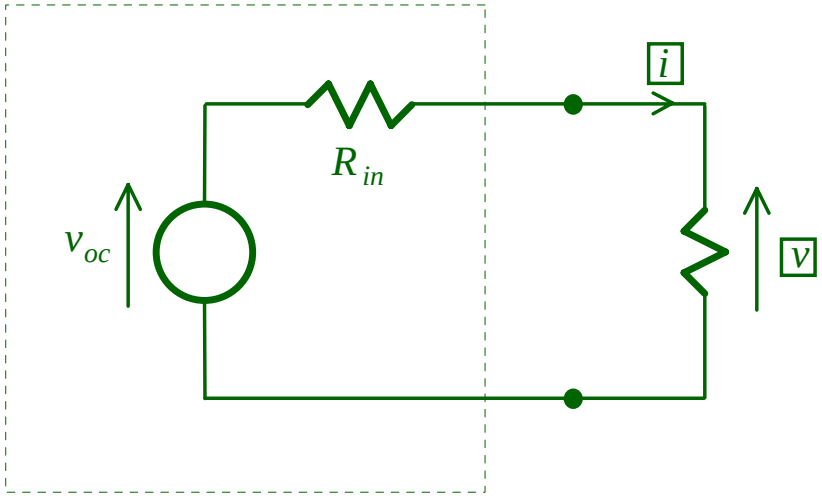
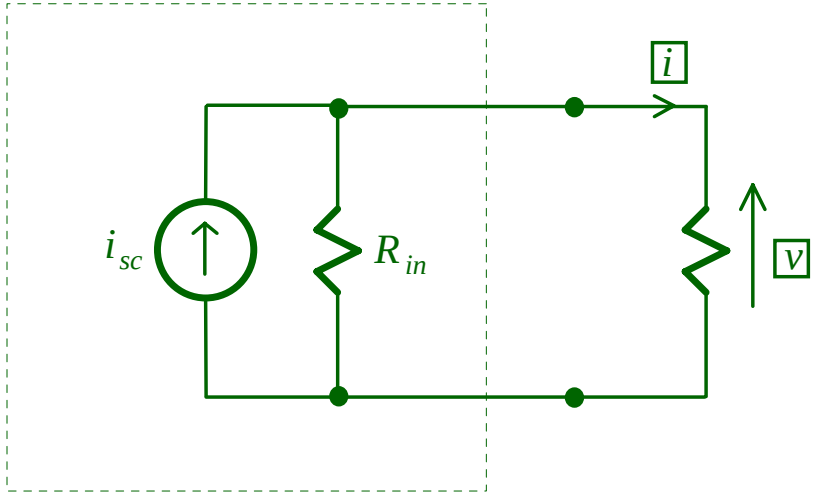
# 2.20 Norton's Equivalent Circuit




Thevenin's equivalent circuit

$$v_{oc} = v + i R_{in}$$

$$\| \| \text{ if } \quad v_{oc} = i_{sc} R_{in}$$




Norton's equivalent circuit

$$i_{sc} = \frac{v}{R_{in}} + i$$

$$i_{sc} R_{in} = v + i R_{in}$$

$$\| \| \text{ if } \quad v_{oc} = i_{sc} R_{in}$$

It is simple to see that if we let

$$v_{oc} = i_{sc} R_{in}$$

then the relationships of voltage/current for both Thevenin's and Norton's equivalent circuits are exactly the same.



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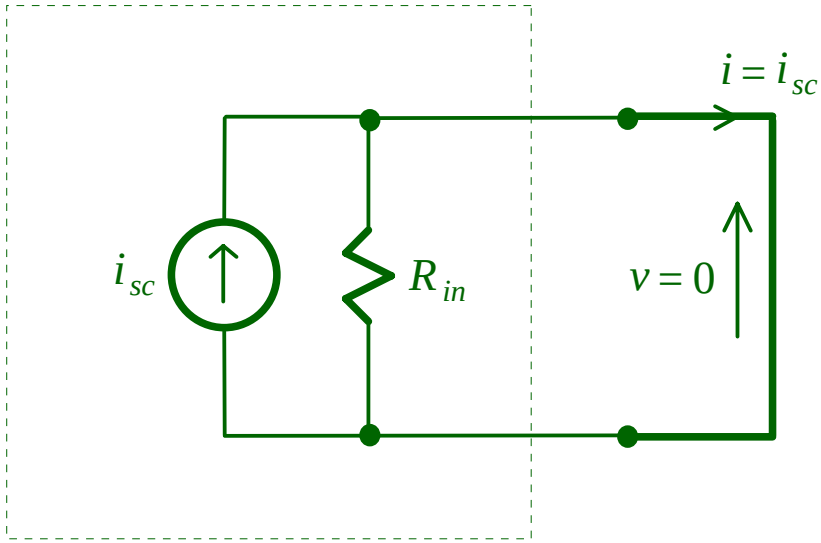
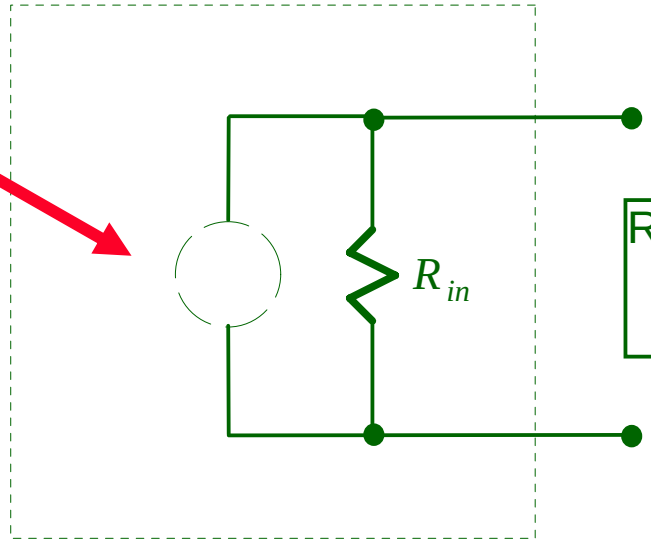
From the Norton's equivalent circuit, the two parameters $i_{sc}$ and $R_{in}$ can be obtained from:




Short circuit current $= i_{sc}$

Open Circuit




Resistance seen with source replace by internal resistance $= R_{in}$



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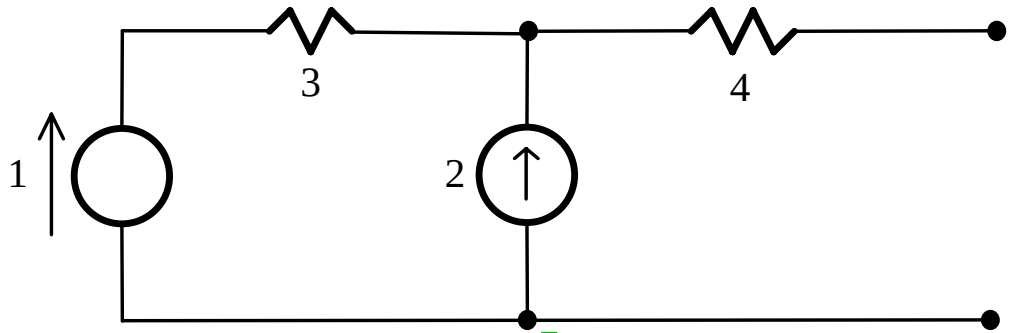
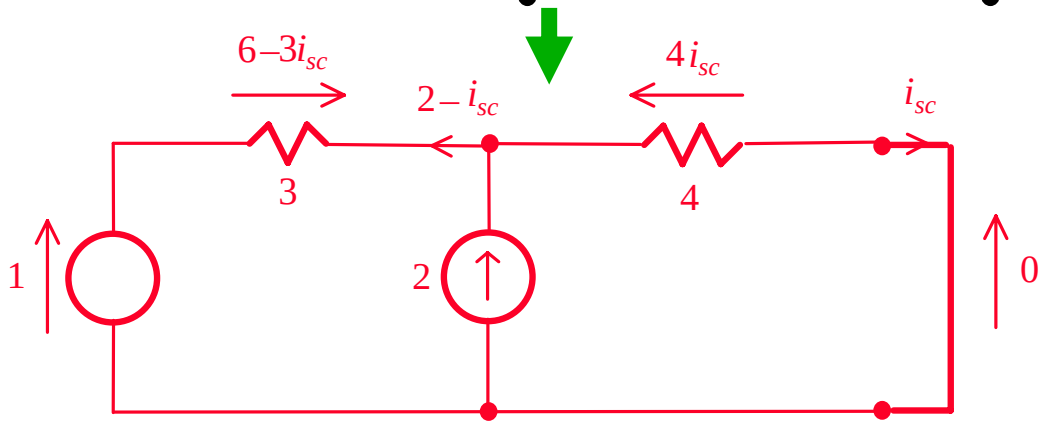
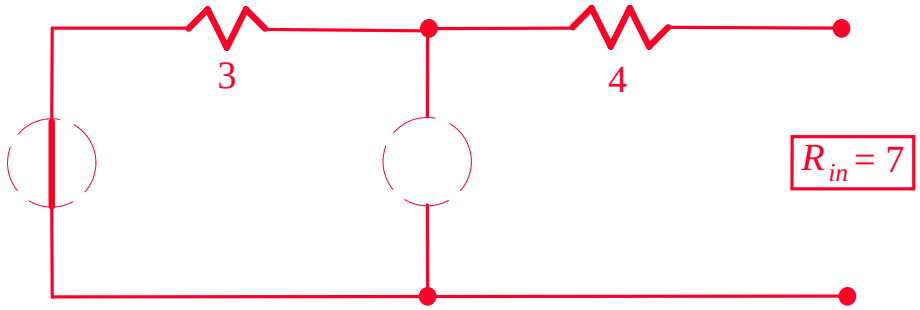
## Example: Reconsider the circuit



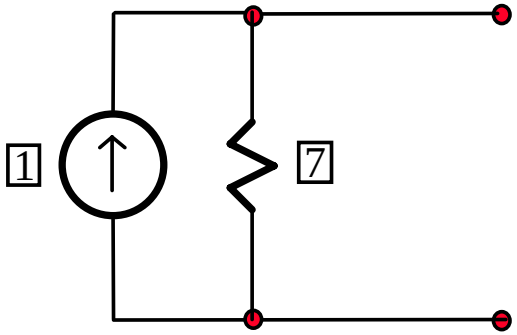




$1+6-3i_{sc}=4i_{sc}$ or $i_{sc}=1$



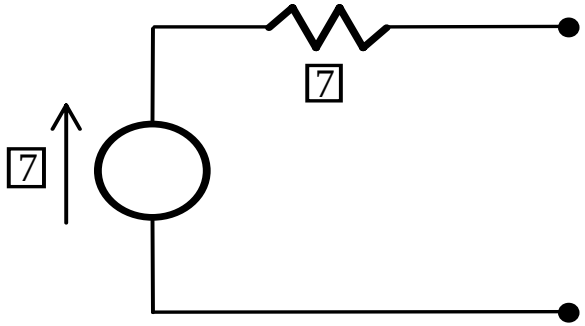

$R_{in}=7$

➡ The Norton's equivalent circuit is therefore:




➡

And the Thevenin's equivalent circuit is:



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## Summary on how to find an equivalent circuit:

Step 1. Identify the circuit or a portion of a complicated circuit that is to be simplified. Be clear in your mind on which two terminals are to be connected to the other network.

Step 2. Short-circuit all the independent voltage sources and open-circuit all independent current sources in the circuit that you are going to simplify. Then, find the equivalent resistance w.r.t. the two terminals identified in Step 1.

Step 3. Find the open circuit voltage at the output terminals (for Thevenin's equivalent circuit) or the short circuit current at the output terminals (for Norton's equivalent circuit).

Step 4. Draw the equivalent circuit (either the Thevenin's or Norton's one).



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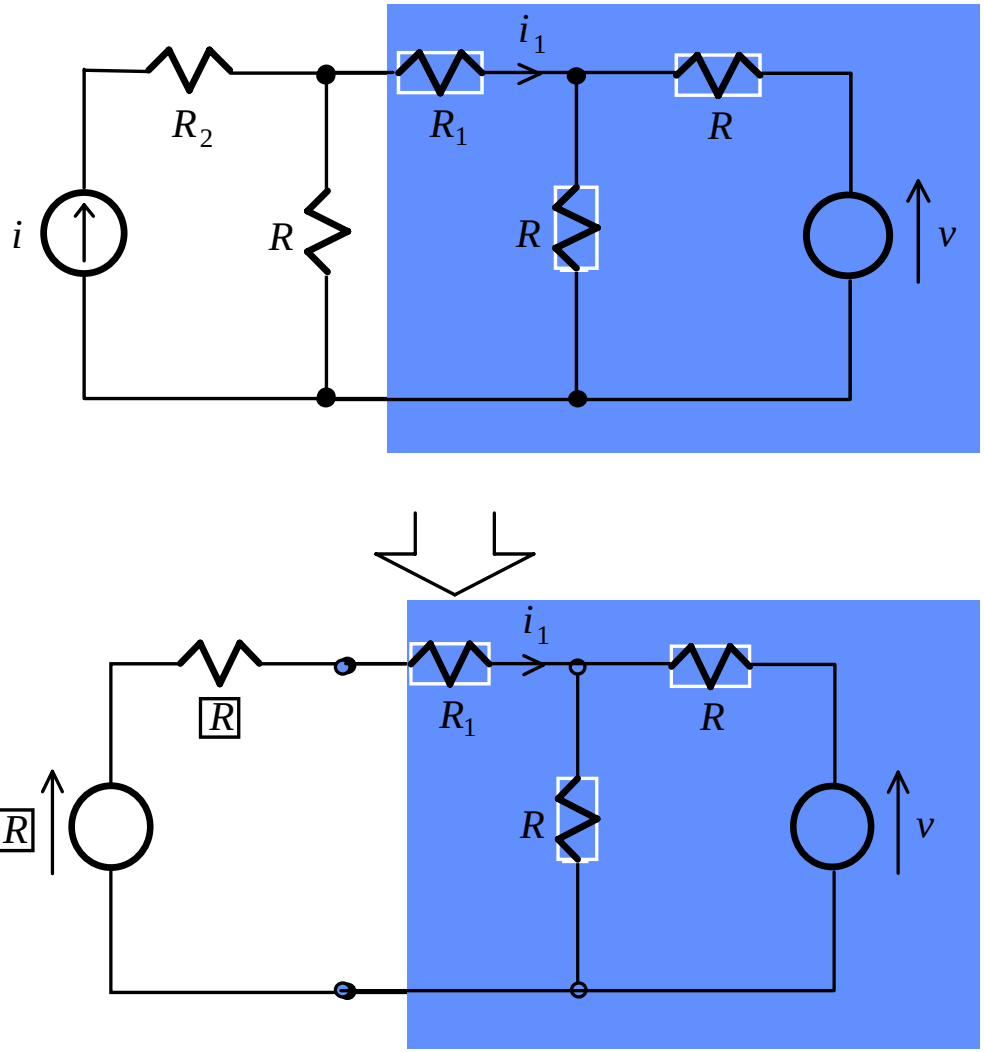
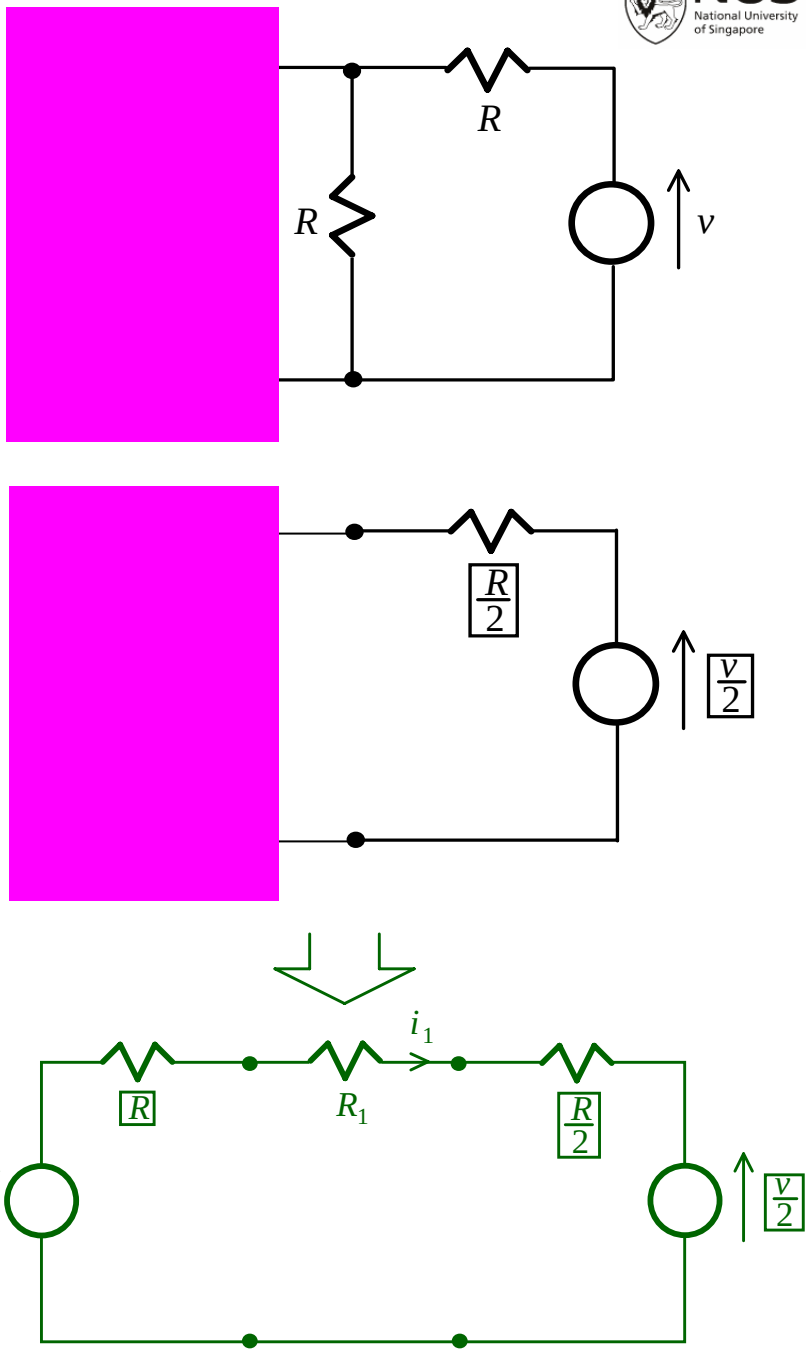
# More Example For Equivalent Circuits:



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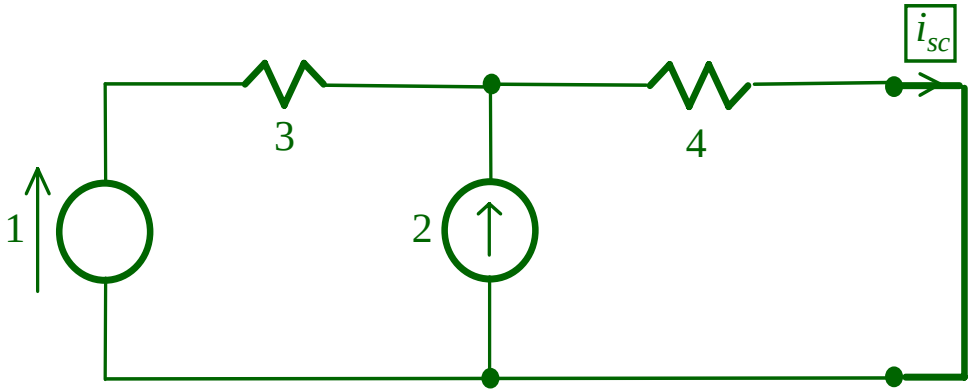
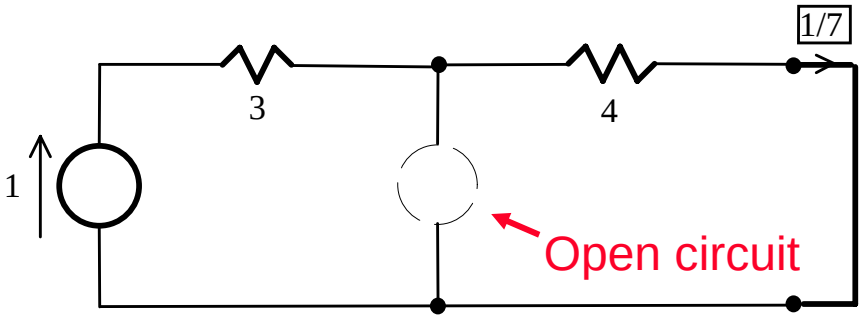
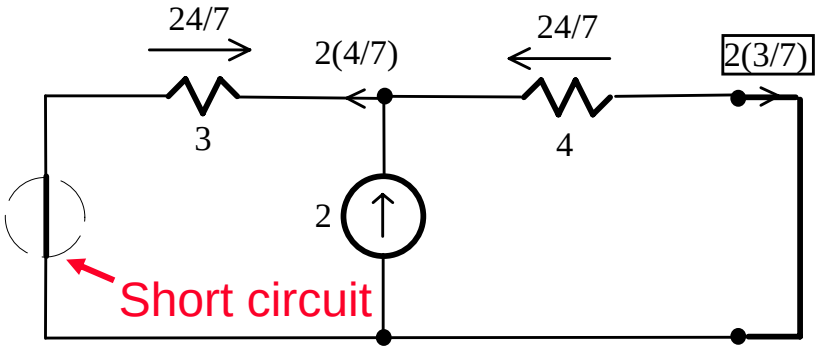
## 2.21 Superposition

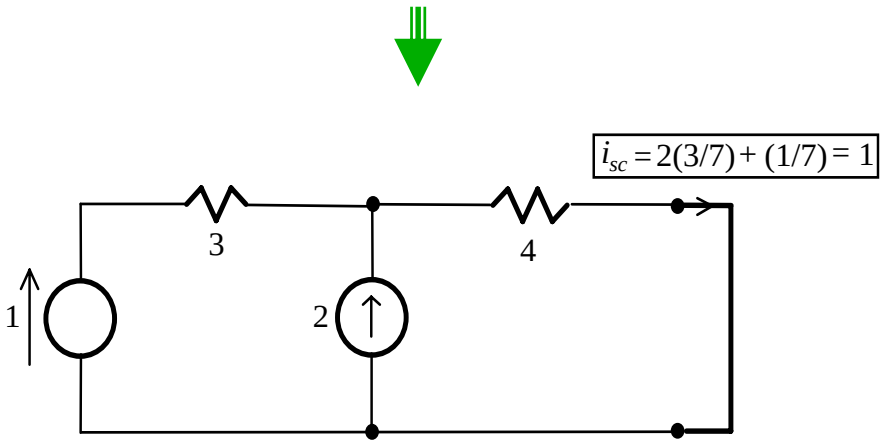
Consider finding $i_{sc}$ in the circuit:




By using the principle of super-position, this can be done by finding the components of $i_{sc}$ due to the 2 ***independent*** sources on their own (with the other sources replaced by their internal resistances):



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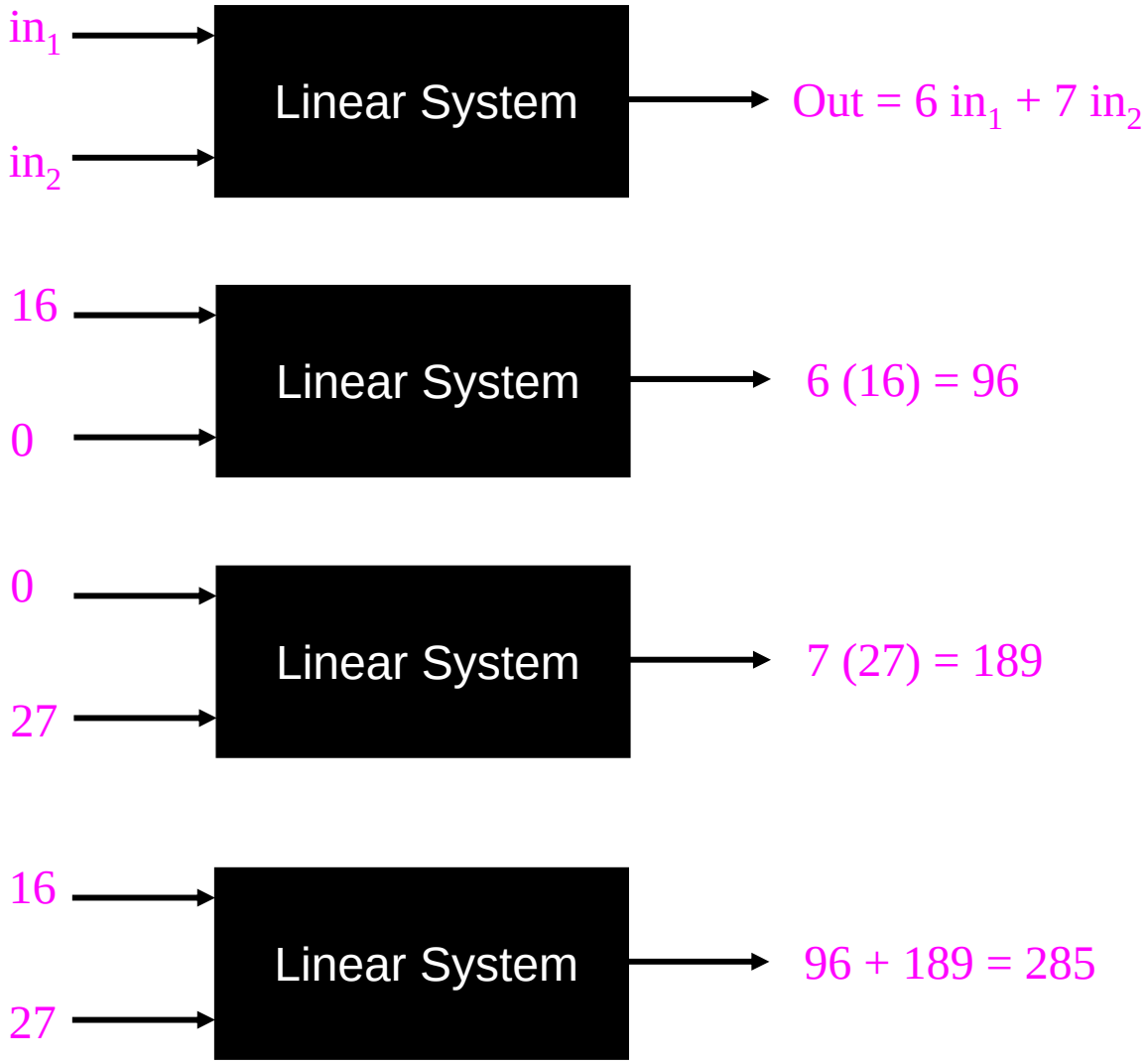
# Linear Systems and Superposition




(the definition of the linear system)

Note that:

Linear system: linear relationship between inputs and outputs

Superposition: Applicable only to linear systems



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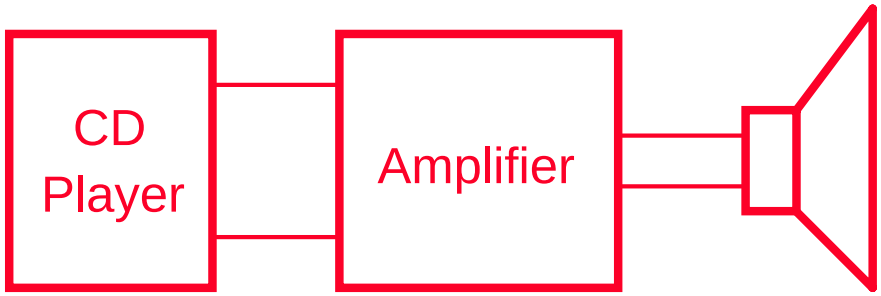
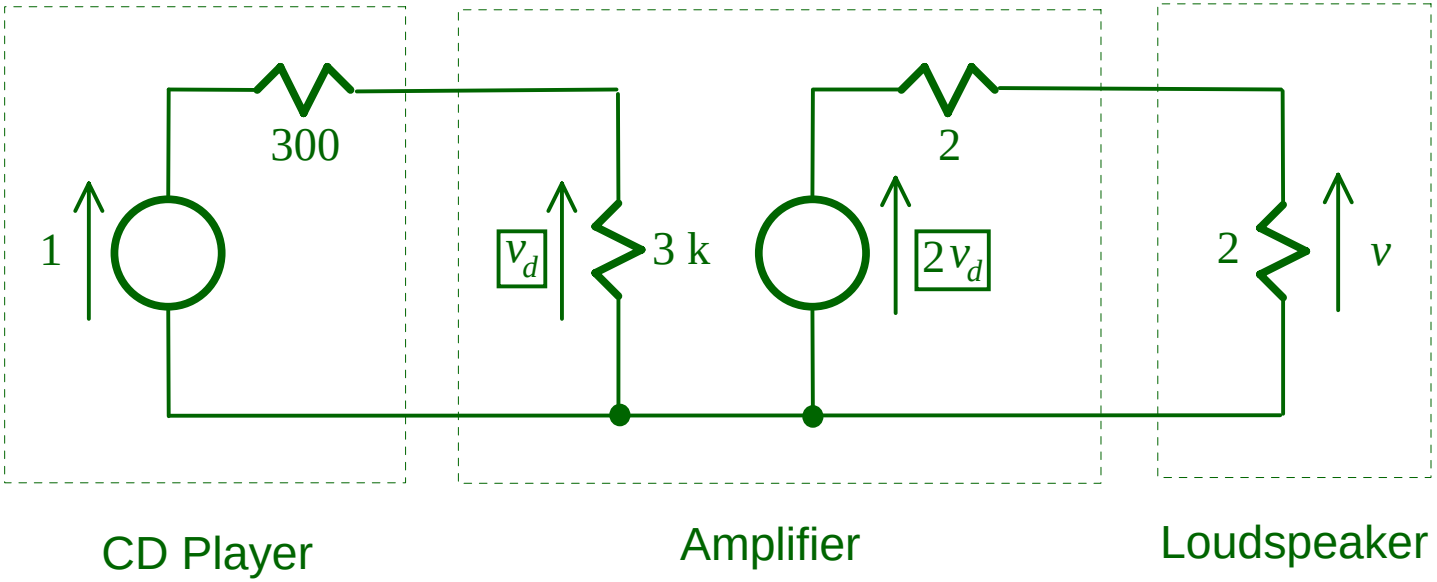
## 2.22 Dependent Source

Consider the following system:




This may be represented by



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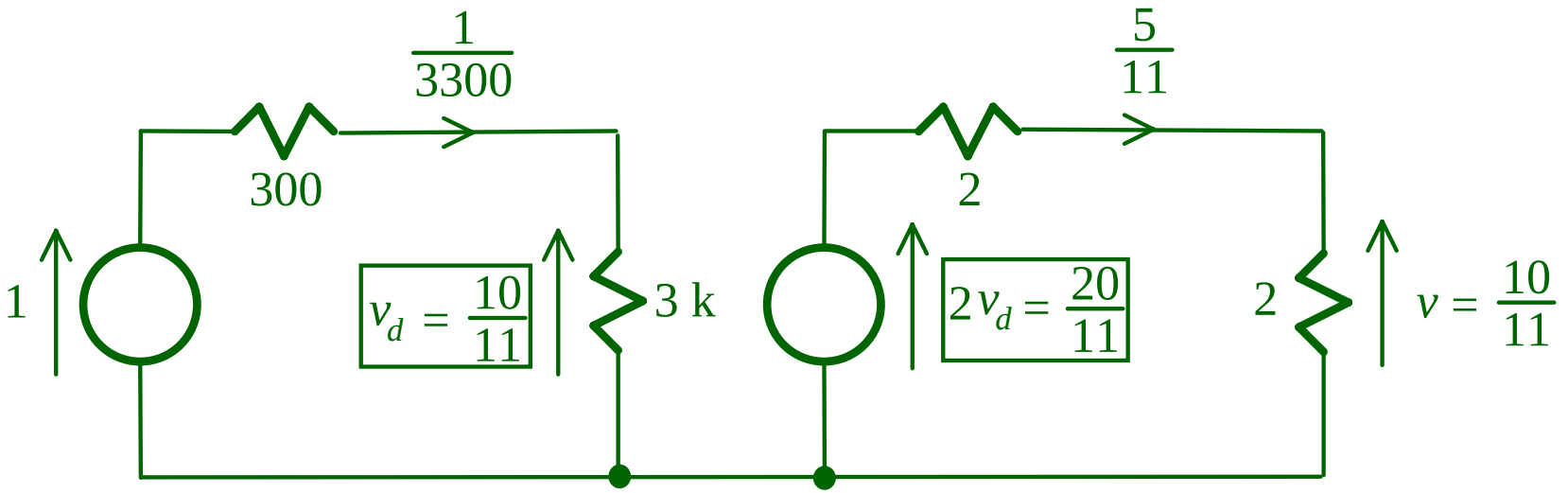
Note that the source in the Amplifier block is a **dependent source**. Its value depends on $v_d$ , the voltage across the inputs of the amplifier. Using KCL and KVL, the voltage $v$ can be easily found:




However, if we use the principle of superposition treating the dependent source as an independent source (which is wrong !), the value of $v$ will be 0:



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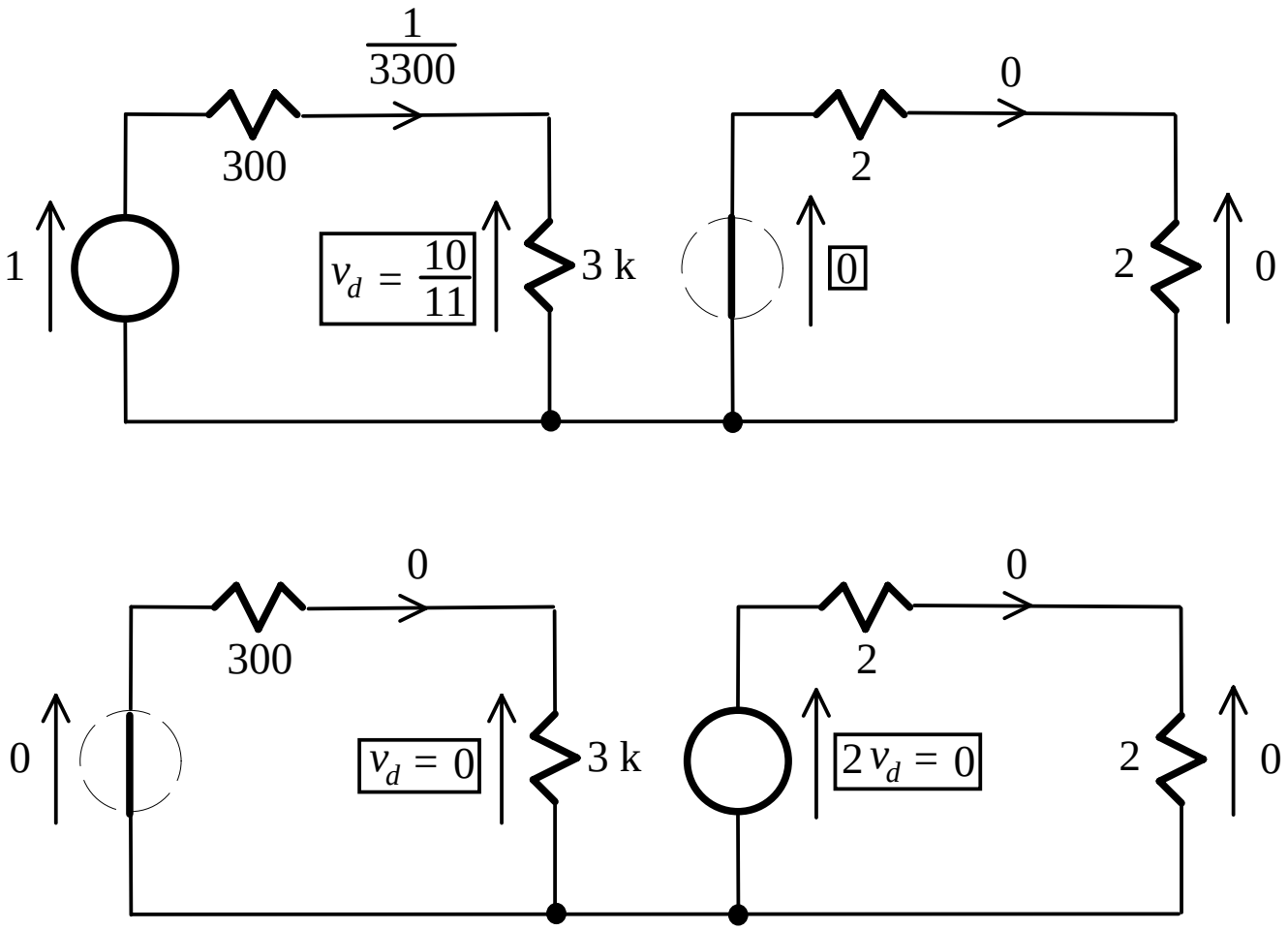


Dependent sources, which depend on other voltages/currents in the circuit and are therefore not independent excitations, cannot be removed when the principle of superposition is used. They should be treated like other passive components such as resistors in circuit analysis.



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# Topics Skipped

- Nonlinear Circuit
- Delta Circuit
- Star Circuit

- All these topics are not examinable in test and examination.



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# Reading Assignment

- **Appendix C.1. Matrix Algebra**
- **Appendix C.2. Complex Number**
- **Appendix C.3. Linear Differential Equation**



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# Chapter 3. AC Circuit Analysis



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# Appendix Materials: Operations of Complex Numbers

Coordinates: Cartesian Coordinate and Polar Coordinate

$$12 + j5 = 13\, e^{j0.39} = \sqrt{12^2 + 5^2}\; e^{j \tan^{-1}\left(\frac{5}{12}\right)}$$

real part — imaginary part — magnitude — argument

Euler's Formula: $e^{j\theta} = \cos(\theta) + j \sin(\theta)$

Additions: It is easy to do additions (subtractions) in Cartesian coordinate.

$$(a + jb) + (v + jw) = (a + v) + j(b + w)$$

Multiplication's: It is easy to do multiplication's (divisions) in Polar coordinate.

$$re^{j\theta} \cdot ue^{j\omega} = (ru)e^{j(\theta+\omega)}$$

$$\frac{re^{j\theta}}{ue^{j\omega}} = \frac{r}{u} e^{j(\theta-\omega)}$$



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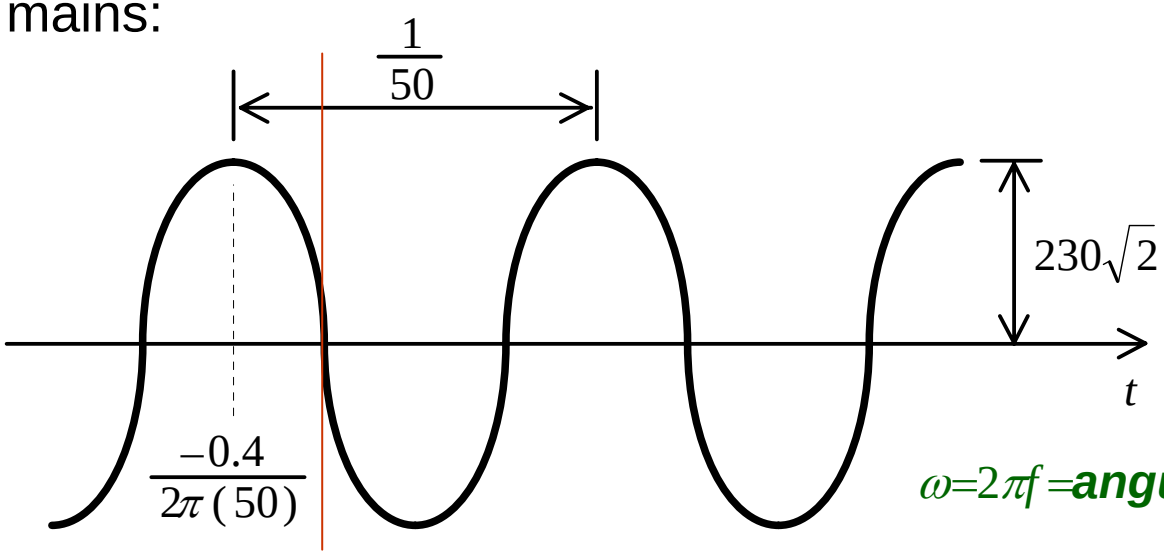
## 3.1 AC Sources

Voltages and currents in DC circuit are constants and do not change with time. In AC (**alternating current**) circuits, voltages and currents change with time in a sinusoidal manner. The most common ac voltage source is the mains:




$\theta = \textbf{phase} = 0.4\text{rad}$

$f = \textbf{frequency} = 50\text{Hz}$

$\omega = 2\pi f = \textbf{angular frequency} = 100\pi = 314\text{rad/s}$

$$T = \frac{1}{f} = \textbf{period} = \frac{1}{50} = 0.02\text{s}$$

$$v(t) = \sqrt{2}r\cos(2\pi ft+\theta) = \sqrt{2}r\cos(\omega t+\theta) = \sqrt{2}r\cos\left(\frac{2\pi t}{T}+\theta\right)$$

$$= 230\sqrt{2}\cos(100\pi t+0.4)$$

$\sqrt{2}r = \textbf{peak value} = 230\sqrt{2} = 324\text{V}$

$r = \textbf{rms (root mean square) value} = 230\text{V}$



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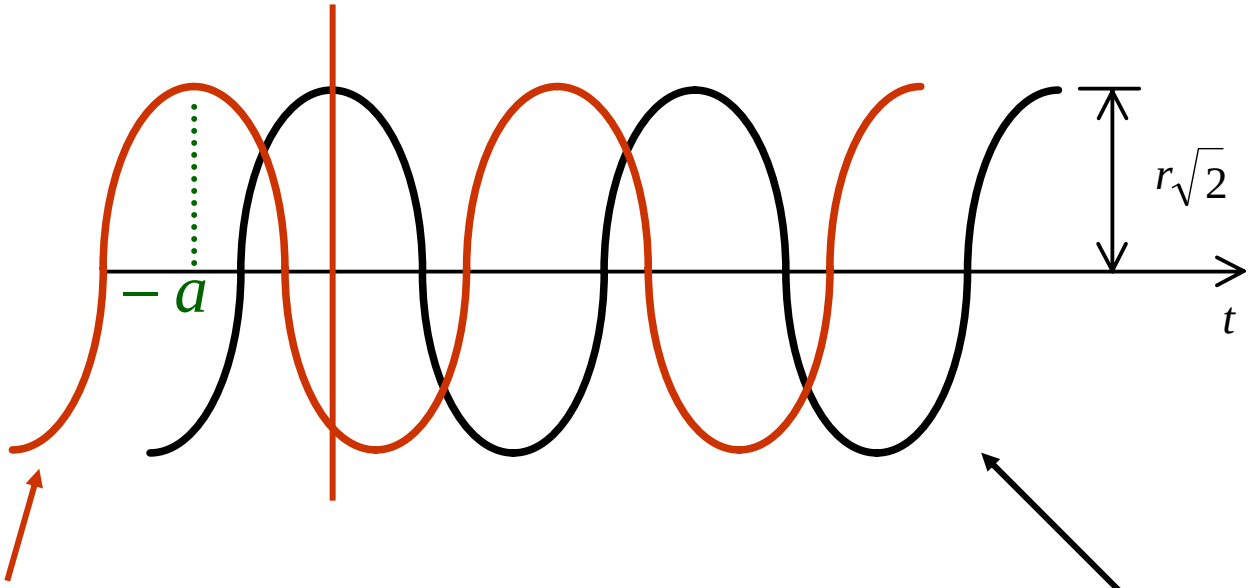
# How to find the phase for a sinusoidal function?




$$v(t) = r\sqrt{2}\cos(\omega[t + a])$$
$$= r\sqrt{2}\cos(\omega t + \omega a)$$

$$v(t) = r\sqrt{2}\cos(\omega t)$$

$$\Longrightarrow \quad \boxed{\text{Phase } \theta = \omega a}$$

$$-\pi \le \theta \le \pi$$

$$\theta = \omega a = 314\left(\frac{0.4}{2\pi(50)}\right) = 0.4$$

**for previous example.**



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Euler's Formula: $e^{j\omega} = \cos(\omega) + j\sin(\omega)$

## 3.2 Phasor

A sinusoidal voltage/current is represented using complex number format:

$$v(t) = \sqrt{2}r\cos(\omega t+\theta) = \sqrt{2}r\,\mathrm{Re}\left[e^{j(\omega t+\theta)}\right] = \mathrm{Re}\left[\left(re^{j\theta}\right)\left(\sqrt{2}e^{j\omega t}\right)\right]$$

The advantage of this can be seen if, say, we have to add 2 sinusoidal voltages given by:

$$v_1(t) = 3\sqrt{2}\cos\left(\omega t + \frac{\pi}{6}\right) \qquad v_2(t) = 5\sqrt{2}\cos\left(\omega t - \frac{\pi}{4}\right)$$

$$v_1(t) = 3\sqrt{2}\cos\left(\omega t + \frac{\pi}{6}\right) = \mathrm{Re}\left[\left(3e^{j\frac{\pi}{6}}\right)\left(\sqrt{2}e^{j\omega t}\right)\right] \qquad v_2(t) = 5\sqrt{2}\cos\left(\omega - \frac{\pi}{4}\right) = \mathrm{Re}\left[\left(5e^{-j\frac{\pi}{4}}\right)\left(\sqrt{2}e^{j\omega t}\right)\right]$$

$$v_1(t) + v_2(t) = \mathrm{Re}\left[\left(3e^{j\frac{\pi}{6}} + 5e^{-j\frac{\pi}{4}}\right)\left(\sqrt{2}e^{\omega t}\right)\right] = \mathrm{Re}\left[\left(6.47e^{-j0.32}\right)\left(\sqrt{2}e^{j\omega t}\right)\right] = 6.47\sqrt{2}\cos(\omega t - 0.32)$$



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Note that the complex time factor $\sqrt{2}e^{j\omega t}$ appears in all the expressions.

If we represent $v_1(t)$ and $v_2(t)$ by the complex numbers or ***phasors***:

$$V_1 = 3e^{j\frac{\pi}{6}} \text{ representing } v_1(t) = 3\sqrt{2}\cos\left(\omega t + \frac{\pi}{6}\right)$$

$$V_2 = 5e^{-j\frac{\pi}{4}} \text{ representing } v_2(t) = 5\sqrt{2}\cos\left(\omega t - \frac{\pi}{4}\right)$$

then the phasor representation for $v_1(t) + v_2(t)$ will be

$$V_1 + V_2 = 3e^{j\frac{\pi}{6}} + 5e^{-j\frac{\pi}{4}} = 6.47e^{-j0.32} \quad \text{representing} \quad v_1(t) + v_2(t) = 6.47\sqrt{2}\cos(\omega t - 0.32)$$

$$3e^{j\frac{\pi}{6}} + 5e^{-j\frac{\pi}{4}} = 3\left(\cos\left(\frac{\pi}{6}\right) + j\sin\left(\frac{\pi}{6}\right)\right) + 5\left(\cos\left(-\frac{\pi}{4}\right) + j\sin\left(-\frac{\pi}{4}\right)\right) = 6.14 - j2.03 = 6.47e^{-j0.32}$$

Euler's Formula: $e^{j\omega} = \cos(\omega) + j\sin(\omega)$



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By using phasors, a time-varying ac voltage

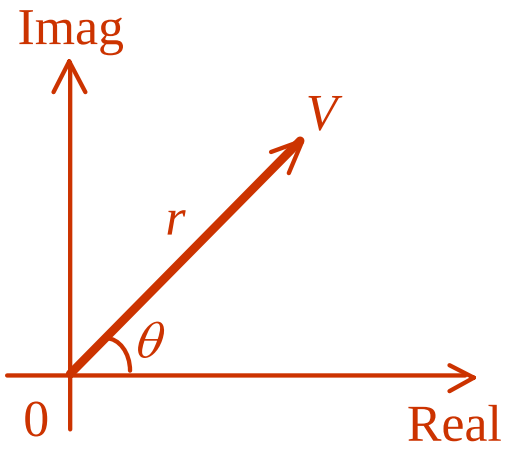
$$v(t) = \sqrt{2}r\cos(\omega t+\theta) = \mathrm{Re}\left[\left(re^{j\theta}\right)\left(\sqrt{2}e^{j\omega t}\right)\right]$$

becomes a simple complex time-invariant number/voltage $V = re^{j\theta} = r\angle\theta$

$r = |V|$ = magnitude/modulus of $V$ = r.m.s. value of $v(t)$

$\theta = \mathrm{Arg}[V]$ = phase of $V$

Graphically, on a phasor diagram:




Complex Plane

Using phasors, all time-varying ac quantities become complex dc quantities and all dc circuit analysis techniques can be employed for ac circuit with virtually no modification.



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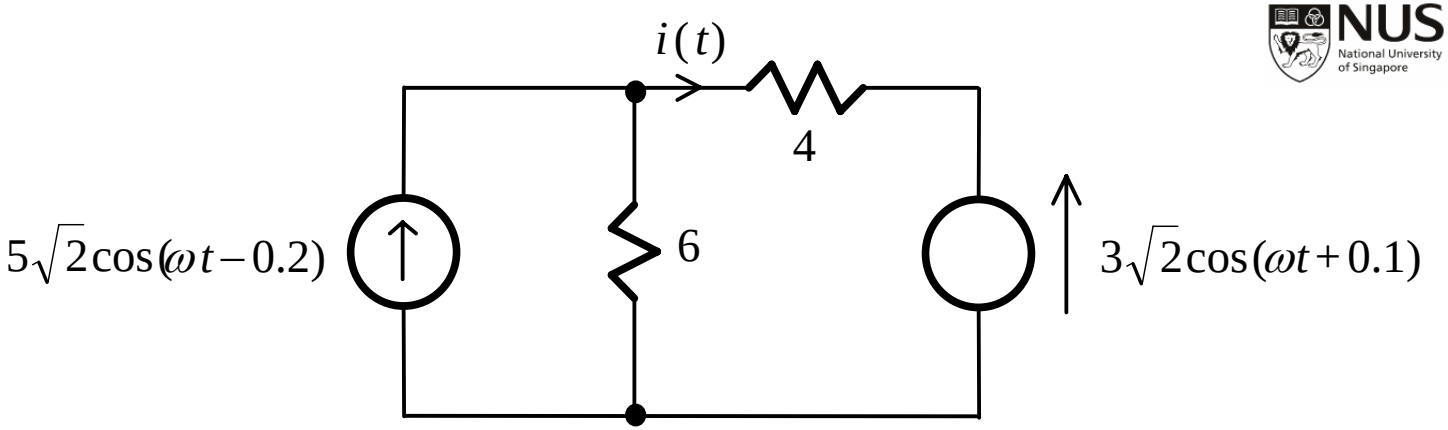
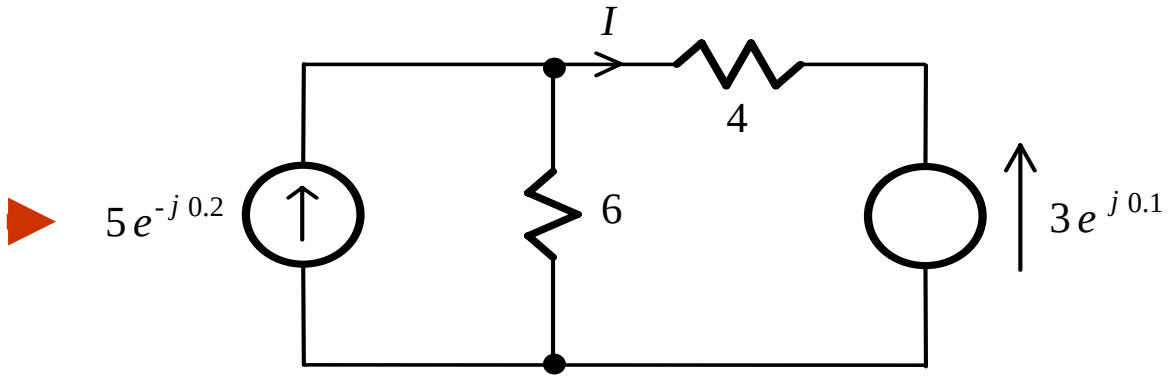
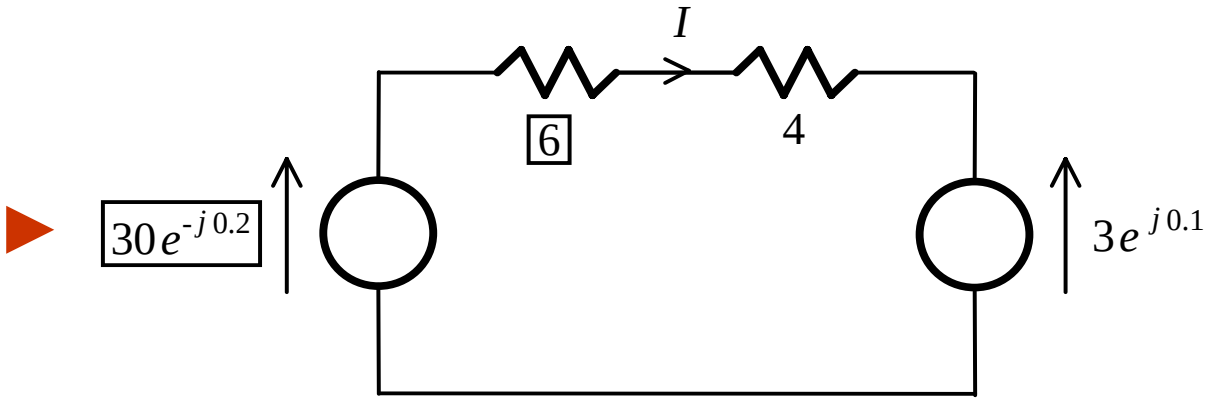
## Example:




Thevenin's equivalent circuit
for current soure and 6 Ω resistor



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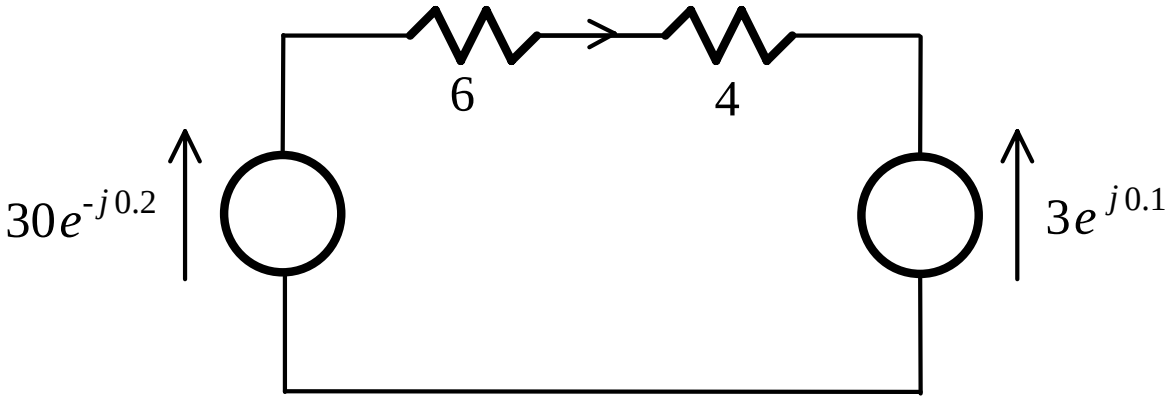
$$\Downarrow$$

$$\boxed{I = \frac{30e^{-j0.2} - 3e^{j0.1}}{10} = 2.64 - j0.63 = 2.71e^{-j0.23}}$$




$$I = \frac{30e^{-j0.2} - 3e^{j0.1}}{10}$$

$$= 3\left[\cos(-0.2) + j\sin(-0.2)\right] - 0.3\left[\cos 0.1 + j\sin 0.1\right]$$

$$= (2.940 - j0.596) - (0.299 + j0.030) = 2.641 - j0.626$$

$$= \sqrt{2.641^2 + 0.626^2}\, e^{j\tan^{-1}\left(\frac{-0.626}{2.641}\right)} = 2.71e^{-j0.23}$$

$$\Rightarrow i(t) = 2.71\sqrt{2}\cos(\omega t - 0.23)$$



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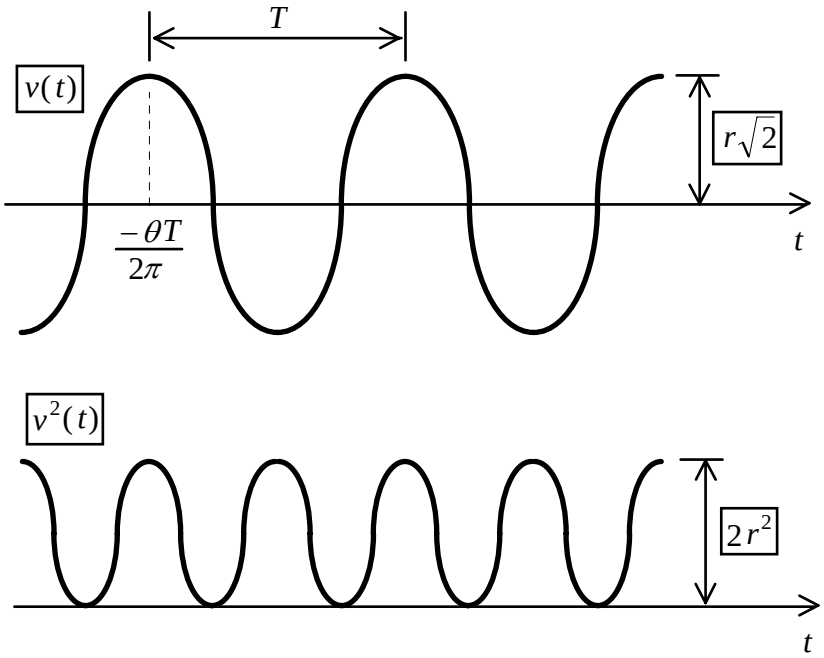
## 3.3 Root Mean Square (rms) Value

For the ac voltage

$$v(t) = \sqrt{2}r\cos(2\pi ft + \theta) = \sqrt{2}r\cos\left(\frac{2\pi t}{T} + \theta\right)$$

$$\cos(2x) = 2\cos^2(x) - 1$$

$$v^2(t) = 2r^2\cos^2\left(\frac{2\pi t}{T} + \theta\right) = r^2\left[1 + \cos\left(\frac{4\pi t}{T} + 2\theta\right)\right]$$




The average or mean of the square value is

$$\frac{1}{1\,\text{period}}\int_{1\,\text{period}} v^2(t)\,dt = \frac{1}{T}\int_0^T v^2(t)dt = \frac{1}{T}\int_0^T r^2\left[1 + \cos\left(\frac{4\pi t}{T} + 2\theta\right)\right]dt = \frac{1}{T}\int_0^T r^2 dt = r^2$$

The square root of this or the ***rms*** value of ***v*** (***t***) is rms value of $\sqrt{2}r\cos(\omega t + \theta) = r$

**Side Note: rms value can be defined for any periodical signal.**



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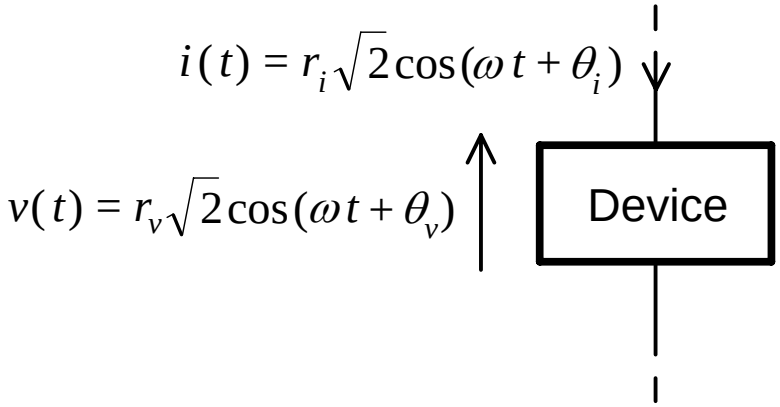
## 3.4 Power

Consider the ac device:




Using $2\cos(x_1)\cos(x_2) = \cos(x_1 - x_2) + \cos(x_1 + x_2)$, the ***instantaneous power*** consumed is

$$p(t) = i(t)v(t) = 2r_i r_v \cos(\omega t + \theta_i)\cos(\omega t + \theta_v) = r_i r_v\left[\cos(\theta_i - \theta_v) + \cos(2\omega t + \theta_i + \theta_v)\right]$$

The ***average power*** consumed is

$$p_{av} = \frac{1}{1\,\text{period}}\int_{1\,\text{period}} p(t)dt = \frac{r_i r_v}{T}\int_0^T \left[\cos(\theta_i - \theta_v) + \cos\left(\frac{4\pi t}{T} + \theta_i + \theta_v\right)\right]dt = r_i r_v \cos(\theta_i - \theta_v)$$



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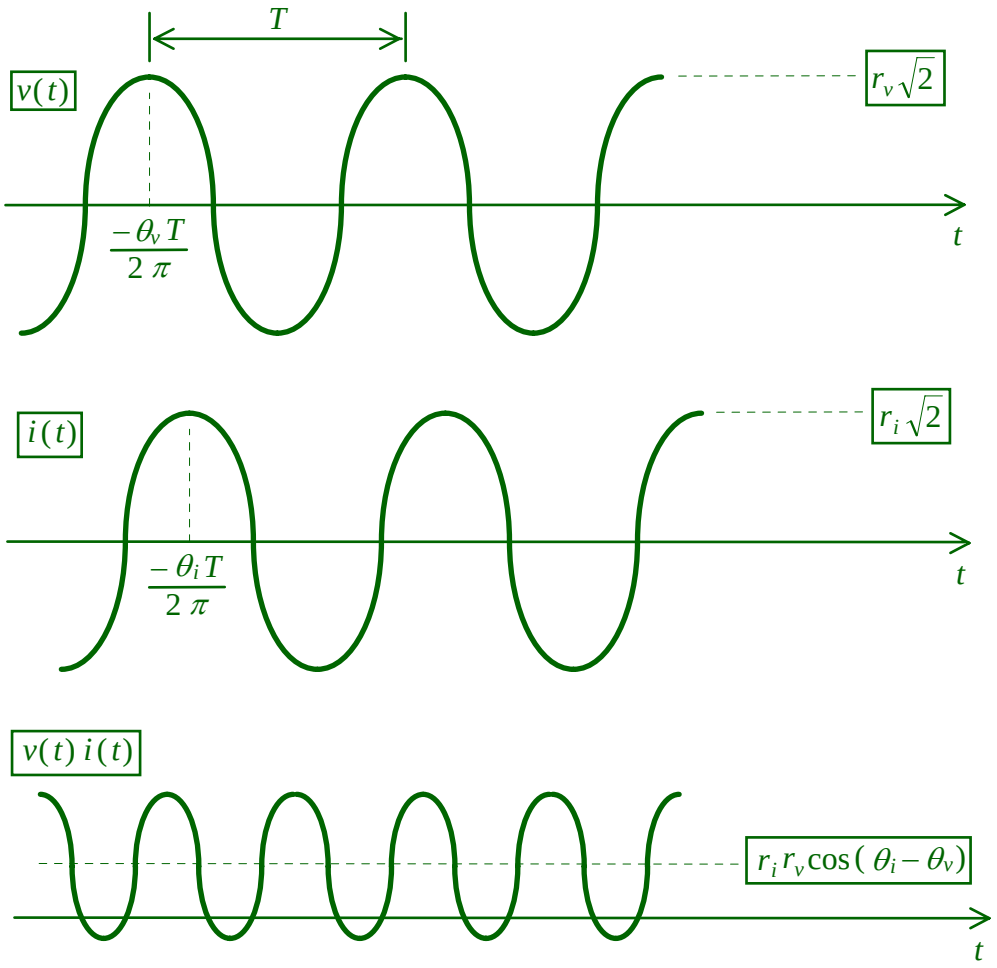


In phasor notation:

$$V = r_v e^{j\theta_v}, \quad V^* = r_v e^{-j\theta_v}$$

$$I = r_i e^{j\theta_i}, \quad I^* = r_i e^{-j\theta_i}$$

$$V^* I = r_v r_i e^{j(\theta_i - \theta_v)}$$

$$V I^* = r_v r_i e^{j(\theta_v - \theta_i)}$$

$$p_{av} = r_v r_i \cos(\theta_v - \theta_i) = r_v r_i \cos(\theta_i - \theta_v) = \mathrm{Re}\left[r_v r_i e^{j(\theta_v - \theta_i)}\right] = \mathrm{Re}\left[V^* I\right] = \mathrm{Re}\left[V I^*\right]$$



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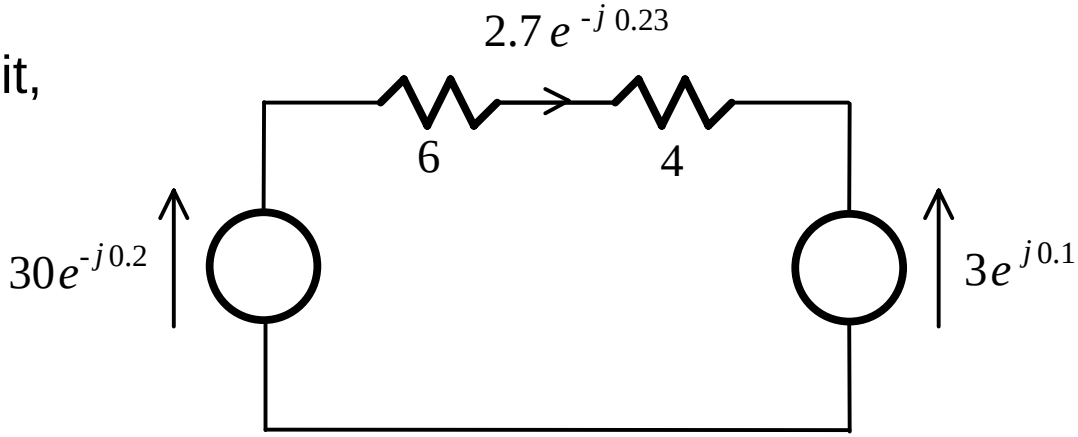
Note that the formula $p_{av}=\mathrm{Re}\left[I^{*}V\right]$ is based on rms voltages and currents. Also, this is valid for dc circuits, which is a special case of ac circuits with $f = 0$ and $V$ and $I$ having real values.

Example: Consider the ac circuit,




$$I=\frac{30e^{-j0.2}-3e^{j0.1}}{4+6}=2.7148e^{-j0.2327}$$

$3e^{j0.1}$ source : $\mathrm{Re}\left[\left(2.7e^{-j0.23}\right)^{*}\left(3e^{j0.1}\right)\right]=\mathrm{Re}\left[8.1e^{j0.33}\right]=8.1\cos(0.33)=7.66$

$30e^{-j0.2}$ source : $\mathrm{Re}\left[-\left(2.7e^{-j0.23}\right)^{*}\left(30e^{-j0.2}\right)\right]=-81\cos(0.03)=-80.96$

$6\,\Omega$ resistor : $\mathrm{Re}\left[\left(2.7e^{-j0.23}\right)^{*}\left(6\times 2.7e^{-j0.23}\right)\right]=6(2.7)^{2}=43.74$

$4\,\Omega$ resistor : $\mathrm{Re}\left[\left(2.7e^{-j0.23}\right)^{*}\left(4\times 2.7e^{-j0.23}\right)\right]=4(2.7)^{2}=29.16$

$= 0$



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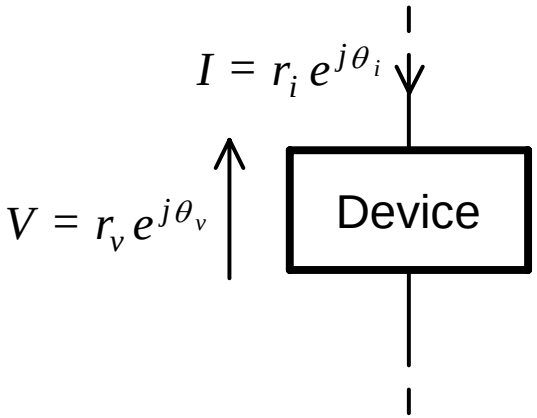
## 3.5 Power Factor

Consider the ac device:




Ignoring the phase difference between $V$ and $I$, the voltage-current rating or ***apparent power*** consumed is

$$\text{Apparent power} = \text{voltage - current rating} = |V||I| = r_v r_i \text{ VA}$$

However, the actual power consumed is

$$\text{Actual power} = \text{Re}\left[V^* I\right] = r_v r_i \cos(\theta_i - \theta_v) \text{ W}$$

The ratio of the these powers is the ***power factor*** of the device:

$$\text{Power factor} = \frac{\text{Actual power}}{\text{Apparent power}} = \cos(\theta_i - \theta_v)$$

This has a maximum value of 1 when Unity power factor $\Leftrightarrow I$ and $V$ in phase $\Leftrightarrow \theta_i = \theta_v$

The power factor is said to be ***leading*** or ***lagging*** if

Leading power factor $\Leftrightarrow I$ leads $V$ in phase $\Leftrightarrow \theta_i > \theta_v$

Lagging power factor $\Leftrightarrow I$ lags $V$ in phase $\Leftrightarrow \theta_i < \theta_v$



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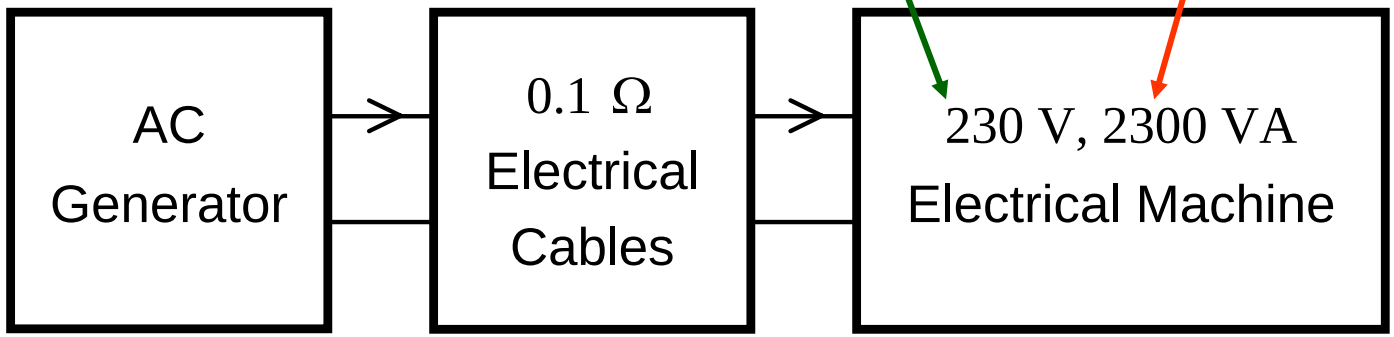
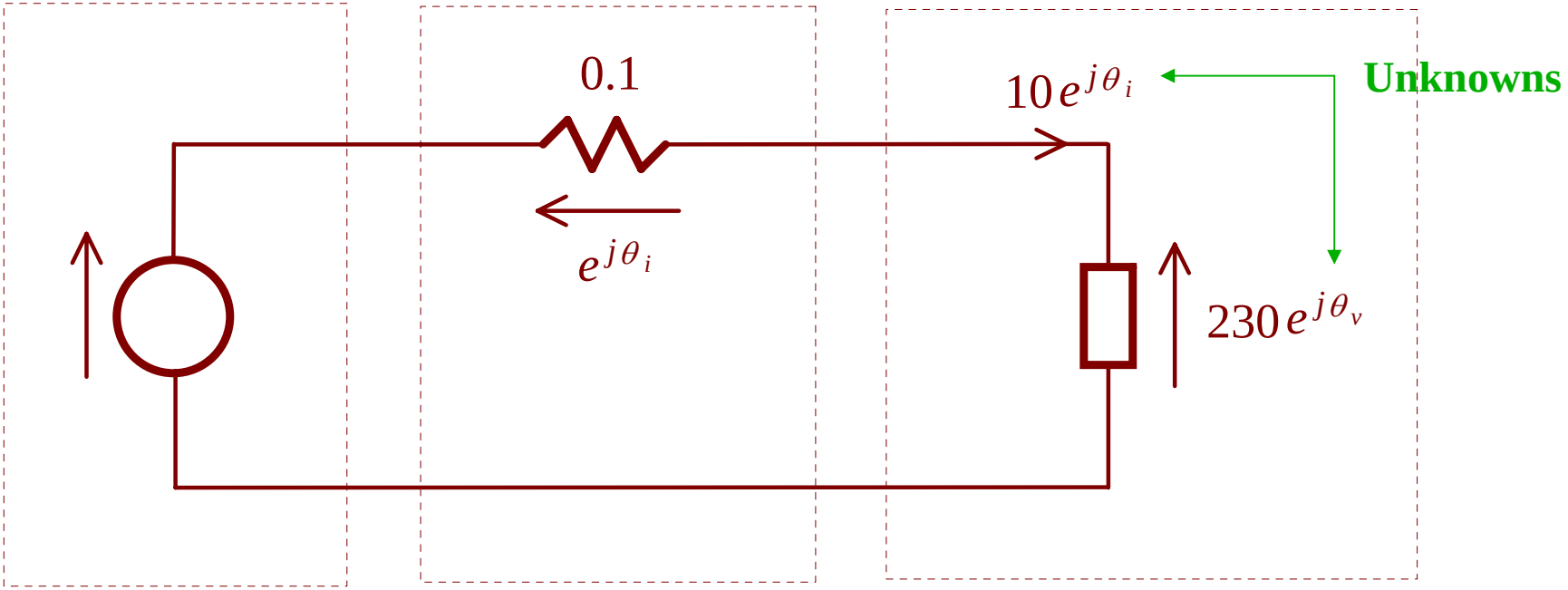
Consider the following ac system:

$r_v = 230$

$r_i \, r_v = 2300 \Rightarrow r_i = 10$



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The power consumed by the machine and power loss at different power factors are:

| | | | |
|---|---|---|---|
| **Voltage-current rating** | 2300 VA | 2300 VA | 2300 VA |
| **Voltage across machine** | 230 V | 230 V | 230 V |
| **Current** | 10 A | 10 A | 10 A |
| **Power factor** | 0.11 leading | 1 | 0.11 lagging |
| $\theta_i - \theta_v$ | $\cos^{-1}(0.11) = 1.4\text{rad}$ | 0 | $-\cos^{-1}(0.11) = -1.46\text{rad}$ |
| **Power consumed by machine** | $(2300)(0.11) = 232$ W | $(2300)(1) = 2300$W | $(2300)(0.11) = 232$W |
| **Power loss in cables** | $(0.1)(10)^2 = 10$W | $(0.1)(10)^2 = 10$W | $(0.1)(10)^2 = 10$W |



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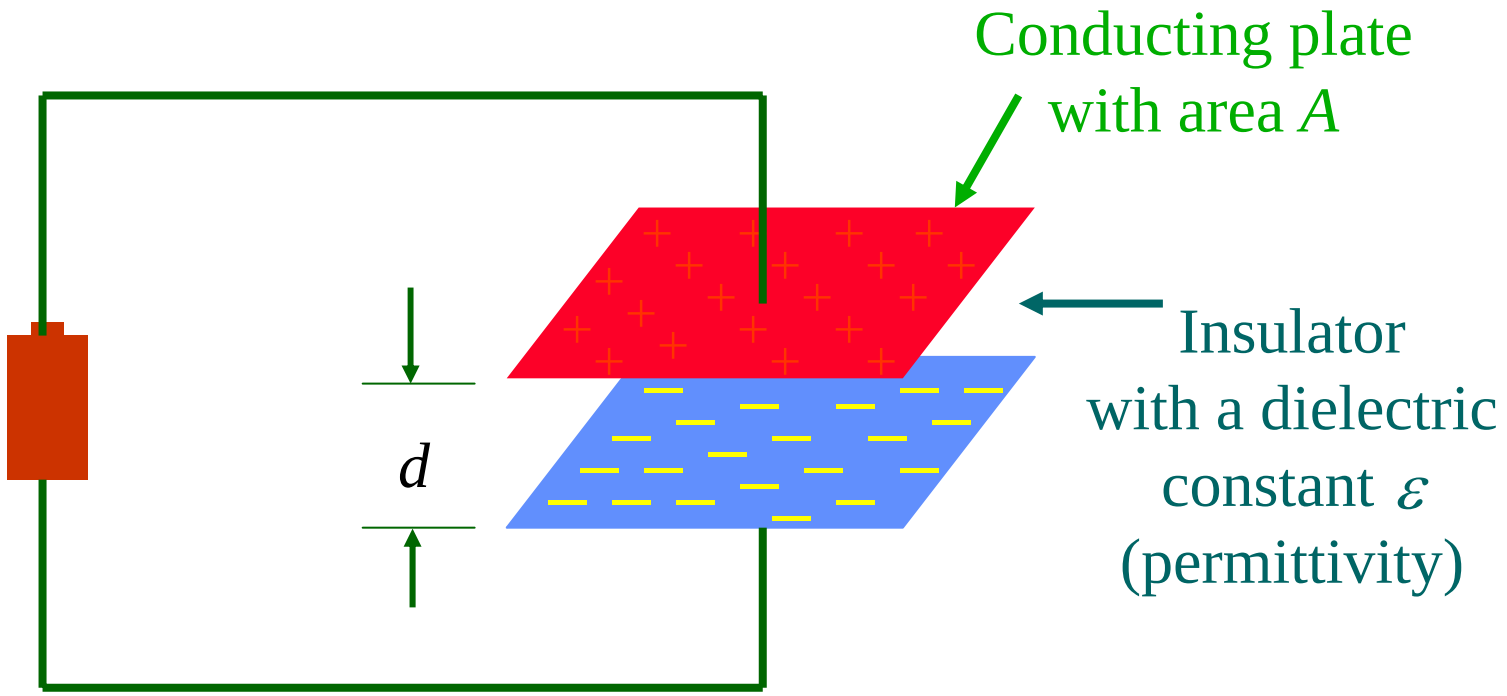
## 3.6 Capacitor

A ***capacitor*** consists of parallel metal plates for storing electric charges.




The capacitance of the capacitor is given by $C = \varepsilon \dfrac{A}{d}$ F or Farad

Area of metal plates required to produce a 1F capacitor in the free space if $d = 0.1$ mm is

$$A = \frac{Cd}{\varepsilon} = \frac{1\text{F} \times 0.0001\text{m}}{8.85 \times 10^{-12}\ \text{F/m}} = 11.3\ (\text{km})^2$$



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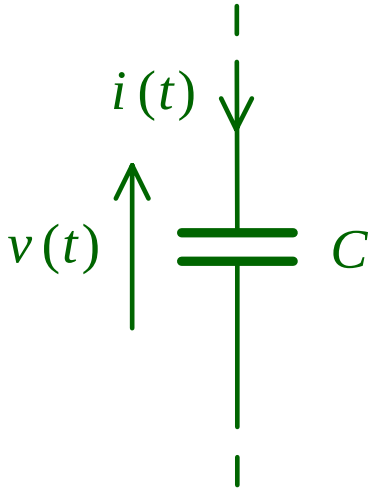
The circuit symbol for an ideal capacitor is:



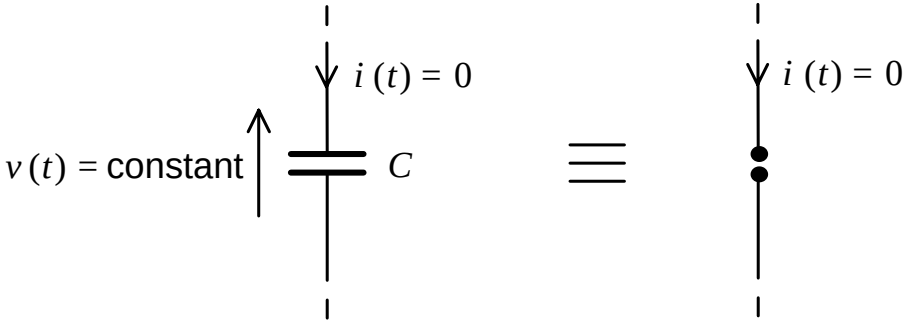

Provided that the voltage and current arrows are in opposite directions, the voltage-current relationship is:

$$i(t) = C\frac{dv(t)}{dt}$$

For dc circuits:

$$v(t) = \text{constant} \Rightarrow \frac{dv(t)}{dt} = 0 \quad \Rightarrow \quad i(t) = 0$$

and the capacitor is equivalent to an open circuit:




This is why we don't consider the capacitor in DC circuits.



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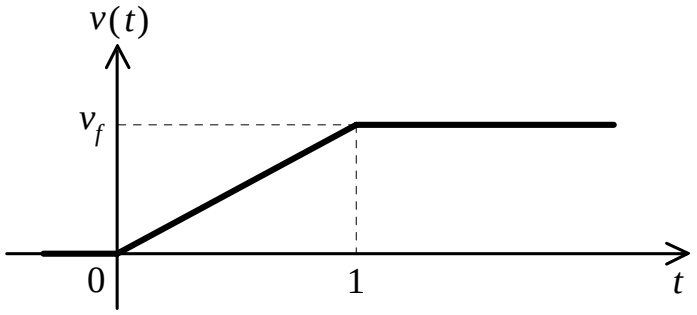
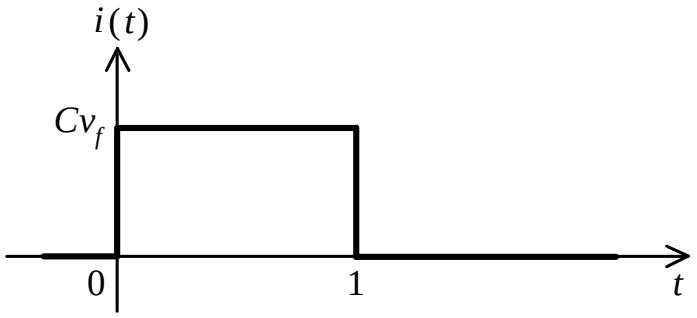
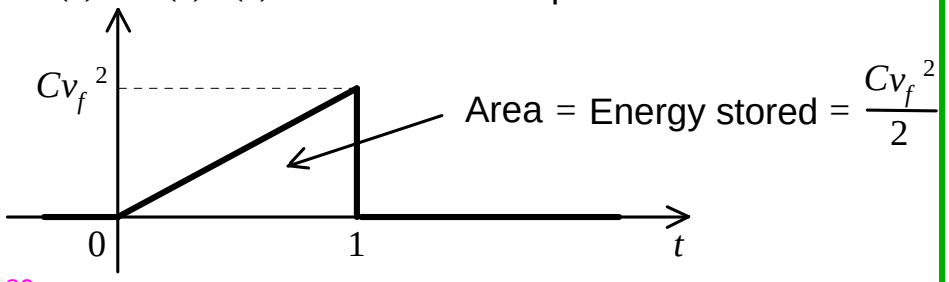
Consider the change in voltage, current and power supplied to the capacitor as indicated below:




$p(t) = v(t)\,i(t)$ = Instantaneous power consumed




In general, the total **energy stored** in the electric field established by the charges on the capacitor plates at time is

$$e(t) = \frac{Cv^2(t)}{2}$$

Proof.

$$e(t) = \int_{-\infty}^{t} p(x)dx = \int_{-\infty}^{t} v(x)i(x)dx$$

$$= \int_{-\infty}^{t} v(x)C\frac{dv(x)}{dx}dx$$

$$= C\int_{-\infty}^{t} v(x)dv(x) = \frac{C}{2}v^2(x)\Big|_{-\infty}^{t}$$

$$= \frac{C}{2}\left[v^2(t) - v^2(-\infty)\right]$$

$$= \frac{Cv^2(t)}{2}, \qquad \text{if } v(-\infty) = 0.$$



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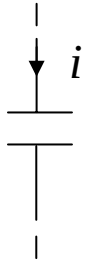
Now consider the operation of a capacitor in an ac circuit:

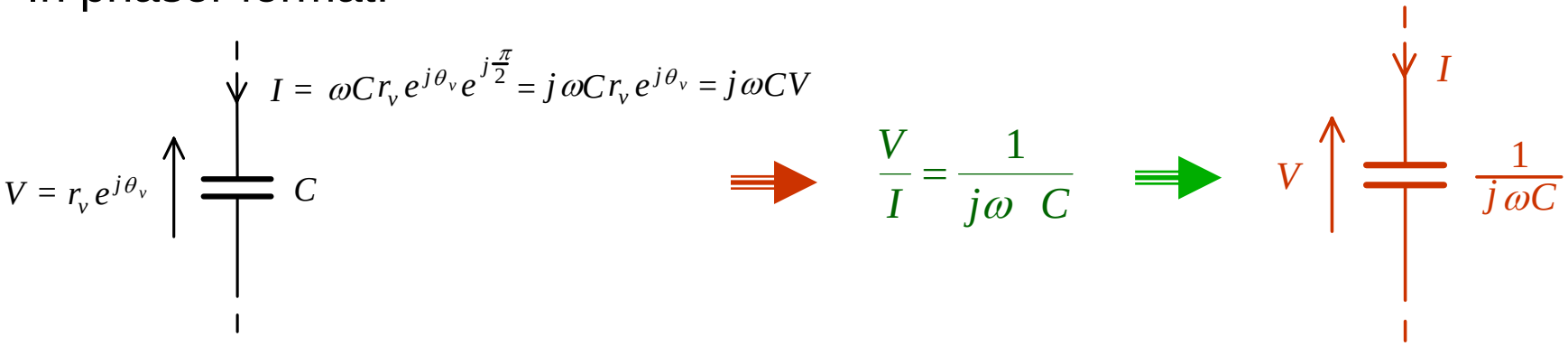
$$v(t) = r_v\sqrt{2}\cos(\omega t + \theta_v)$$



$$i(t) = C\frac{dv(t)}{dt} = -\omega C r_v\sqrt{2}\sin(\omega t + \theta_v)$$

$$= \omega C r_v\sqrt{2}\cos(\omega t + \theta_v + \frac{\pi}{2})$$

In phasor format:




With phasor representation, the capacitor behaves as if it is a resistor with a "complex resistance" or an **impedance** of

$$Z_C = \frac{1}{j\omega C} \quad \Rightarrow \quad p_{av} = \mathrm{Re}\left[I^* V\right] = \mathrm{Re}\left[I^* I Z_C\right] = \mathrm{Re}\left[\frac{|I|^2}{j\omega C}\right] = 0$$

An ideal capacitor is a non-dissipative but energy-storing device.



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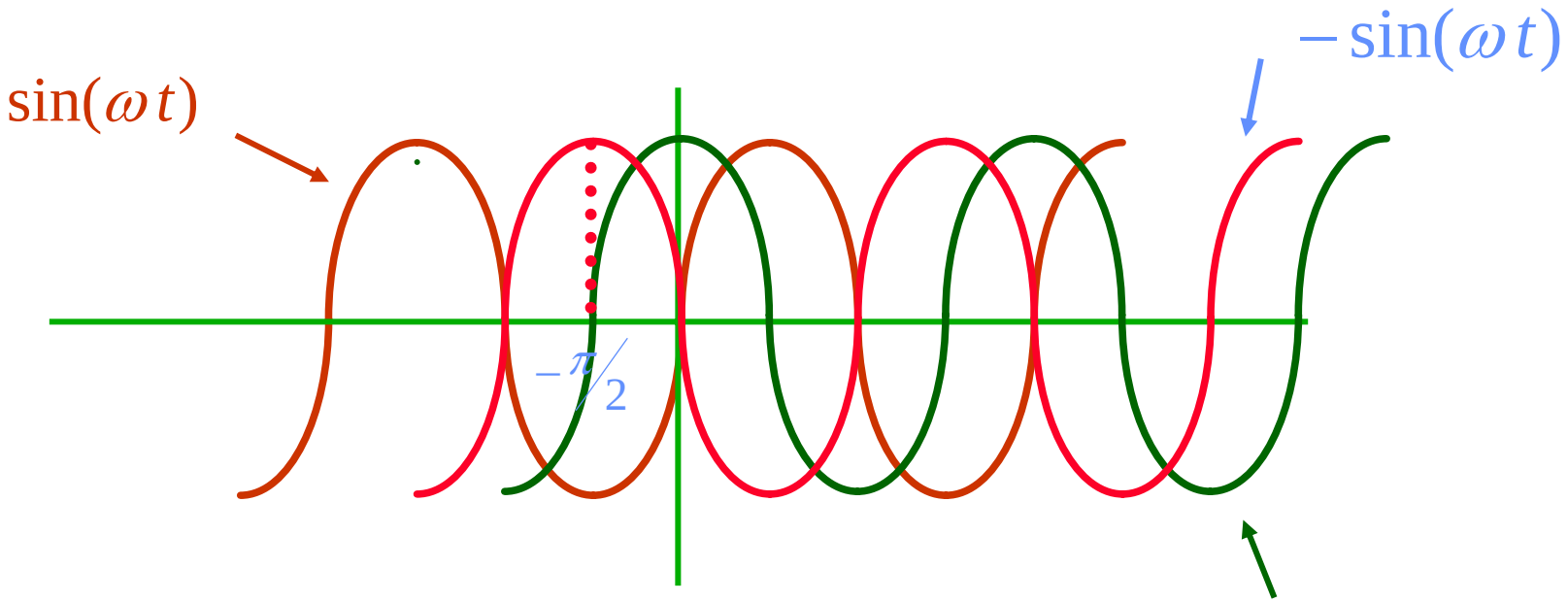
Since the phase of $I$ relative to $V$ that of is

$$\text{Arg}[I]-\text{Arg}[V]=\text{Arg}\left[\frac{I}{V}\right]=\text{Arg}\left[\frac{1}{Z_C}\right]=\text{Arg}[j\omega\ C]=90^0$$

the ac current $i(t)$ of the capacitor leads the voltage $v(t)$ by 90°.




$$-\sin\left(\omega t-\frac{\pi}{2}\right)=\cos(\omega t)=\sin\left(\omega t+\frac{\pi}{2}\right)$$



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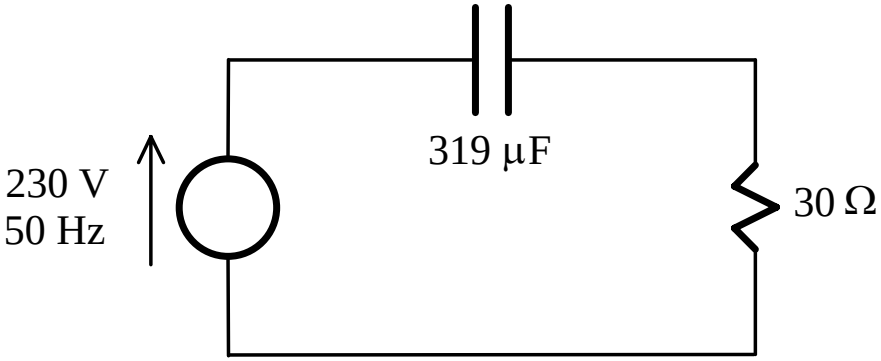
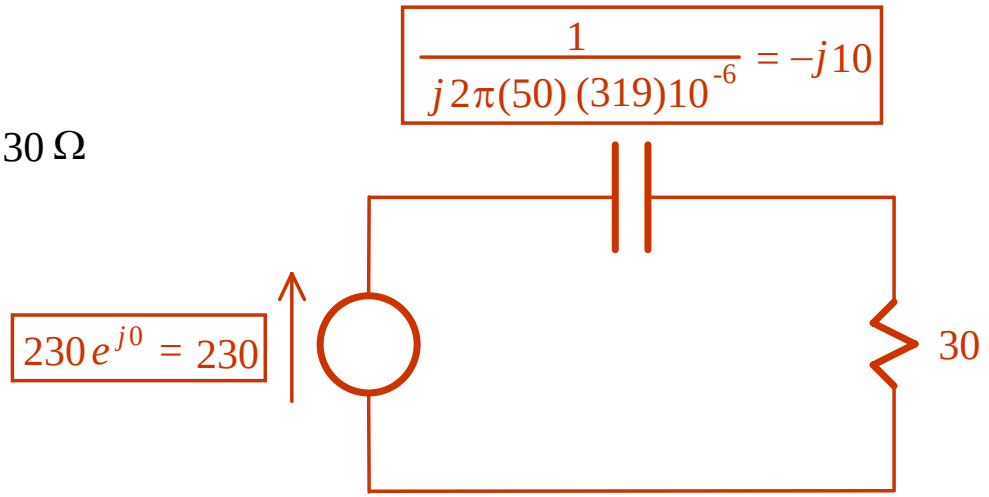
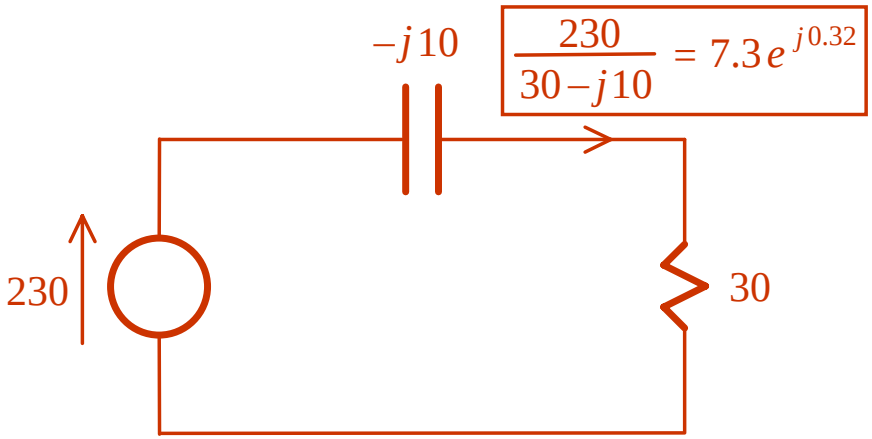
Example: Consider the following ac circuit:




In phasor notation (taking the source to have a reference phase of 0):



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| | |
|---|---|
| **Total circuit impedance** | $Z = (30 - j10)\,\Omega$ |
| **Total circuit *reactance*** | $X = \mathrm{Im}[Z] = \mathrm{Im}[30 - j10] = -10\,\Omega$ |
| **Total circuit resistance** | $R = \mathrm{Re}[Z] = \mathrm{Re}[30 - j10] = 30\,\Omega$ |
| **Current (rms)** | $\lvert I \rvert = 7.3\mathrm{A}$ |
| **Current (peak)** | $\sqrt{2}\lvert I \rvert = 7.3\sqrt{2} = 10\mathrm{A}$ |
| **Source *V-I* phase relationship** | $I$ leads by $0.32\mathrm{rad}$ |
| **Power factor of entire circuit** | $\cos(0.32) = 0.95$ leading |
| **Power supplied by source** | $\mathrm{Re}\left[(230)^*\left(7.3e^{j0.32}\right)\right] = (230)(7.3)\cos(0.32) = 1.6\ \mathrm{kW}$ |
| **Power consumed by resistor** | $\mathrm{Re}\left[\left(7.3e^{j0.32}\right)^*\left(30 \times 7.3e^{j0.32}\right)\right] = (7.3)^2 30 = 1.6\ \mathrm{kW}$ |



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**Impedance**, **Resistance,** **Reactance**, } Relations?

**Admittance**, **Conductance,** and **Susceptance**

**Impedance:** $Z = R + jX$

**Admittance:**

$$Y = \frac{1}{Z} = \frac{1}{R + jX} = \frac{R - jX}{(R + jX)(R - jX)}$$

$$= \frac{R - jX}{R^2 + X^2} = \frac{R}{R^2 + X^2} + j\frac{-X}{R^2 + X^2}$$

$$= G + jB$$

**Conductance** **Susceptance**



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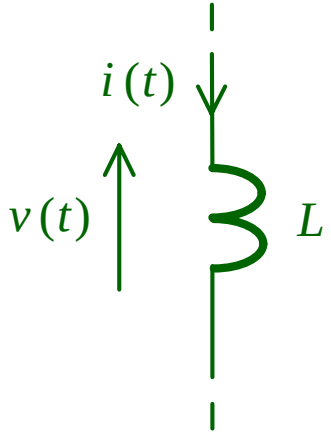
## 3.7 Inductor

An **inductor** consists of a coil of wires for establishing a magnetic field. The circuit symbol for an ideal inductor is:



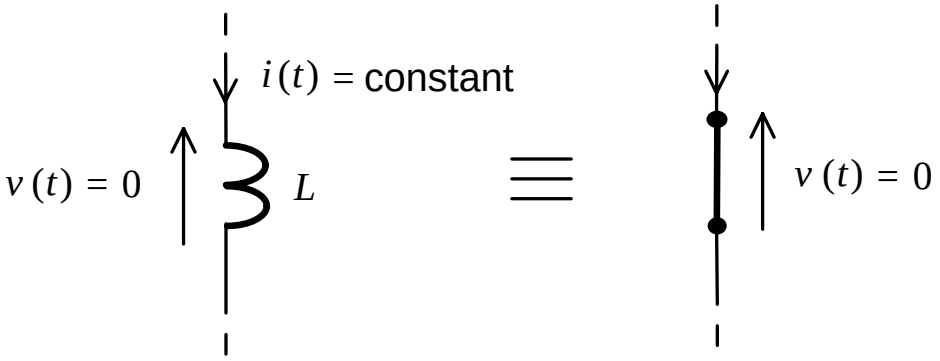

Provided that the voltage and current arrows are in opposite directions, the voltage-current relationship is:

$$v(t) = L\frac{di(t)}{dt}$$

For dc circuits:

$$i(t) = \text{constant} \Rightarrow \frac{di(t)}{dt} = 0 \Rightarrow v(t) = 0$$

and the inductor is equivalent to a short circuit:




That is why there is nothing interesting about the inductor in DC circuits.



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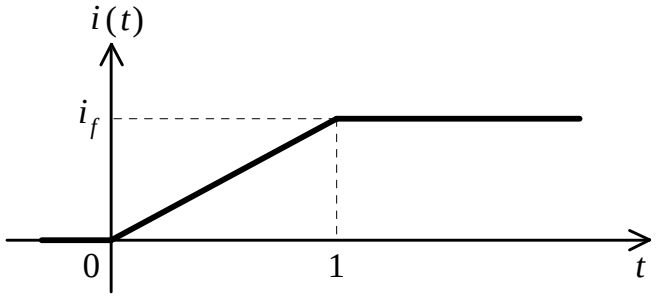
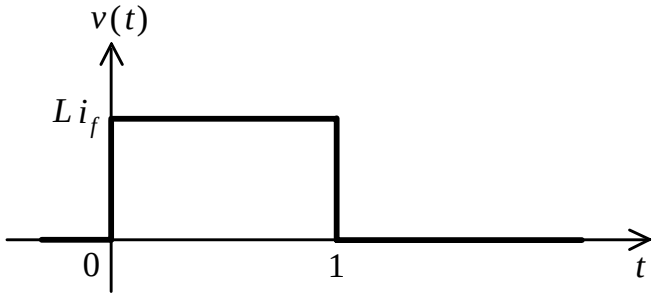
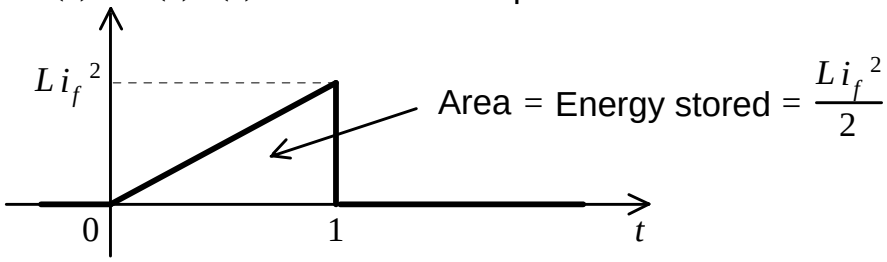
Consider the change in voltage, current and power supplied to the inductor as indicated below:




$p(t) = v(t)\,i(t)$ =Instantaneous power consumed




In general, the total energy stored in the magnetic field established by the current $i(t)$ in the inductor at time $t$ is given by

$$e(t) = \frac{Li^2(t)}{2}$$

$$e(t) = \int_{-\infty}^{t} p(x)dx = \int_{-\infty}^{t} v(x)i(x)dx$$

$$= \int_{-\infty}^{t} i(x)L\frac{di(x)}{dx}dx$$

$$= L\int_{-\infty}^{t} i(x)di(x) = \frac{L}{2}i^2(x)\Big|_{-\infty}^{t}$$

$$= \frac{L}{2}\left[i^2(t) - i^2(-\infty)\right]$$

$$= \frac{Li^2(t)}{2}, \qquad \text{if} \quad i(-\infty) = 0.$$



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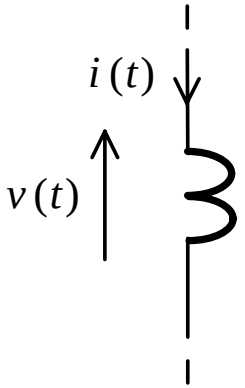
Now consider the operation of an inductor in an ac circuit:

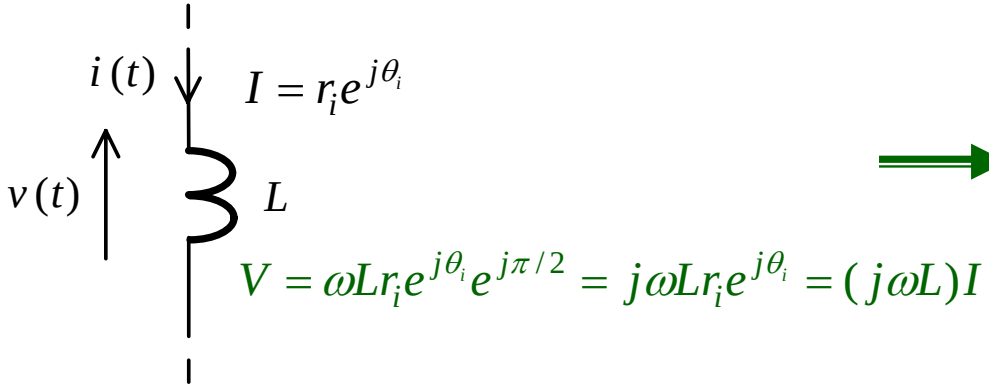



$$i(t) = r_i\sqrt{2}\cos(\omega t + \theta_i)$$

$$v(t) = L\frac{di(t)}{dt} = -\omega L r_i\sqrt{2}\sin(\omega t + \theta_i)$$

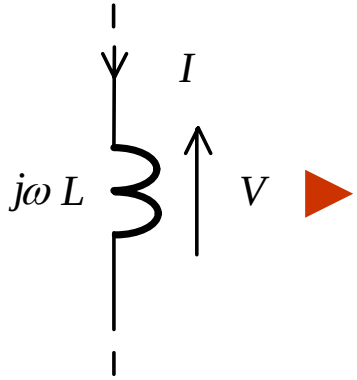
$$= \omega L r_i\sqrt{2}\cos(\omega t + \theta_i + \frac{\pi}{2})$$

In phasor:




$$I = r_i e^{j\theta_i}$$

$$V = \omega L r_i e^{j\theta_i} e^{j\pi/2} = j\omega L r_i e^{j\theta_i} = (j\omega L) I$$




$$Z_L = \frac{V}{I} = j\omega L$$

$Z_L$ is the impedance of the inductor. The ave. power absorbed by the inductor:

$$p_{av} = \mathrm{Re}\left[I^* V\right] = \mathrm{Re}\left[I^* Z_L I\right] = \mathrm{Re}\left[j\omega L I^* I\right] = \mathrm{Re}\left[j\omega L \mid I \mid^2\right] = 0$$



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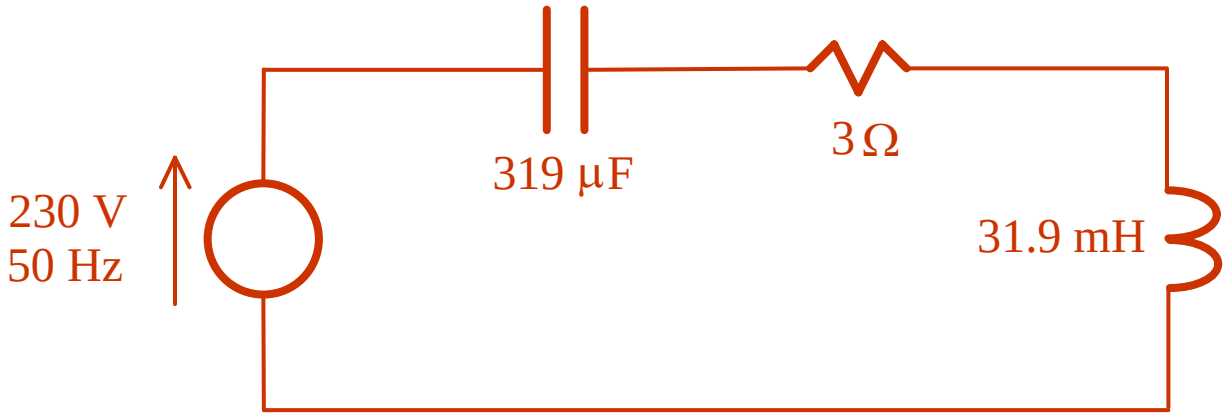
Since the phase of $I$ relative to that of $V$ is

$$\text{Arg}[I] - \text{Arg}[V] = \text{Arg}\left[\frac{I}{V}\right] = \text{Arg}\left[\frac{1}{Z_L}\right] = \text{Arg}\left[\frac{1}{j\omega L}\right] = -90^0$$

the ac current $i(t)$ lags the voltage $v(t)$ by $90^o$.

As an example, consider the following series ac circuit:




We can use the phasor representation to convert this ac circuit to a 'DC' circuit with complex voltage and resistance.



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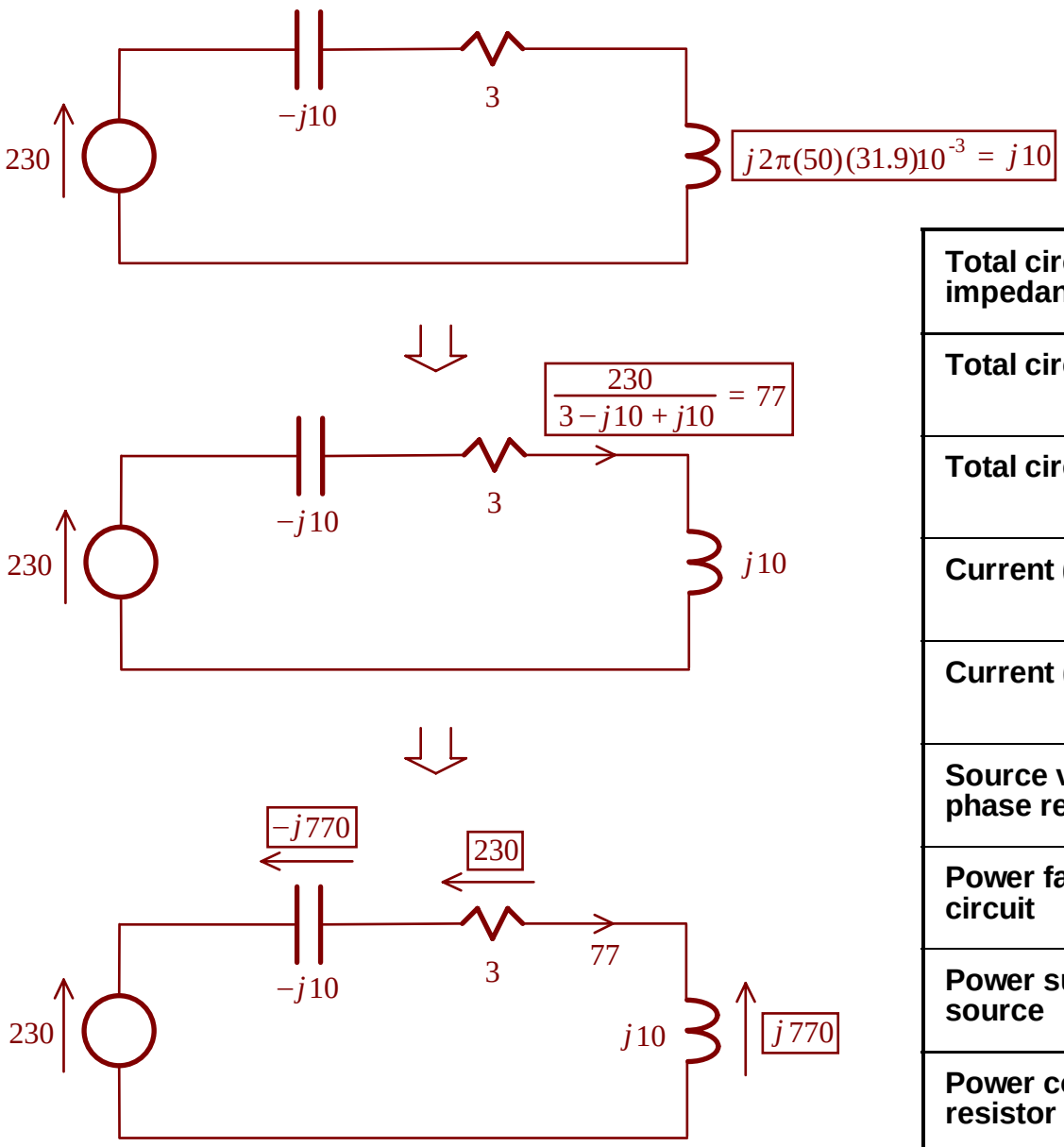


## Summary of the circuit:

| | |
|---|---|
| **Total circuit impedance** | $Z = 3 - j10 + j10 = 3\Omega$ |
| **Total circuit reactance** | $X = \mathrm{Im}[Z] = \mathrm{Im}[3] = 0\Omega$ |
| **Total circuit resistance** | $R = \mathrm{Re}[Z] = \mathrm{Re}[3] = 3\Omega$ |
| **Current (rms)** | $\lvert I \rvert = 77\mathrm{A}$ |
| **Current (peak)** | $\sqrt{2}\lvert I \rvert = 77\sqrt{2} = 108\mathrm{A}$ |
| **Source voltage-current phase relationship** | $0\ (\text{in phase})$ |
| **Power factor of entire circuit** | $\cos(0) = 1$ |
| **Power supplied by source** | $\mathrm{Re}\left[(77)^*(230)\right] = 18\mathrm{kW}$ |
| **Power consumed by resistor** | $\mathrm{Re}\left[(77)^*(3\times 77)\right] = 18\mathrm{kW}$ |



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Note that the rms voltages across the inductor and capacitor are larger than the source voltage. This is possible in ac circuits because the reactances of capacitors and inductors, and so the voltages developed across them, may cancel out one another:

| Source voltage | = | Voltage across capacitor | + | Voltage across resistor | + | Voltage across inductor |
|---|---|---|---|---|---|---|
| $230$ | | $-j770$ | | $230$ | | $j770$ |

In dc circuits, it is not possible for a passive resistor (with positive resistance) to cancel out the effect of another passive resistor (with positive resistance).



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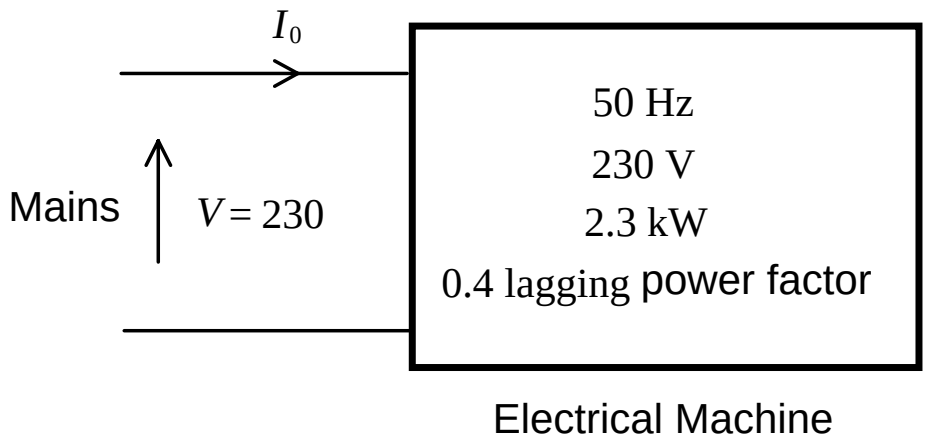
## 3.8 Power Factor Improvement

Consider the following system:




Due to the small power factor, the machine cannot be connected to standard 13A outlets even though it consumes only 2.3 kW of power.

The current $I_0$ can be found as follows:

$$\frac{2300\text{W}}{(230\text{V})(|I_0|\text{A})} = 0.4 \Rightarrow |I_0| = \frac{2300}{(230)(0.4)} = 25$$

**Can we improve it?**

$$\left.\begin{aligned} &\cos\{\text{Arg}[I_0] - \text{Arg}[V]\} = 0.4 \\ &\text{Arg}[I_0] - \text{Arg}[V] < 0 \end{aligned}\right\} \Rightarrow \text{Arg}[I_0] = -\cos^{-1}(0.4) = -1.16 \Longrightarrow I_0 = |I_0| e^{j\text{Arg}[I_0]} = 25e^{-j1.16}$$



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# EG1103 Mid-term Test

- **When?** The time of your tutorial class in the week right after the recess week.
- **Where?** In your tutorial classroom.
- **Why?** To collect some marks for your final grade for EG1103.
- **What?** Two questions cover materials up to DC circuit analysis.



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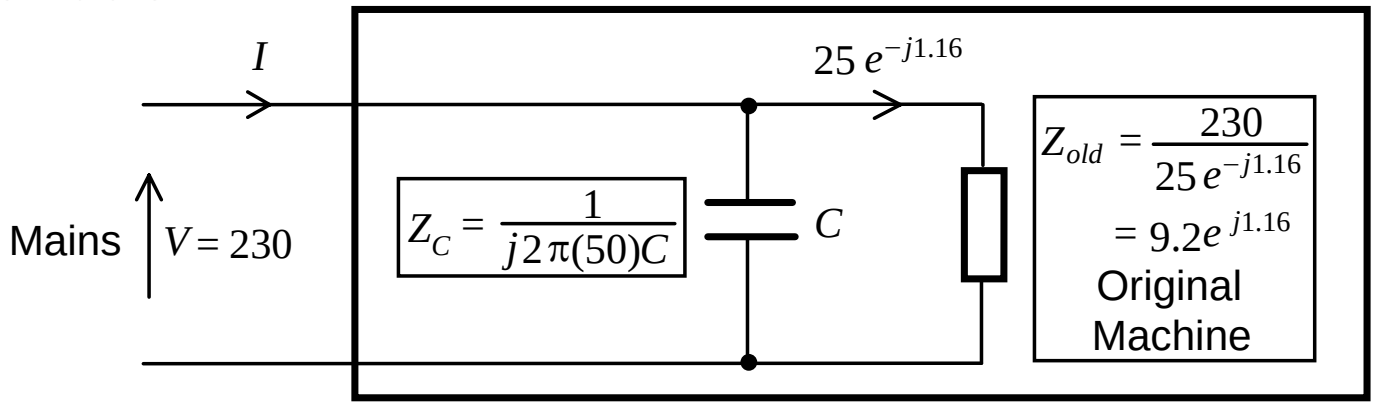
To overcome this problem, a parallel capacitor can be used to improve the power factor:




New Electrical Machine

$$I = \frac{V}{Z_C} + 25e^{-j1.16} = j23000\pi C + 10 - j23 = 10 + j(23000\pi C - 23)$$

Thus, if we choose $23000\pi C = 23 \Rightarrow C = 0.32\text{mF}$ then $I = 10\text{A}$ and

$$\text{Power factor of new machine} = \cos[\text{Arg}(I) - \text{Arg}(V)] = 1$$

By changing the power factor, the improved machine can now be connected to standard 13A outlets. The price to pay is the use of an additional capacitor.



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To reduce cost, we may wish to use a capacitor which is as small as possible.

To find the smallest capacitor that will satisfy the 13A requirement:

$$|I|^2 = 10^2 + (72200C - 23)^2 = 13^2 \quad \Rightarrow \quad 13^2 = 10^2 + (72200C - 23)^2$$

$$\Downarrow$$

$$0 = 10^2 - 13^2 + (72200C - 23)^2 = (72200C - 23)^2 - 8.3^2$$

$$\Downarrow$$

$$0 = (72200C - 23 - 8.3)(72200C - 23 + 8.3)$$

$$\Downarrow$$

$$C = 0.2\,\text{mF} \quad \text{or} \quad 0.44\,\text{mF}$$

There are 2 possible values for $C$, one giving a lagging overall power factor, the other giving a leading overall power factor. To save cost, $C$ should be

$$\boxed{C = 0.2\,\text{mF}}$$



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# Chapter 4. Frequency Response



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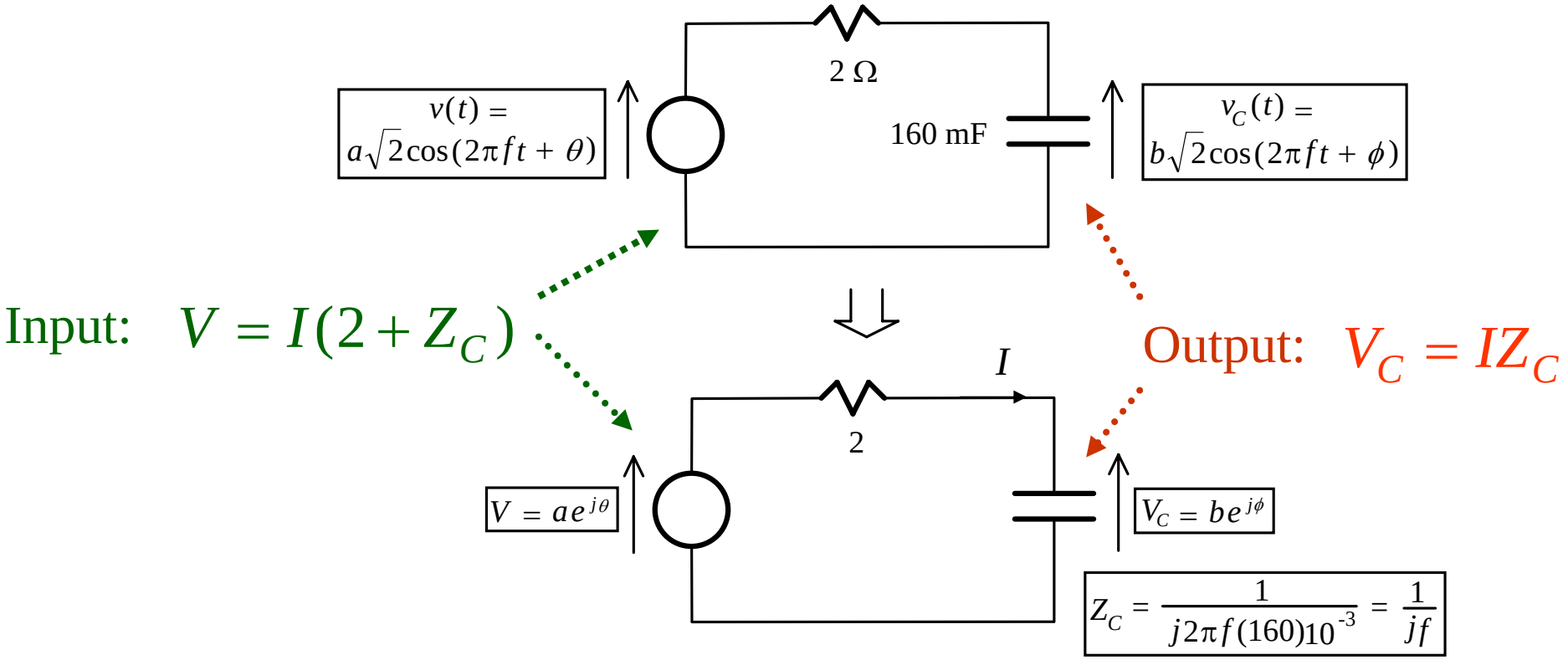
## 4.1 RC Circuit

Consider the series RC circuit:




$$H(f) = \frac{V_C}{V} = \frac{be^{j\phi}}{ae^{j\theta}} = \frac{b}{a}e^{j(\phi-\theta)} = \frac{Z_C}{2+Z_C} = \frac{\frac{1}{jf}}{2+\frac{1}{jf}} = \frac{1}{1+j2f}$$

⇐ Frequency Response



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The magnitude of $H(f)$ is

$$|H(f)| = \left|\frac{V_C}{V}\right| = \frac{|V_C|}{|V|} = \frac{b}{a}$$
$$= \sqrt{\frac{1}{1+(2f)^2}} = \sqrt{\frac{1}{1+4f^2}}$$

and is called the ***magnitude response***.

The phase of $H(f)$ is

$$\text{Arg}[H(f)] = \text{Arg}\left[\frac{V_C}{V}\right] = \text{Arg}[V_C] - \text{Arg}[V]$$
$$= \phi - \theta = \text{Arg}\left[\frac{1}{1+j2f}\right]$$
$$= -\text{Arg}[1+j2f] = -\tan^{-1}(2f)$$

and is called the ***phase response***.

The physical significance of these responses is that $H(f)$ gives the ratio of output to input phasors, $|H(f)|$ gives the ratio of output to input magnitudes, and $\text{Arg}[H(f)]$ gives the output to input phase difference at a frequency $f$.



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| | | |
|---|---|---|
| **Input** | $v(t) = 3\sqrt{2}\cos[2\pi(5)t+7]$<br>$V = 3e^{j7}$ | $v(t) = r\sin[2\pi(4)t] = r\cos\left[2\pi(4)t-\frac{\pi}{2}\right]$<br>$V = \frac{r}{\sqrt{2}}e^{-j\pi/2}$ |
| **Frequency** | $f = 5$ | $f = 4$ |
| **Frequency response** | $H(5) = \frac{1}{1+j10}$ | $H(4) = \frac{1}{1+j8}$ |
| **Magnitude response** | $\lvert H(5)\rvert = \frac{1}{\sqrt{101}}$ | $\lvert H(4)\rvert = \frac{1}{\sqrt{65}}$ |
| **Phase response** | $\text{Arg}[H(5)] = -\tan^{-1}(10)$ | $\text{Arg}[H(4)] = -\tan^{-1}(8)$ |
| **Output** | $v_C(t) = \frac{3\sqrt{2}}{\sqrt{101}}\cos[2\pi(5)t+7-\tan^{-1}(10)]$<br>$V_C = \frac{3}{\sqrt{101}}e^{j[7-\tan^{-1}(10)]}$ | $v_C(t) = \frac{r}{\sqrt{65}}\sin[2\pi(4)t-\tan^{-1}(8)]$<br>$= \frac{r}{\sqrt{65}}\cos\left[2\pi(4)t-\tan^{-1}(8)-\frac{\pi}{2}\right]$<br>$V_C = \frac{r}{\sqrt{130}}e^{j[-\tan^{-1}(8)-\pi/2]}$ |



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Due to the presence of components such as capacitors and inductors with frequency-dependent impedances, $H(f)$ is usually frequency-dependent and the characteristics of the circuit is often studied by finding how $H(f)$ changes as $f$ is varied. Numerically, for the series RC circuit:

| $f$ | $\lvert H(f)\rvert = \frac{1}{\sqrt{(1+4f^2)}}$ | $\text{Arg}[H(f)] = -\tan^{-1}(2f)$ |
|---|---|---|
| 0 | $1 = 20\log(1) = 0\ \text{dB}$ | $0\ \text{rad} = 0^0$ |
| 0.5 | $\frac{1}{\sqrt{1+4(0.5)^2}} = \frac{1}{\sqrt{2}} = 20\log\left(\frac{1}{\sqrt{2}}\right) = -3\text{dB}$ | $-\tan^{-1}(2\times 0.5) = -\frac{\pi}{4}\text{rad} = -45^0$ |
| $\to\infty$ | $\to 0 = -\infty\text{dB}$ | $-\tan^{-1}(\infty) = -\frac{\pi}{2}\text{rad} = -90^0$ |



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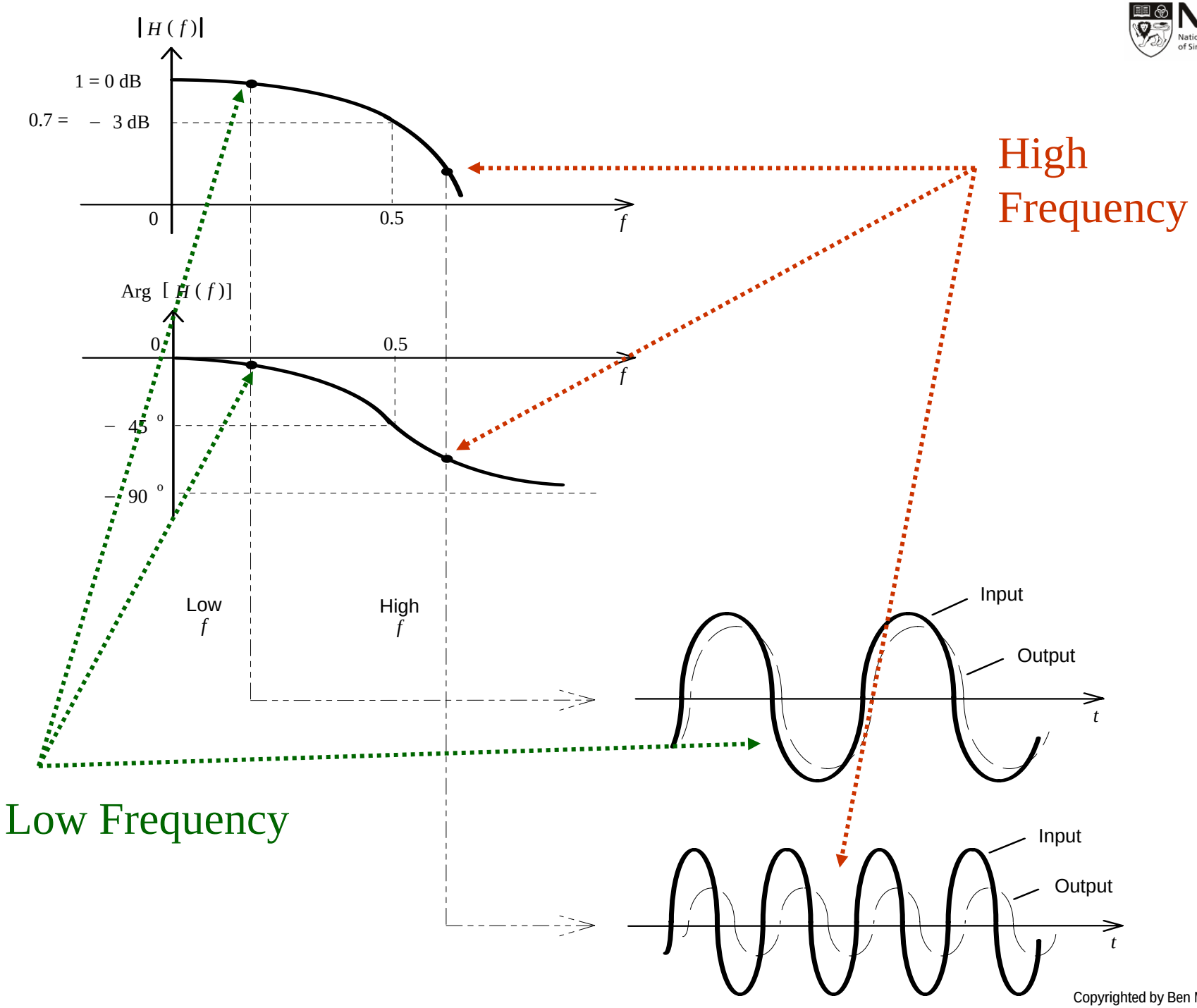




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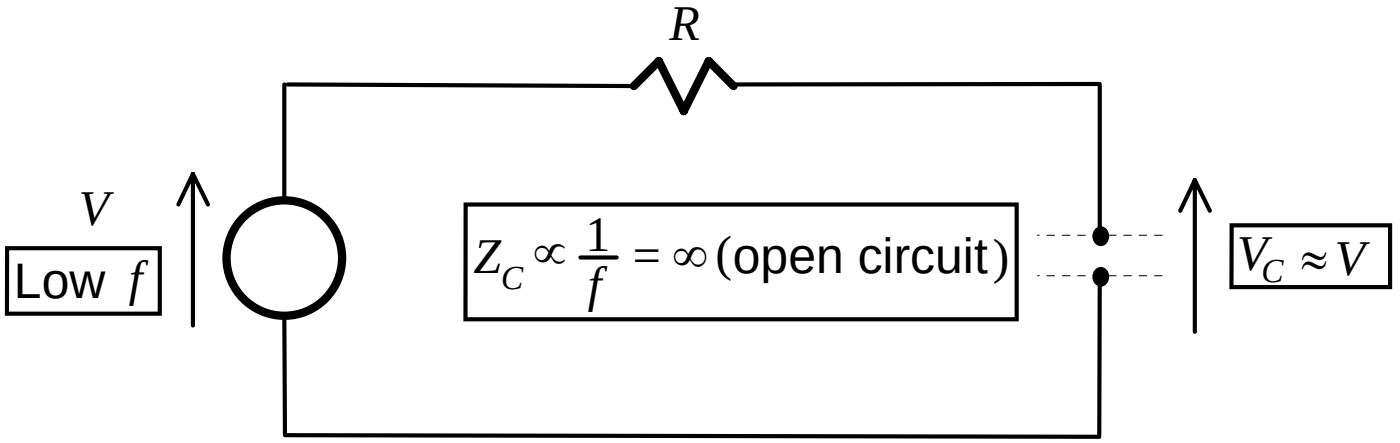
At small $f$, the output approximates the input. However, at high $f$, the output will become attenuated. Thus, the circuit has a ***low pass*** characteristic (low frequency input will be passed, high frequency input will be rejected).

The frequency at which $|H(f)|$ falls to –3 dB of its maximum value is called the ***cutoff*** frequency. For the above example, the cutoff frequency is 0.5 Hz.

To see why the circuit has a low pass characteristic, note that at low $f$, $C$ has large impedance (approximates an open circuit) when compare with $R$ (2 in the above example). Thus, $V_C$ will be approximately equal to $V$:



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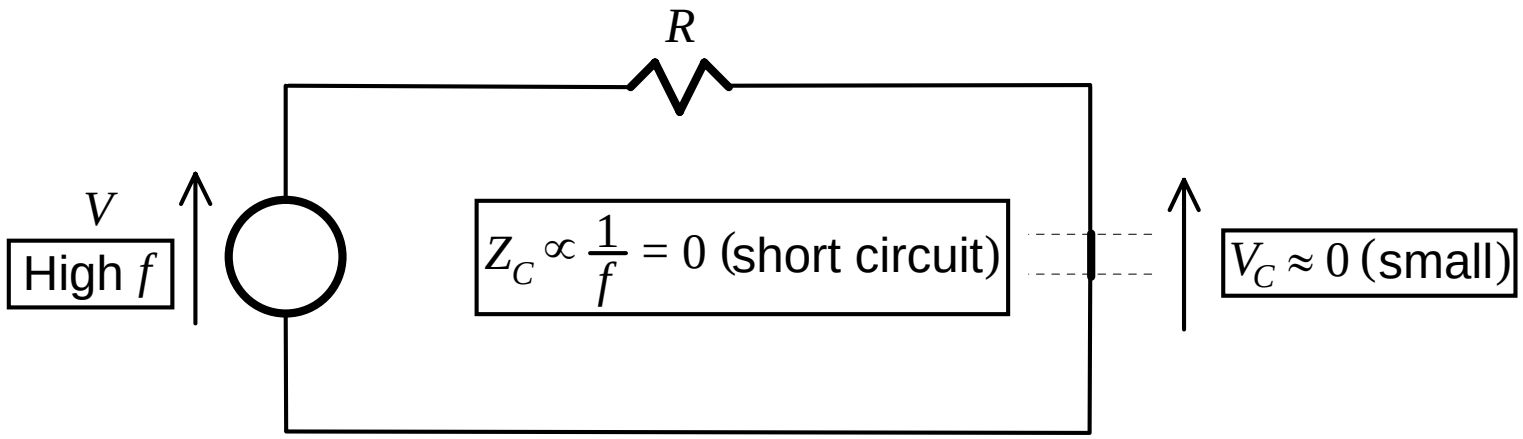
However, at high $f$, $C$ has small impedance (approximates a short circuit) when compare with $R$. Thus, $V_C$ will be small:




**Key Notes: The capacitor is acting like a short circuit at high frequencies and an open circuit at low frequencies. It is totally open for a dc circuit.**



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# An Electric Joke

**Q: Why does a capacitor block DC but allow AC to pass through?**

**A: You see, a capacitor is like this ——–| |——– , OK. DC Comes straight, like this ————, and the capacitor stops it. But AC, goes up, down, up and down and jumps right over the capacitor!**



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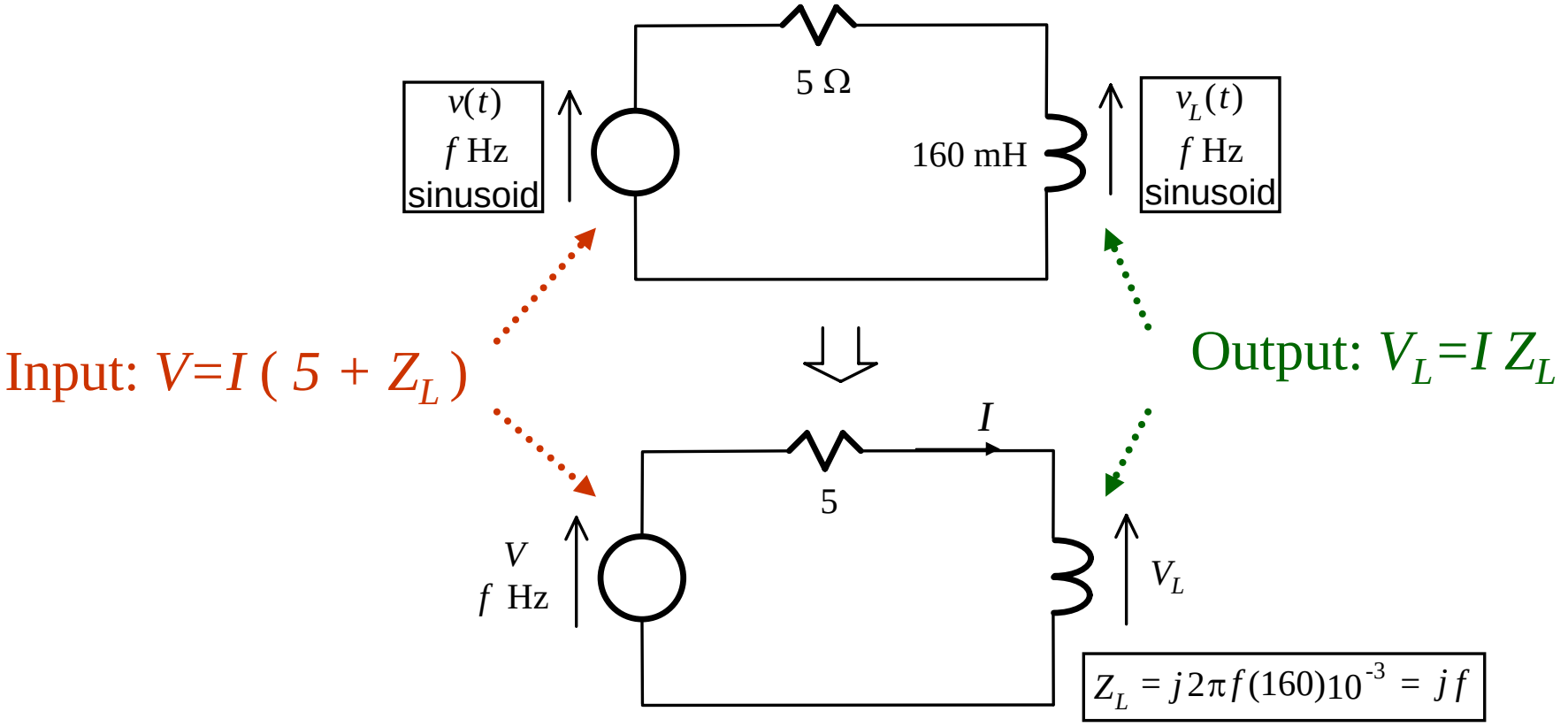
## 4.2 RL Circuit

Consider the ***series RL circuit***:




$$\frac{V_L}{V} = H(f) = \frac{Z_L}{5 + Z_L} = \frac{jf}{5 + jf}$$

**Frequency Response**



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The magnitude response is

$$|H(f)| = \sqrt{\frac{f^2}{5^2+f^2}} = \sqrt{\frac{f^2}{25+f^2}}$$

The phase response is

$$\begin{aligned}\text{Arg}[H(f)] &= \text{Arg}\left[\frac{jf}{5+jf}\right] \\ &= \text{Arg}[jf] - \text{Arg}[5+jf] \\ &= \frac{\pi}{2} - \tan^{-1}\left(\frac{f}{5}\right)\end{aligned}$$

Numerically:

| $f$ | $\lvert H(f)\rvert = \sqrt{\frac{f^2}{25+f^2}}$ | $\text{Arg}[H(f)] = \frac{\pi}{2} - \tan^{-1}\left(\frac{f}{5}\right)$ |
|---|---|---|
| 0 | $0 = 20\log(0) = -\infty$ dB | $\frac{\pi}{2}$ rad $= 90^0$ |
| 5 | $\sqrt{\frac{5^2}{25+5^2}} = \frac{1}{\sqrt{2}} = 20\log\left(\frac{1}{\sqrt{2}}\right) = -3$dB | $\frac{\pi}{2} - \tan^{-1}\left(\frac{5}{5}\right) = \frac{\pi}{4}$ rad $= 45^0$ |
| $\to\infty$ | $\to 1 = 0$ dB | $\frac{\pi}{2} - \tan^{-1}(\infty) = 0$ rad $= 0^0$ |



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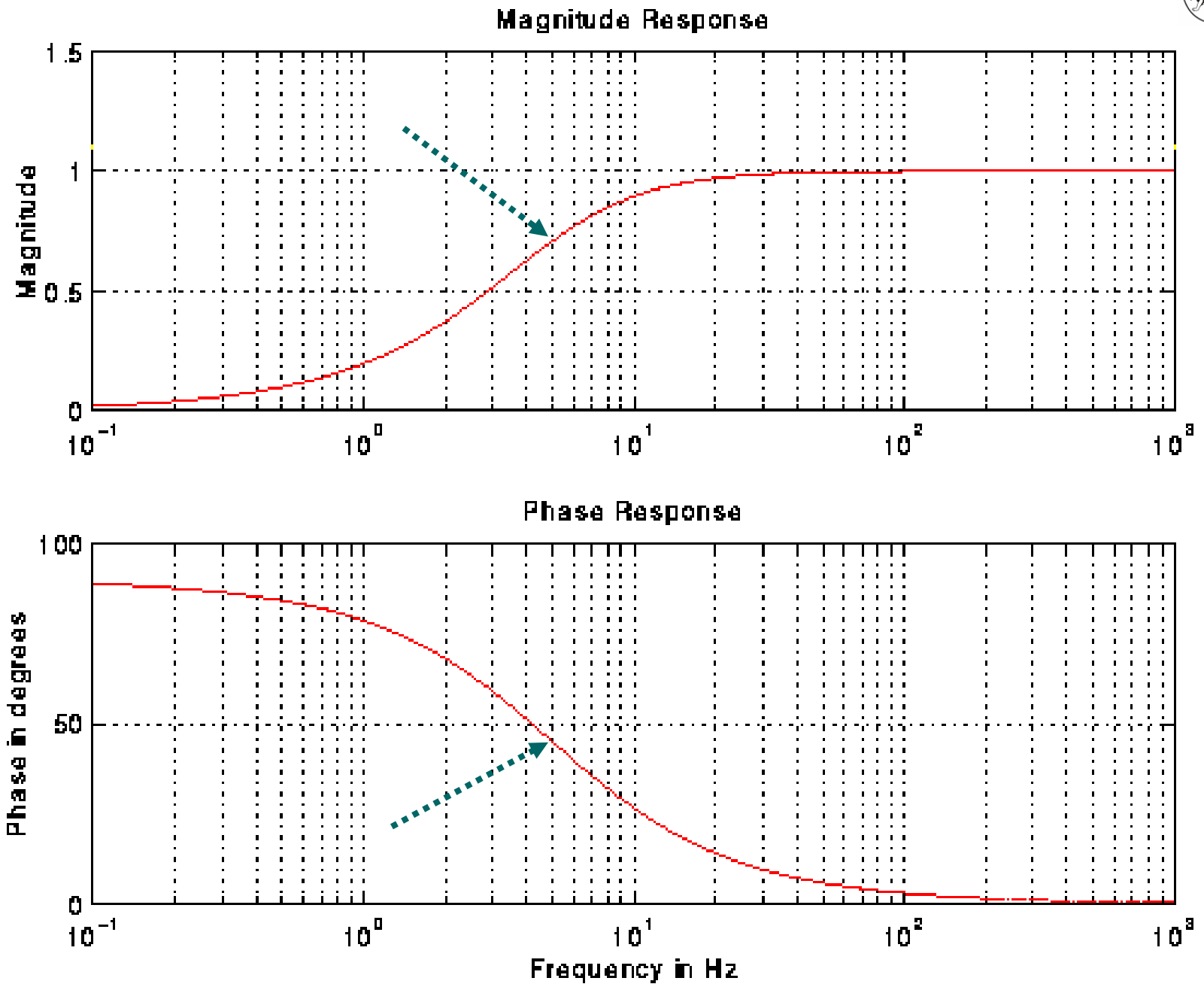




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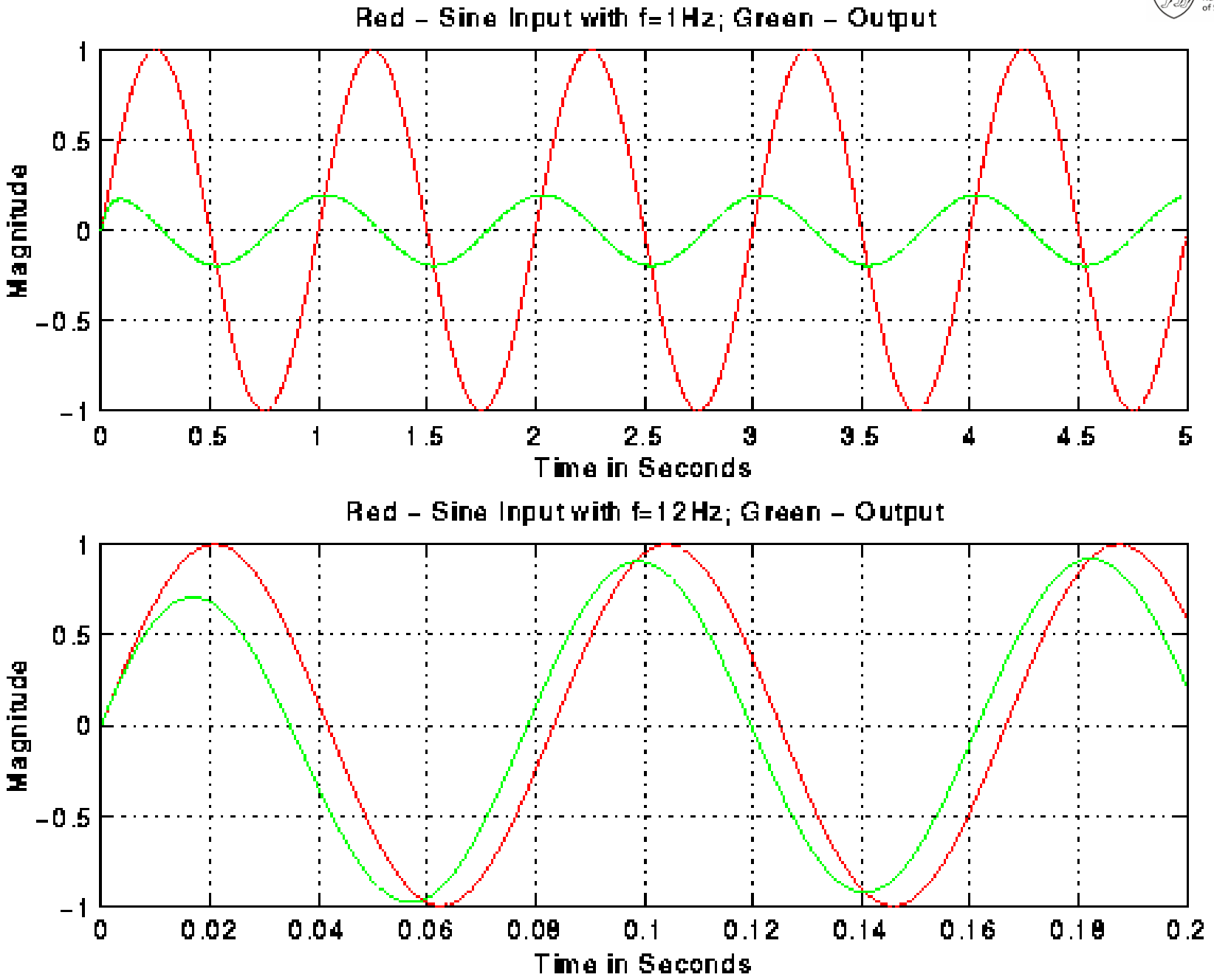




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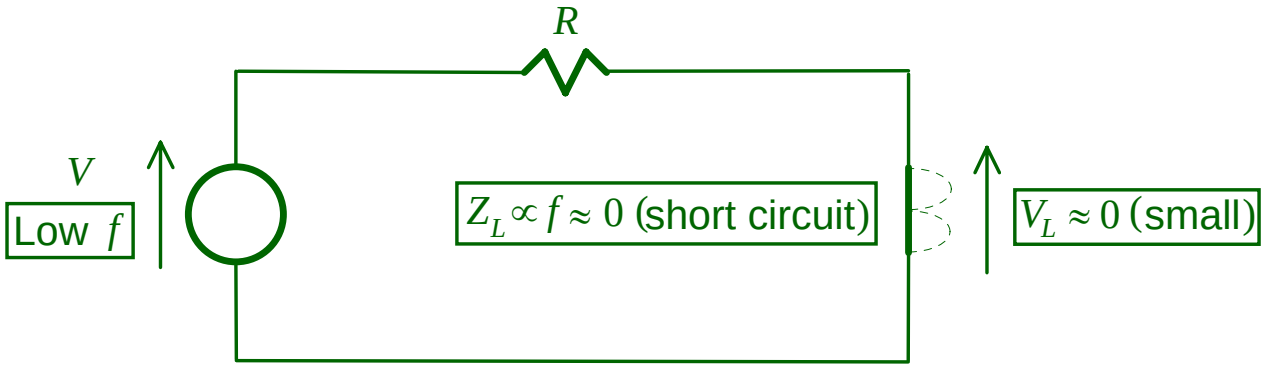
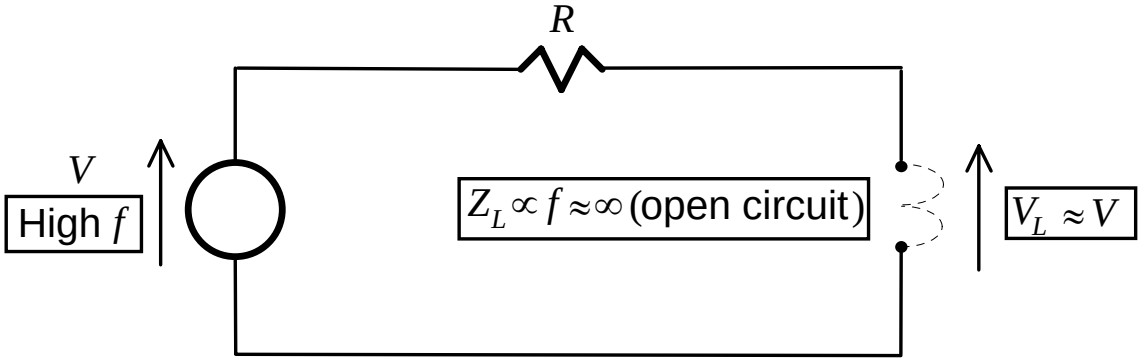
Physically, at small $f$, $L$ has small impedance (approximates a short circuit) when compare with $R$ (5 in the above example). Thus, $V_L$ will be small:




However, at high $f$, $L$ has large impedance (approximates an open circuit) when compare with $R$. Thus, $V_L$ will approximates $V$:




**Due to these characteristics, the circuit is *highpass* in nature.**



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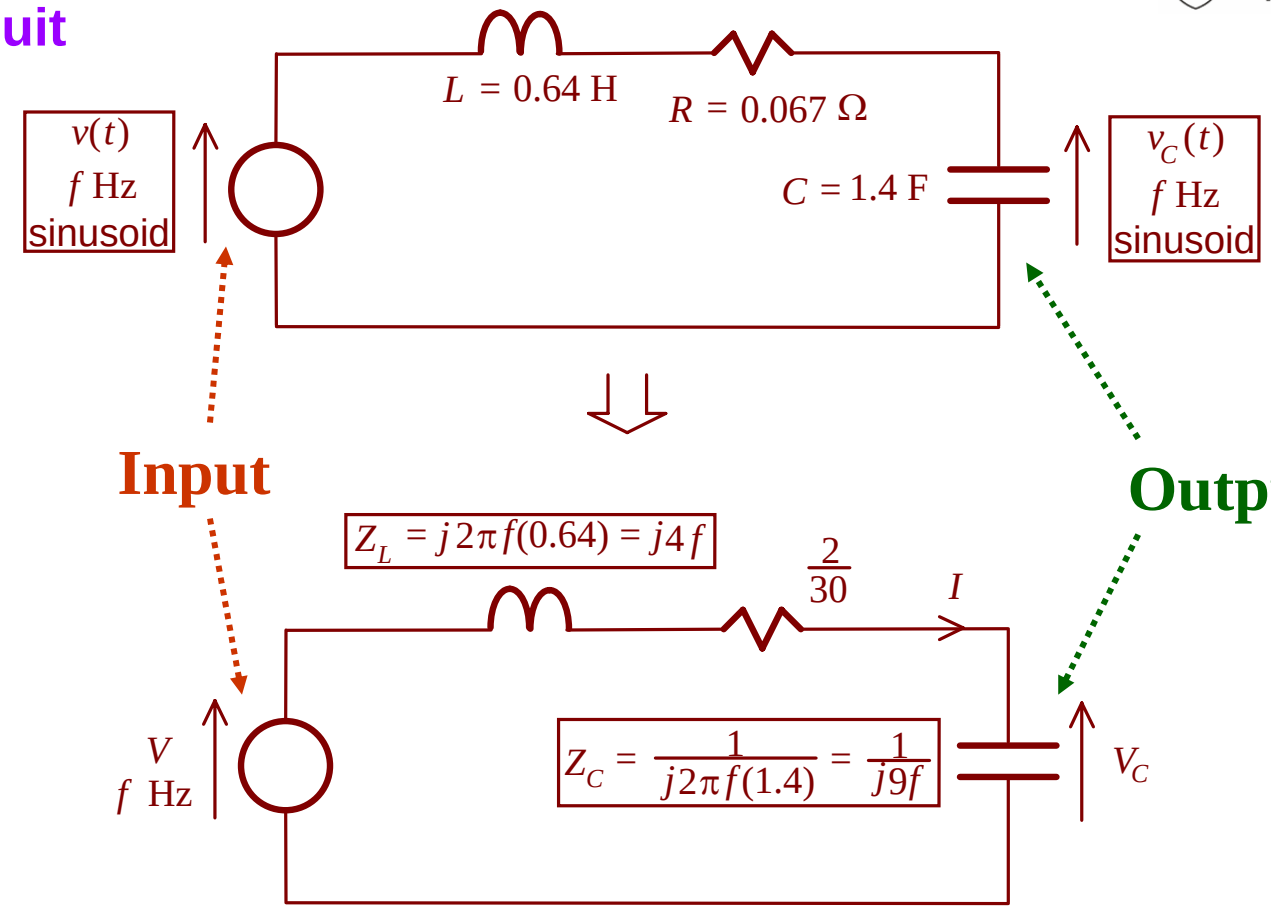
## 4.3 Series Tune Circuit




**The total impedance is**

$$Z = R + Z_L + Z_C$$

$$= \frac{2}{30} + j4f + \frac{1}{j9f}$$

$$= \frac{2}{30} + j\left(4f - \frac{1}{9f}\right)$$

$$= \frac{2}{30}$$

**Resonance Frequency** ⟹

$$f_0 = \frac{1}{\sqrt{(4)(9)}} = \frac{1}{6} \Leftrightarrow f_0 = \frac{1}{\sqrt{(2\pi L)(2\pi C)}} = \frac{1}{2\pi\sqrt{LC}}$$

***Q* factor** ⟹

$$Q = \frac{\text{Reactance of inductor at } f_0}{\text{Resistance}} = \frac{2/3}{2/30} = 10 \Leftrightarrow Q = \frac{2\pi f_0 L}{R}$$



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The frequency response is

$$H(f)=\frac{V_C}{V}=\frac{Z_C}{Z}=\frac{\frac{1}{j9f}}{\frac{2}{30}+j4f+\frac{1}{j9f}}=\frac{1}{1-36f^2+j0.6f}$$

The magnitude response is

$$|H(f)|=\frac{1}{\sqrt{(1-36f^2)^2+(0.6f)^2}}=\frac{1}{\sqrt{(36f^2)^2-(72-0.6^2)f^2+1}}$$

$$=\frac{1}{\sqrt{(36f^2)^2-2(36f^2)\left(1-\frac{0.6^2}{72}\right)+\left(1-\frac{0.6^2}{72}\right)^2+1-\left(1-\frac{0.6^2}{72}\right)^2}}$$

$$=\frac{1}{\sqrt{\left[36f^2-\left(1-\frac{0.6^2}{72}\right)\right]^2+\left(\frac{0.6^2}{72}\right)\left(2-\frac{0.6^2}{72}\right)}}$$

$$x^2-bx+c$$

$$=x^2-2\left(\frac{b}{2}\right)x+\left(\frac{b}{2}\right)^2-\left(\frac{b}{2}\right)^2+c$$

$$=\left(x-\frac{b}{2}\right)^2+c-\left(\frac{b}{2}\right)^2$$

Since $f$ only appears in the $[\bullet]^2$ term in the denominator and $[\bullet]^2 >= 0$, $|H(f)|$ will increase if $[\bullet]^2$ becomes smaller, and vice versa.



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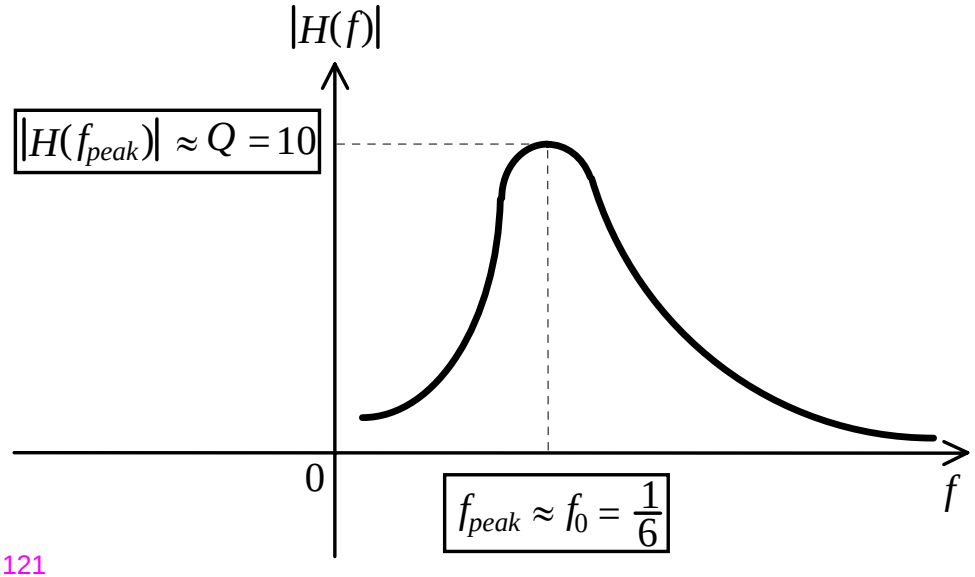
The maximum value for $|H(f)|$ corresponds to the situation of $[\bullet]^2$ or at a frequency $f = f_{peak}$ given by:

$$36 f_{peak}^2 = 1 - \frac{0.6^2}{72} \approx 1 \quad \Leftrightarrow \quad f_{peak} \approx \frac{1}{6} \quad \Leftrightarrow \quad f_{peak} \approx f_0$$

At $f = f_{peak}$ , $[\bullet]^2$ and the maximum value for $|H(f)|$ is

$$\left|H\left(f_{peak}\right)\right| = \frac{1}{\sqrt{\left(\frac{0.6^2}{72}\right)\left(2 - \frac{0.6^2}{72}\right)}} \approx \frac{1}{\sqrt{\left(\frac{0.6^2}{72}\right)(2)}} = 10 \quad \Leftrightarrow \quad \left|H\left(f_{peak}\right)\right| \approx Q$$




The series tuned circuit has a ***bandpass*** characteristic. Low- and high-frequency inputs will get attenuated, while inputs close to the resonant frequency will get amplified by a factor of approximately $Q$.



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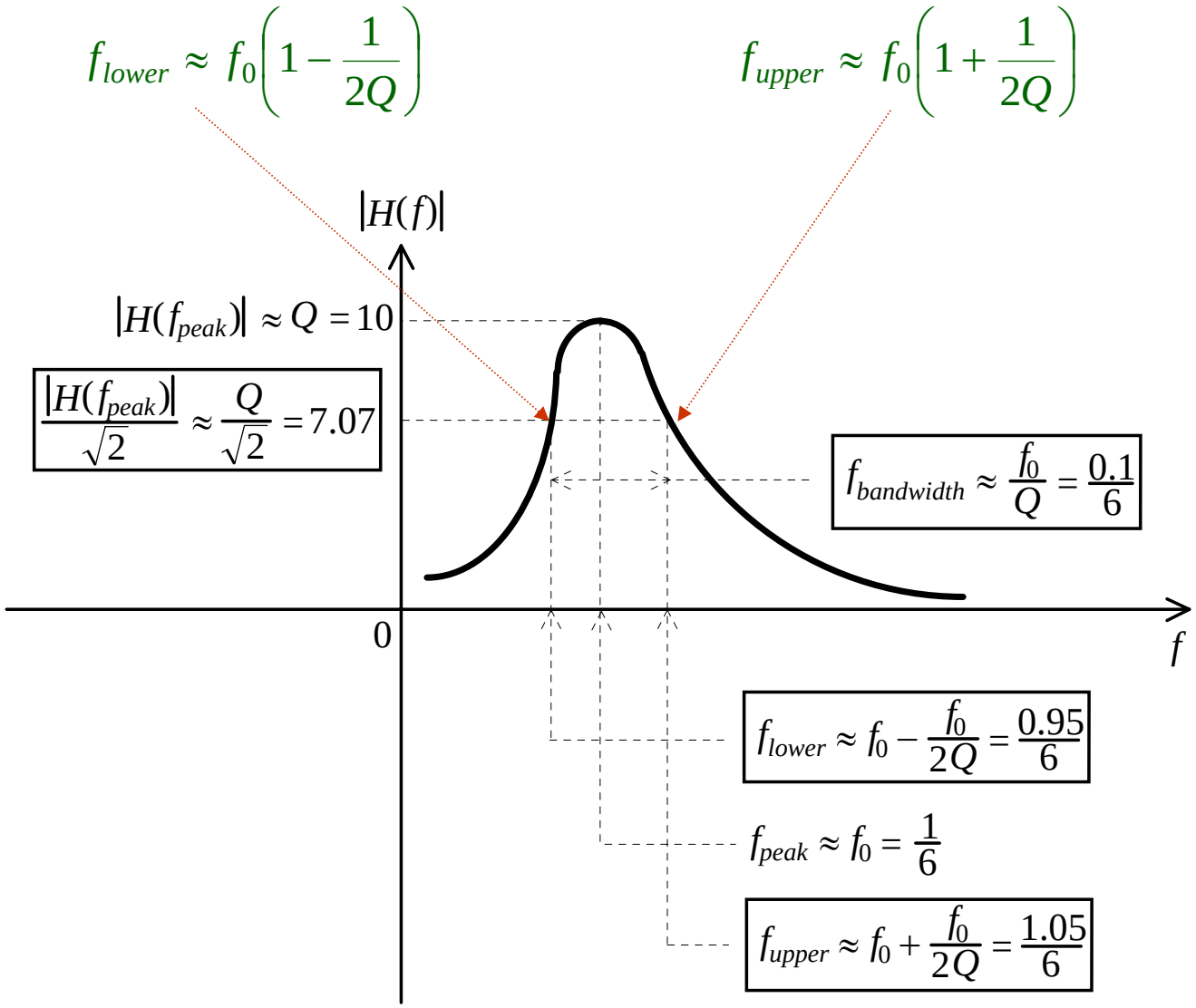
The cutoff frequencies, at which $|H(f)|$ decrease by a factor of 0.7071 or by 3 dB from its peak value $|H(f_{peak})|$, can be shown to be given by




Very roughly, the circuit will pass inputs with frequency between $f_{lower}$ and $f_{upper}$. The ***bandwidth*** of the circuit is

$$f_{bandwidth} = f_{upper} - f_{lower} \approx \frac{f_0}{Q}$$

and the ***fractional bandwidth*** is

$$\frac{f_{bandwidth}}{f_0} \approx \frac{1}{Q}$$



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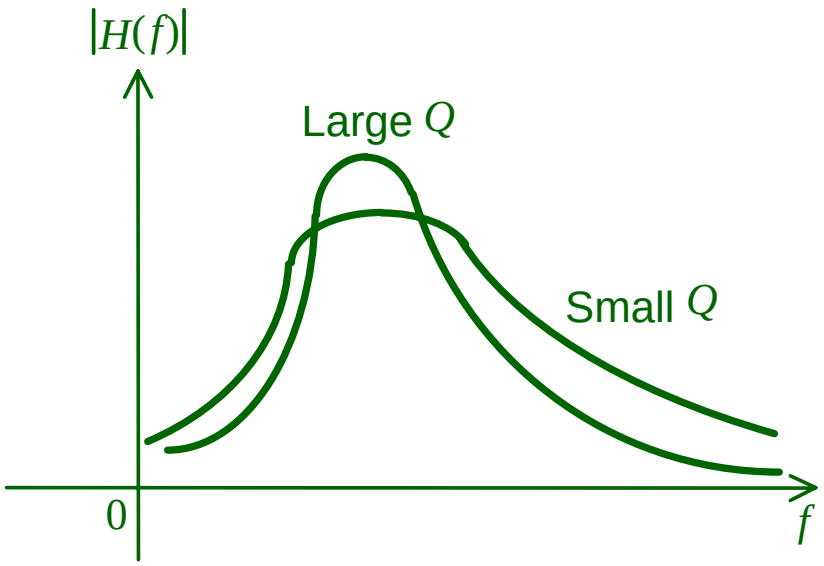
The larger the $Q$ factor, the sharper the magnitude response, the bigger the amplification, and the narrower the fractional bandwidth:




In practice, a series tune circuit usually consists of a practical inductor or coil connected in series with a practical capacitor. Since a practical capacitor usually behaves quite closely to an ideal one but a coil will have winding resistance, such a circuit can be represented by:



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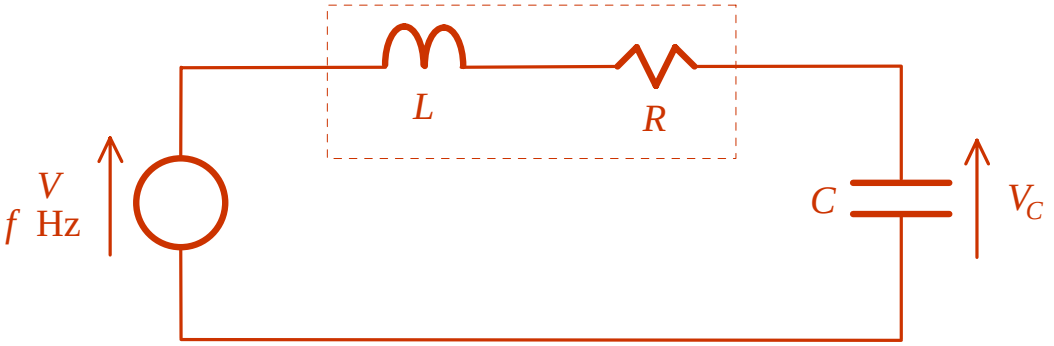
Equivalent circuit for coil or practical inductor




## The main features are:

| | |
|---|---|
| **Circuit impedance** | $Z = R + j2\pi fL + \dfrac{1}{j2\pi fC}$ |
| **Resonance frequency** | $f_0 = \dfrac{1}{2\pi\sqrt{LC}}$ |
| $Q$ **factor** | $Q = \dfrac{2\pi f_0 L}{R}$ |
| **Frequency response** | $H(f) = \dfrac{1}{1 - 4\pi^2 f^2 LC + j2\pi fCR} = \dfrac{1}{1 - \left(\dfrac{f}{f_0}\right)^2 + \dfrac{j}{Q}\left(\dfrac{f}{f_0}\right)}$ |



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For the usual situation when $Q$ is large:

| | |
|---|---|
| **Magnitude response** | Bandpass with $\lvert H(f)\rvert$ decreasing as $f \to 0$ and $f \to \infty$ |
| **Response peak** | $\lvert H(f)\rvert$ peaks at $f = f_{peak} \approx f_0$ with $\lvert H(f_{peak})\rvert \approx Q$ |
| **Cutoff frequencies** | $\lvert H(f)\rvert = \frac{\lvert H(f_{peak})\rvert}{\sqrt{2}} \approx \frac{Q}{\sqrt{2}}$ at<br>$f = f_{lower},\ f_{upper} \approx f_0\left(1 - \frac{1}{2Q}\right),\ f_0\left(1 + \frac{1}{2Q}\right)$ |
| **Bandwidth** | $f_{bandwidth} = f_{upper} - f_{lower} \approx \frac{f_0}{Q}$ |
| **Fractional bandwidth** | $\frac{f_{bandwidth}}{f_0} \approx \frac{1}{Q}$ |



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The $Q$ factor is an important parameter of the circuit.

$$Q = \frac{2\pi f_0 L}{R} = \frac{\text{Inductor reactance at } f_0}{\text{Circuit resistance}}$$

However, since $R$ is usually the winding resistance of the practical coil making up the tune circuit:

$$Q = \frac{\text{Reactance of practical coil at } f_0}{\text{Resistance of practical coil}}$$

As a good practical coil should have low winding resistance and high inductance, the $Q$ factor is often taken to be a characteristic of the practical inductor or coil. The higher the $Q$ factor, the higher the ***quality*** of the coil.



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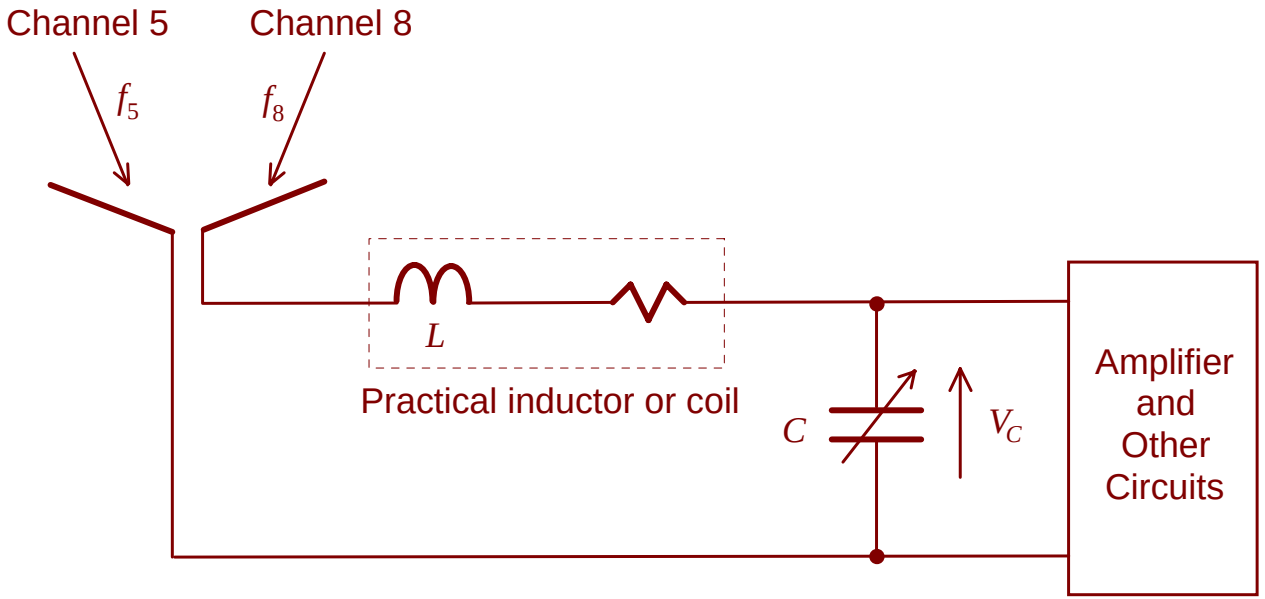
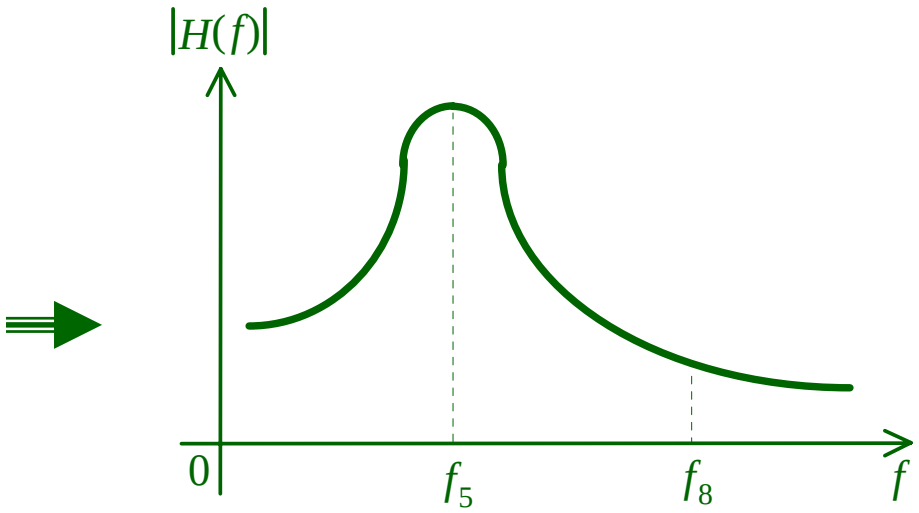
Due to its bandpass characteristic, tune circuits are used in radio and TV tuners for selecting the frequency channel of interest:




To tune in to channel 5, $C$ has to be adjusted to a value of $C_5$ so that the circuit resonates at a frequency given by

$$f_5 = \frac{1}{2\pi\sqrt{LC_5}}$$



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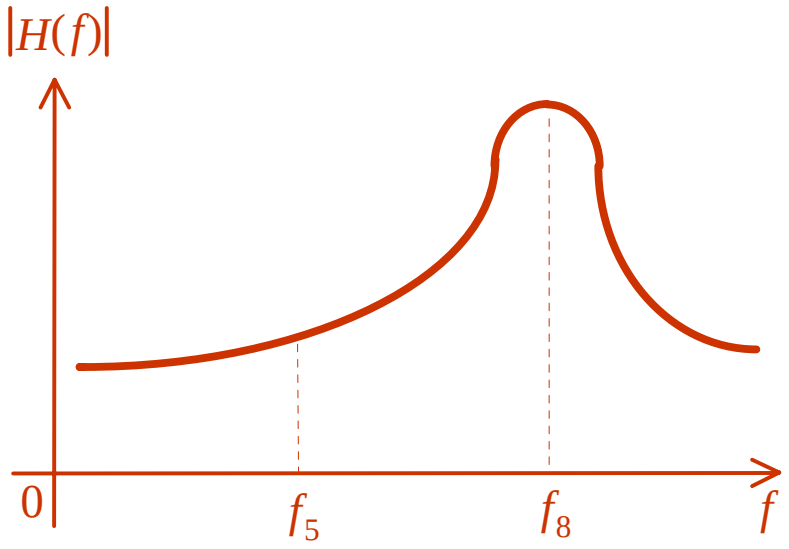
To tune in to channel 8, $C$ has to be adjusted to a value of $C_8$ so that the circuit resonates at a frequency given by

$$f_8 = \frac{1}{2\pi\sqrt{LC_8}}$$

and has a magnitude response of:



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## Additional Notes on Frequency Response

Frequency response is defined as the ratio of the phasor of the output to the phasor of the input. Note that both the input and output could be voltage and/or current. Thus, frequency response could have

$$\frac{V(output)}{I(input)}, \quad \frac{V(output)}{V(input)}, \quad \frac{I(output)}{I(input)}, \quad \frac{I(output)}{V(input)}.$$



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# Chapter 5. Periodic Signals



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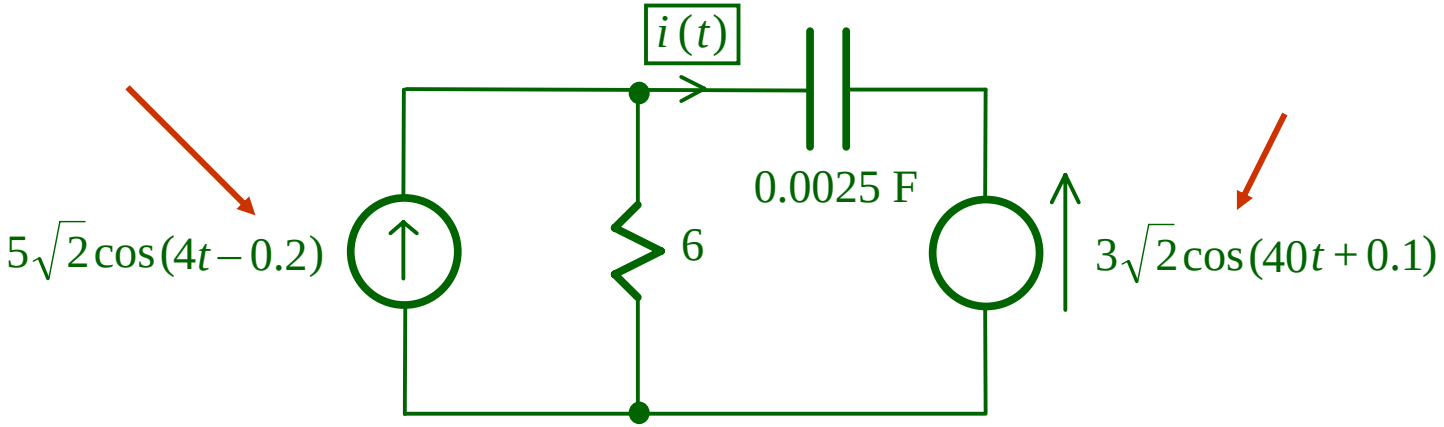
## 5.1 Superposition

In analyzing ac circuits, we have assumed that the voltages and currents are sinusoids and have the same frequency $f$. When this is not the case but the circuit is linear (consisting of resistors, inductors and capacitors), the principle of superposition may be used. Consider the following system:




The current $i(t)$ can be found by summing the contributions due to the two sources on their own (with the other sources replaced by their internal resistances).



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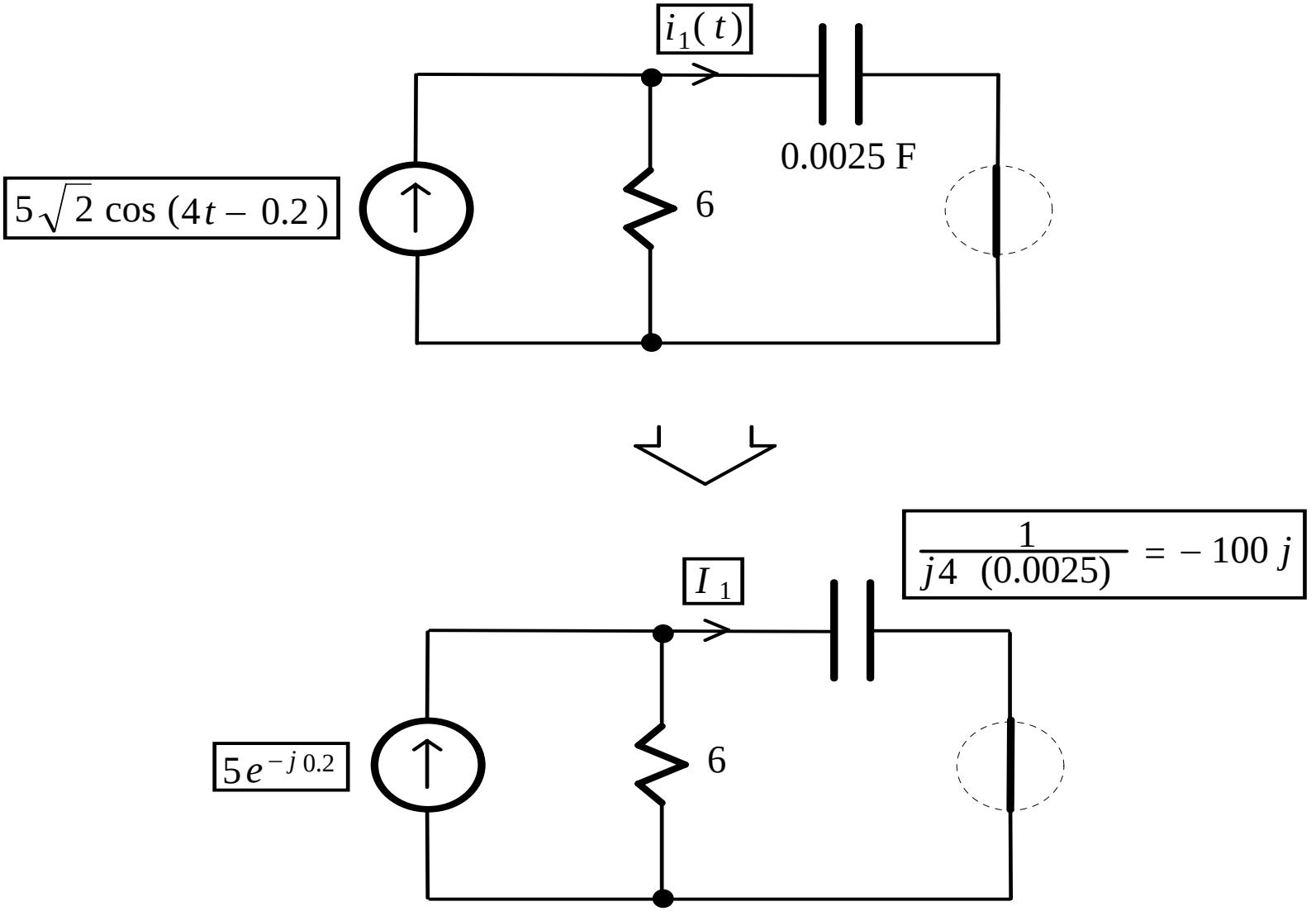




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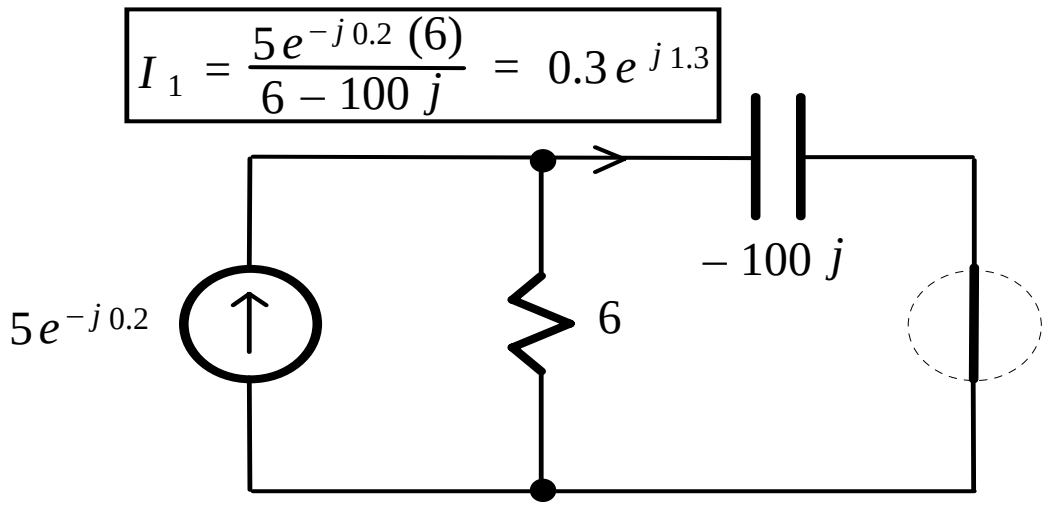
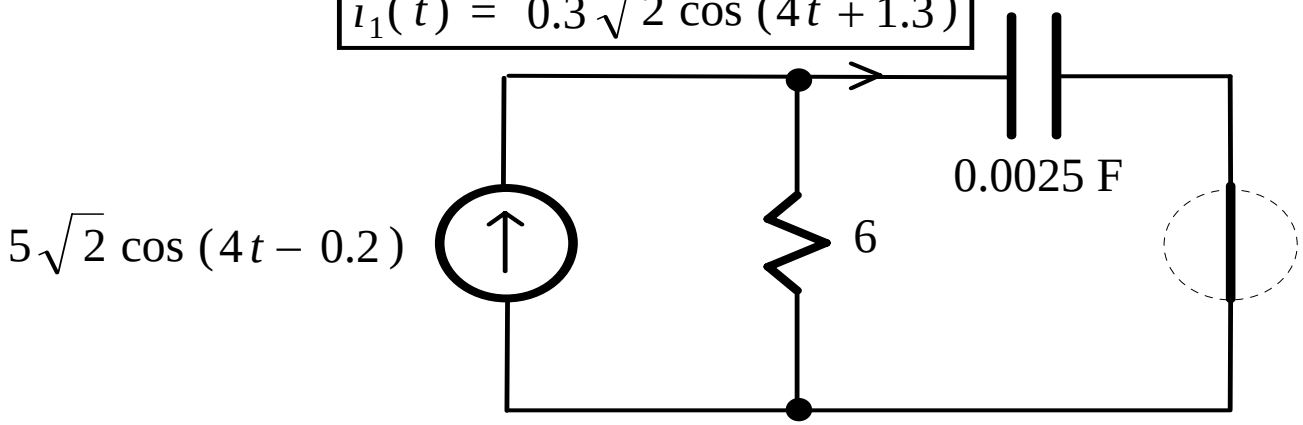
$$I_1 = \frac{5e^{-j\,0.2}\,(6)}{6 - 100\,j} = 0.3e^{j\,1.3}$$




$$i_1(t) = 0.3\sqrt{2}\cos(4t + 1.3)$$



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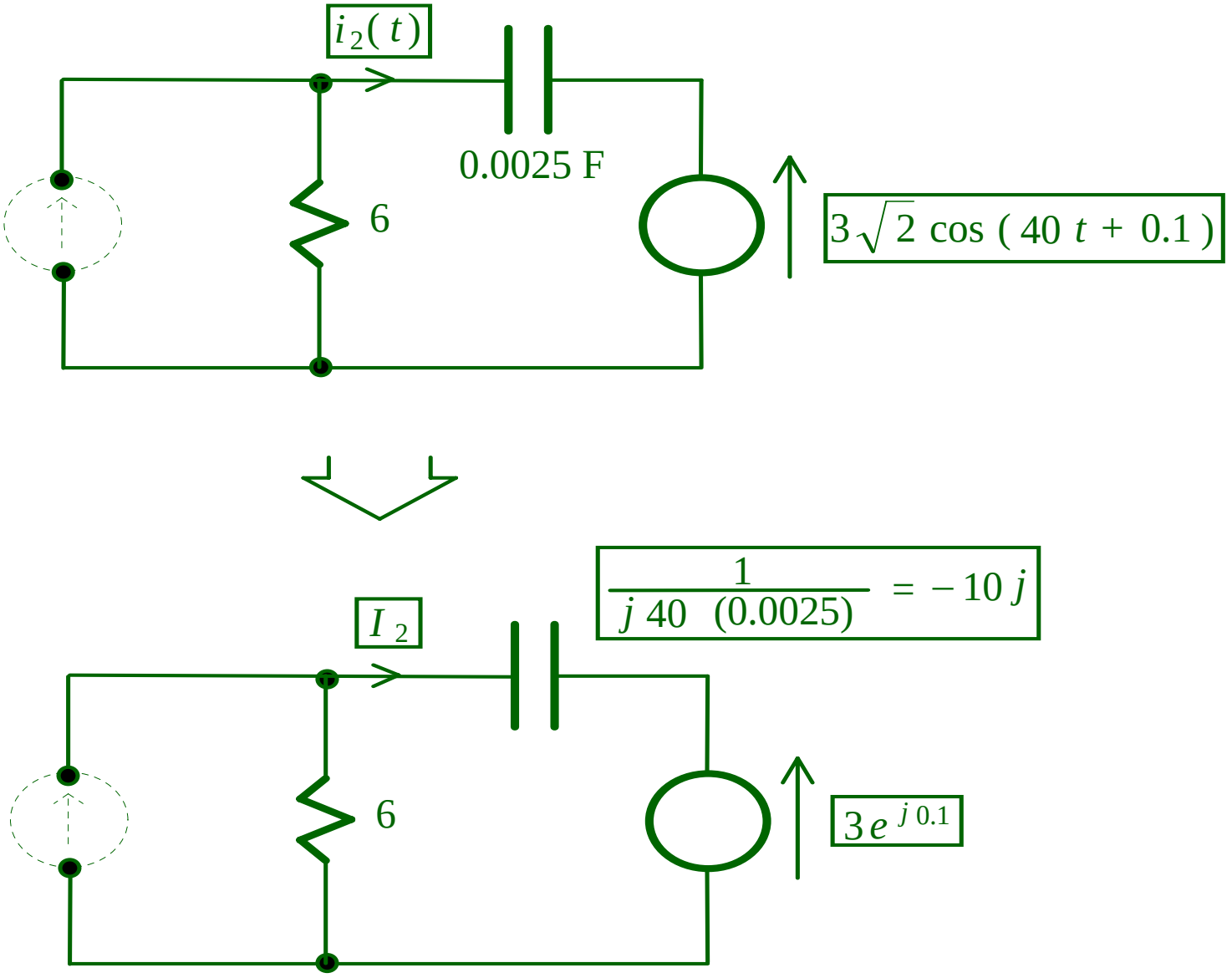




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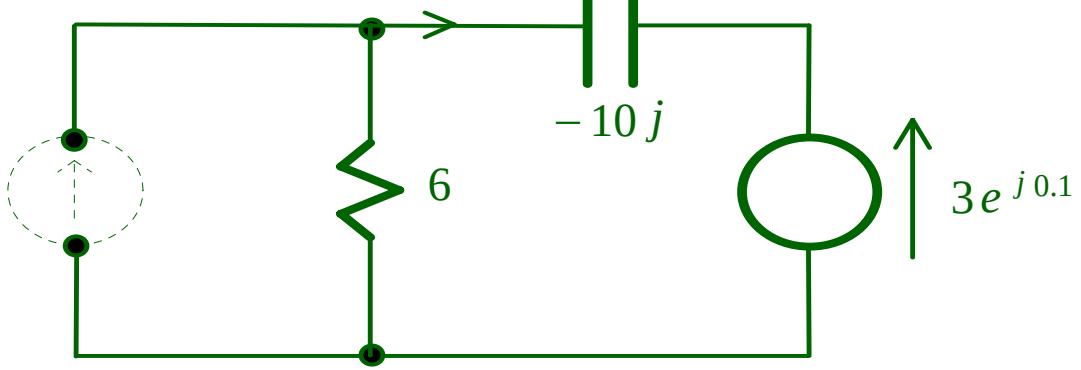
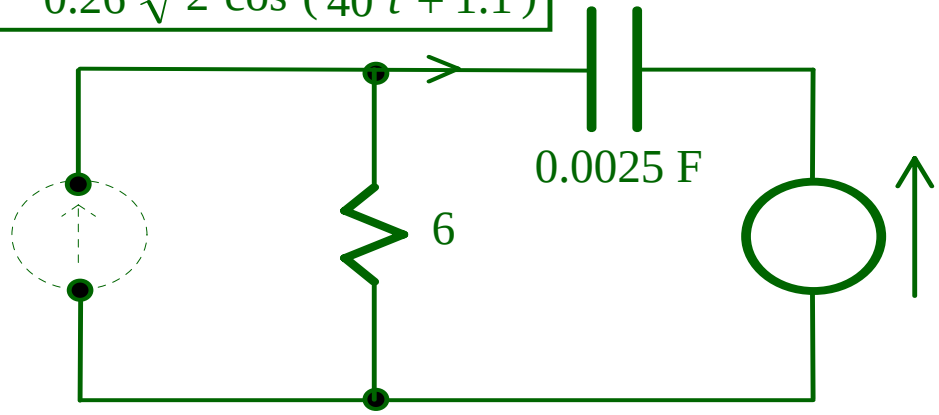
$$I_2 = \frac{-3e^{j\,0.1}}{6 - 10\,j} = -\,0.26\;e^{j\,1.1}$$




$$i_2(t) = -\,0.26\sqrt{2}\cos(40\,t + 1.1)$$



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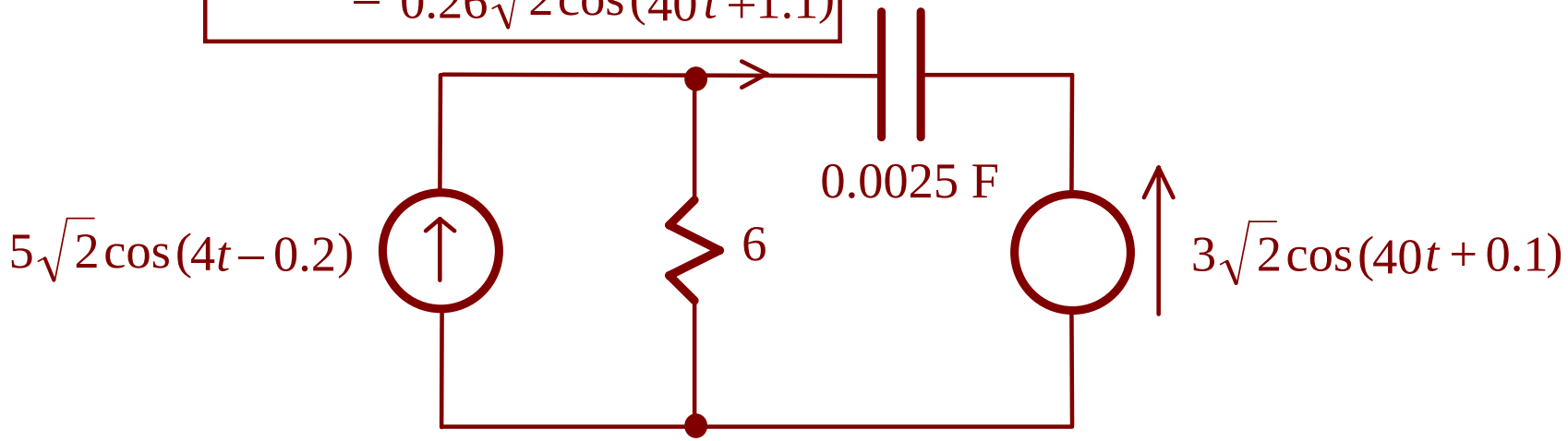
Lastly, the actual current when both sources are present:

$$i(t) = i_1(t) + i_2(t)$$
$$= 0.3\sqrt{2}\cos(4t + 1.3) - 0.26\sqrt{2}\cos(40t + 1.1)$$



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## 5.2 Circuit Analysis using Fourier Series

Using superposition, the voltages and currents in circuits with sinusoidal signals at different frequencies can be found.

Circuits with non-sinusoidal but periodic signals can also be analyzed by first representing these signals as sums of sinusoids or ***Fourier series***.

The following example shows how a periodic square signal can be represented as a sum of sinusoidal components of different frequencies:



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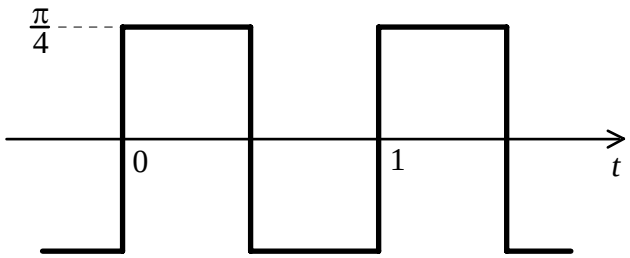
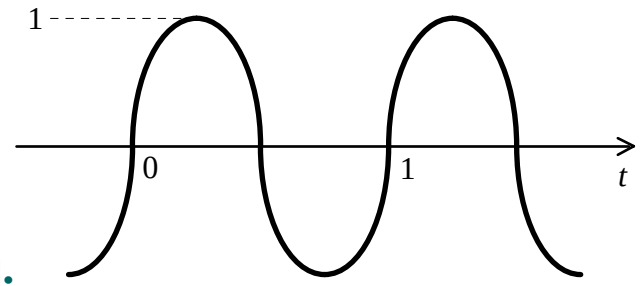
Original periodic square waveform: $v(t)$




||

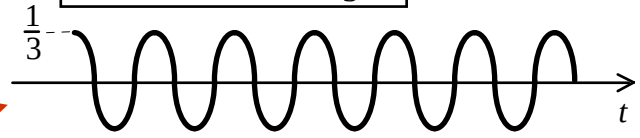
Fundamental component: $\sin(2\pi t)$




**Same freq. as the orginal.**

+

Third harmonic: $\frac{\sin(6\pi t)}{3}$




**Freq. is triple.**

+

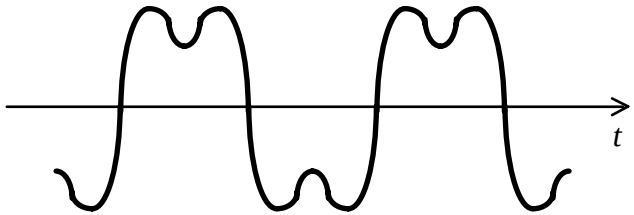
⋮

Any periodic signal can be represented as an infinite sum of sine signals.

$$v(t) = \sin(2\pi t) + \frac{\sin(6\pi t)}{3} + \frac{\sin(10\pi t)}{5} + \cdots$$

Fundamental + third harmonic:

$\sin(2\pi t) + \frac{\sin(6\pi t)}{3}$



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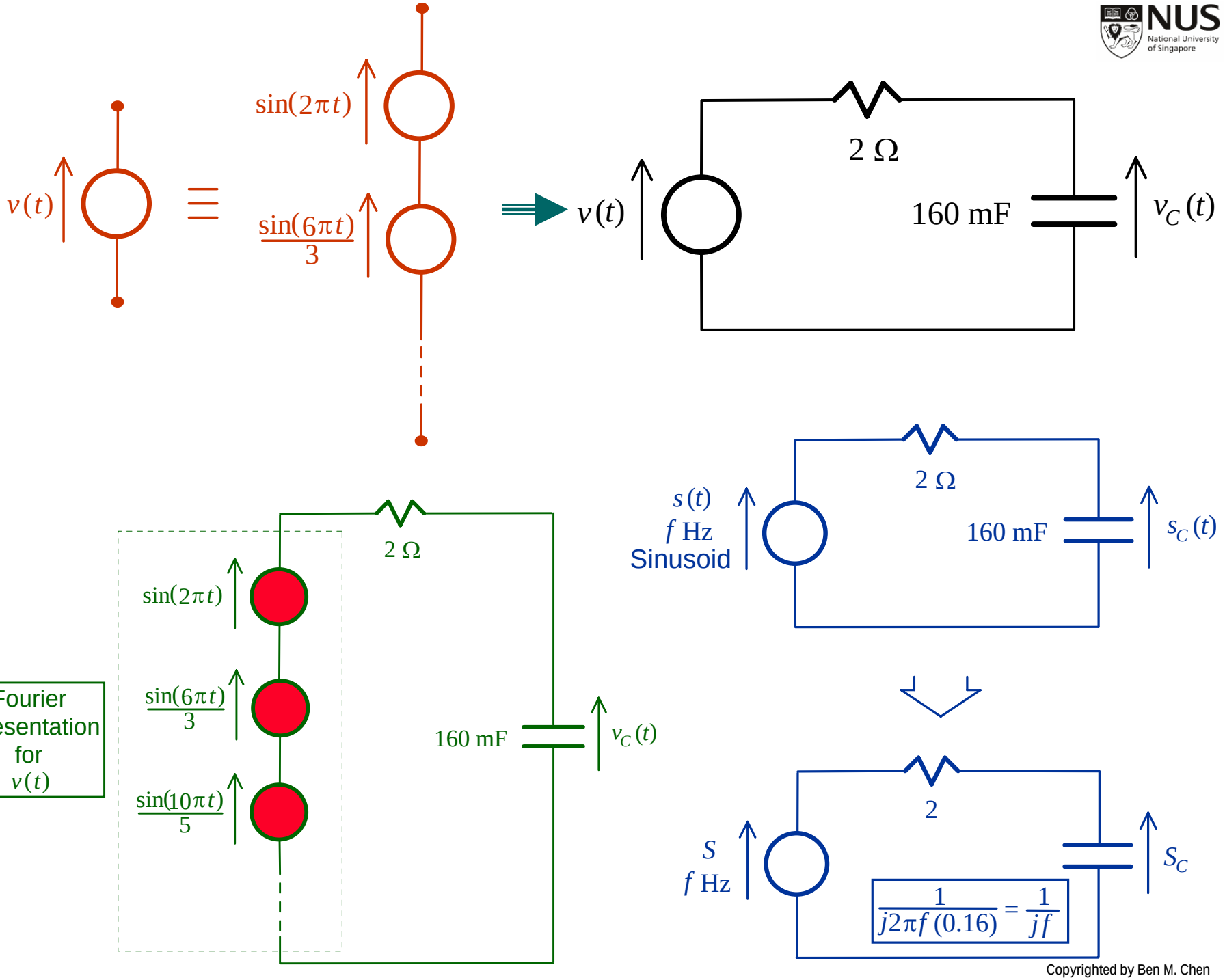




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The frequency response:

$$H(f)=\frac{S_C}{S}=\frac{\frac{1}{jf}}{2+\frac{1}{jf}}=\frac{1}{1+j2f}$$

From superposition, if the input is the periodic square signal

$$v(t)=\sin(2\pi t)+\frac{\sin(6\pi t)}{3}+\frac{\sin(10\pi t)}{5}+\cdots$$

then the output will be

$$v_C(t)=0.44\sin(2\pi t-1.1)+0.05\sin(6\pi t-1.4)+\cdots$$

$$=\frac{\sin\left[2(1)\pi t-\tan^{-1}(2\times 1)\right]}{(1)\sqrt{1+(2\times 1)^2}}+\frac{\sin\left[2(3)\pi t-\tan^{-1}(2\times 3)\right]}{(3)\sqrt{1+(2\times 3)^2}}+\cdots$$

$$=\sum_{n=1,3,5,\cdots}^{\infty}\frac{\sin\left[2n\pi t-\tan^{-1}(2n)\right]}{n\sqrt{1+(2n)^2}}$$

## Side Notes:

**Superposition for infinite series will not be examined.**

**Topic on Fourier Representation of periodic signal is skipped and hence ...**



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# Chapter 6. Transient Circuit Analysis



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## C.3 Linear Differential Equation

General solution:

| | |
|---|---|
| $n$th order linear differential equation | $\frac{d^n x(t)}{dt^n} + a_{n-1} \frac{d^{n-1} x(t)}{dt^{n-1}} + \cdots + a_0 x(t) = u(t)$ |
| General solution | $x(t) = x_{ss}(t) + x_{tr}(t)$ |
| Steady state response with no arbitrary constant | $x_{ss}(t)$ = particular integral obtained from assuming solution to have the same form as $u(t)$ |
| Transient response with $n$ arbitrary constants | $x_{tr}(t)$ = general solution of homogeneous equation $\frac{d^n x_{tr}(t)}{dt^n} + a_{n-1} \frac{d^{n-1} x_{tr}(t)}{dt^{n-1}} + \cdots + a_0 x_{tr}(t) = 0$ |



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## General solution of homogeneous equation:

| | |
|---|---|
| $n$ th order linear homogeneous equation | $\frac{d^n x_{tr}(t)}{dt^n} + a_{n-1} \frac{d^{n-1} x_{tr}(t)}{dt^{n-1}} + \cdots + a_0 x_{tr}(t) = 0$ |
| Roots of polynomial from homogeneous equation | Roots : $z_1, \cdots, z_n$<br>given by $(z - z_1) \cdots (z - z_n) = z^n + a_{n-1} z^{n-1} + \cdots + a_0$ |
| General solution (distinct roots) | $x_{tr}(t) = k_1 e^{z_1 t} + \cdots + k_n e^{z_n t}$ |
| General solution (non-distinct roots) | $x_{tr}(t) = (k_1 + k_2 t + k_3 t^2) e^{13t} + (k_4 + k_5 t) e^{22t} + k_6 e^{31t} + k_7 e^{41t}$<br>if roots are $13, 13, 13, 22, 22, 31, 41$ |



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# Particular integral:

| | |
|---|---|
| $x_{ss}(t)$ | Any specific solution (with no arbitrary constant) of $\frac{d^n x(t)}{dt^n}+a_{n-1}\frac{d^{n-1}x(t)}{dt^{n-1}}+\cdots+a_0 x(t)=u(t)$ |
| Method to determine $x_{ss}(t)$ | Trial and error approach: assume $x_{ss}(t)$ to have the same form as $u(t)$ and substitute into differential equation |
| Example to find $x_{ss}(t)$ for $\frac{dx(t)}{dt}+2x(t)=e^{3t}$ | Try a solution of $he^{3t}$<br>$\frac{dx(t)}{dt}+2x(t)=e^{3t}\Rightarrow 3he^{3t}+2he^{3t}=e^{3t}\Rightarrow h=0.2$<br>$x_{ss}(t)=0.2e^{3t}$ |
| Standard trial solutions | $u(t)$ — trial solution for $x_{ss}(t)$<br>$e^{\alpha t}$ — $he^{\alpha t}$<br>$t$ — $ht$<br>$te^{\alpha t}$ — $(h_1+h_2 t)e^{\alpha t}$<br>$a\cos(\omega t)+b\sin(\omega t)$ — $h_1\cos(\omega t)+h_2\sin(\omega t)$ |



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## 6.1 Steady State and Transient Analyses

So far, we have discussed the DC and AC circuit analyses. DC analysis can be regarded as a special case of AC analysis when the signals have frequency $f = 0$. Using Fourier series, the situation of having periodic signals can be handled using AC analysis and superposition. These analyses are often called ***steady state*** analyses, as the signals are assumed to exist at all time.

In order for the results obtained from these analyses to be valid, it is necessary for the circuit to have been working for a considerable period of time. This will ensure that all the ***transients*** caused by, say, the switching on of the sources have died out, the circuit is working in the steady state, and all the voltages and currents are as if they exist from all time.

However, when the circuit is first switched on, the circuit will not be in the steady state and it will be necessary to go back to first principle to determine the ***behavior*** of the system.



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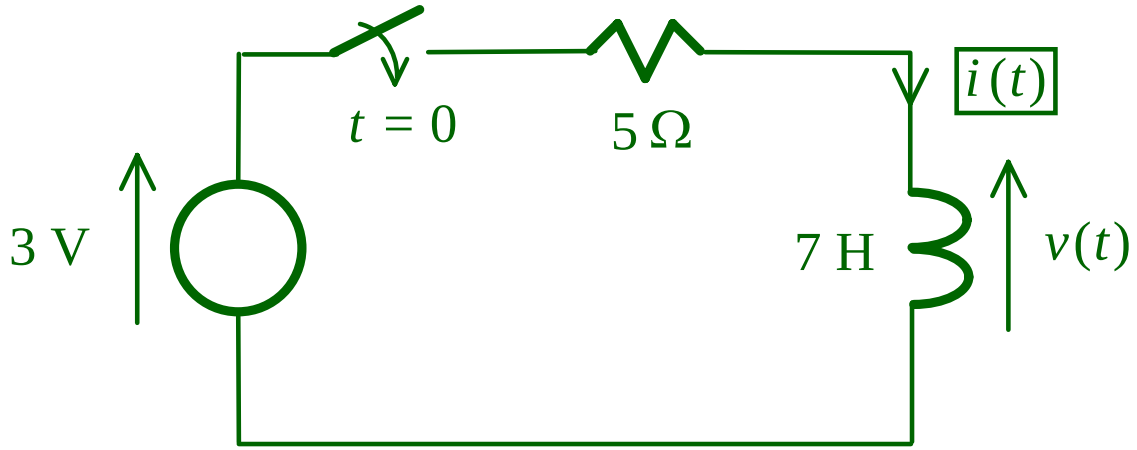
## 6.2 RL Circuit and Governing Differential Equation

Consider determining $i(t)$ in the following series RL circuit:




where the switch is open for $t < 0$ and is closed for $t \geq 0$.

Since $i(t)$ and $v(t)$ will not be equal to constants or sinusoids for all time, these cannot be represented as constants or phasors. Instead, the basic general voltage-current relationships for the resistor and inductor have to be used:



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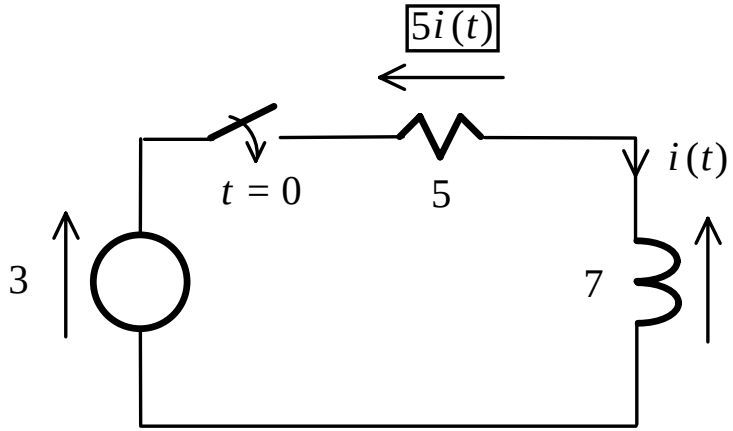
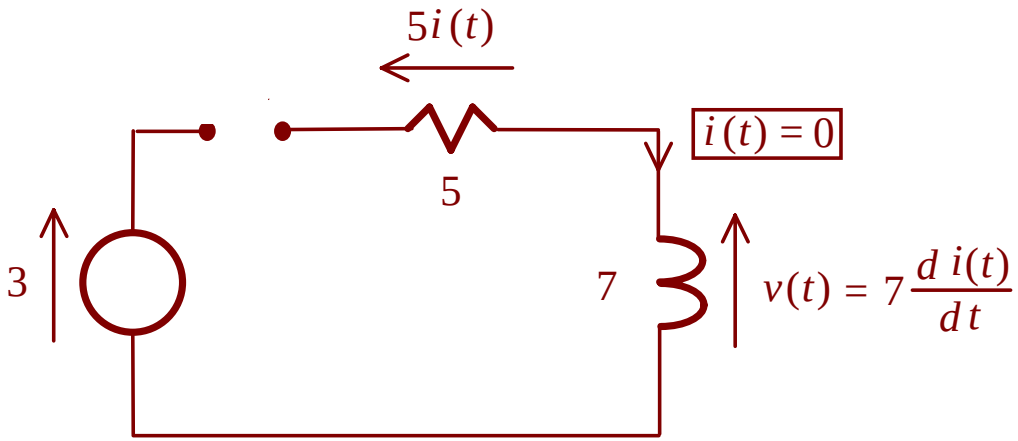
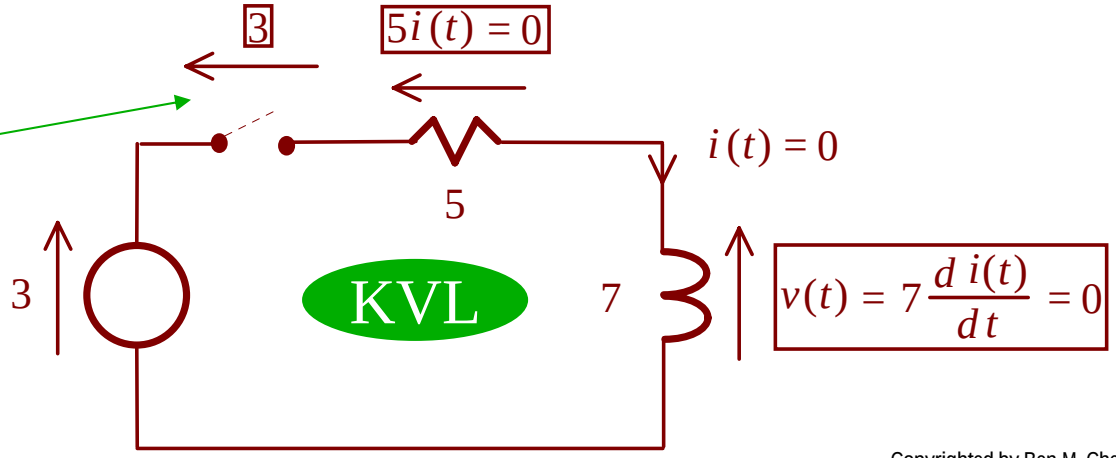



## For t < 0




**voltage cross over the switch**



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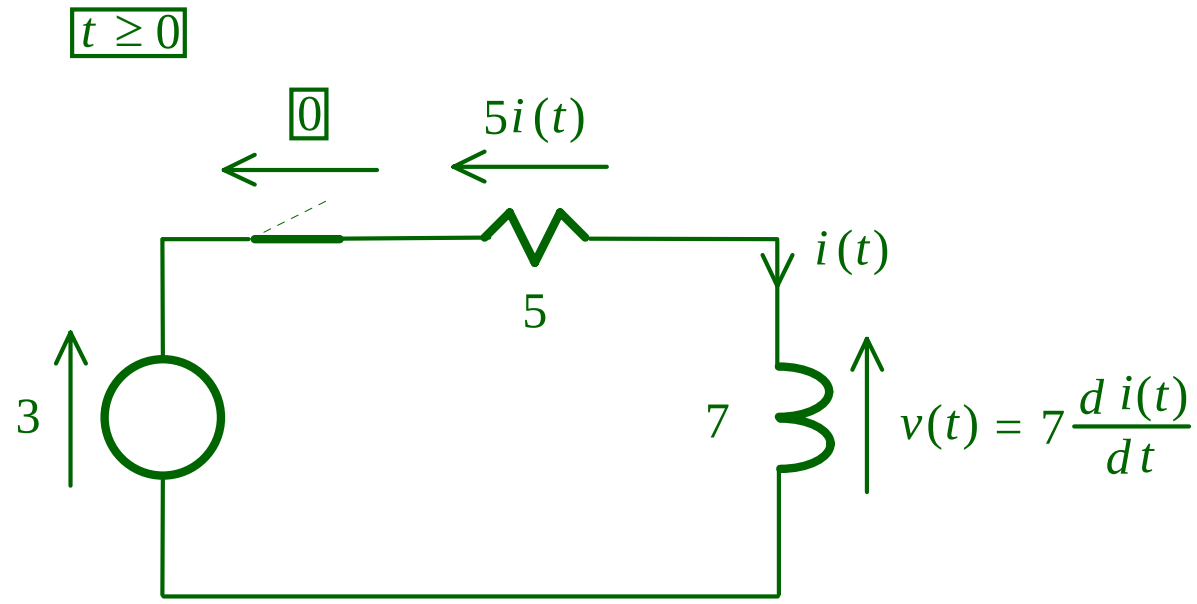


Applying KVL:

$$7\frac{di(t)}{dt}+5i(t)=3, \quad t\geq 0$$

and $i(t)$ can be found from determining the ***general solution*** to this first order linear differential equation (d.e.) which governs the behavior of the circuit for $t \geq 0$.

Mathematically, the above d.e. is often written as

$$7\frac{di(t)}{dt}+5i(t)=u(t),\ t\geq 0$$

where the r.h.s. is $u(t)=3,\ t\geq 0$ and corresponds to the dc source or excitation in this example.



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## 6.3 Steady State Response

Since the r.h.s. of the governing d.e.

$$7\frac{di(t)}{dt}+5i(t)=u(t)=3,\ \ t\geq 0$$

Let us try a steady state solution of

$$i_{ss}(t)=k,\quad t\geq 0$$

which has the same form as $u(t)$, as a possible solution.

$$7\frac{di_{ss}(t)}{dt}+5i_{ss}(t)=3$$

$$\Rightarrow 7(0)+5(k)=3$$

$$\Rightarrow k=\frac{3}{5}$$

$$i_{ss}(t)=\frac{3}{5},\quad t\geq 0$$

$$7\frac{di_{ss}(t)}{dt}+5i_{ss}(t)=7\frac{d}{dt}\left(\frac{3}{5}\right)+5\left(\frac{3}{5}\right)=3,\quad t\geq 0$$

and is a solution of the governing d.e.

In mathematics, the above solution is called the ***particular integral*** or solution and is found from letting the answer to have the same form as $u(t)$. The word "particular" is used as the solution is only one possible function that satisfy the d.e.



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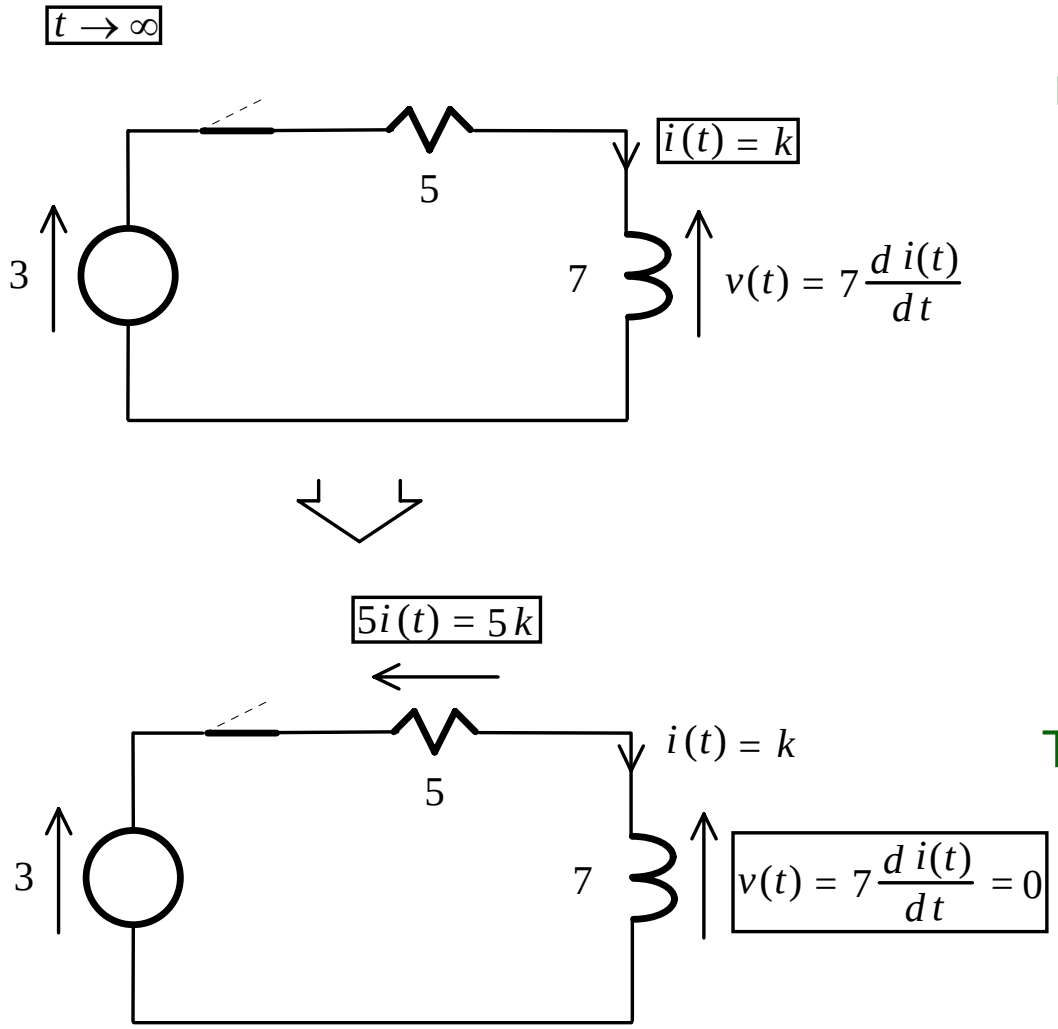
In circuit analysis, the derivation of $i_{ss}(t)$ by letting the answer to have the same form as $u(t)$ can be shown to give the ***steady state response*** of the circuit as $t \to \infty$.




Using KVL, the steady state response is

$$3 = 0 + 5k + 0 = 5k$$

$$\Rightarrow k = \frac{3}{5}$$

$$\Rightarrow i(t) = \frac{3}{5}, \quad t \to \infty$$

This is the same as $i_{ss}(t)$.



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## 6.4 Transient Response

To determine $i(t)$ for all $t$, it is necessary to find the complete solution of the governing d.e.

$$7\frac{di(t)}{dt}+5i(t)=u(t)=3,\ \ t\geq 0$$

From mathematics, the complete solution can be obtained from summing a particular solution, say, $i_{ss}(t)$, with $i_{tr}(t)$: $i(t)=i_{ss}(t)+i_{tr}(t),\quad t\geq 0$

where $i_{tr}(t)$ is the general solution of the ***homogeneous*** equation

$$7\frac{di(t)}{dt}+5i(t)=0,\ \ t\geq 0$$

$$7\frac{di_{tr}(t)}{dt}+5i_{tr}(t)\Big|_{\frac{di_{tr}(t)}{dt}\text{ replaced by } z}$$

$$=7z^1+5z^0=7z+5$$

$$\Longrightarrow \quad z_1=-\frac{5}{7}$$

$$\Longrightarrow \quad i_{tr}(t)=k_1e^{z_1t}=k_1e^{-\frac{5}{7}t},\ t\geq 0$$

where $k_1$ is a constant (unknown now).

$$i_{tr}(t)=k_1e^{-\frac{5}{7}t}\to 0,\ t\to\infty$$

Thus, it is called ***transient response***.



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## 6.5 Complete Response

To see that summing $i_{ss}(t)$ and $i_{tr}(t)$ gives the general solution of the governing d.e.

$$7\frac{di(t)}{dt}+5i(t)=3,\ \ t\geq 0$$

note that

$$i_{ss}(t)=\frac{3}{5},\quad t\geq 0 \quad \text{satisfies} \quad 7\frac{d}{dt}\left(\frac{3}{5}\right)+5\left(\frac{3}{5}\right)=3,\quad t\geq 0$$

$$i_{tr}(t)=k_1e^{-\frac{5}{7}t},\quad t\geq 0 \quad \text{satisfies} \quad 7\frac{d}{dt}\left(k_1e^{-\frac{5}{7}t}\right)+5\left(k_1e^{-\frac{5}{7}t}\right)=0,\quad t\geq 0$$

$$\Downarrow$$

$$i_{ss}(t)+i_{tr}(t)=\frac{3}{5}+k_1e^{-\frac{5}{7}t},\quad t\geq 0 \quad \text{satisfies} \quad 7\frac{d}{dt}\left(\frac{3}{5}+k_1e^{-\frac{5}{7}t}\right)+5\left(\frac{3}{5}+k_1e^{-\frac{5}{7}t}\right)=3$$

$$\Rightarrow$$

$$i(t)=i_{ss}(t)+i_{tr}(t)=\frac{3}{5}+k_1e^{-\frac{5}{7}t},\quad t\geq 0$$

is the general solution of the d.e.



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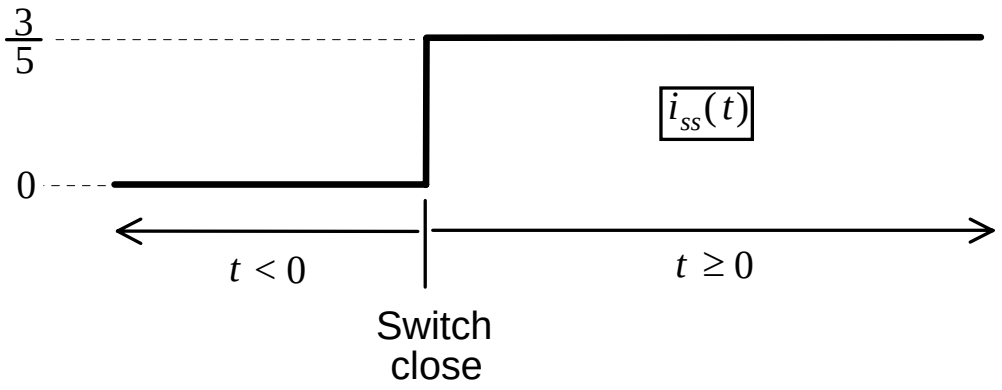
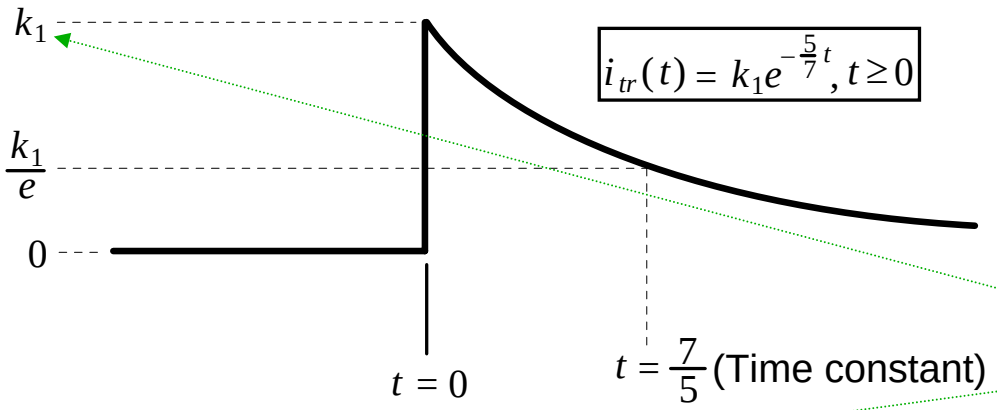
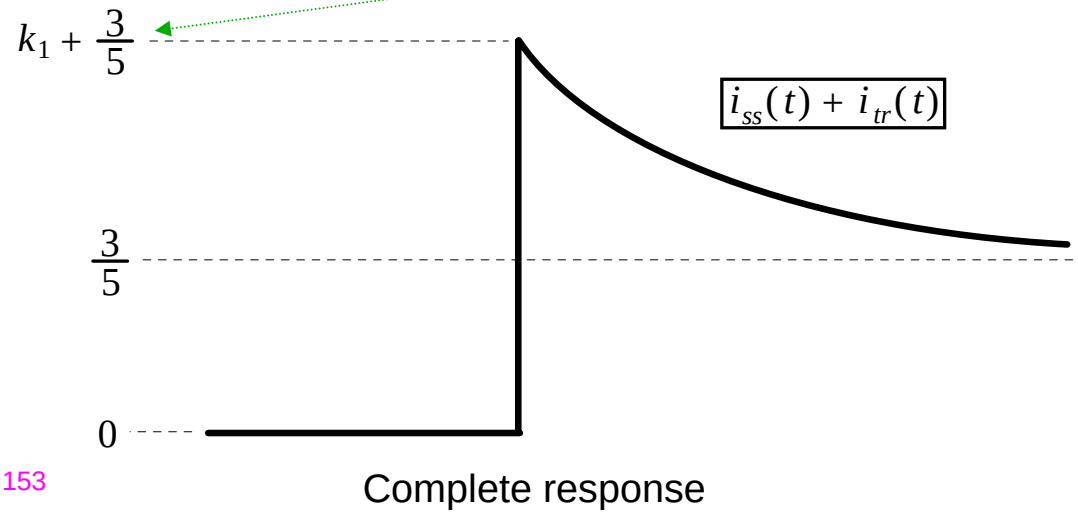



**Steady State Response**




**Transient Response**

**$k_1$ is to be determined later**




**Complete Response**



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Note that the time it takes for the transient or zero-input response $i_{tr}(t)$ to decay to $1/e$ of its initial value is

$$\text{Time taken for } i_{tr}(t) \text{ to decay to } 1/e \text{ of initial value} = \frac{7}{5}$$

and is called the ***time constant*** of the response or system.

We can take the transient response to have died out after a few time constants.



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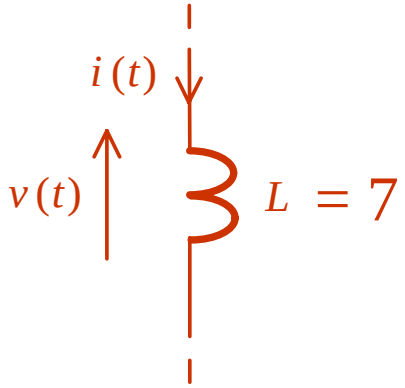
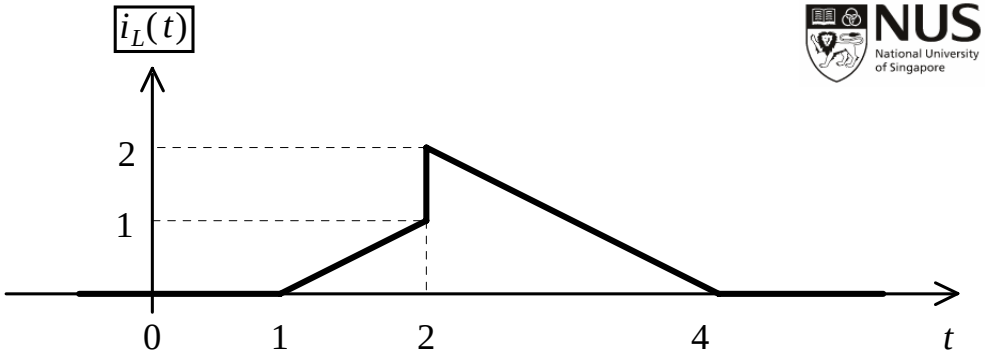
## 6.6 Current Continuity for Inductor

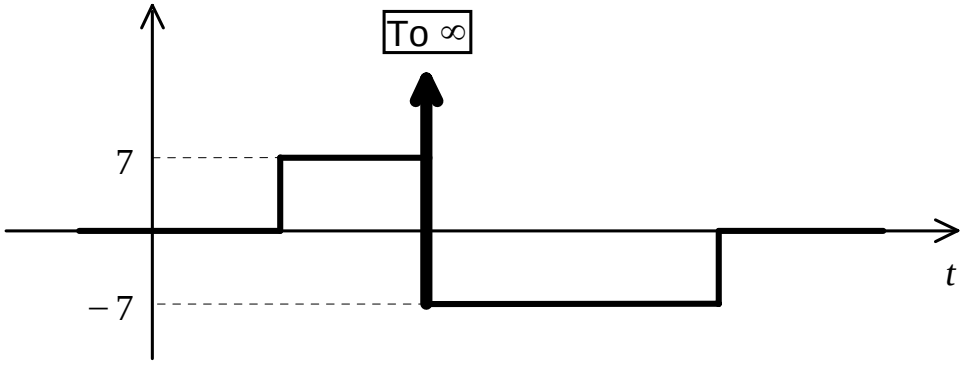
To determine the constant $k_1$ in the transient response of the RL circuit, the concept of ***current continuity for an inductor*** has to be used.

Consider the following example:

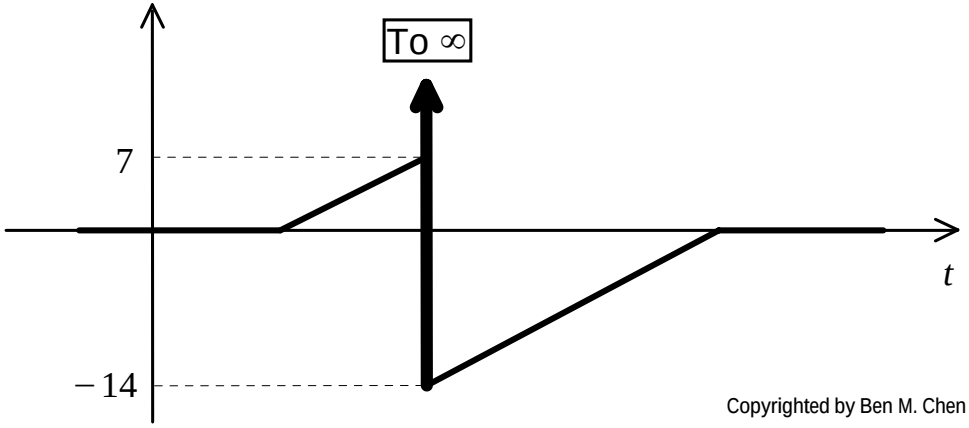



$$v_L(t) = 7\frac{di_L(t)}{dt}$$




$i_L(t)\,v_L(t)$ =Instantaneous power supplied



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Due to the step change or discontinuity in $i_L(t)$ at $t = 2$, and the power supplied to the inductor at $t = 2$ will go to infinity. Since it is impossible for any system to deliver an infinite amount of power at any time, it is impossible for $i_L(t)$ to change in the manner shown.

In general, the **current through an inductor must be a continuous function of time** and cannot change in a step manner.



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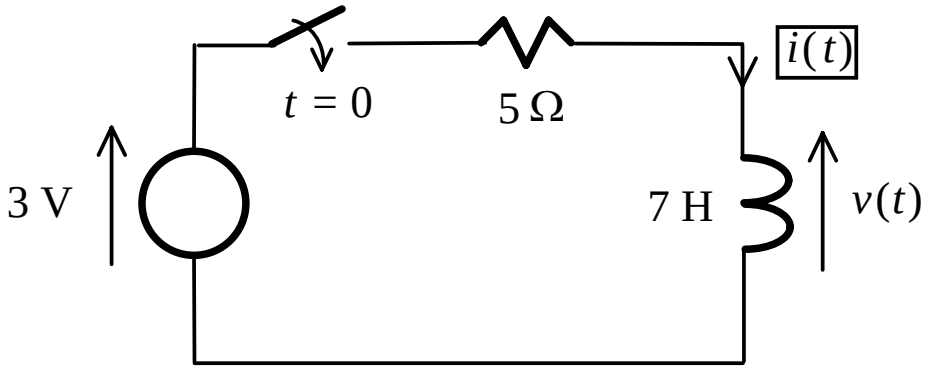
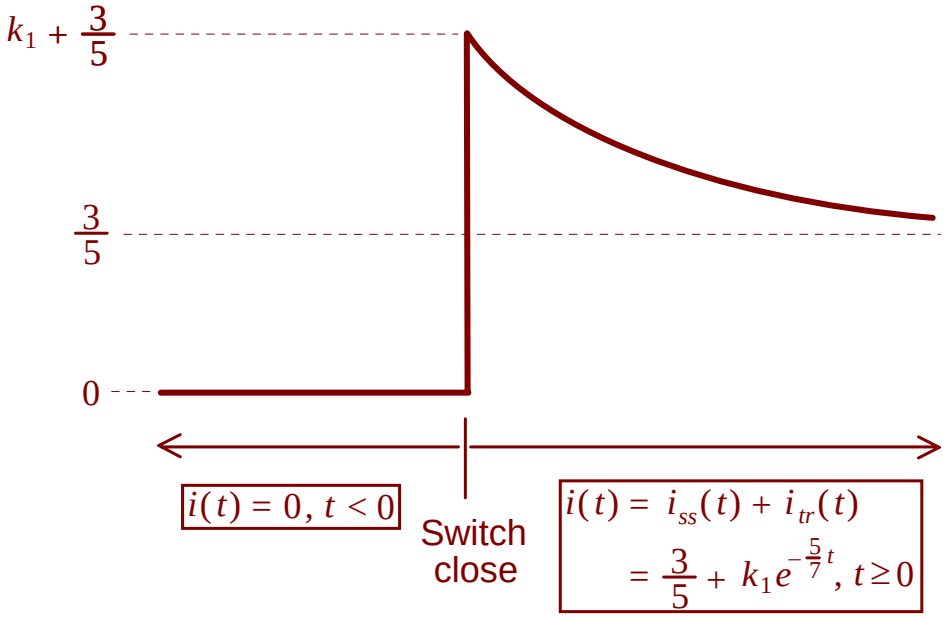
# Now back to our RL Circuit:

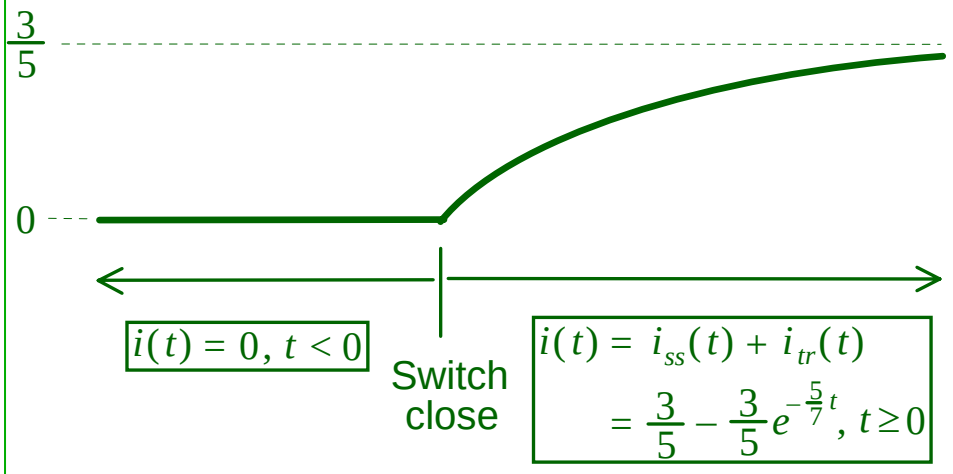



Using current continuity for an inductor at $t = 0$:

$$i(t=0) = \frac{3}{5} + k_1 = 0 \quad \Rightarrow \quad k_1 = -\frac{3}{5}$$

$$\Rightarrow \quad i(t) = \begin{cases} 0, & t < 0 \\ \frac{3}{5} - \frac{3}{5} e^{-\frac{5}{7}t}, & t \geq 0 \end{cases}$$



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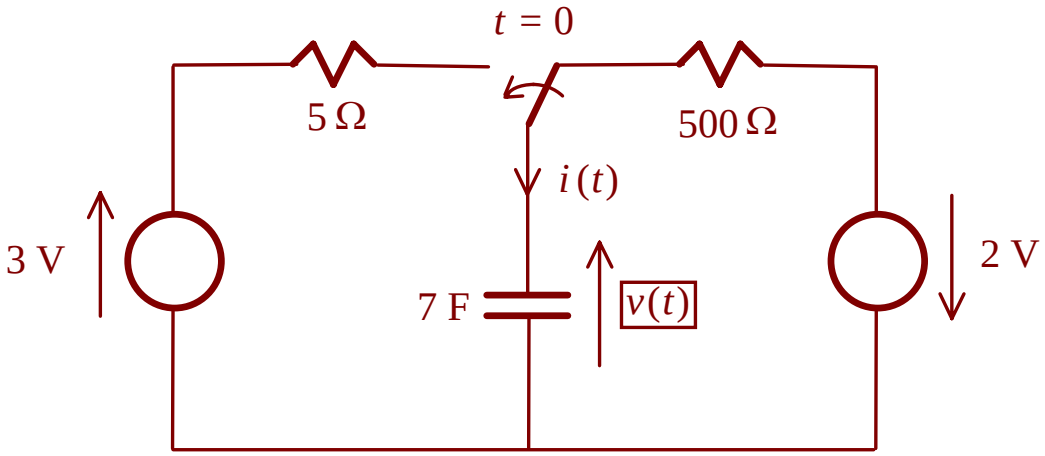
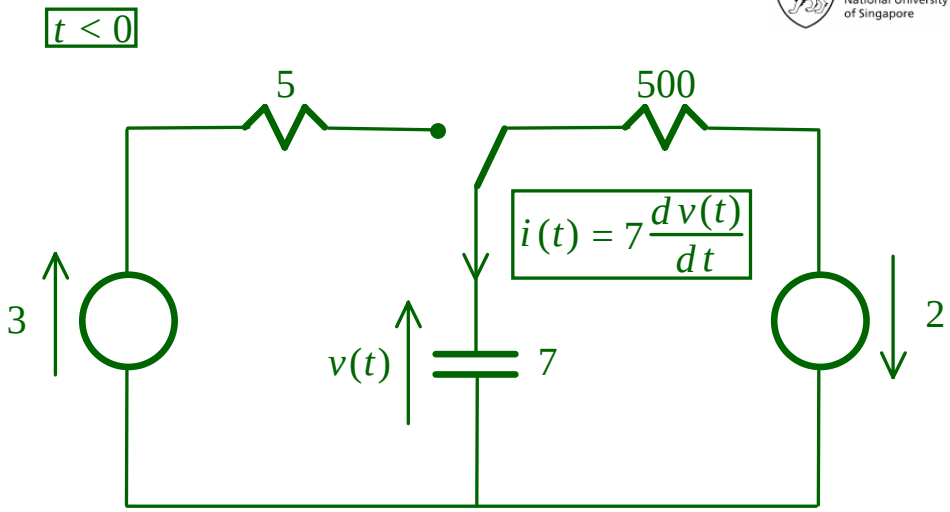
## 6.7 RC Circuit

Consider finding $v(t)$ in the following RC circuit:




where the switch is in the position shown for $t < 0$ and is in the other position for $t \geq 0$.




Taking the switch to be in this position starting from $t = -\infty$, the voltages and currents will have settled down to constant values for practically all $t < 0$.

$$i(t) = 7\frac{dv(t)}{dt} = 7\frac{d(\text{constant})}{dt} = 0, \quad t < 0$$



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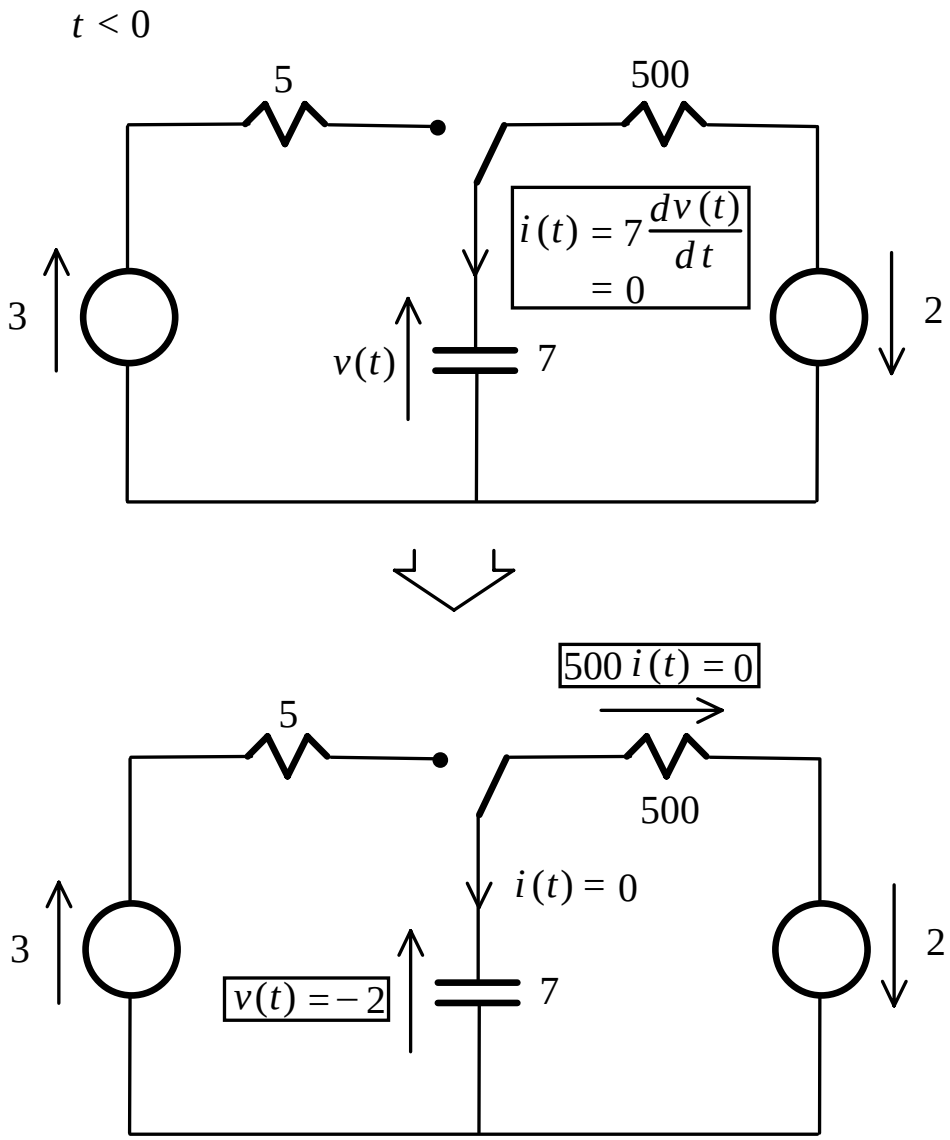


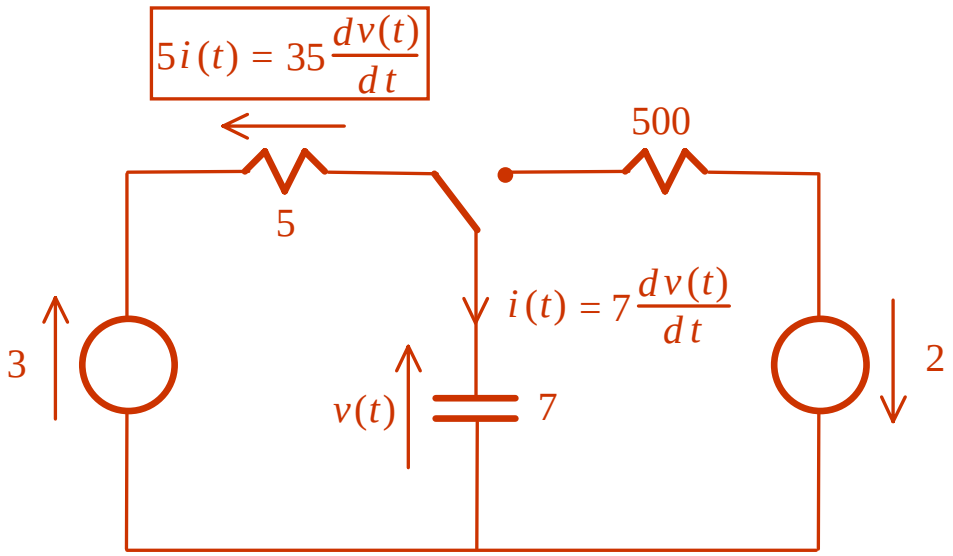


Applying KVL:

$$35\frac{dv(t)}{dt} + v(t) = u(t) = 3, \quad t \geq 0$$

which has a solution

$$v(t) = v_{ss}(t) + v_{tr}(t), \quad t \geq 0$$



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## (1) Steady State Response

$$u(t) = 3, \quad t \geq 0$$

$$\Downarrow$$

$$v_{ss}(t) = k, \quad t \geq 0$$

$$\Downarrow$$

$$35\frac{dv_{ss}(t)}{dt} + v_{ss}(t) = 3$$

$$\Rightarrow 0 + k = 3 \Rightarrow k = 3$$

$$\Downarrow$$

$$v_{ss}(t) = 3, \quad t \geq 0$$

## (2) Transient Response

$$35\frac{dv_{tr}(t)}{dt} + v_{tr}(t) = 0, \quad t \geq 0$$

$$\Downarrow$$

$$35\frac{dv_{tr}(t)}{dt} + v_{tr}(t)\Big|_{\frac{dv_{tr}(t)}{dt} \text{ replaced by } z}$$

$$= 35z^1 + z^0 = 35z + 1$$

$$\Longrightarrow \quad z_1 = -\frac{1}{35}$$

$$\Longrightarrow \quad v_{tr}(t) = k_1 e^{z_1 t} = k_1 e^{-\frac{t}{35}}, \quad t \geq 0$$

$$v(t) = \begin{cases} -2, & t<0 \\ v_{ss}(t) + v_{tr}(t), & t \geq 0 \end{cases} = \begin{cases} -2, & t<0 \\ 3 + k_1 e^{-\frac{t}{35}}, & t \geq 0 \end{cases} \Longleftarrow \textbf{Complete Response}$$



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## 6.8 Voltage Continuity for Capacitor

To determine $k_1$ in the transient response of the RC circuit, the concept of ***voltage continuity for a capacitor*** has to be used.

Similar to current continuity for an inductor, the voltage $v(t)$ across a capacitor $C$ must be continuous and cannot change in a step manner.

Thus, for the RC circuit we consider, the complete solution was derived as:

$$v(t) = \begin{cases} -2, & t < 0 \\ v_{ss}(t) + v_{tr}(t), & t \geq 0 \end{cases} = \begin{cases} -2, & t < 0 \\ 3 + k_1 e^{-\frac{t}{35}}, & t \geq 0 \end{cases}$$

At $t = 0$,

$$v(0) = 3 + k_1 = -2 \;\Rightarrow\; k_1 = -5 \quad \Longrightarrow \quad v(t) = \begin{cases} -2, & t < 0 \\ 3 - 5e^{-\frac{t}{35}}, & t \geq 0 \end{cases}$$



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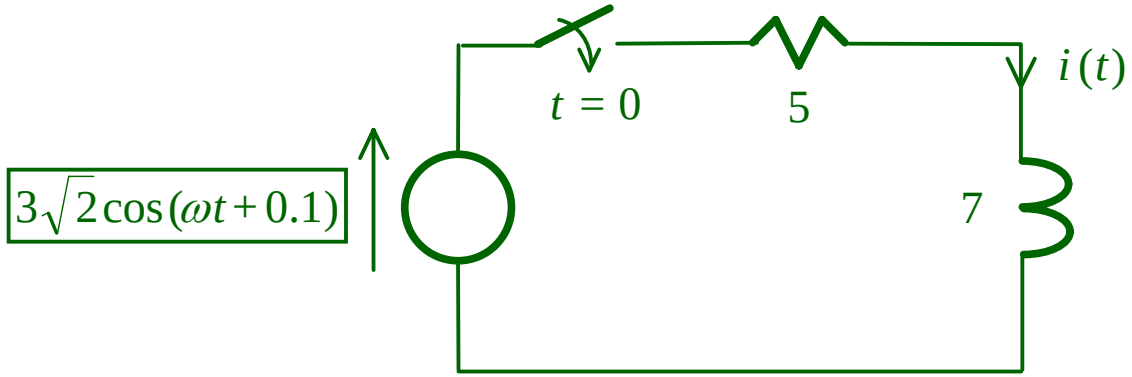
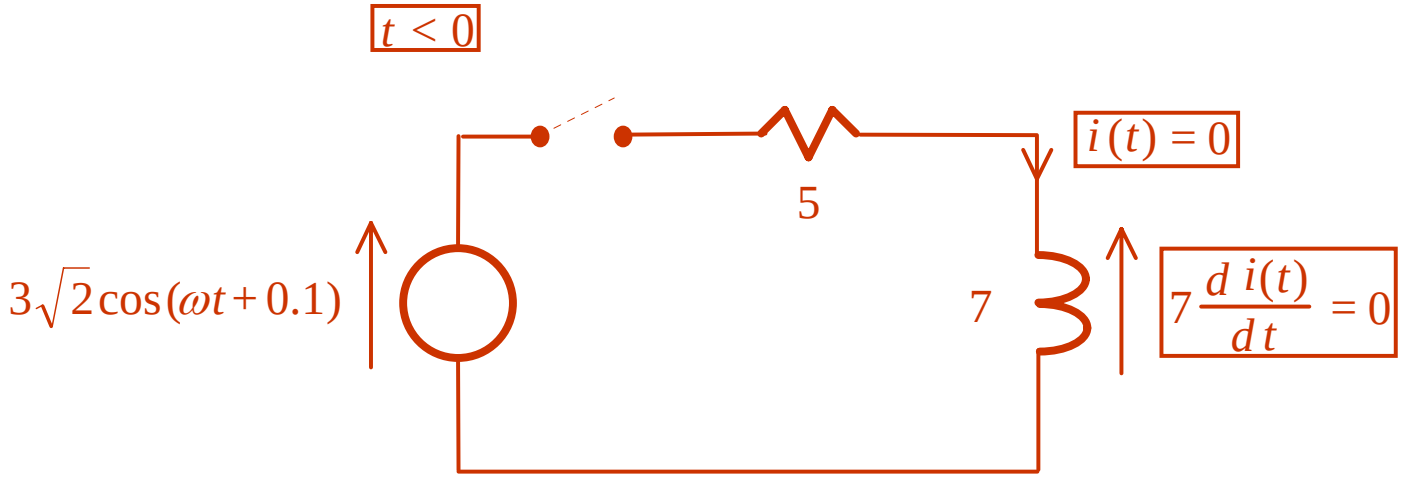
## 6.9 Transient with Sinusoidal Source

Consider the RL circuit with the dc source changed to a sinusoidal one:




For $t < 0$ when the switch is open:



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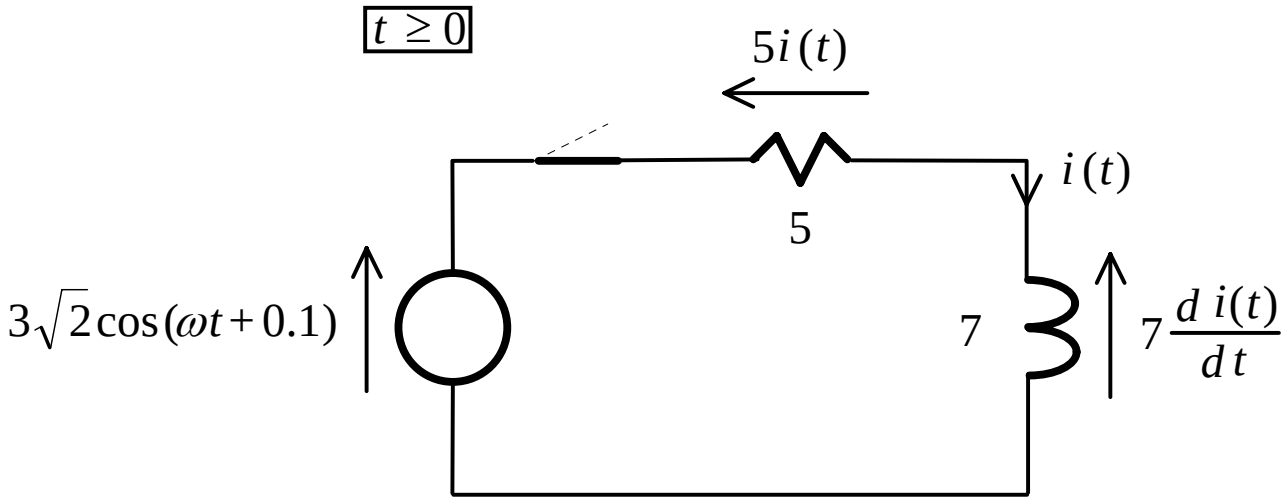
For $t \geq 0$ when the switch is closed:




The governing d.e. is

$$7\frac{d\,i(t)}{dt} + 5\,i(t) = u(t), \quad t \geq 0$$

**Looking for general solution**

$$i(t) = i_{ss}(t) + i_{tr}(t), \quad t \geq 0$$

with

$$u(t) = 3\sqrt{2}\cos(\omega t + 0.1) = \mathrm{Re}\left[3\sqrt{2}e^{j(\omega t + 0.1)}\right] = \mathrm{Re}\left[\left(3e^{j0.1}\right)\left(\sqrt{2}e^{j\omega t}\right)\right], \quad t \geq 0$$



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Since $u(t)$ is sinusoidal in nature, a trial solution for the steady state response or particular integral $i_{ss}(t)$ may be

$$i_{ss}(t) = r\sqrt{2}\cos(\omega t + \theta) = \mathrm{Re}\left[\left(re^{j\theta}\right)\left(\sqrt{2}e^{j\omega t}\right)\right], \quad t \geq 0$$

$$\Longrightarrow \quad 7\frac{di_{ss}(t)}{dt} + 5i_{ss}(t) = 7\frac{d\,\mathrm{Re}\left[\left(re^{j\theta}\right)\left(\sqrt{2}e^{j\omega t}\right)\right]}{dt} + 5\,\mathrm{Re}\left[\left(re^{j\theta}\right)\left(\sqrt{2}e^{j\omega t}\right)\right]$$

$$= 7\,\mathrm{Re}\left[\left(re^{j\theta}\right)\left(j\omega\right)\left(\sqrt{2}e^{j\omega t}\right)\right] + 5\,\mathrm{Re}\left[\left(re^{j\theta}\right)\left(\sqrt{2}e^{j\omega t}\right)\right]$$

$$= \mathrm{Re}\left[\left(re^{j\theta}\right)\left(j\omega 7 + 5\right)\left(\sqrt{2}e^{j\omega t}\right)\right]$$

$$= \mathrm{Re}\left[\left(3e^{j0.1}\right)\left(\sqrt{2}e^{j\omega t}\right)\right] = u(t)$$

**This is Method One:** $\Longrightarrow$ $\boxed{\left(j\omega 7+5\right)re^{j\theta}=3e^{j0.1}}$



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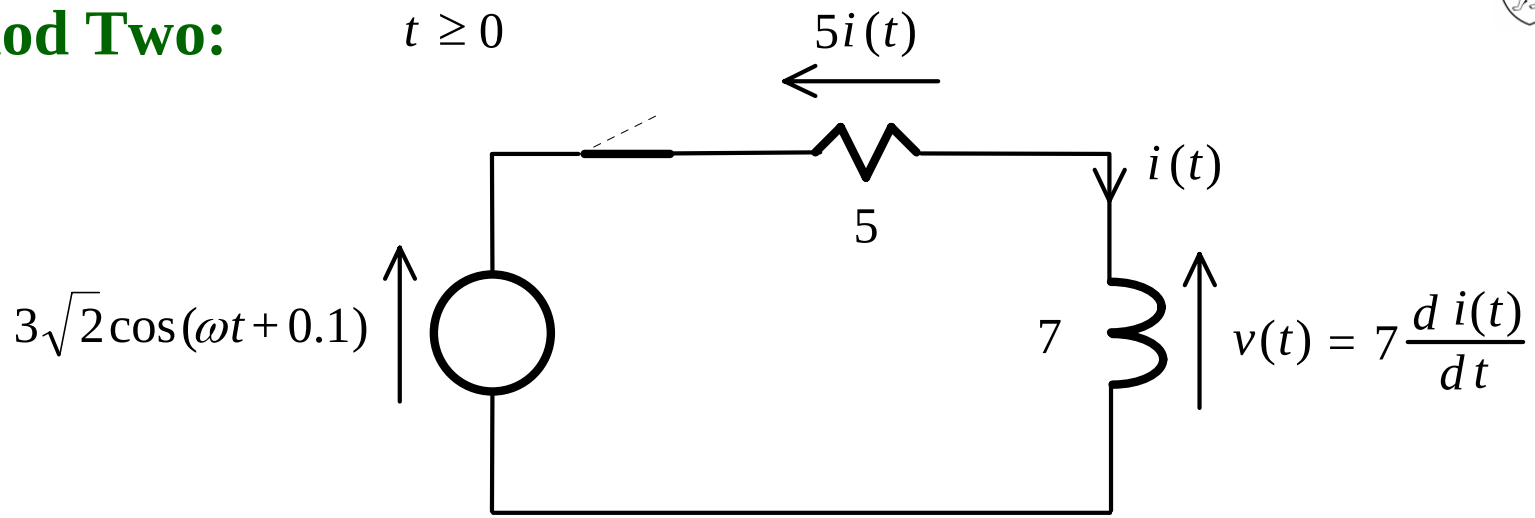
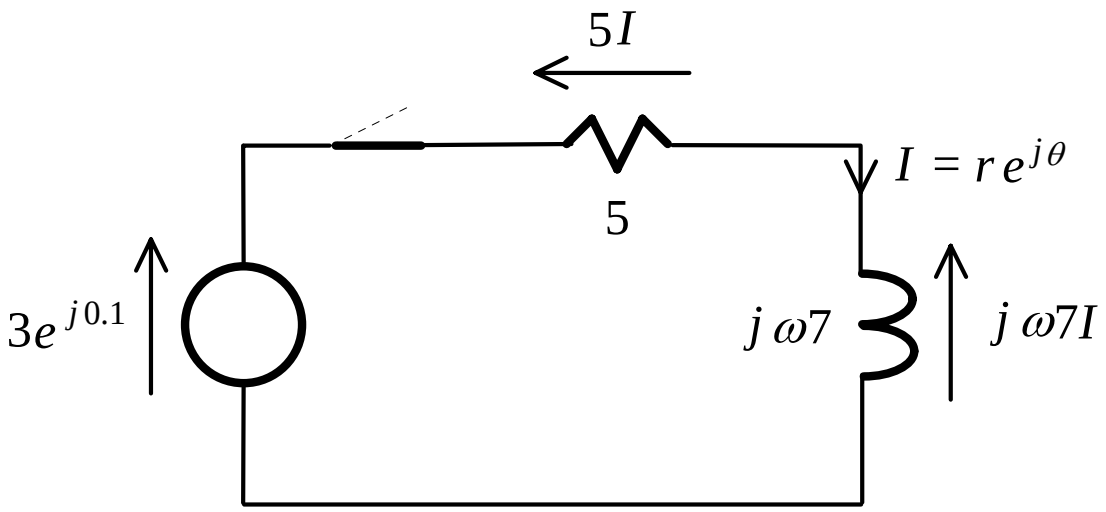
## Method Two:

$t \geq 0$




$$(j\omega 7 + 5)I = (j\omega 7 + 5)re^{j\theta} = 3e^{j0.1}$$



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## (1) Steady State Response

$$(j\omega 7+5)re^{j\theta} = 3e^{j0.1}$$

$$\Rightarrow \quad re^{j\theta} = \frac{3e^{j0.1}}{5+j\omega 7}$$

$$\Rightarrow \quad r = \frac{\left|3e^{j0.1}\right|}{\left|5+j\omega 7\right|} = \frac{3}{\sqrt{5^2+7^2\omega^2}}$$

$$\theta = \text{Arg}\left[e^{j0.1}\right] - \text{Arg}\left[5+j\omega 7\right]$$

$$= 0.1 - \tan^{-1}\left(\frac{7\omega}{5}\right)$$

$$\Downarrow$$

$$i_{ss}(t) = r\sqrt{2}\cos(\omega t+\theta) = \frac{3\sqrt{2}}{\sqrt{25+49\omega^2}}\cos\left[\omega t+0.1-\tan^{-1}\left(\frac{7\omega}{5}\right)\right], \quad t\geq 0$$

## (2) Transient Response

$$7\frac{di_{tr}(t)}{dt} + 5i_{tr}(t) = 0, \quad t\geq 0$$

$i_{tr}(t)$ will have the same form as the dc source case:

$$i_{tr}(t) = k_1 e^{-\frac{5}{7}t}, \quad t\geq 0$$

$$\Downarrow$$

**Complete Response**

$$\Uparrow$$



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# Complete Response

$$i(t) = i_{ss}(t) + i_{tr}(t), \quad t \geq 0$$

$$= \frac{3\sqrt{2}}{\sqrt{25+49\omega^2}} \cos\left[\omega t + 0.1 - \tan^{-1}\left(\frac{7\omega}{5}\right)\right] + k_1 e^{-\frac{5}{7}t}, \quad t \geq 0$$

To determine $k_1$, the continuity of $i(t)$, the current through the inductor, can be used.

$$i(t) = 0, \quad t < 0 \rightarrow i(0) = i_{ss}(0) + i_{tr}(0) = \frac{3\sqrt{2}}{\sqrt{25+49\omega^2}} \cos\left[0.1 - \tan^{-1}\left(\frac{7\omega}{5}\right)\right] + k_1$$

$$\rightarrow k_1 = -\frac{3\sqrt{2}}{\sqrt{25+49\omega^2}} \cos\left[0.1 - \tan^{-1}\left(\frac{7\omega}{5}\right)\right] \rightarrow$$

$$i(t) = \begin{cases} 0, & t < 0 \\ \dfrac{3\sqrt{2}}{\sqrt{25+49\omega^2}} \left\{ \cos\left[\omega t + 0.1 - \tan^{-1}\left(\dfrac{7\omega}{5}\right)\right] - \cos\left[0.1 - \tan^{-1}\left(\dfrac{7\omega}{5}\right)\right] e^{-\frac{5}{7}t} \right\}, & t \geq 0 \end{cases}$$



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**An FAQ:** Can we apply KVL to the rms values of voltages in AC circuits?

**Answer:** No. In an AC circuit, KVL is valid for the phasors of the voltages in a closed-loop, i.e., the sum of the phasors of voltages in a closed-loop is equal to 0 provided that they are all assigned to the same direction. KVL cannot be applied to the magnitudes or rms values of the voltages alone. For example, a closed-loop circuit containing a series of an AC source, a resistor and a capacitor could have the following situation: The source has a voltage with a rms value of 20V, while the resistor and the capacitor have their voltages with the rms values of 9V and 15V, respectively. All in all, if you want to apply KVL in AC circuits, apply it to the phasors of its voltages.

**By the way, KVL is valid as well when the voltages are specified as functions of time. This is true for any type of circuits.**



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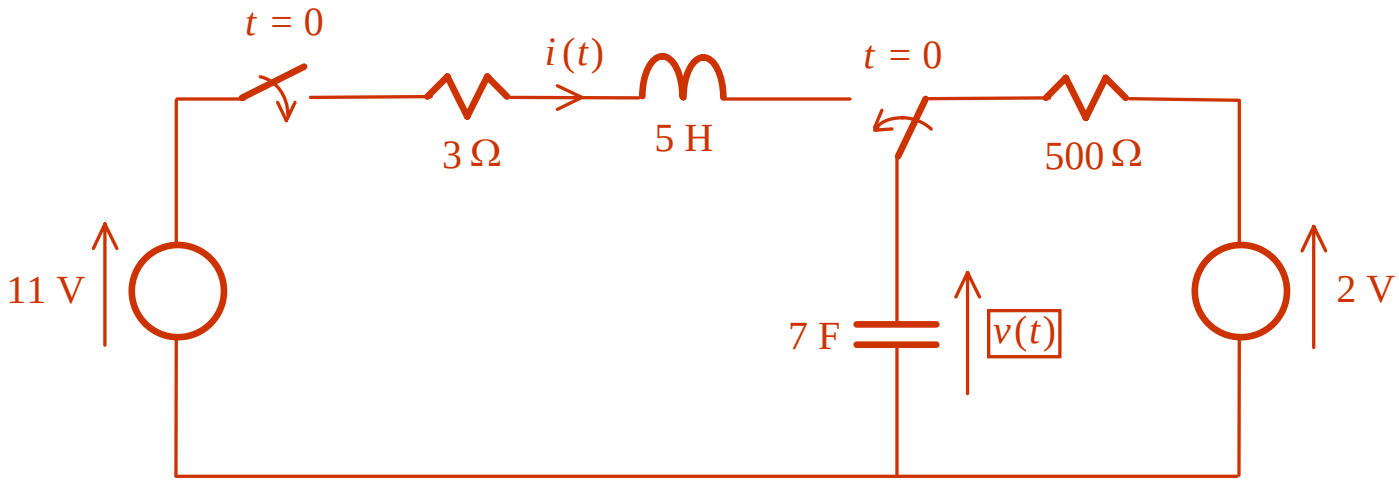
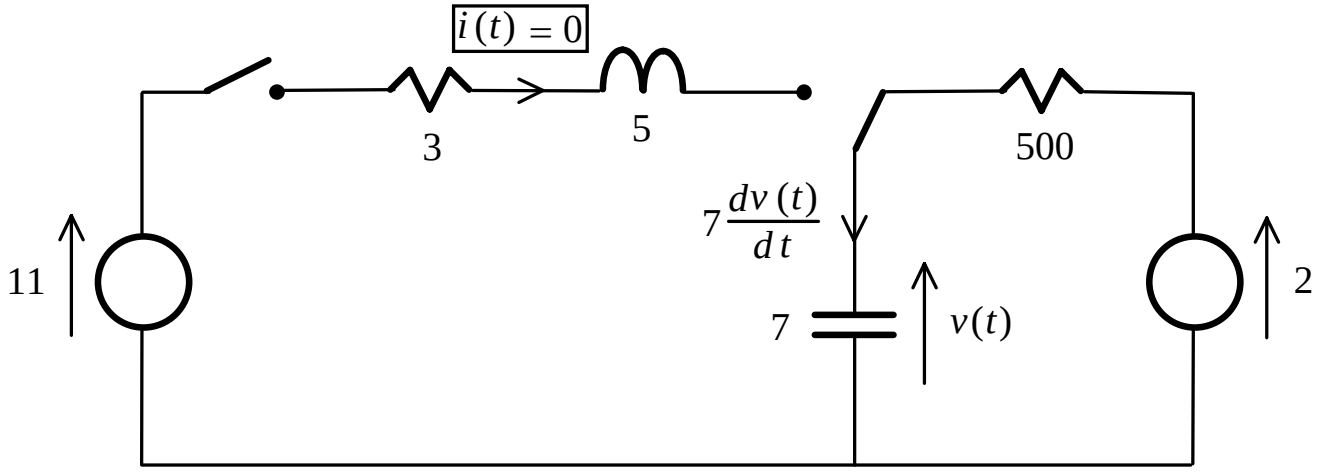
# 6.10 Second Order RLC Circuit

Consider determining $v(t)$ in the following series RLC circuit:




Both switches are in the position shown for $t < 0$ & are in the other positions for $t \geq 0$.

For $t < 0$



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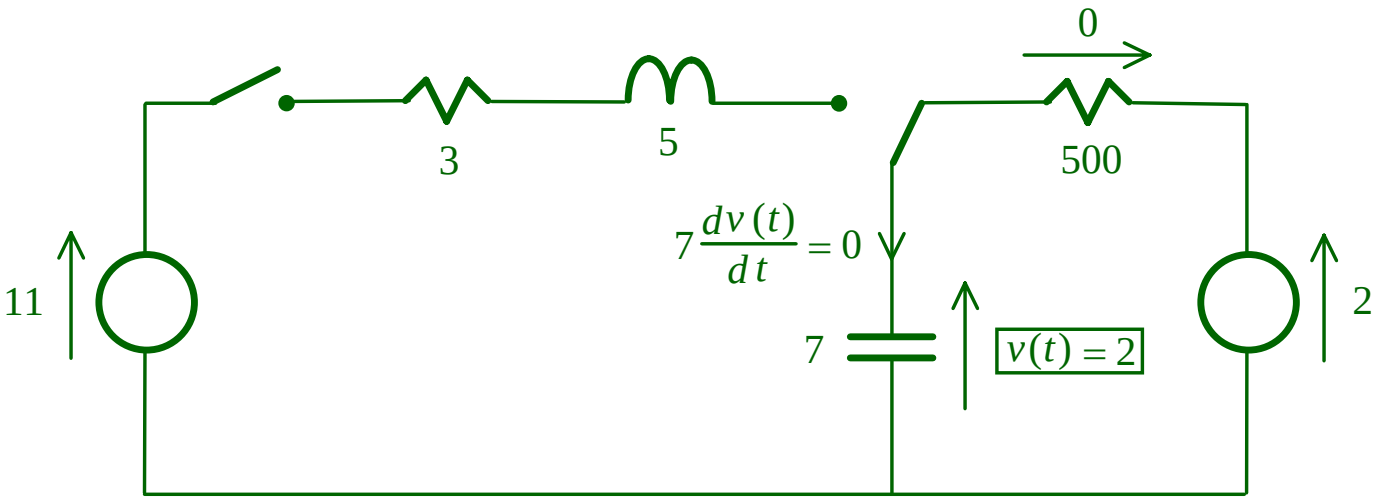
Taking the switches to be in the positions shown starting from $t = -\infty$, the voltages and currents will have settled down to constant values for practically all $t < 0$ and the important voltages and currents are given by:




Mathematically:

$$v(t) = 2, \quad t < 0 \quad \& \quad i(t) = 0, \quad t < 0$$



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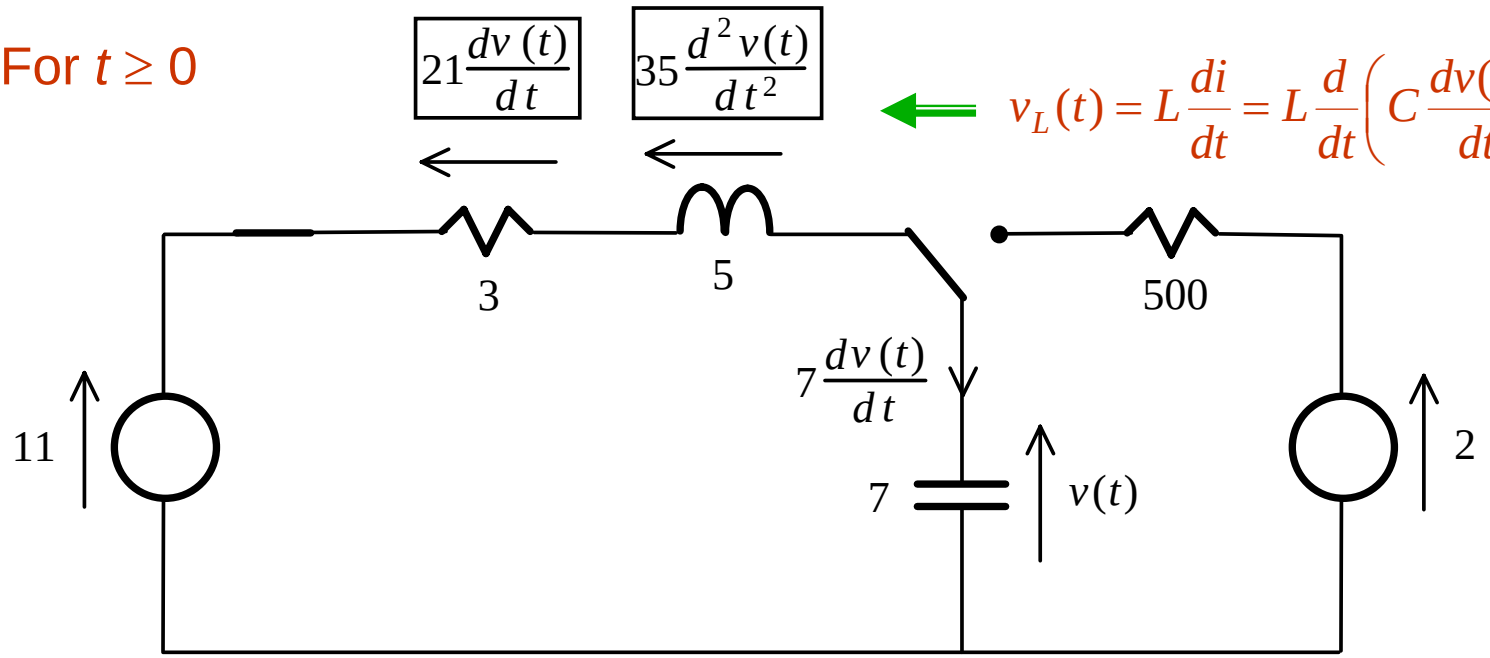
For $t \geq 0$




$$v_L(t) = L\frac{di}{dt} = L\frac{d}{dt}\left(C\frac{dv(t)}{dt}\right) = LC\frac{d^2v(t)}{dt^2}$$

Applying KVL:

$$35\frac{d^2v(t)}{dt^2} + 21\frac{dv(t)}{dt} + v(t) = u(t) = 11, \quad t \geq 0$$

Due to the presence of 2 energy storage elements, the governing d.e. is a second order one and the general solution is

$$v(t) = v_{ss}(t) + v_{tr}(t), \quad t \geq 0$$



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## (1) Steady State Response

$$u(t)=11, \quad t\geq 0 \quad \Longrightarrow \quad v_{ss}(t)=k, \quad t\geq 0 \quad \Longrightarrow$$

$$35\frac{d^2v_{ss}(t)}{dt^2}+21\frac{dv_{ss}(t)}{dt}+v_{ss}(t)=0+0+k=11 \quad \Longrightarrow \quad \boxed{v_{ss}(t)=11, \quad t\geq 0}$$

## (2) Transient Response

$$35\frac{d^2v_{tr}(t)}{dt^2}+21\frac{dv_{tr}(t)}{dt}+v_{tr}(t)=0, \quad t\geq 0 \quad \Longrightarrow$$

$$35\frac{d^2v_{tr}(t)}{dt^2}+21\frac{dv_{tr}(t)}{dt}+v_{tr}(t)\Bigg|_{\frac{dv_{tr}(t)}{dt}\text{ replaced by } z}=35z^2+21z^1+z^0=35z^2+21z+1$$

$$\Longrightarrow \quad z_1, z_2=\frac{-21\pm\sqrt{21^2-4(35)(1)}}{2(35)}=\frac{-21\pm 17}{2(35)}=-0.54, -0.06$$

$$\boxed{v_{tr}(t)=k_1e^{z_1t}+k_2e^{z_2t}=k_1e^{-0.54t}+k_2e^{-0.06t}, \quad t\geq 0} \quad \Longleftarrow$$



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# Complete Solution (Response)

To be determined

$$v(t)=\begin{cases}2, & t<0\\ v_{ss}(t)+v_{tr}(t), & t\geq 0\end{cases}=\begin{cases}2, & t<0\\ 11+k_1e^{-0.54t}+k_2e^{-0.06t}, & t\geq 0\end{cases}$$

$$\Longrightarrow\quad i(t)=7\frac{dv(t)}{dt}=\begin{cases}0, & t<0\\ 7(-0.54k_1e^{-0.54t}-0.06\,k_2e^{-0.06t}), & t\geq 0\end{cases}$$

To determine $k_1$ and $k_2$, voltage continuity for the capacitor and current continuity for the inductor have to be used.

The voltage across the capacitor at $t$ = 0:

$$v(0)=11+k_1+k_2=2\quad\Rightarrow\quad k_1+k_2=-9$$

The current passing through the inductor at $t$ = 0:

$$i(0)=-0.54k_1-0.06k_2=0\quad\Rightarrow\quad 0.54k_1+0.06k_2=0$$

$$k_1=9/8$$

$$k_2=-81/8$$



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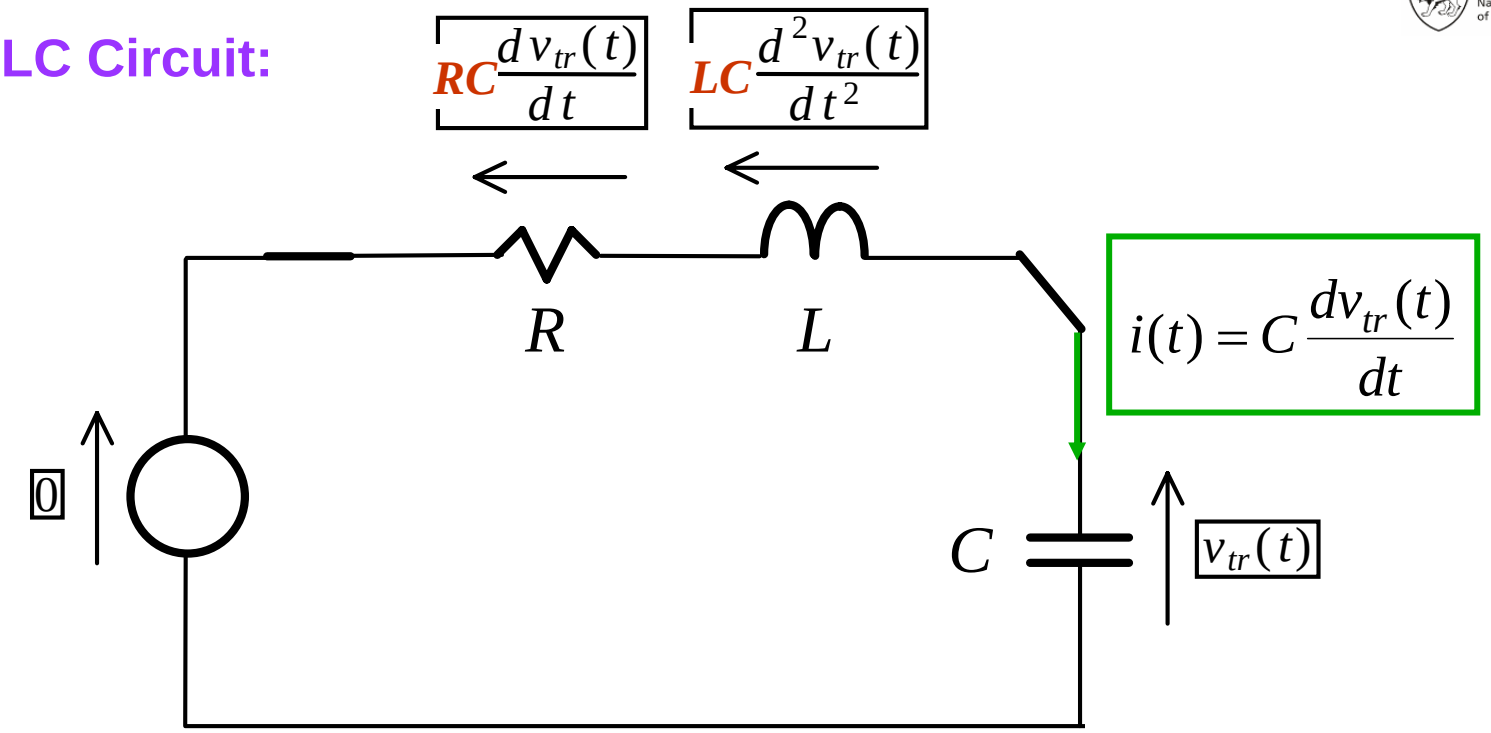
## General RLC Circuit:




**for $t \geq 0$**

**By KVL:**

$$LC\frac{d^2v_{tr}(t)}{dt^2}+RC\frac{dv_{tr}(t)}{dt}+v_{tr}(t)=0, \quad t\geq 0$$

$$LC\frac{d^2v_{tr}(t)}{dt^2}+RC\frac{dv_{tr}(t)}{dt}+v_{tr}(t)\Bigg|_{\frac{dv_{tr}(t)}{dt}\text{ replaced by } z} = LCz^2+RCz^1+1=0$$

$$\Longrightarrow \quad z_1, z_2 = \frac{-RC \pm \sqrt{(RC)^2-4LC}}{2LC}$$



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Recall that for RLC circuit, the $Q$ factor is defined as

$$Q = \frac{2\pi f_0 L}{R} = \frac{2\pi L}{R}\frac{1}{2\pi\sqrt{LC}} = \frac{L}{R\sqrt{LC}} = \frac{\sqrt{LC}}{RC}$$

Thus,

$$z_1, z_2 = \frac{-RC \pm \sqrt{(RC)^2 - 4LC}}{2LC} = \frac{-RC \pm RC\sqrt{1-4\dfrac{LC}{(RC)^2}}}{2LC} = \frac{-R \pm R\sqrt{1-4Q^2}}{2L}$$

$=$ {

two real roots if $1 - 4Q^2 > 0$ or $Q^2 < 1/4$ or $Q < 1/2$

two complex conjugate roots if $1 - 4Q^2 < 0$ or $Q > 1/2$

two identical roots if $1 - 4Q^2 = 0$ or $Q = 1/2$



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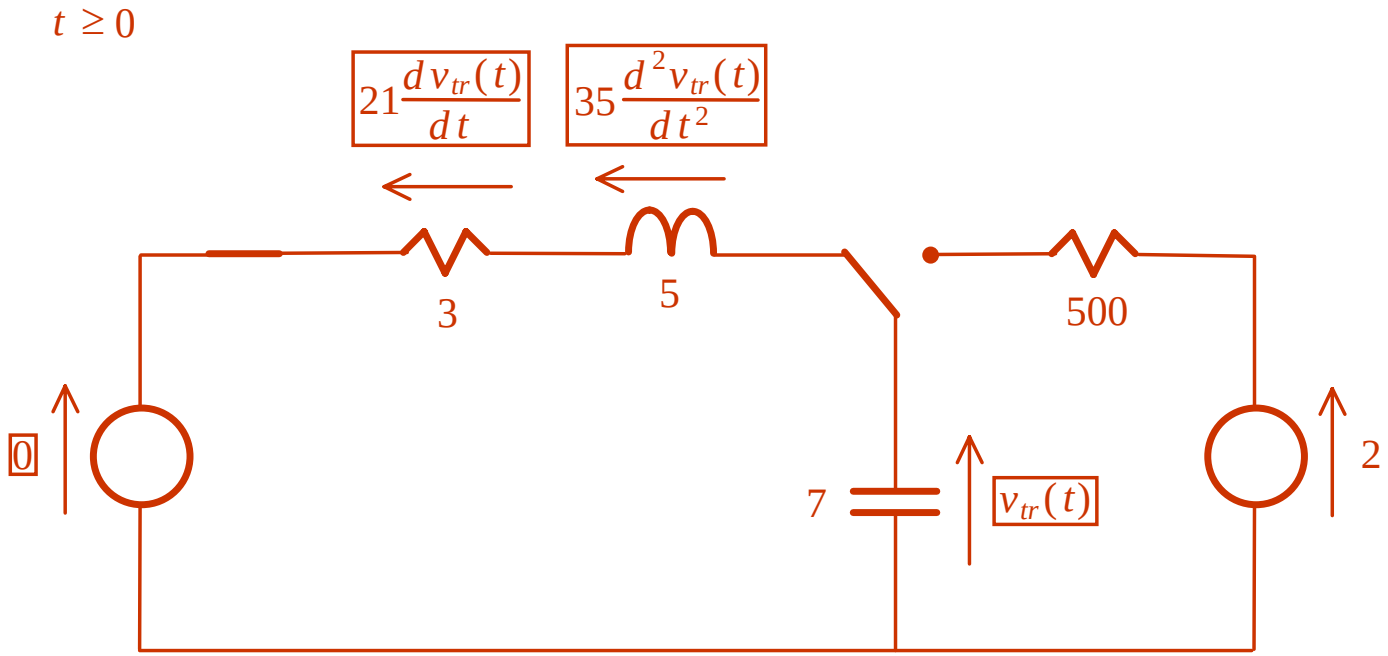
## 6.11 Overdamped Response

Reconsider the previous RLC example, i.e.,




$$35\frac{d^2v_{tr}(t)}{dt^2} + 21\frac{dv_{tr}(t)}{dt} + v_{tr}(t) = 0, \quad t \geq 0 \quad \Longrightarrow \quad Q = \frac{\sqrt{LC}}{RC} = \frac{\sqrt{35}}{21} = 0.2817 < \frac{1}{2}$$

$$z_1, z_2 = \frac{-21 \pm \sqrt{21^2 - 4(35)(1)}}{2(35)} = \frac{-21 \pm 17}{2(35)} = -0.54, -0.06$$



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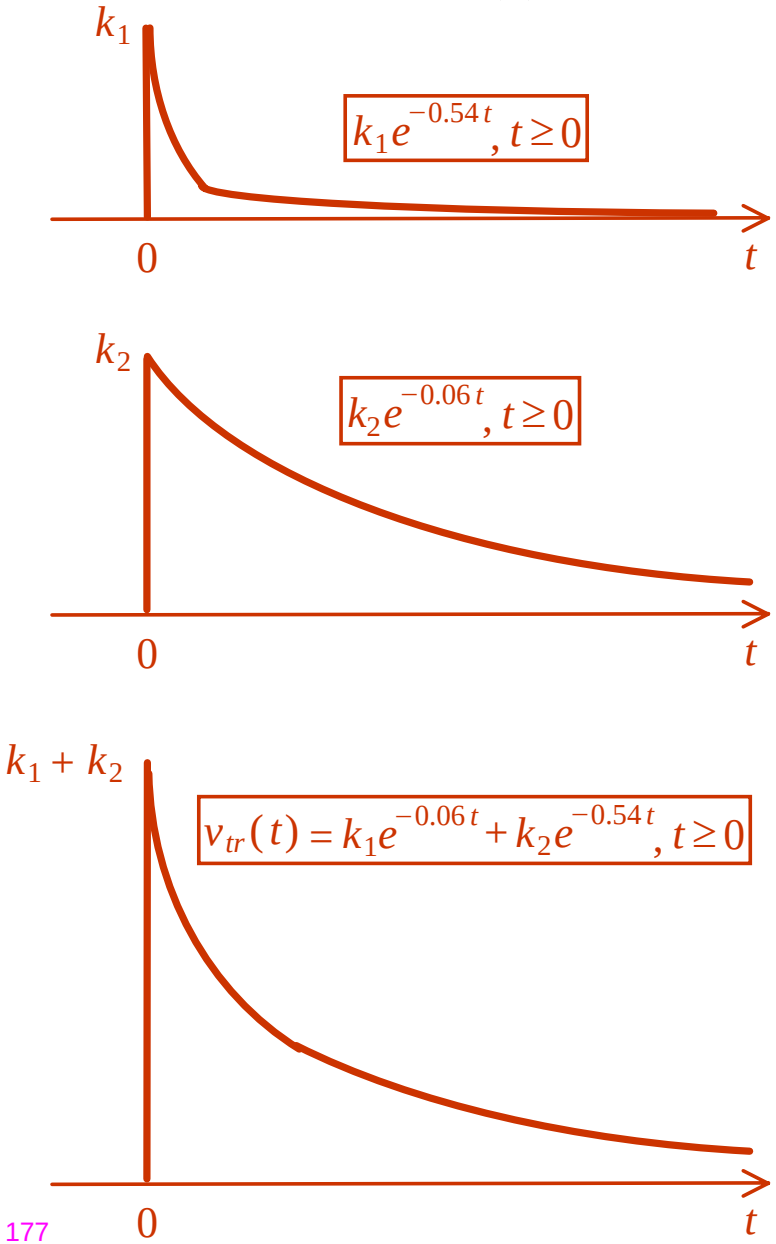
$$v_{tr}(t) = k_1 e^{z_1 t} + k_2 e^{z_2 t} = k_1 e^{-0.54t} + k_2 e^{-0.06t}, \quad t \geq 0$$




Due to its exponentially decaying nature, the response $i_{tr}(t)$ and the RLC circuit are said to be **overdamped**.

Typically, when an external input is suddenly applied to an overdamped system, the system will take a long time to move in an exponentially decaying manner to the steady state position.

The response is slow and sluggish, and the *Q* factor is small.



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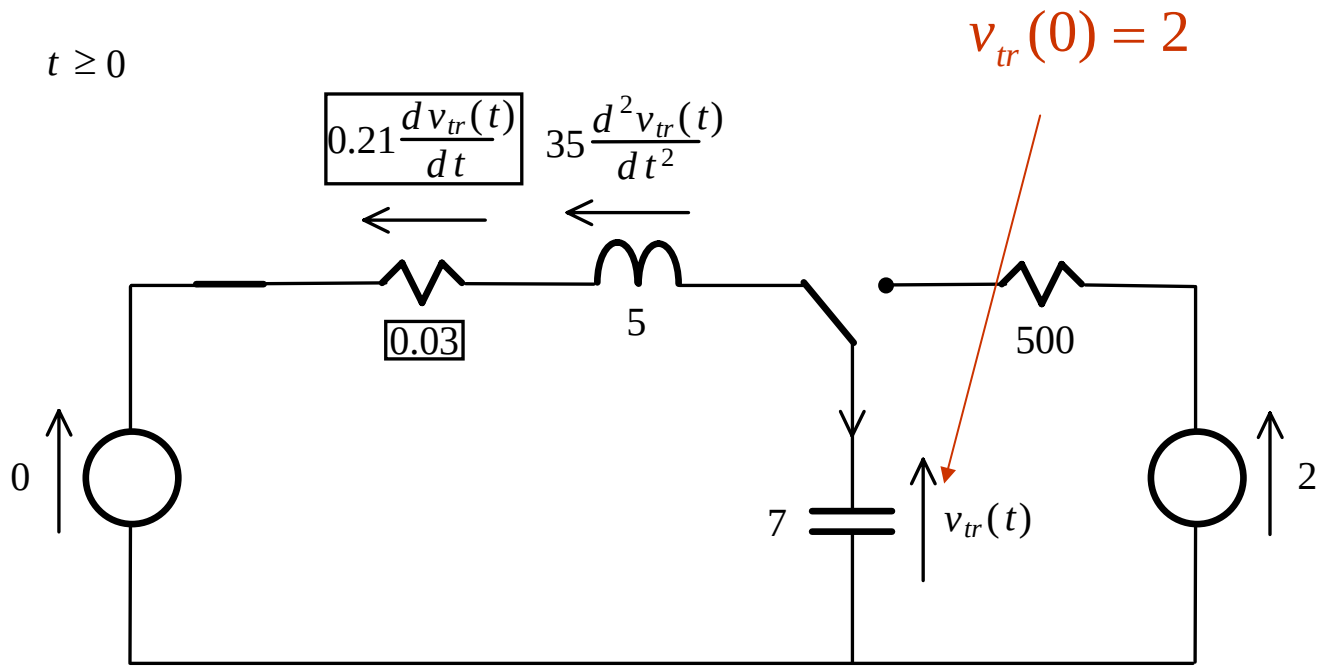
## 6.12 Underdamped Response




$$Q = \frac{\sqrt{5}}{0.03\sqrt{7}} = 28 > \frac{1}{2} \qquad 35\frac{d^2v_{tr}(t)}{dt^2} + 0.21\frac{dv_{tr}(t)}{dt} + v_{tr}(t) = 0, \quad t \geq 0$$

$$z_1, z_2 = \frac{-0.21 \pm \sqrt{0.21^2 - 4(35)(1)}}{2(35)} = \frac{-0.21 \pm \sqrt{-139.96}}{2(35)} = -0.003 \pm j0.17$$



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$$v_{tr}(t) = k_1 e^{z_1 t} + k_2 e^{z_2 t} = k_1 e^{(-0.003 + j0.17)t} + k_2 e^{(-0.003 - j0.17)t}$$

$$\Longrightarrow \quad v_{tr}(0) = k_1 + k_2 = 2$$

$$i_{tr}(t) = 7k_1(-0.003 + j0.17)e^{(-0.003 + j0.17)t} + 7k_2(-0.003 - j0.17)e^{(-0.003 - j0.17)t}$$

$$= -0.042 + j1.19(k_1 - k_2) = 0$$

$$= -0.021(k_1 + k_2) + j1.19(k_1 - k_2)$$

$$i_{tr}(0) = 7k_1(-0.003 + j0.17) + 7k_2(-0.003 - j0.17)$$

$$\Longrightarrow \quad k_1 - k_2 = -j0.0353$$



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$$v_{tr}(t) = k_1 e^{z_1 t} + k_2 e^{z_2 t} = k_1 e^{(-0.003 + j0.17)t} + k_2 e^{(-0.003 - j0.17)t}$$

$$= e^{-0.003t}\left(k_1 e^{j0.17t} + k_2 e^{-j0.17t}\right)$$

$$= e^{-0.003t}\left\{k_1[\cos(0.17t) + j\sin(0.17t)] + k_2[\cos(0.17t) - j\sin(0.17t)]\right\}$$

$$= e^{-0.003t}\left[(k_1 + k_2)\cos(0.17t) + j(k_1 - k_2)\sin(0.17t)\right]$$

$$= e^{-0.003t}\left[2\cos(0.17t) + 0.0353\sin(0.17t)\right], \quad t \geq 0$$

$$= e^{-0.003t}\sqrt{2^2 + 0.0353^2}\left[\frac{2}{\sqrt{2^2 + 0.0353^2}}\cos(0.17t) + \frac{0.0353}{\sqrt{2^2 + 0.0353^2}}\sin(0.17t)\right]$$

$$= 2e^{-0.003t}\left[\cos 1^\circ \cos(0.17t) + \sin 1^\circ \sin(0.17t)\right], \quad t \geq 0$$

$$= 2e^{-0.003t}\cos\left(0.17t - 1^\circ\right), \quad t \geq 0$$



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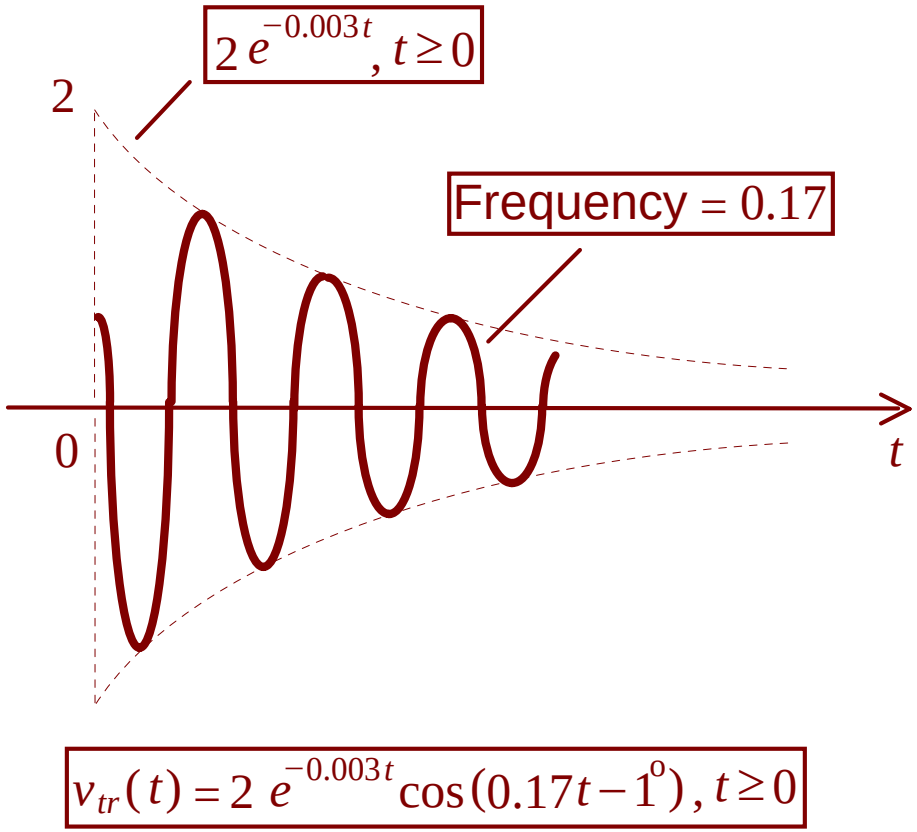


$$v_{tr}(t) = 2\,e^{-0.003t}\cos(0.17t - 1^{o}),\ t \geq 0$$

Since this is an exponentially decaying sinusoid, the response $v_{tr}(t)$ and the RLC circuit are said to be **underdamped**.

When an external input is applied to an underdamped system, the system will oscillate. The oscillation will decay exponentially but it may take some time for the system to reach its steady state position.

Underdamped systems have large *Q* factors and are used in systems such as tune circuit. However, they will be not be suitable in situations such as car suspensions or instruments with moving pointers.

It will take too long for the pointer to oscillate and settle down to its final position if the damping system for the pointer is highly underdamped in nature.



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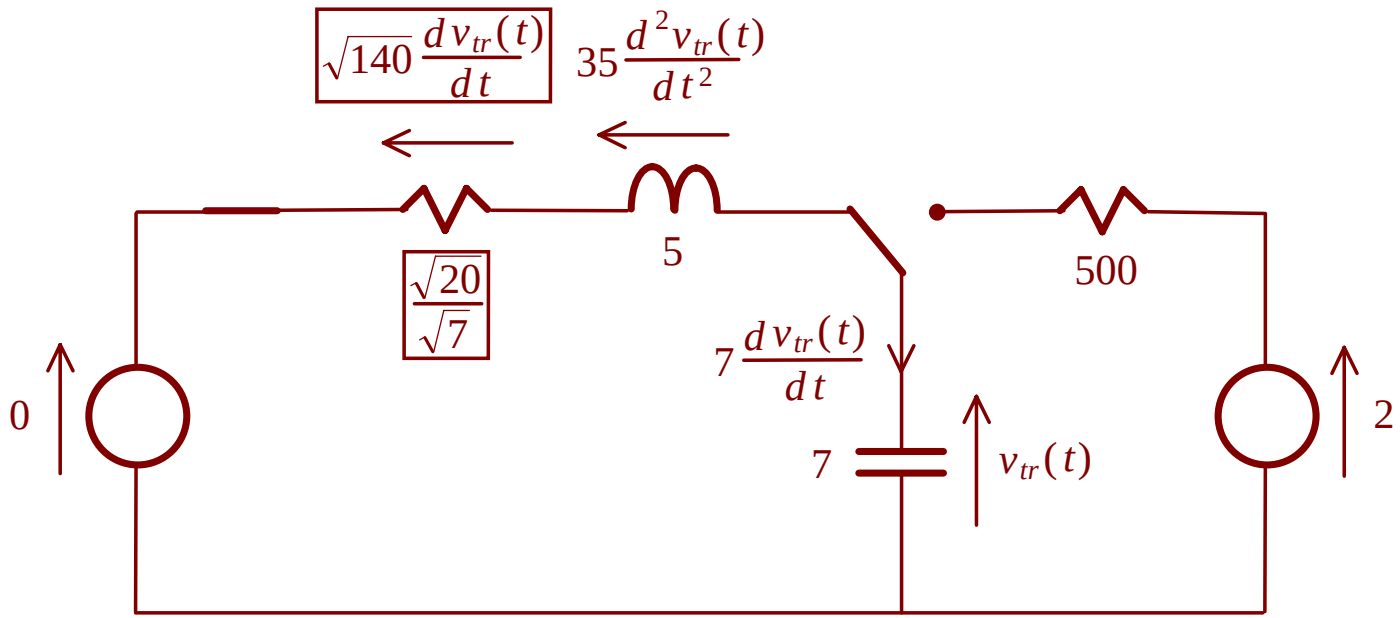
## 6.13 Critically Damped Response

$$Q = \frac{\sqrt{5}}{\sqrt{20/7}\sqrt{7}} = \frac{1}{2}$$

$t \geq 0$




$$35\frac{d^2v_{tr}(t)}{dt^2} + \sqrt{140}\frac{dv_{tr}(t)}{dt} + v_{tr}(t) = 0, \quad t \geq 0$$

$$z_1, z_2 = -\frac{1}{\sqrt{35}}$$

$$v_{tr}(t) = (k_1 + k_2 t)\,e^{z_1 t} = (k_1 + k_2 t)\,e^{-\frac{t}{\sqrt{35}}} \quad \Rightarrow \quad v_{tr}(0) = k_1 = 2$$

$$\frac{i_{tr}(0)}{7} = \frac{d}{dt}(k_1 + k_2 t)\,e^{-\frac{t}{\sqrt{35}}}\bigg|_{t=0} = \left(k_2 - \frac{k_1}{\sqrt{35}} - \frac{k_2 t}{\sqrt{35}}\right)e^{-\frac{t}{\sqrt{35}}}\bigg|_{t=0} = 0 \Rightarrow k_2 = \frac{2}{\sqrt{35}}$$



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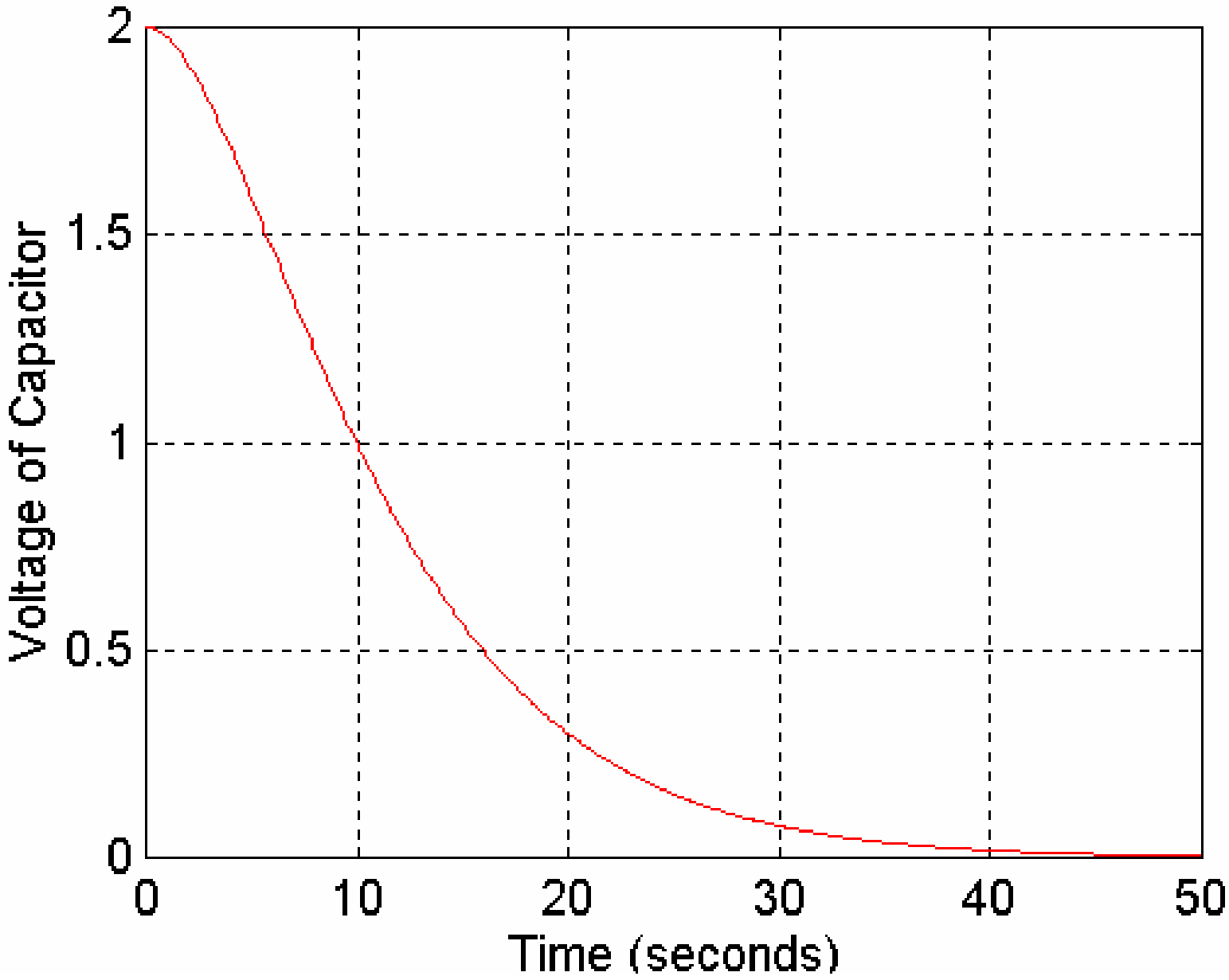




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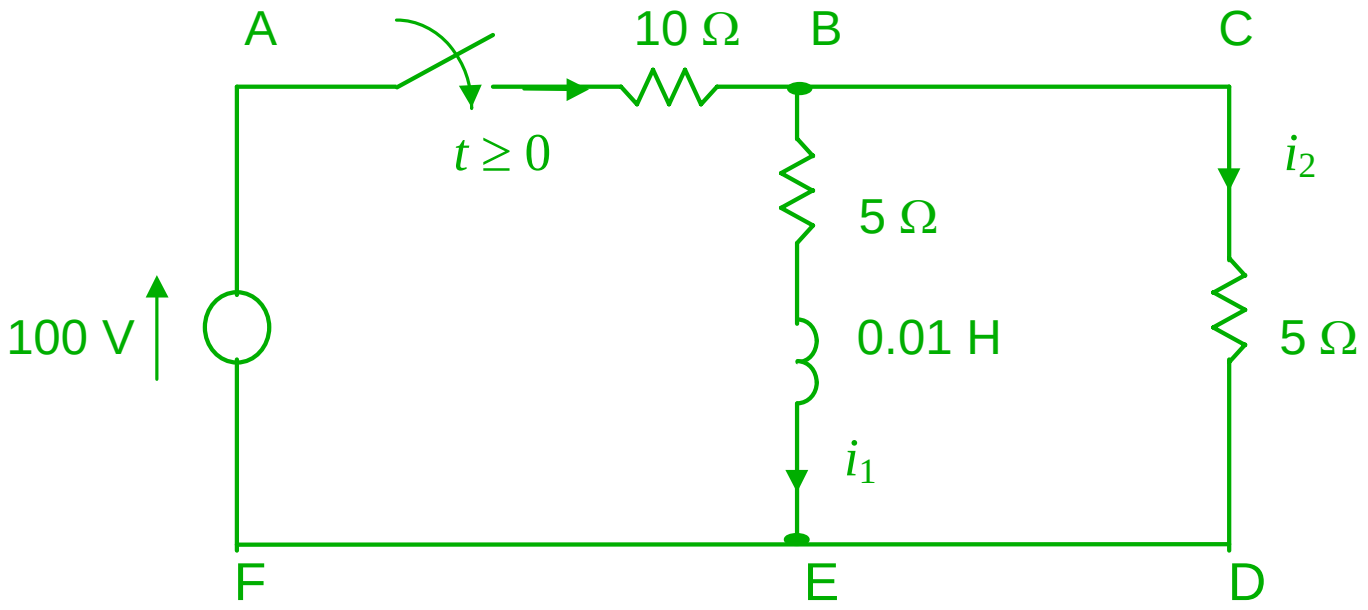
Example: The switch in the circuit shown in the following circuit is closed at time $t = 0$. Obtain the current $i_2(t)$ for $t > 0$.




After the switch is closed, the current passing through the source or the 10Ω resistor is $i_1 + i_2$. Applying the KVL to the loops, ABEFA and ABCDEFA, respectively, we obtain



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$$10(i_1 + i_2) + 5i_1 + 0.01\frac{di_1}{dt} = 100$$

$$10(i_1 + i_2) + 5i_2 = 100 \quad \Rightarrow \quad i_2 = (100 - 10i_1)/15$$

$$\Downarrow$$

$$\frac{di_1}{dt} + 833i_1 = 3333$$

$$\Rightarrow \quad i_1(\infty) = 3333/833 = 4.0\text{A} \quad \Rightarrow \quad i_1(t) = \alpha e^{-833t} + 4.0$$

$$i_1(0) = 0 \quad \Rightarrow \quad \alpha = -4.0 \quad \Rightarrow \quad i_1(t) = 4.0\left(1 - e^{-833t}\right)$$

$$\Downarrow$$

$$i_2(t) = 4.0 + 2.67e^{-833t}$$



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# Chapter 7: Magnetic Circuit



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## 7.1 Magnetic Field and Material

In electrostatic, an electric field is formed by static charges. It is described in terms of the electric field intensity. The permittivity is a measure of how easy it is for the field to be established in a medium given the same charges.

Similarly, a magnetic field is formed by moving charges or electric currents. It is described in terms of the magnetic flux density *B*, which has a unit of tesla(T) = (N/A)m. The permeability $\mu$ is a measure of how easy it is for a magnetic field to be formed in a material. The higher the $\mu$, the greater the *B* for the same currents.

In free space, $\mu$ is $\mu_0 = 4\pi \times 10^{-7}\,\mathrm{H/m}$. The relative permeability $\mu_r$ is

$$\mu_r = \frac{\mu}{\mu_0}$$

Most "non-magnetic" materials such as air and wood have $\mu_r \approx 1$. However, "magnetic materials" such as iron and steel may have $\mu_r \approx 1000$.



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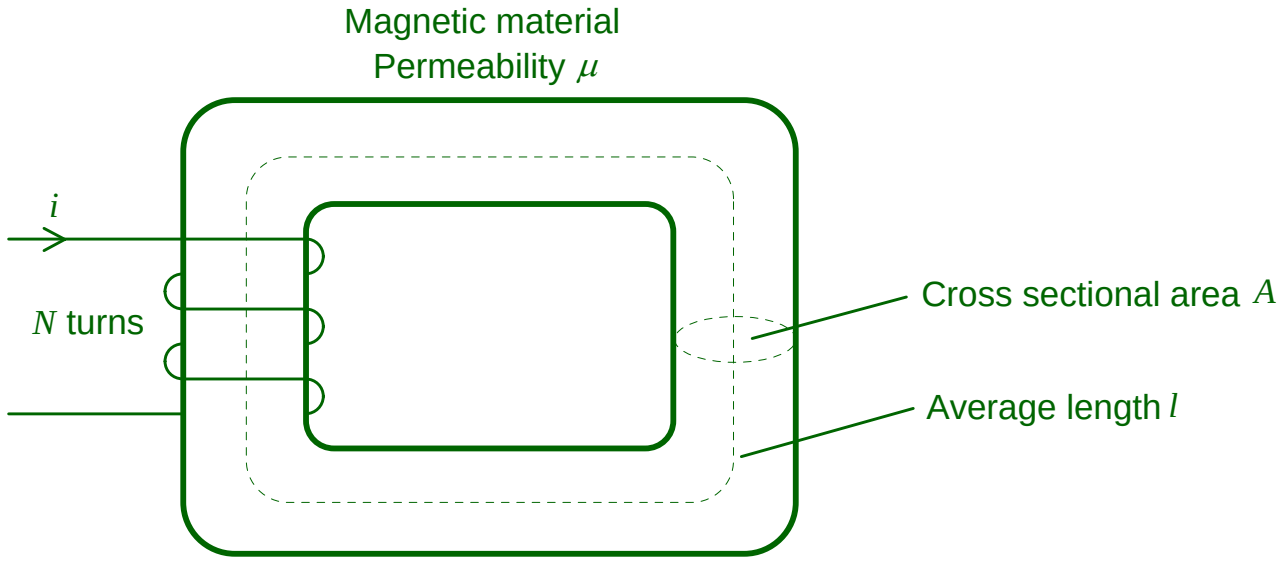
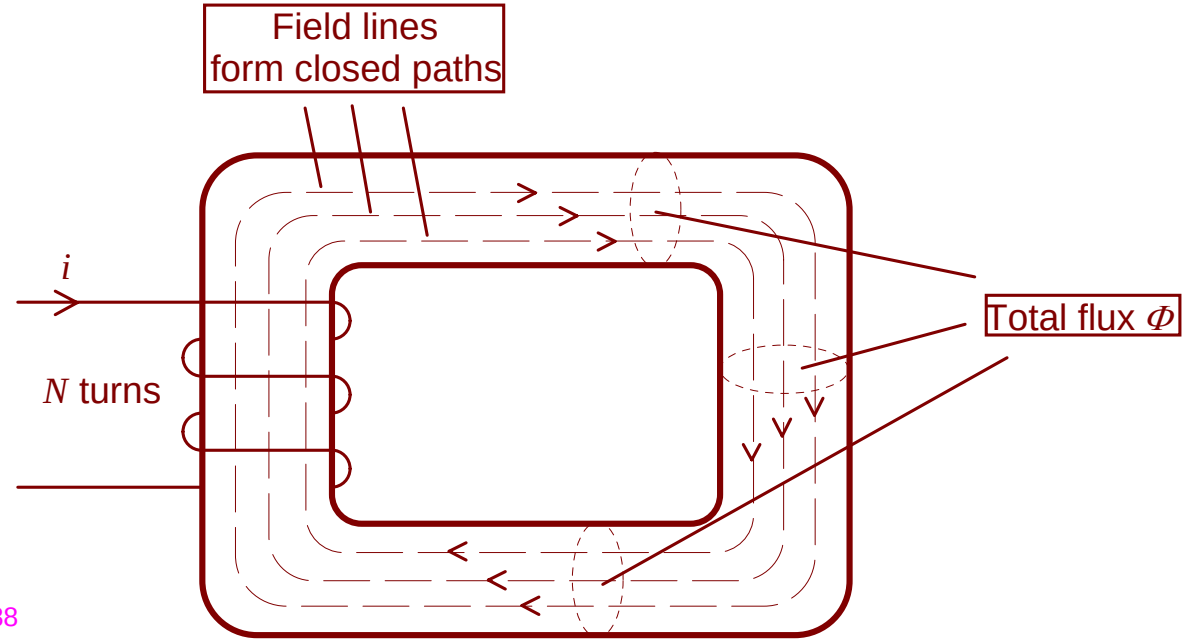
## 7.2 Magnetic Flux ---- Consider the following magnetic system:




If μ is large, almost the entire magnetic field will be concentrated inside the material and there will be no flux leakage.

The distribution of flux density *B* or field lines will be




Since the field lines form closed paths and there is no leakage, the total flux Φ passing through any cross section of the material is the same.



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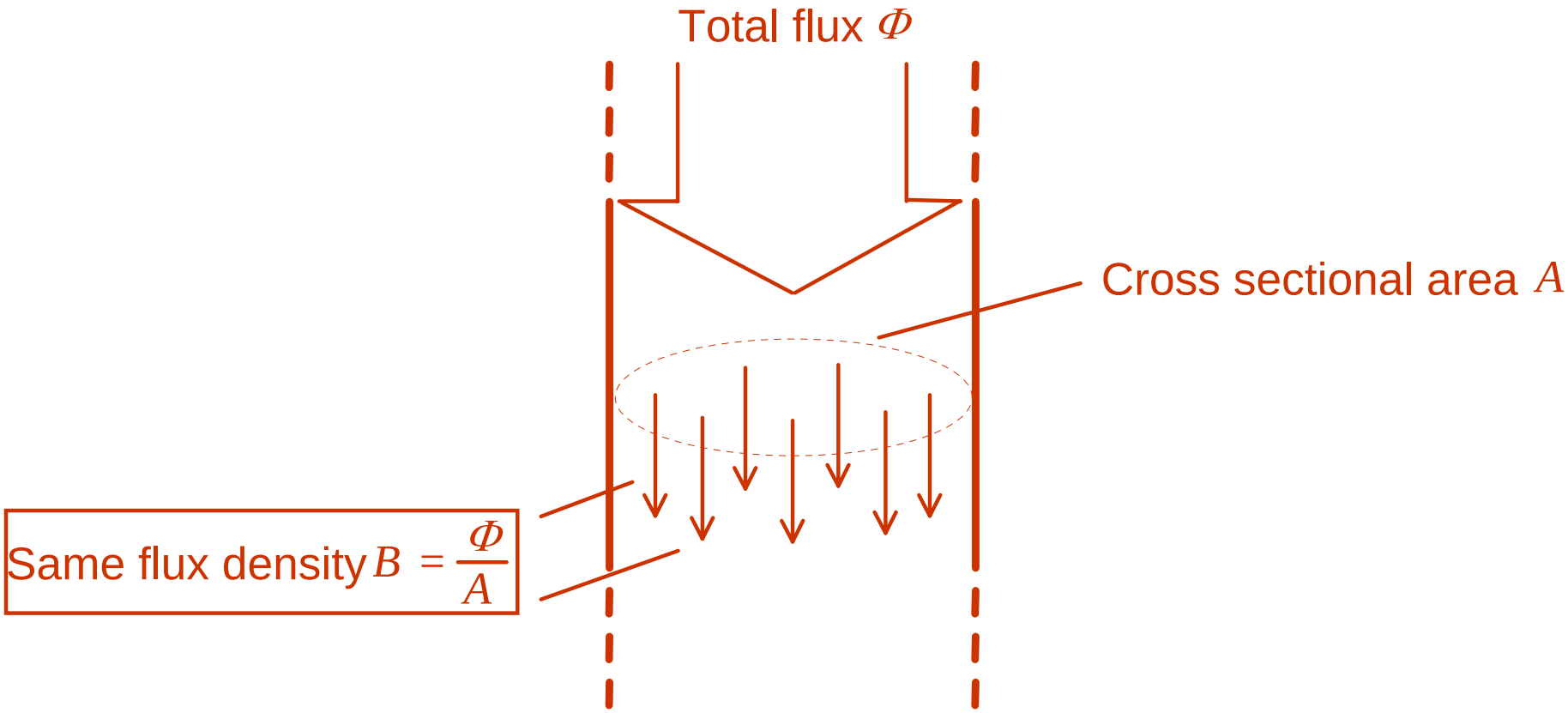
Assuming the flux to be uniformly distributed so that the flux density $B$ have the same value over the entire cross sectional area $A$:




$$B = \frac{\Phi}{A} \quad \text{with units} \quad \text{tesla (T)} = \frac{\text{weber (Wb)}}{\text{m}^2}$$



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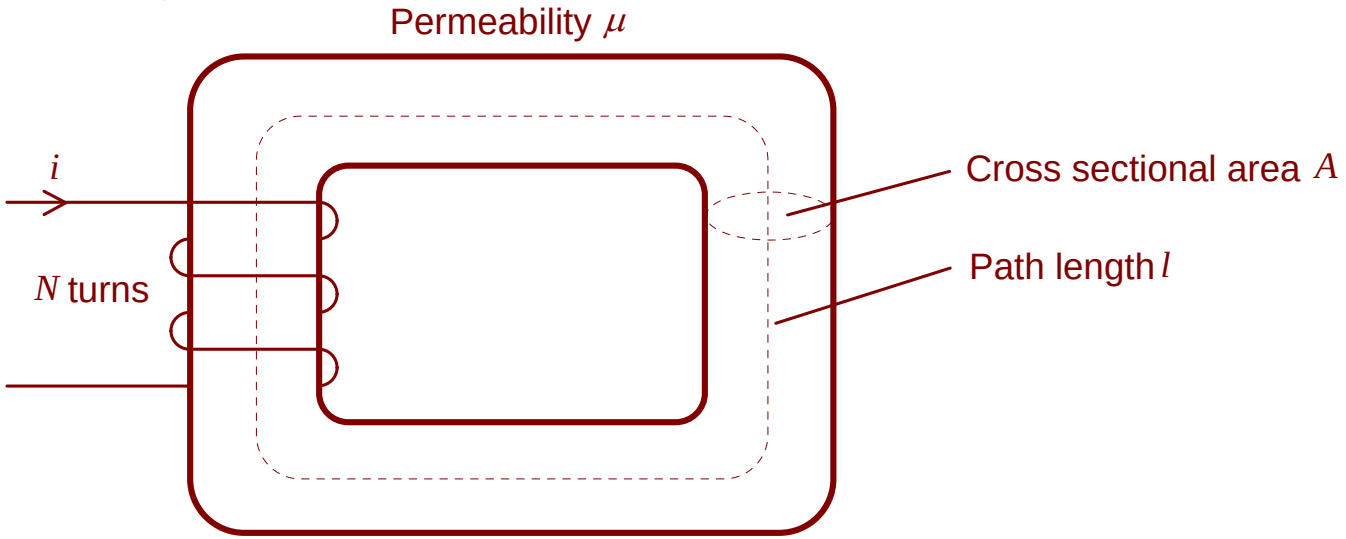
## 7.3 Ampere's Law

The values of $\Phi$ or $B$ can be calculated using **Ampere's law**:

**Line integral of $\frac{B}{\mu}$ along any closed path = current enclosed by path**




Line integral of $\frac{B}{\mu}$ along dotted path $= \frac{Bl}{\mu} = \left(\frac{l}{\mu A}\right)\Phi$

$\Longrightarrow$ $Ni = \left(\frac{l}{\mu A}\right)\Phi$

= Current enclosed by dotted path $Ni$



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Note that the ratio $H = B/\mu$ is called the magnetic field intensity and Ampere's law is usually stated in terms of $H$. Stating the law in terms of $H$ has the advantage that the effects of magnetic materials, which influence $\mu$, is not in the main equation. In a certain sense, characterizes the magnetic field due to only current distributions. By multiplying $H$ with $\mu$ to end up with the most important flux density $B$, the effect of the medium is taken into consideration.

$$Ni = \left(\frac{l}{\mu A}\right)\Phi \quad \Longrightarrow \quad H = \frac{B}{\mu} = \frac{\Phi}{\mu A} = \frac{Ni\mu A}{l\mu A} = \left(\frac{N}{l}\right)i$$



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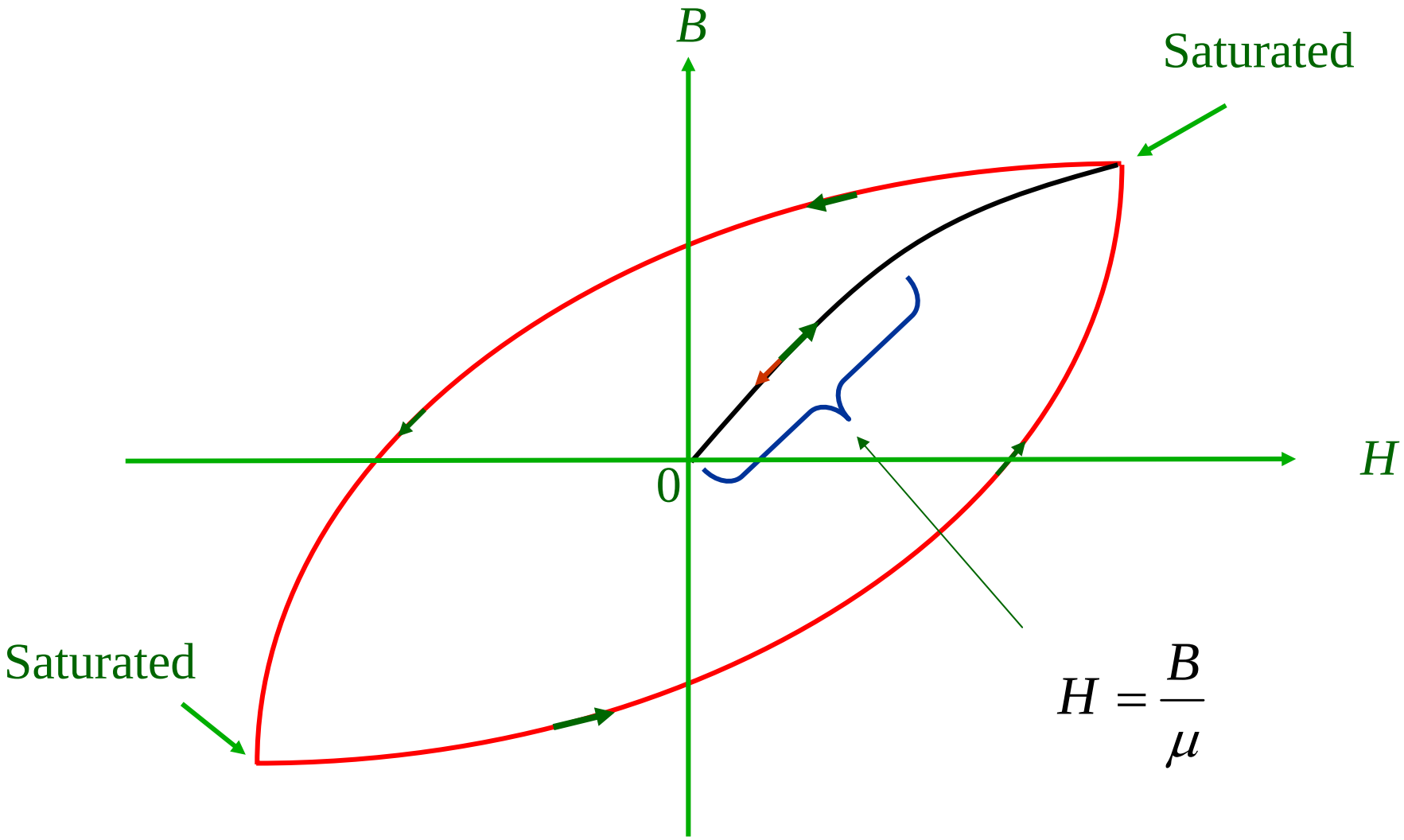


$$H = \left(\frac{N}{l}\right) i, \text{ which is propotional to the current } i.$$



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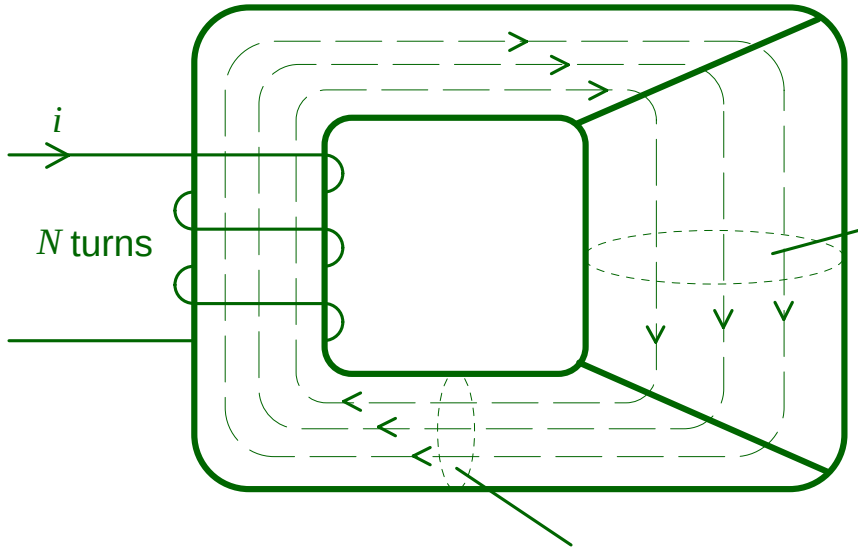
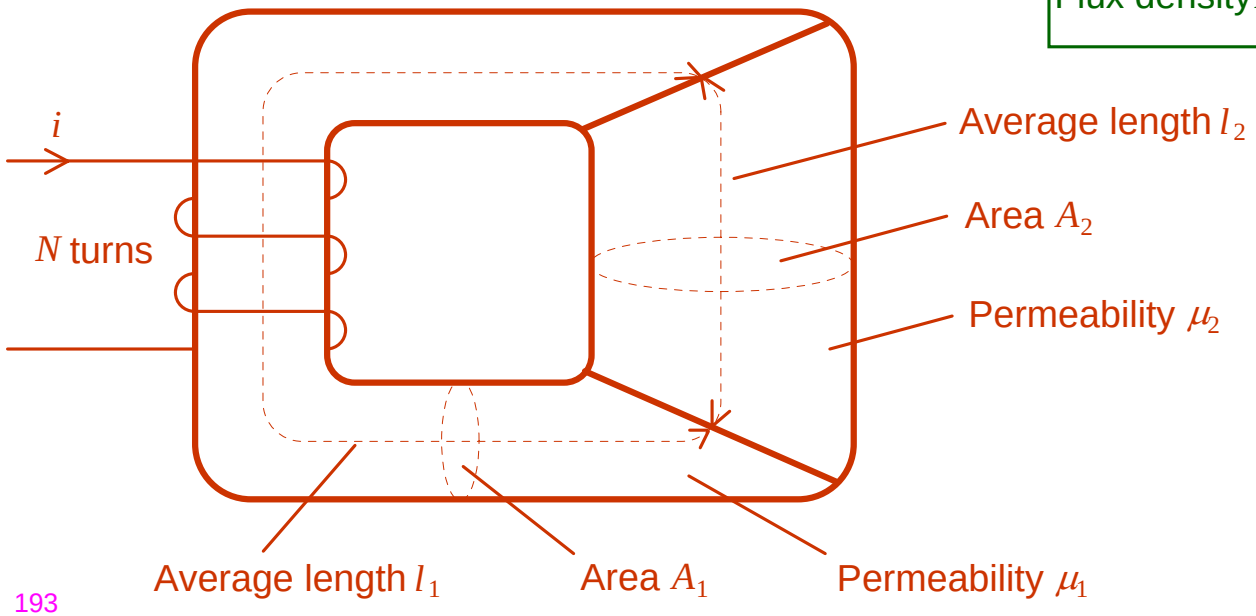
# 7.4 Magnetic and Electric Circuits --- A more complicated example:

**Assuming no flux leakage and uniform flux distribution, the field lines, total flux and flux densities are:**




**Ampere's Law:**

**Line integral of $\frac{B}{\mu}$ along any closed path = current enclosed by path**



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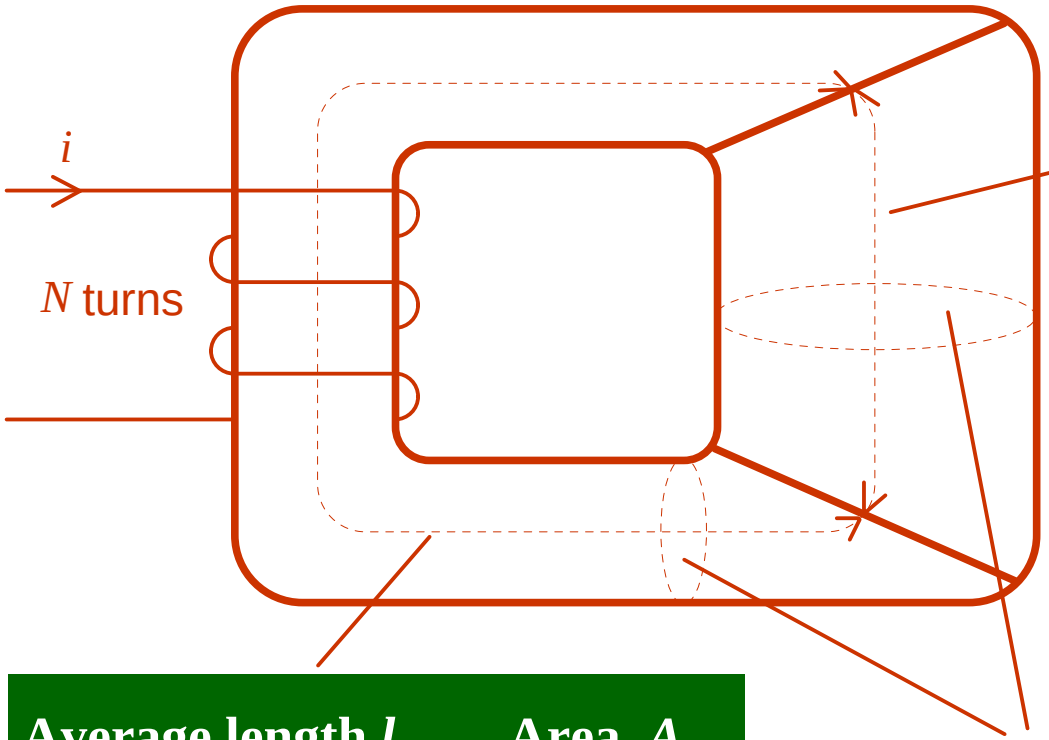


**Average length $l_2$ Area $A_2$**

**Permeability $\mu_2$ Flux density $B_2$**

**Line integral =** $\dfrac{l_2 B_2}{\mu_2} = \dfrac{l_2}{\mu_2 A_2}\Phi$

**Average length $l_1$ Area $A_1$**

**Permeability $\mu_1$ Flux density $B_1$**

**Line integral =** $\dfrac{l_1 B_1}{\mu_1} = \dfrac{l_1}{\mu_1 A_1}\Phi$

$$Ni = \left(\frac{l_1}{\mu_1 A_1} + \frac{l_2}{\mu_2 A_2}\right)\Phi$$

= Current enclosed by path = $Ni$

Line integral of $\dfrac{B}{\mu}$ along entire path $= \dfrac{B_1 l_1}{\mu_1} + \dfrac{B_2 l_2}{\mu_2} = \left(\dfrac{l_1}{\mu_1 A_1} + \dfrac{l_2}{\mu_2 A_2}\right)\Phi$



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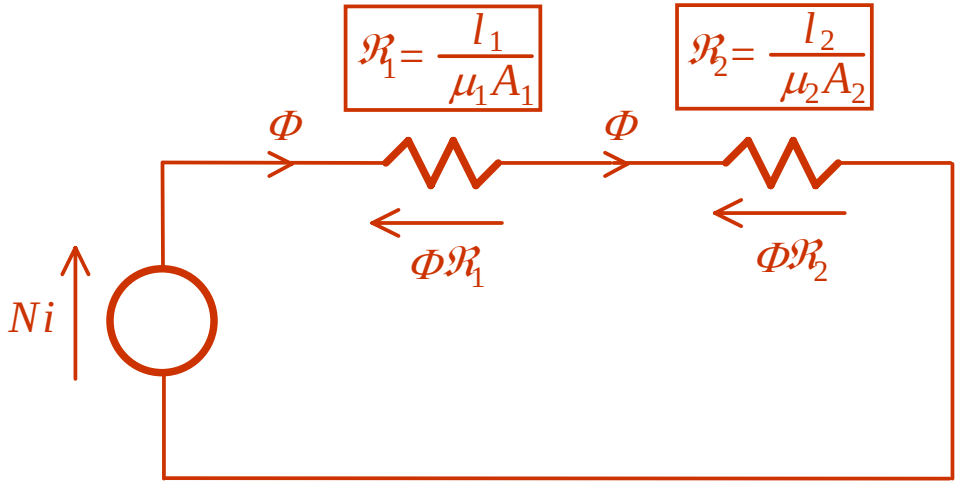
Note that the above process of calculation magnetic flux is the same as the calculation of current in the following electric circuit:




From KVL, the same equation can be obtained:

$$Ni = \Phi\mathcal{R}_1 + \Phi\mathcal{R}_2 = \left(\frac{l_1}{\mu_1 A_1} + \frac{l_2}{\mu_2 A_2}\right)\Phi$$

This is not surprising because the two basic laws in electric circuits are equivalent to the two basic laws in magnetic circuits and the following quantities are equivalent:



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| Electric circuits | Magnetic circuits |
|---|---|
| KCL:<br>total current entering a closed surface equals total current leaving the surface | Flux lines form closed path:<br>total flux entering a closed surface equals total flux leaving the surface |
| KVL:<br>sum of voltages along a closed path equals zero | Ampere's law:<br>integral or sum of $B/\mu$ ("magnetic voltage drops") along a closed path equals currents enclosed ("magnetic voltage sources") |
| Voltage | $Ni$ (***magnetomotive force or mmf***) |
| Current | $\Phi$ (flux) |
| Resistance | $\mathcal{R} = \dfrac{l_1}{\mu_1 A_1}$ (***reluctance***) |



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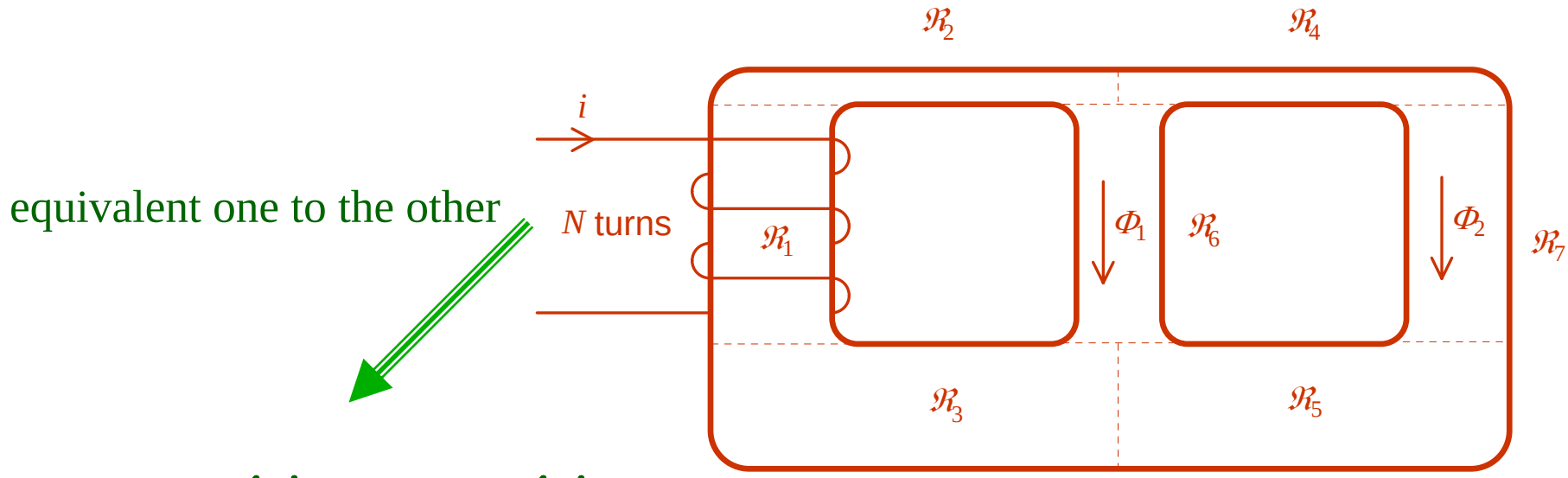
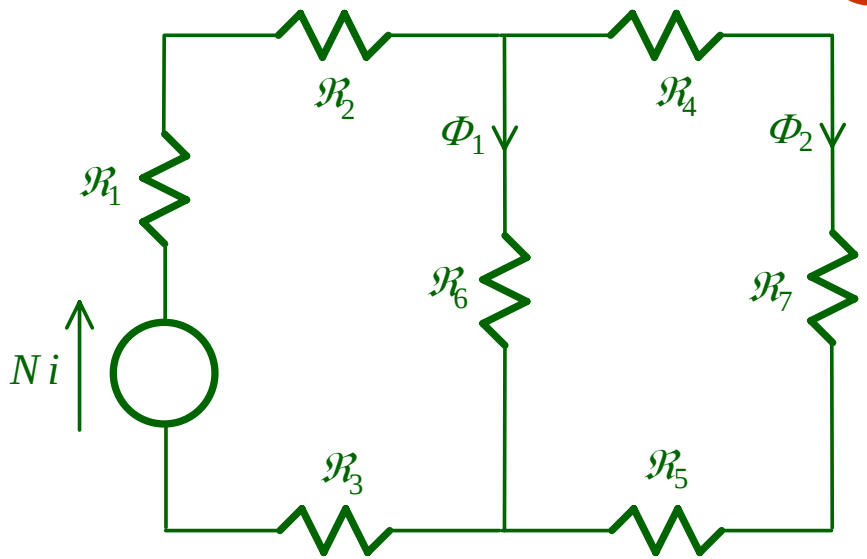
Thus, provided there is no flux leakage and uniform distribution of flux across any cross section, the parallel magnetic circuit with reluctances as indicated:




equivalent one to the other




**You can use the DC circuit techniques to solve this problem.**



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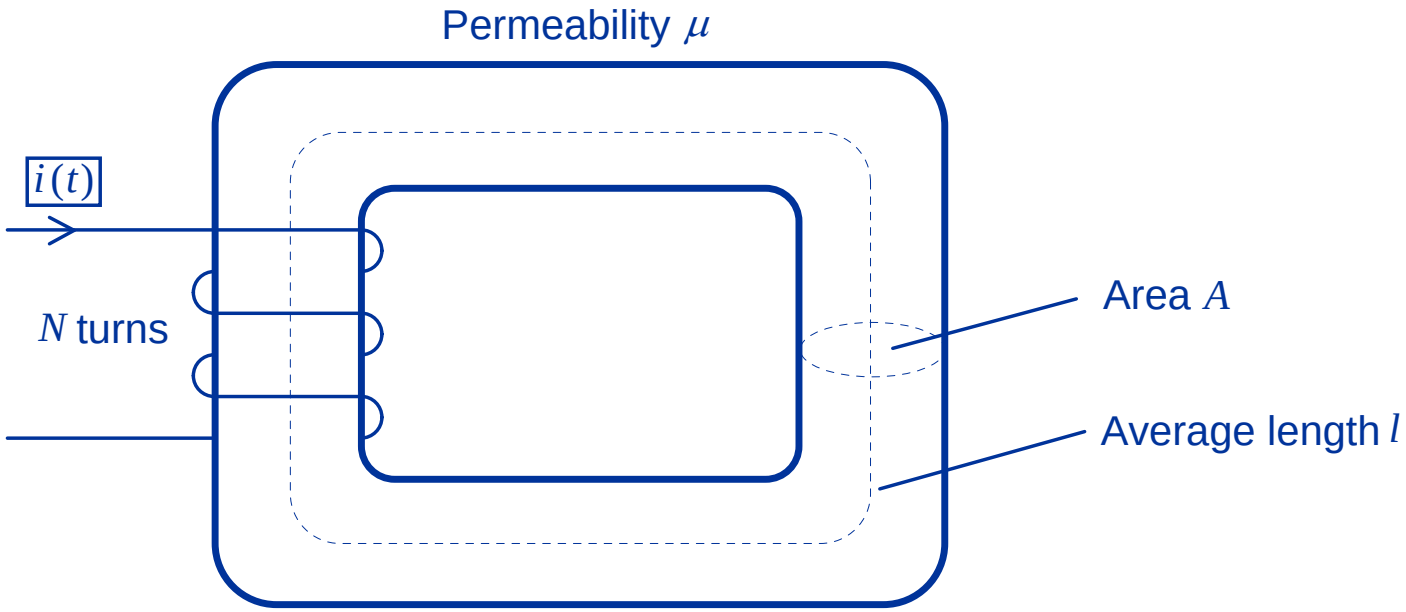
## 7.5 Inductance --- Consider the magnetic circuit:




Assuming no flux leakage and uniform flux distribution, the reluctance and the flux linking or enclosed by the winding is

$$\Re = \frac{l}{\mu A} \qquad \text{and} \qquad \Phi(t) = \frac{\text{mmf}}{\text{reluctance}} = \frac{Ni(t)}{\Re}$$



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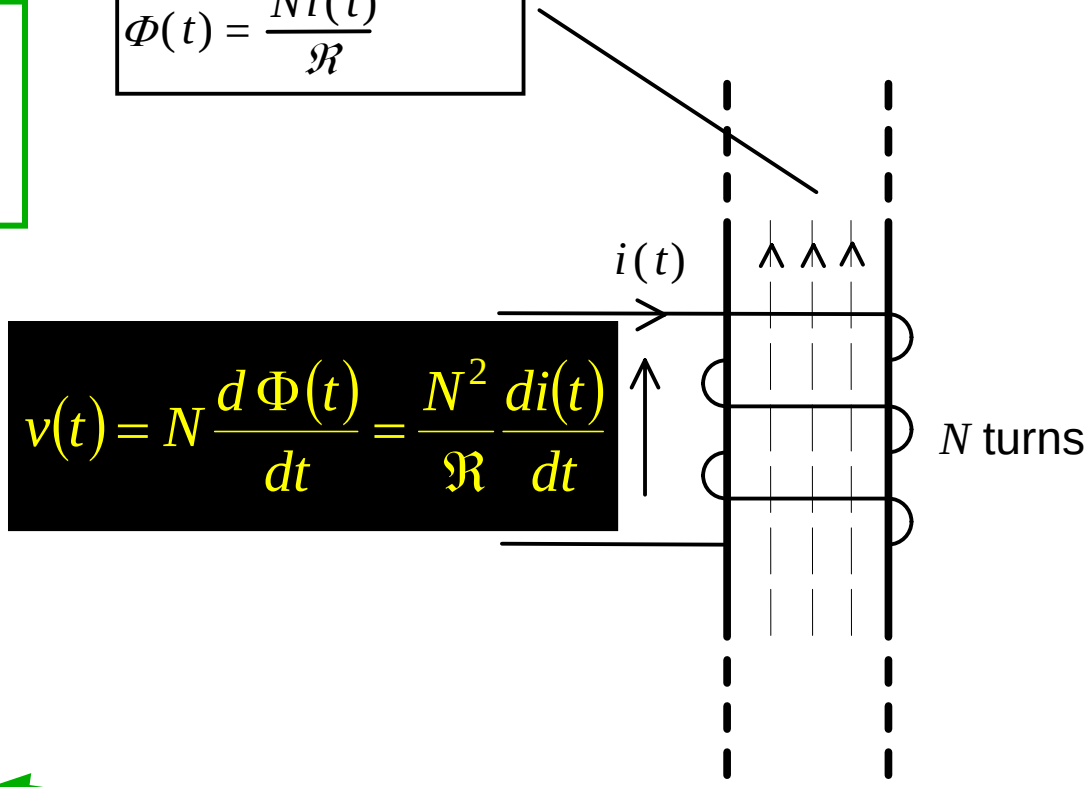
From **Faraday's law of induction**, a voltage will be induced in the winding if the flux linking the winding changes as a function of time. This induced voltage, called the **back emf (electromotive force)** will attempt to oppose the change and is given by

$$v(t) = N\frac{d\Phi(t)}{dt} = \frac{N^2}{\Re}\frac{di(t)}{dt}$$

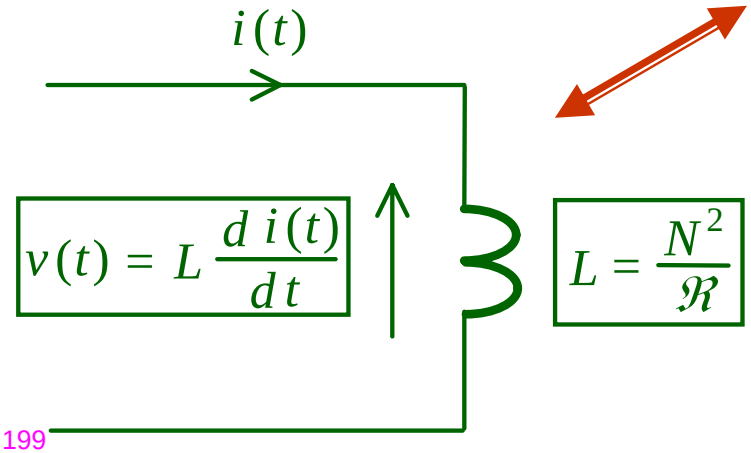
Flux linking winding

$$\Phi(t) = \frac{Ni(t)}{\Re}$$




$$v(t) = N\frac{d\Phi(t)}{dt} = \frac{N^2}{\Re}\frac{di(t)}{dt}$$

**An equivalent inductor:**




**The inductance**



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The following summarize the main features of an ideal magnetic system with no flux leakage:

| | |
|---|---|
| Number of turns | $N$ |
| Current at time $t$ | $i(t)$ |
| Reluctance | $\Re$ |
| MMF at time $t$ | $Ni(t)$ |
| Flux at time $t$ | $\Phi(t) = \dfrac{Ni(t)}{\Re}$ |
| Back emf at time $t$ | $v(t) = N\dfrac{d\,\Phi(t)}{dt}$ |
| Inductance | $L = \dfrac{N^2}{\Re}$ |
| Energy stored in magnetic field at time $t$ | $e(t) = \dfrac{Li^2(t)}{2} = \dfrac{[Ni(t)]^2}{2\,\Re} = \dfrac{\Re\,\Phi^2(t)}{2}$ |



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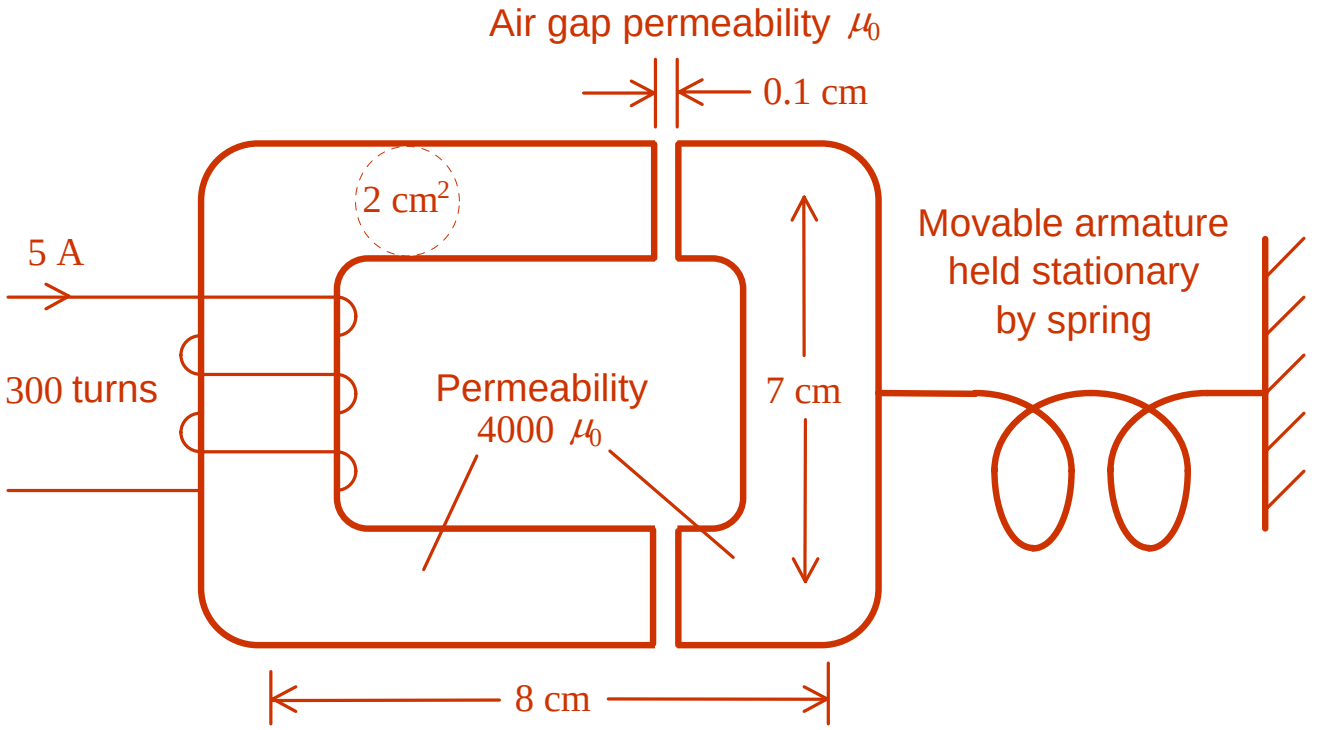
## 7.6 Force --- Consider the magnetic relay:




**With no flux leakage and uniform flux distribution (even in air gaps)**

Reluctance of entire magnetic material

$$= \frac{2(7\text{cm}+8-0.1\text{cm})}{4000\mu_0(2\text{cm}^2)} = \frac{0.298\text{m}}{4000\mu_0(2\times10^{-4}\text{m}^2)} = \frac{2.98}{8\mu_0}$$

Total reluctance $= \Re = \frac{3}{8\mu_0} + \frac{10}{\mu_0} = \frac{10.375}{\mu_0}$

Reluctance of two air gaps

$$= \frac{2(0.1\text{cm})}{\mu_0(2\text{cm}^2)} = \frac{0.1}{\mu_0(1\text{cm})} = \frac{10}{\mu_0}$$

Flux

$$\Phi = \frac{\text{mmf}}{\Re} = \frac{300(5)}{10.375/\mu_0}$$

$$= 145(4\pi\times10^{-7}) = 0.182\times10^{-3}\text{Wb}$$



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Due to the much smaller permeability, the reluctance of the air gaps is much larger than that of the entire magnetic material. The inductance and energy stored in the system are

$$\text{Inductance:} \qquad L = \frac{(\text{no. of turns})^2}{\Re} = \frac{(300)^2}{10.375/\mu_0} = 10.9\text{mH}$$

$$\text{Energy stored in magnetic field} = \frac{\Re\Phi^2}{2} = \frac{10.375\left(0.182\times10^{-3}\right)^2}{2\mu_0} = 0.137\text{J}$$

To determine the force of attraction $f$ on the armature, suppose the armature moves in the direction of $f$ by $\delta l$ so that the total reluctance changes by $\delta\Re$. Also, suppose the current is changed by $\delta i$ but the flux is not changed:



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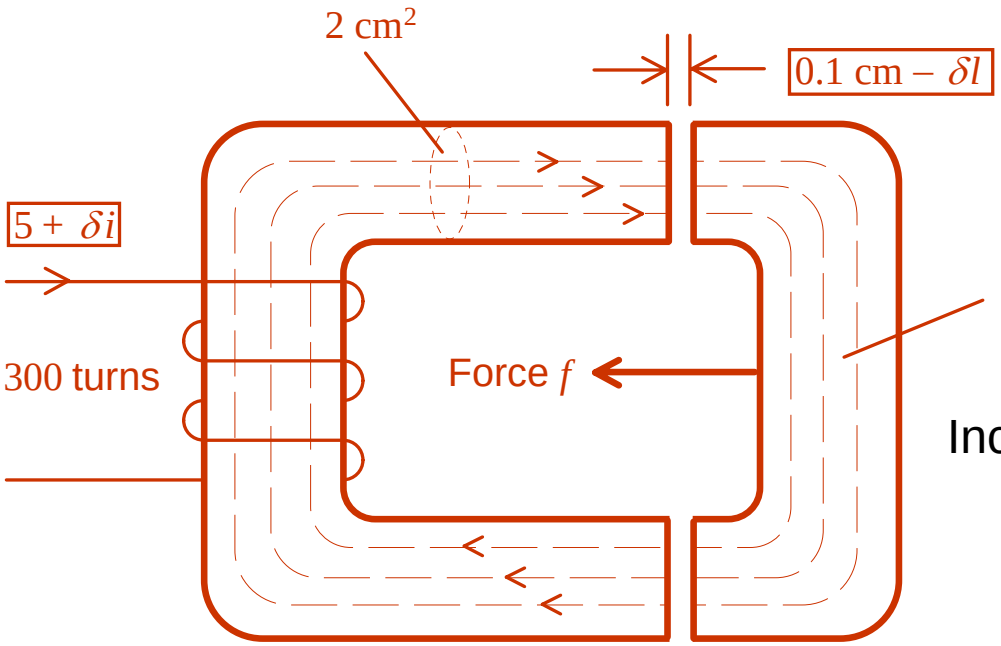


Work done by armature = $f\,(\delta l)$

Increase in energy stored in magnetic field

$$= \frac{(\Re + \delta\Re)\Phi^2}{2} - \frac{\Re\Phi^2}{2} = \frac{\delta\Re\,\Phi^2}{2}$$

From energy conservation:

$$f(\delta l) = -\frac{\delta\Re\,\Phi^2}{2} \Rightarrow f = -\left(\frac{\Phi^2}{2}\right)\frac{\delta\Re}{\delta l}$$

For our example

$$\delta\Re = 2\left[\frac{0.1\text{cm}-\delta l}{\mu_0(2\text{cm}^2)} - \frac{0.1\text{cm}}{\mu_0(2\text{cm}^2)}\right] = -\frac{10^4\delta l}{\mu_0}$$

As there is no change in flux linkage (which will be the case if the magnetic system is close to saturation), there is no back emf and there is no energy supplied by the electrical system. Then

$$f = -\left(\frac{\Phi^2}{2}\right)\frac{\delta\Re}{\delta l} = \frac{10^4\Phi^2}{2\mu_0} = \frac{10^4\left(0.182\times10^{-3}\right)^2}{2\left(4\pi\times10^{-7}\right)} = 132\text{N}$$



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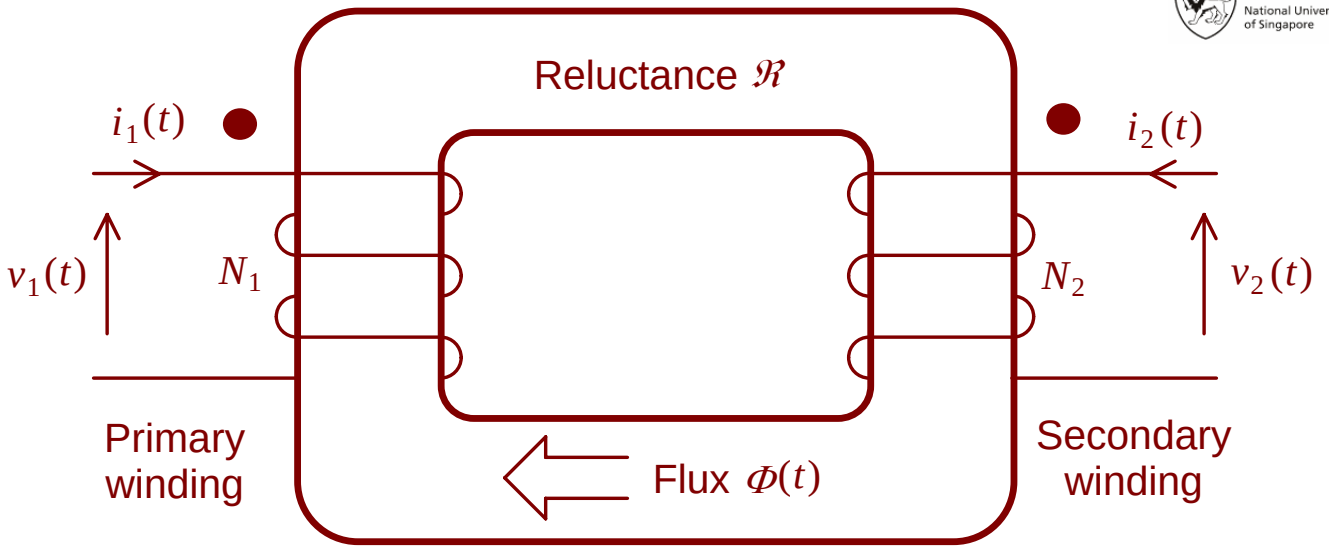
## 7.7 Mutual Inductor

The two dots are associated with the directions of the windings. The fields produced by the two windings will be constructive If the currents going into the dots have the same sign.




$$\text{Total mmf} = N_1 i_1(t) + N_2 i_2(t)$$

$$\text{Total flux} = \Phi(t) = \frac{\text{total mmf}}{\Re} = \frac{N_1 i_1(t) + N_2 i_2(t)}{\Re}$$

$$v_1(t) = N_1 \frac{d\Phi(t)}{dt} = \left(\frac{N_1^2}{\Re}\right)\frac{di_1(t)}{dt} + \left(\frac{N_1 N_2}{\Re}\right)\frac{di_2(t)}{dt}$$

$$v_2(t) = N_2 \frac{d\Phi(t)}{dt} = \left(\frac{N_1 N_2}{\Re}\right)\frac{di_1(t)}{dt} + \left(\frac{N_2^2}{\Re}\right)\frac{di_2(t)}{dt}$$

$$\frac{N_1 N_2}{\Re} = \sqrt{\frac{N_1^2}{\Re} \bullet \frac{N_2^2}{\Re}}$$



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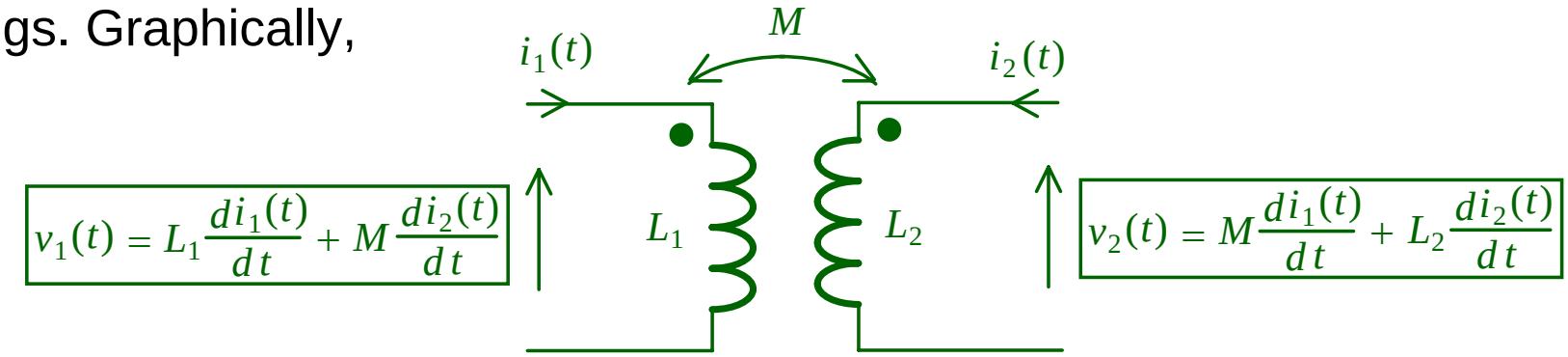
Inductance of primary winding on its own $= L_1 = \dfrac{N_1^{\,2}}{\mathfrak{R}}$

Inductance of secondary winding on its own $= L_2 = \dfrac{N_2^{\,2}}{\mathfrak{R}}$

$$\Longrightarrow \quad v_1(t) = L_1 \frac{di_1(t)}{dt} + \sqrt{L_1 L_2}\,\frac{di_2(t)}{dt} = L_1 \frac{di_1(t)}{dt} + M \frac{di_2(t)}{dt}$$

$$v_2(t) = \sqrt{L_1 L_2}\,\frac{di_1(t)}{dt} + L_2 \frac{di_2(t)}{dt} = M \frac{di_1(t)}{dt} + L_2 \frac{di_2(t)}{dt}$$

where $M = \sqrt{L_1 L_2}$ is called the **mutual inductance** between the two windings. Graphically,




$L_1 = \dfrac{N_1^{\,2}}{\mathfrak{R}}$, $L_2 = \dfrac{N_2^{\,2}}{\mathfrak{R}}$, $M^2 = L_1 L_2$ for no flux leakage and perfect coupling



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In an ac environment when the currents $i_1(t)$ and $i_2(t)$ are given by:

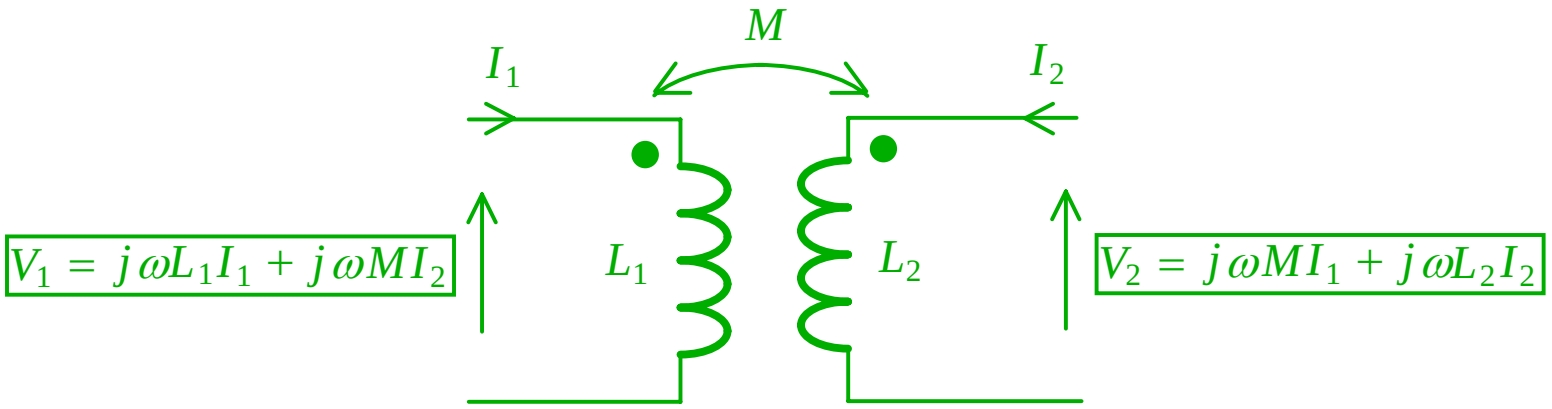
$$i_1(t) = |I_1|\sqrt{2}\cos[\omega t + \text{Arg}(I_1)] = \text{Re}\left[I_1\left(\sqrt{2}e^{j\omega t}\right)\right] \qquad i_2(t) = \text{Re}\left[I_2\left(\sqrt{2}e^{j\omega t}\right)\right]$$

$$v_1(t) = L_1\frac{di_1(t)}{dt} + M\frac{di_2(t)}{dt} = L_1\frac{d\,\text{Re}\left[I_1\left(\sqrt{2}e^{j\omega t}\right)\right]}{dt} + M\frac{d\,\text{Re}\left[I_2\left(\sqrt{2}e^{j\omega t}\right)\right]}{dt}$$

$$= L_1\,\text{Re}\left[j\omega I_1\left(\sqrt{2}e^{j\omega t}\right)\right] + M\,\text{Re}\left[j\omega I_2\left(\sqrt{2}e^{j\omega t}\right)\right]$$

$$= \text{Re}\left[\left(j\omega L_1 I_1 + j\omega M I_2\right)\left(\sqrt{2}e^{j\omega t}\right)\right] \Longrightarrow V_1 = j\omega L_1 I_1 + j\omega M I_2$$

$$v_2(t) = \text{Re}\left[\left(j\omega M I_1 + j\omega L_2 I_2\right)\left(\sqrt{2}e^{j\omega t}\right)\right] \Longrightarrow V_2 = j\omega M I_1 + j\omega L_2 I_2$$



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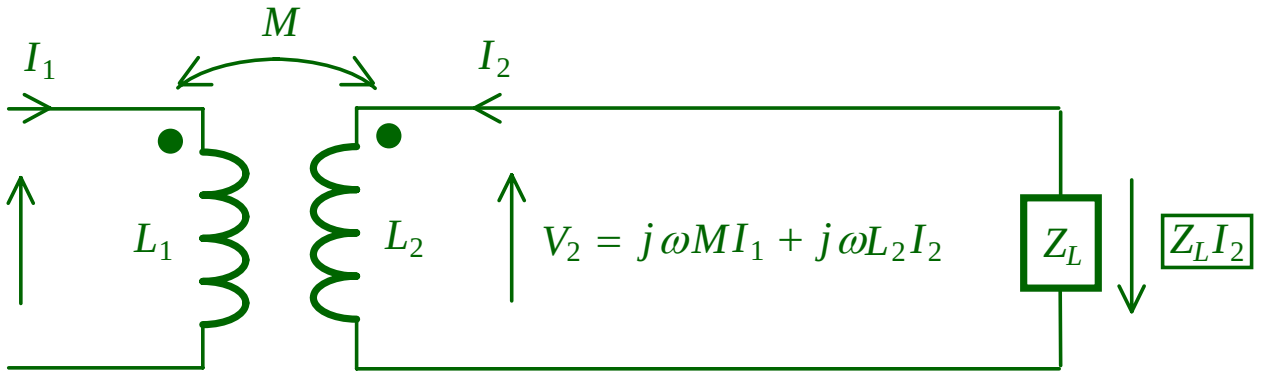
Now consider connecting a mutual inductor to a load with impedance $Z_L$

$$L_1 = \frac{N_1^2}{\Re}$$

$$V_1 = j\omega L_1 I_1 + j\omega M I_2$$

$$L_2 = \frac{N_2^2}{\Re}$$




$$M = \sqrt{L_1 L_2}$$

$$V_2 = -Z_L I_2 = j\omega M I_1 + j\omega L_2 I_2$$

$$\frac{V_2}{V_1} = \frac{MI_1 + L_2 I_2}{L_1 I_1 + MI_2} = \frac{\sqrt{L_1 L_2} I_1 + L_2 I_2}{L_1 I_1 + \sqrt{L_1 L_2} I_2}$$

$$= \frac{\sqrt{L_2}\left(\sqrt{L_1} I_1 + \sqrt{L_2} I_2\right)}{\sqrt{L_1}\left(\sqrt{L_1} I_1 + \sqrt{L_2} I_2\right)}$$

$$= \sqrt{\frac{L_2}{L_1}} = \sqrt{\frac{N_2^2/\Re}{N_1^2/\Re}} = \frac{N_2}{N_1} = n, \text{ turn ratio}$$

$$\Rightarrow \quad -j\omega M I_1 = (j\omega L_2 + Z_L) I_2$$

$$\Rightarrow \quad \frac{I_1}{I_2} = -\frac{j\omega L_2 + Z_L}{j\omega M} \approx -\frac{j\omega L_2}{j\omega M}$$

$$= -\frac{L_2}{\sqrt{L_1 L_2}} = -\sqrt{\frac{L_2}{L_1}} = -n$$

if $\quad |\, j\omega L_2 \,| >> Z_L$



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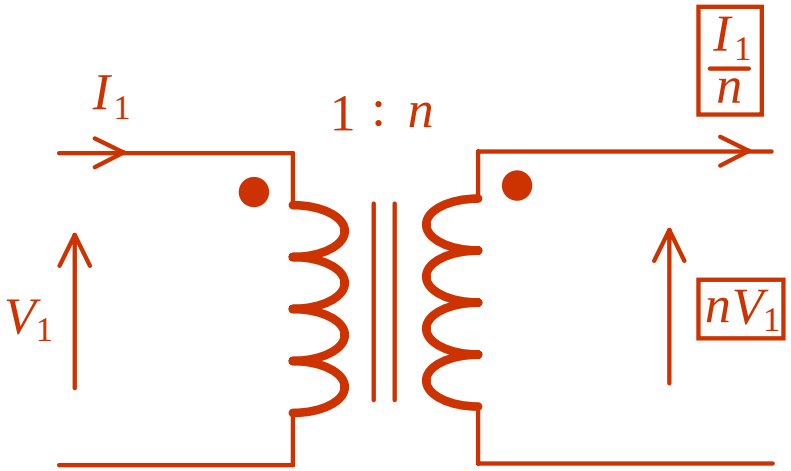
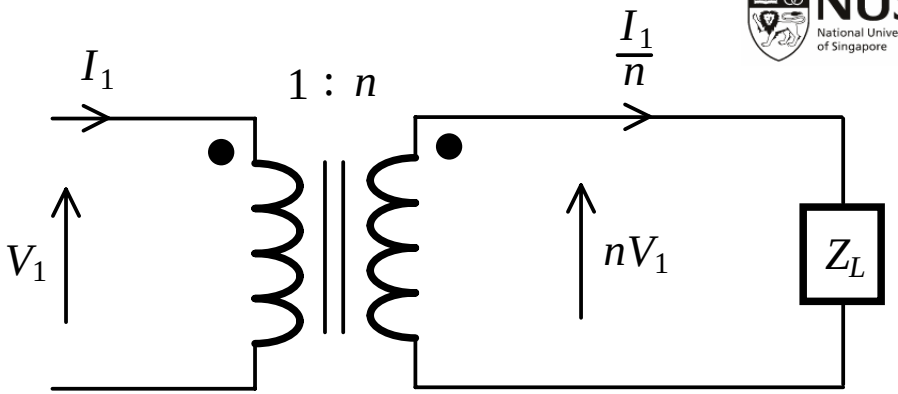
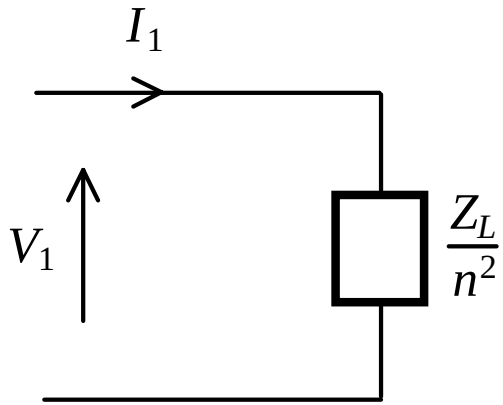
Voltages and currents of the primary and secondary windings of the ideal transformer with $|j\omega L_2| >> Z_L$




$$nV_1 = \frac{I_1}{n} Z_L \quad \Rightarrow \quad \frac{V_1}{I_1} = \frac{Z_L}{n^2}$$

**Equivalent Load:** A load connected to the secondary of a transformer can be replaced by an equivalent load directly connected to the primary



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# Annex G.7. A Past Year Exam Paper

Appendix C.4 will be attached to this year's paper!



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Q.1 (a) Using nodal analysis, derive (but DO NOT simplify or solve) the equations for determining the nodal voltages in the circuit of Fig. 1(a).

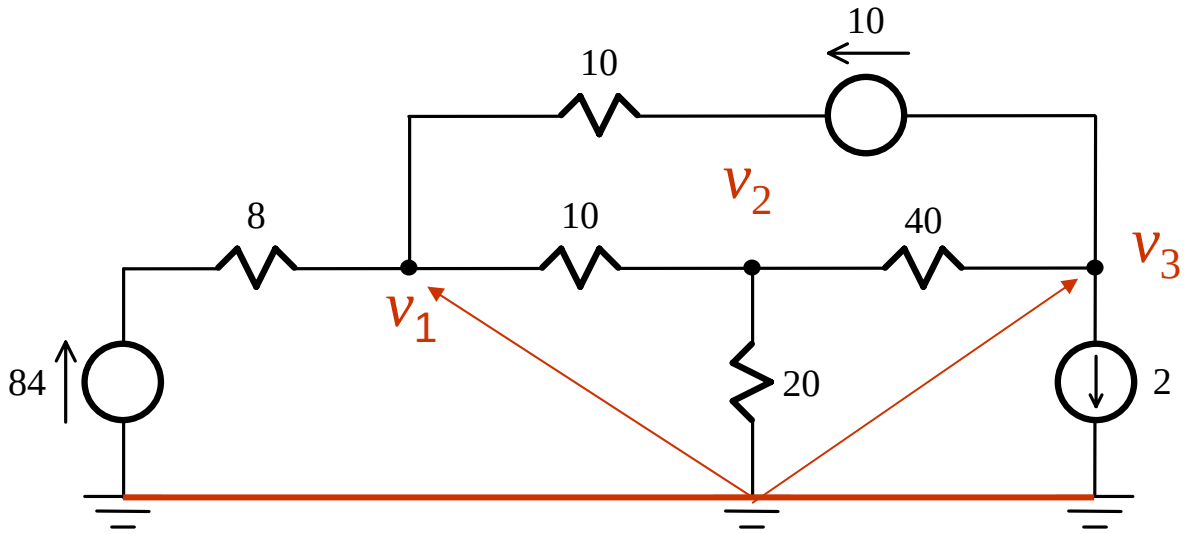


Fig. 1(a)

Numbering the nodes in the circuit by 1, 2 and 3 from left to right, and applying KCL:

$$\frac{v_1-84}{8}+\frac{v_1-v_2}{10}+\frac{v_1-v_3-10}{10}=0 \qquad \frac{v_2}{20}+\frac{v_2-v_1}{10}+\frac{v_2-v_3}{40}=0 \qquad \frac{v_3-v_2}{40}+2+\frac{v_3-v_1+10}{10}=0$$



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(b) Using mesh analysis, derive (but DO NOT solve) the matrix equation for determining the loop currents in the circuit of Fig. 1(b). Note that the circuit has a dependent source.

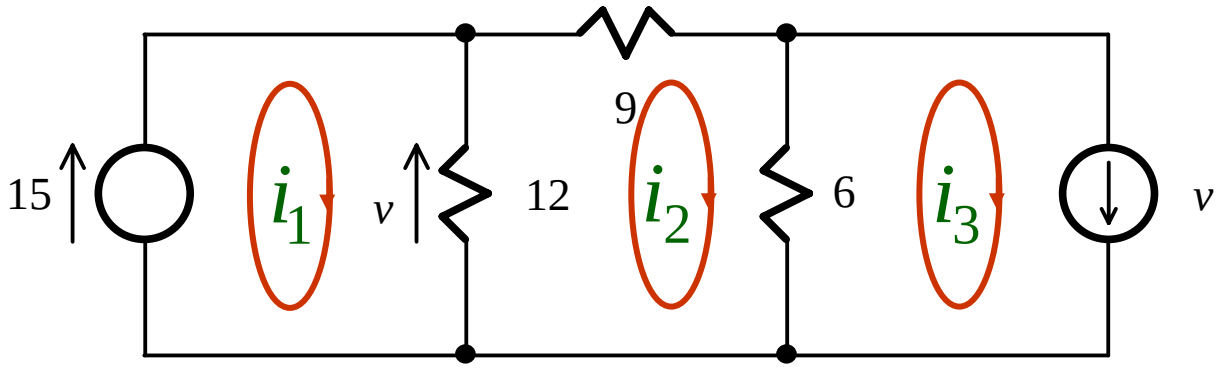


Fig. 1(b)

Relating loop to branch currents and applying KVL:

$$15 = v = 12(i_1 - i_2)$$

$$6(i_2 - i_3) + 9i_2 + 12(i_2 - i_1) = 0$$

$$\Rightarrow \quad -12i_1 + 27i_2 - 6i_3 = 0$$

$$i_3 = v$$

$$\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & -12 & 27 & -6 \\ 1 & 0 & 0 & 0 \\ 0 & 12 & -12 & 0 \end{bmatrix} \begin{bmatrix} v \\ i_1 \\ i_2 \\ i_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 15 \\ 15 \end{bmatrix}$$



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(c) Determine the Thevenin or Norton equivalent circuits as seen from terminals A and B of the network of Fig. 1(c). What is the maximum power that can be obtained from these two terminals?

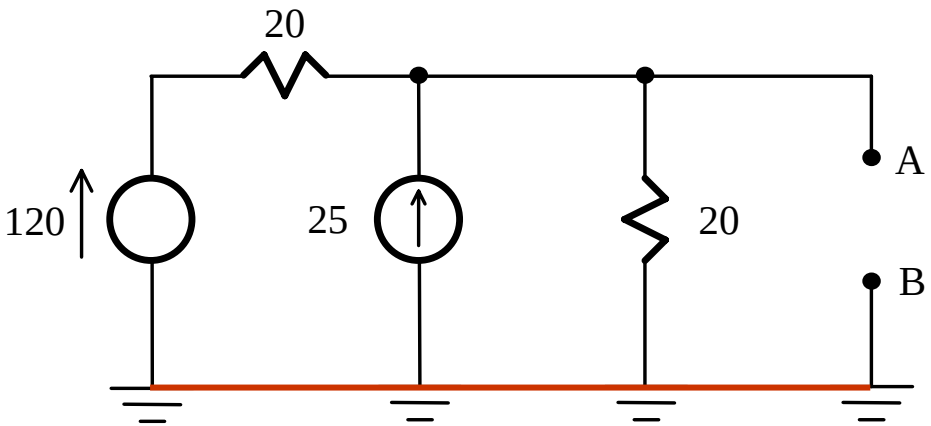



Fig. 1(c)

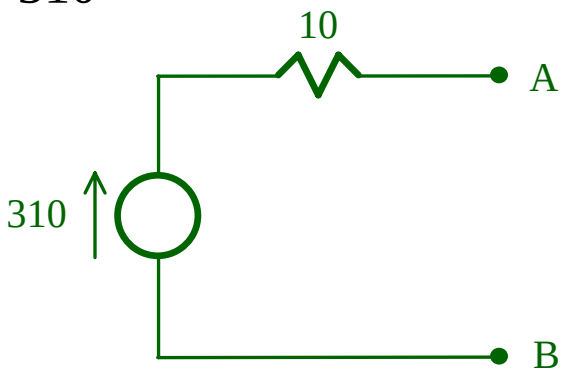
Replacing all independent sources with their internal resistances, the resistance across A and B is

$$R = 20 \,||\, 20 = 10$$

Using superposition, the open circuit voltage across A and B is

$$v_{AB} = 120\left(\frac{20}{20+20}\right) + 25\left(\frac{20}{20+20}\right)20$$

$$= 310$$




The maximum power $p = \dfrac{310^2}{4(10)}$



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Q.2 (a) A 5 kW electric motor is operating at a lagging power factor of 0.5. If the input voltage is

$$v(t) = 500\sin\left(\omega t + 10^0\right)$$

determine the apparent power, and find the phasor and sinusoidal expression for the input current.

Letting $V$ and $I$ to be the voltage and current phasors, the apparent power is

$$\frac{5000}{0.5} = 10000\text{VA} = |VI| = |V||I|$$

where $V = \frac{500}{\sqrt{2}} e^{j\left(10^0 - 90^0\right)} = \frac{500}{\sqrt{2}} e^{-j80^0}$ and $|I| = \frac{10000}{|V|} = \frac{10000\sqrt{2}}{500} = 20\sqrt{2}$

$$\arg(I) - \arg(V) = -\cos^{-1}(0.5)$$

$$i(t) = 20\sqrt{2}\sqrt{2}\cos\left(\omega t - 80^0 - \cos^{-1} 0.5\right)$$
$$= 40\cos\left(\omega t - 80^0 - \cos^{-1} 0.5\right)$$

$\Leftarrow$ $I = 20\sqrt{2}e^{j\left(-80^0 - \cos^{-1} 0.5\right)}$



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(b) In the circuit of Fig. 2(b), the current $i(t)$ is the excitation and the voltage $v(t)$ is the response. Determine the frequency response of the circuit. Derive (but DO NOT solve) an equation for finding the "resonant" frequency at which the frequency response becomes purely real.

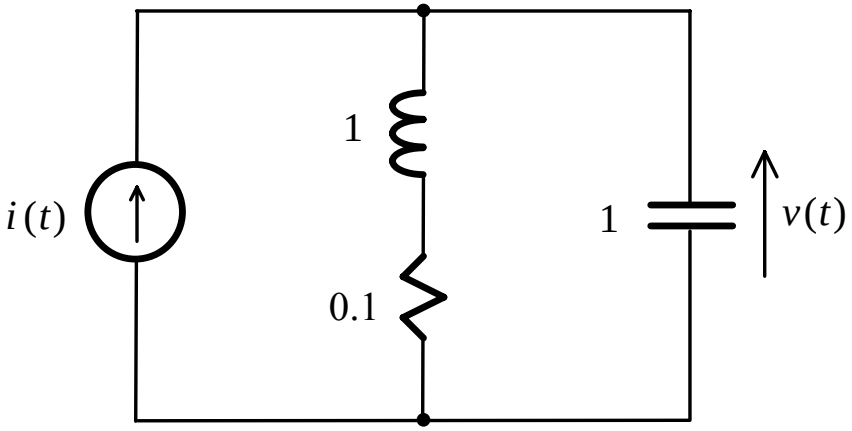


Fig. 2(b)

Using phasor analysis

$$H(f)=\frac{V}{I}=\frac{1}{j\omega}\left(\frac{0.1+j\omega}{0.1+j\omega+\frac{1}{j\omega}}\right)$$

$$=\frac{0.1+j\omega}{1+j0.1\omega-\omega^2}$$

The phase response is

$$\arg[H(f)]=\tan^{-1}\left(\frac{\omega}{0.1}\right)-\tan^{-1}\left(\frac{0.1\omega}{1-\omega^2}\right)$$

The resonant frequency is therefore given by

$$\tan^{-1}(10\omega)=\tan^{-1}\left(\frac{0.1\omega}{1-\omega^2}\right)$$



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(c) A series RLC resonant circuit is to be designed for use in a communication receiver. Based on measurements using an oscilloscope, the coil that is available is found to have an inductance of 25.3mH and a resistance of 2 Ω. Determine the value of the capacitor that will give a resonant frequency of 1 kHz. If a *Q* factor of 100 is required, will the coil be good enough?

$$f_0 = \frac{1}{2\pi\sqrt{LC}} \quad \Rightarrow \quad C = \left(\frac{1}{2\pi\sqrt{L}\, f_0}\right)^2 = \left(\frac{1}{2\pi\sqrt{25.3\times10^{-3}}\,1000}\right)^2 = 1\mu\text{F}$$

$$Q = \frac{\omega_0 L}{R} = \frac{2\pi 1000\left(25.3\right)10^{-3}}{2} = 79.5$$

**Since this is less than 100, the coil is not good enough.**



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Q.3 (a) In the circuit of Fig. 3(a), the switch has been in the position shown for a long time and is thrown to the other position for time $t \geq 0$. Determine the values of $i(t)$, $v_C(t)$, $v_R(t)$, $v_L(t)$, and $di(t)/dt$ just after the switch has been moved to the final position?

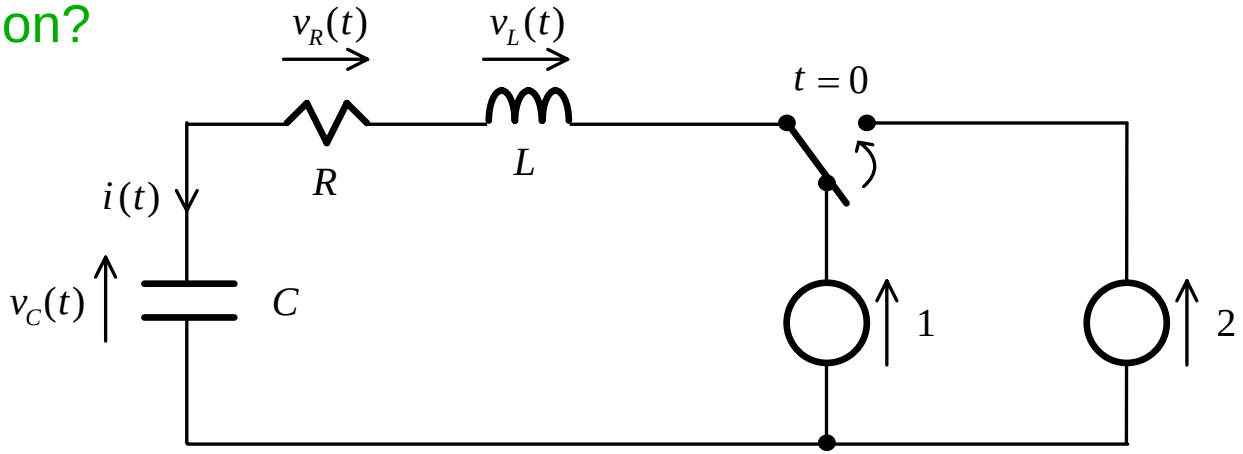


Fig. 3(a)

Taking all the voltages and currents to be constants for $t < 0$:

$$i(t) = C\frac{dv_C(t)}{dt} = 0$$

$$v_R(t) = Ri(t) = 0$$

$$v_L(t) = L\frac{di(t)}{dt} = 0$$

$$v_C(t) + v_R(t) + v_L(t) = 1 \Rightarrow v_C(t) = 1$$

Applying continuity for $i(t)$ and $v_C(t)$:

$$i(0) = 0 \quad v_R(0) = Ri(0) = 0 \quad v_C(0) = 1$$

$$v_C(0) + v_R(0) + v_L(0) = 2 \quad \Rightarrow \quad v_L(0) = 1$$

$$v_L(t) = L\frac{di(t)}{dt} \Rightarrow \left.\frac{di(t)}{dt}\right|_{t=0} = \frac{v_L(0)}{L} = \frac{1}{L}$$



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(b) For $v_S(t) = \cos(t+1)$, derive (but DO NOT solve) the differential equation from which $i(t)$ can be found in the circuit of Fig. 3(b). Is this differential equation sufficient for $i(t)$ to be determined?

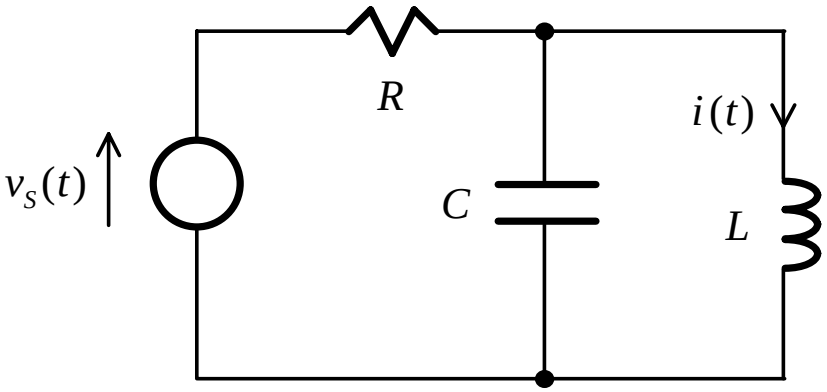


Fig. 3(b)

$$v_L(t) = L\frac{di(t)}{dt}$$

$$i_C(t) = C\frac{dv_C(t)}{dt} = C\frac{dv_L(t)}{dt} = CL\frac{d^2i(t)}{dt^2}$$

$$i_R(t) = i_C(t) + i(t) = CL\frac{d^2i(t)}{dt^2} + i(t)$$

$$v_R(t) = Ri_R(t) = RCL\frac{d^2i(t)}{dt^2} + Ri(t)$$

Applying KVL:

$$v_S(t) = v_R(t) + v_L(t)$$

$$= RCL\frac{d^2i(t)}{dt^2} + Ri(t) + L\frac{di(t)}{dt} = \cos(t+1)$$

This is not sufficient for $i(t)$ to be determined.



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(c) The differential equation characterizing the current $i(t)$ in a certain RCL circuit is

$$\frac{d^2 i(t)}{dt^2} + \frac{1}{CR}\frac{di(t)}{dt} + \frac{i(t)}{CL} = e^{jt}$$

Determine the condition for $R$, $L$ and $C$ such that the circuit is critically damped.

The characteristic equation for the transient response is

$$z^2 + \frac{z}{CR} + \frac{1}{CL} = 0$$

$$z_{1,2} = \frac{-\frac{1}{CR} \pm \sqrt{\frac{1}{C^2R^2} - \frac{4}{CL}}}{2}$$

Thus, the circuit will be critically damped if

$$\frac{1}{C^2R^2} = \frac{4}{CL}$$



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Q.4 (a) Determine the mean and rms values of the voltage waveform in Fig. 4(a). If this waveform is applied to a 20 Ω resistor, what is the power absorbed by the resistor?

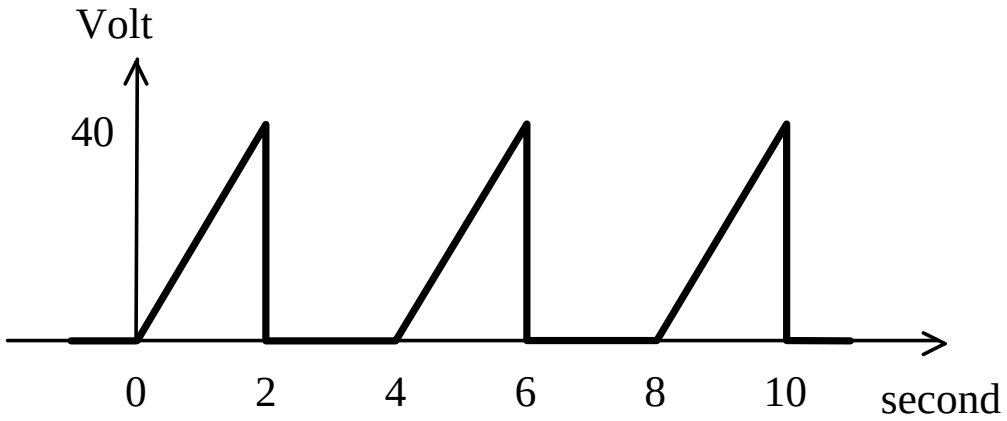


Fig. 4(a)

One period of the waveform is

$$v(t)=\begin{cases}20t, & 0\le t<2\\ 0, & 2\le t<4\end{cases}$$

$$v_m=\frac{\int_0^2 20t\,dt}{4}=\frac{20\left(\frac{2^2}{2}\right)}{4}=10$$

$$v_{ms}=\frac{\int_0^2 (20t)^2\,dt}{4}=\frac{400\left(\frac{2^3}{3}\right)}{4}=\frac{800}{3}$$

$$v_{rms}=\sqrt{\frac{800}{3}}$$

$$p=\frac{v_{rms}^{\ 2}}{20}=\frac{800}{3(20)}=\frac{40}{3}$$



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(b) In the circuit of Fig. 4(b), a transformer is used to couple a loudspeaker to a amplifier. The loudspeaker is represented by an impedance of value $Z_L = 6 + j\,2$, while the amplifier is represented by a Thevenin equivalent circuit consisting of a voltage source in series with an impedance of $Z_S = 3 + j\,a$. Determine the voltage across the loudspeaker. Hence, find the value of $a$ such that this voltage is maximized. Will maximum power be delivered to the loudspeaker under this condition?

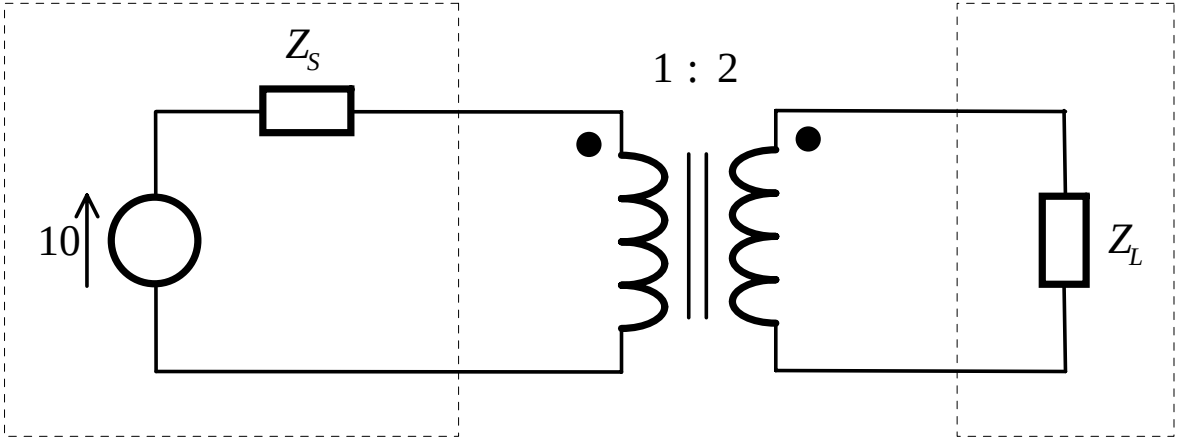


Fig. 4(b)



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If $V$ is the voltage across the loudspeaker, the currents in the primary & secondary windings are

$$I_2 = \frac{V}{Z_L} \qquad I_1 = 2I_2 = \frac{2V}{Z_L}$$

The primary voltage is

$$V_1 = \frac{V}{2}$$

Applying KVL to the primary circuit:

$$10 = I_1 Z_S + V_1 = \frac{2VZ_S}{Z_L} + \frac{V}{2}$$

$$\Longrightarrow \quad V = \frac{10}{\frac{2Z_S}{Z_L} + \frac{1}{2}} = \frac{20}{1 + 4\left(\frac{3 + aj}{6 + 2j}\right)} = \frac{20(6 + 2j)}{18 + j(4a + 2)}$$

For the magnitude of this to be maximized, the denominator has to be minimized:

$$\max\left[\left|\frac{20(6 + 2j)}{18 + j(4a + 2)}\right|\right] = \min\left[\left|18 + j(4a + 2)\right|\right] \quad \Longrightarrow \quad a = -\frac{2}{4} = -\frac{1}{2}$$

Maximum power will be delivered since power is proportional to $|V|^2$.



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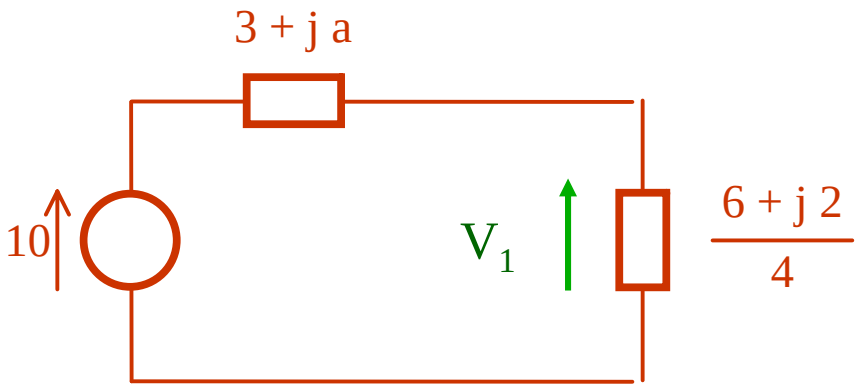
**Method 2:** The given circuit is equivalent to the following one,




Then, we have

$$V_1 = \left( \frac{10}{(3+ja)+\left(\frac{3}{2}+j\frac{1}{2}\right)} \right)\left(\frac{3}{2}+j\frac{1}{2}\right) = \frac{10(3+j)}{9+j(2a+1)}$$

$$\Rightarrow \quad V_{load} = V_2 = nV_1 = \frac{20(3+j)}{9+j(2a+1)}$$

The rest follows ……



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