# EE5110/6110 Special Topics in Automation and Control

## Autonomous Systems: Unmanned Aerial Vehicles

## Modeling and Control System Design

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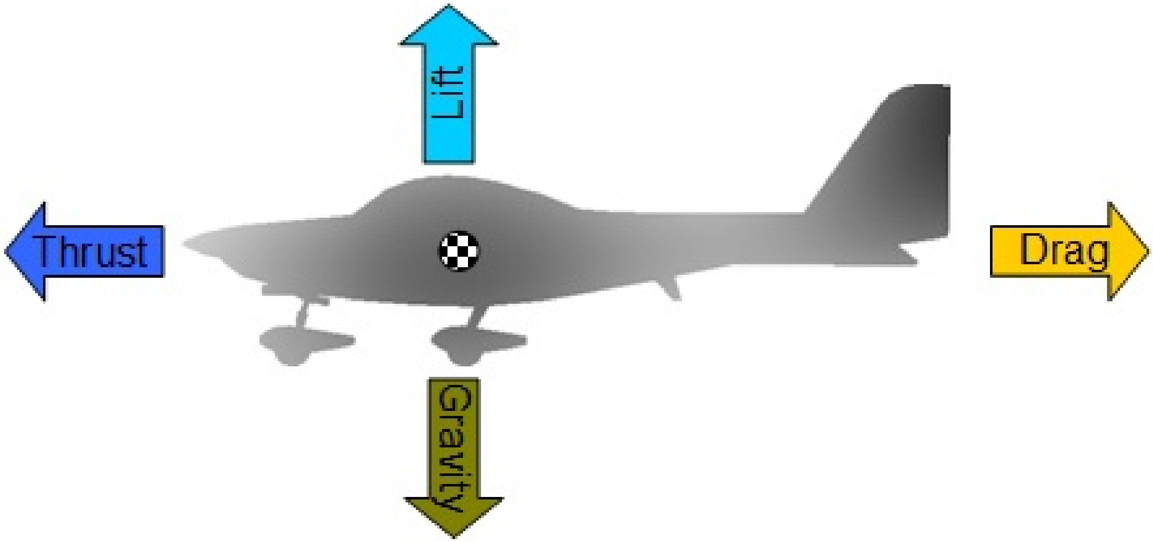
What is a **flight dynamics model**?

A flight dynamics model is a set of equations that explains

1. How aerodynamic forces are generated on the aircraft body?
2. How actuators and external factors affect these forces?
3. How these forces affect the aircraft's motion?



Why do we need a **flight dynamics model**?

1. Design and tuning of autonomous flight control law

Modern control techniques are mostly model based. An accurate flight dynamics model can assist flight control law design efficiently in terms of parameter tuning and performance evaluation.

2. Design and modification of aircraft structure

Aerodynamic stability, physical limits and characteristics can be analyzed based on the obtained model. Mechanical modification or re-structuring can be applied if there is inherent deficiencies in the original structure.

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How do we obtain a **flight dynamics model**?

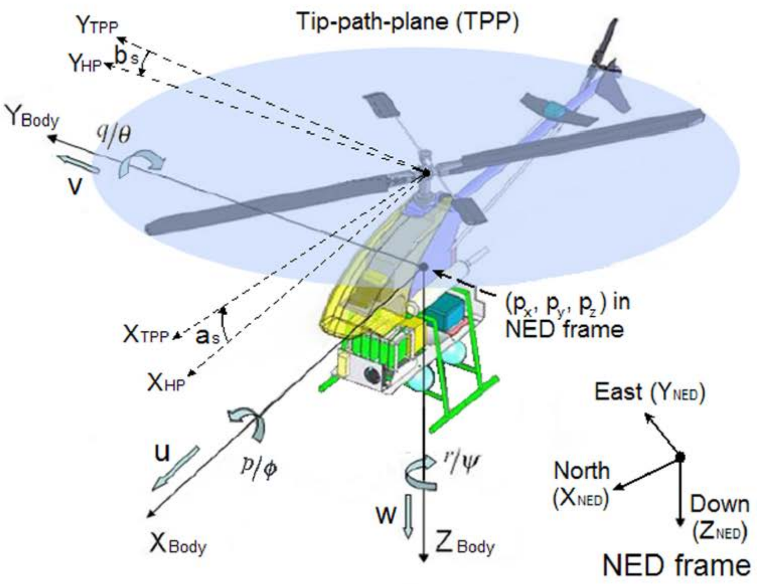
1) **System Identification Approach**
   - Build mathematical model by fitting measurement data
   - Simplified, linear, black-box
   - Flight test experiments

2) **First-principles Modeling Approach**
   - Build mathematical model by manipulating equations of physics
   - Complex, nonlinear, rigorous with physical meanings
   - Test-bench experiments

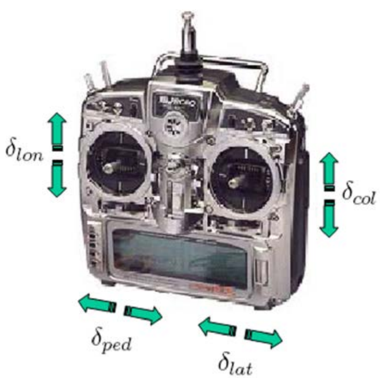
3) **Combination of the above two approaches**

# System Identification Approach

## Single-rotor helicopter



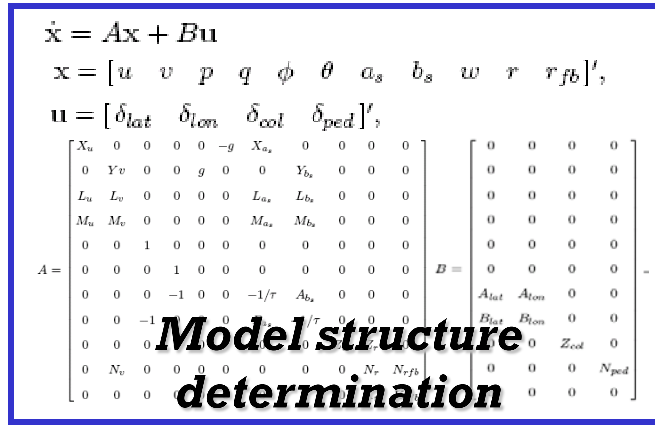

| Variable | Physical description | Unit |
|---|---|---|
| $p_x, p_y, p_z$ | Position vector along NED-frame x, y, and z axes | m |
| $u, v, w$ | Velocity vector along body-frame x, y, and z axes | m/s |
| $p, q, r$ | Roll, pitch, and yaw angular rates | rad/s |
| $\phi, \theta, \psi$ | Euler angles | rad |
| $a_s, b_s$ | Longitudinal and lateral tip-path-plane (TPP) flapping angle | rad |
| $\delta_{ped,int}$ | Intermediate state in yaw rate gyro dynamics | NA |
| $\delta_{lat}$ | Normalized aileron servo input $(-1, 1)$ | NA |
| $\delta_{lon}$ | Normalized elevator servo input $(-1, 1)$ | NA |
| $\delta_{col}$ | Normalized collective pitch servo input $(-1, 1)$ | NA |
| $\delta_{ped}$ | Normalized rudder servo input $(-1, 1)$ | NA |

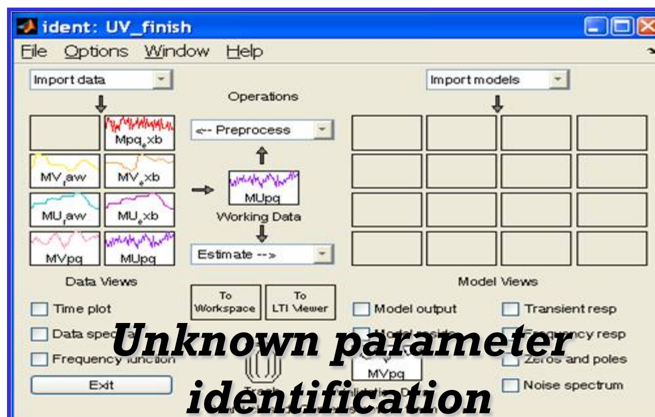


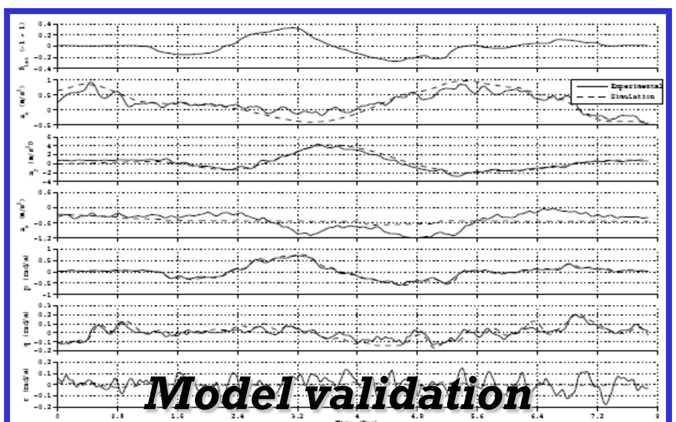

*Despite the modeling approach adopted, we always need to first*

1. *Understand the basic working principles*
2. *Define coordinate frames*
3. *Define input and output variables (or state variables)*

**System Identification Approach** is suitable for obtaining linear dynamics model, and can be conducted in both time-domain and frequency-domain.

# System Identification Approach

## Single-rotor helicopter

Toolkits:

Time-domain: IDENT        Frequency-domain: CIFER

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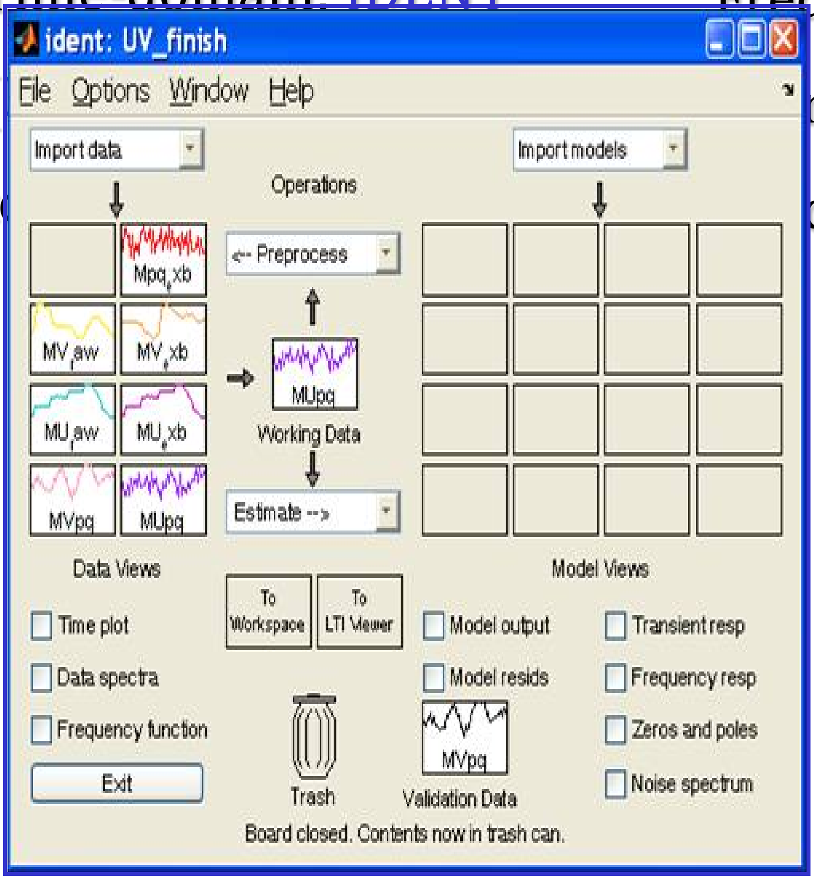
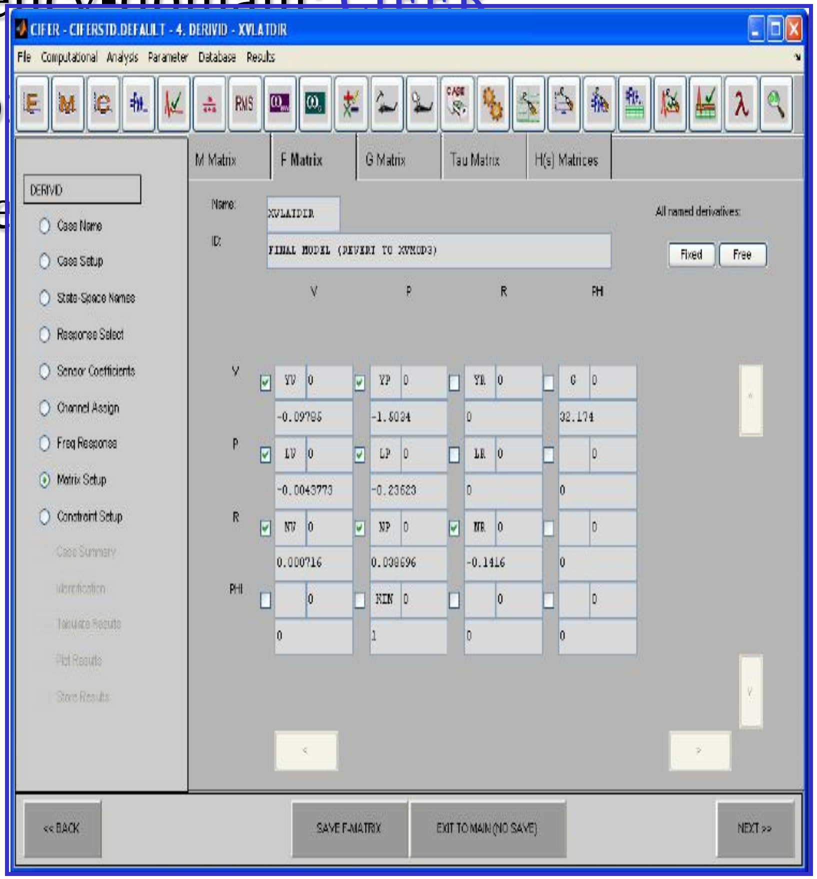
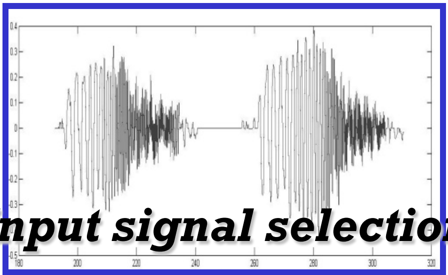


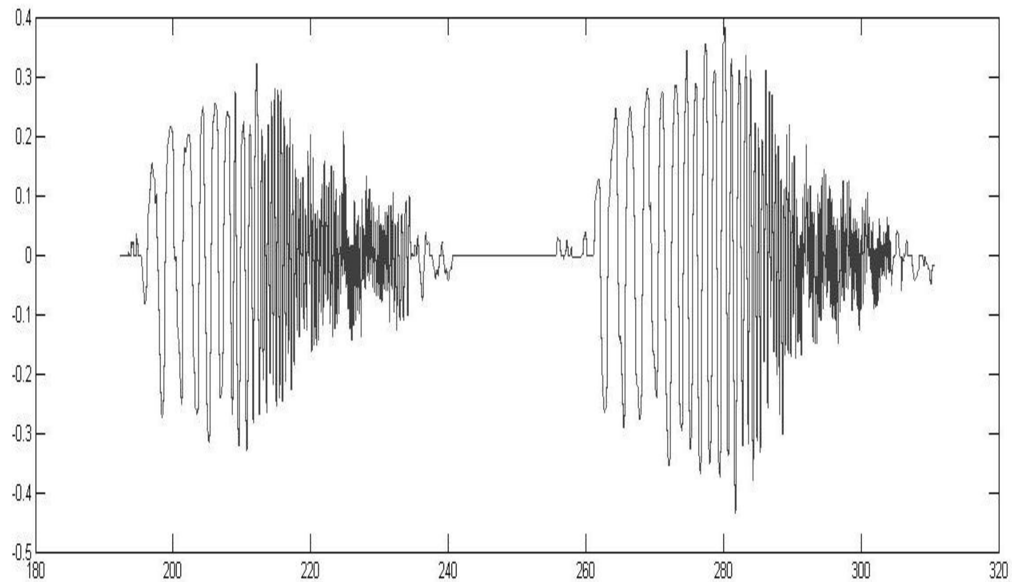

IDENT        CIFER

## Step 1: Data collection and preprocessing:





Data preprocessing

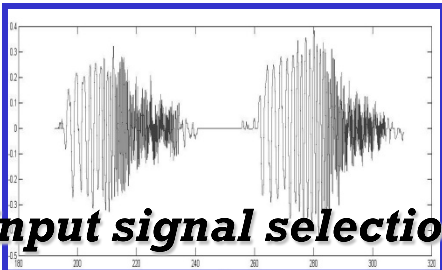



***Frequency sweep input signal***

# System Identification Approach

*Single-rotor helicopter*

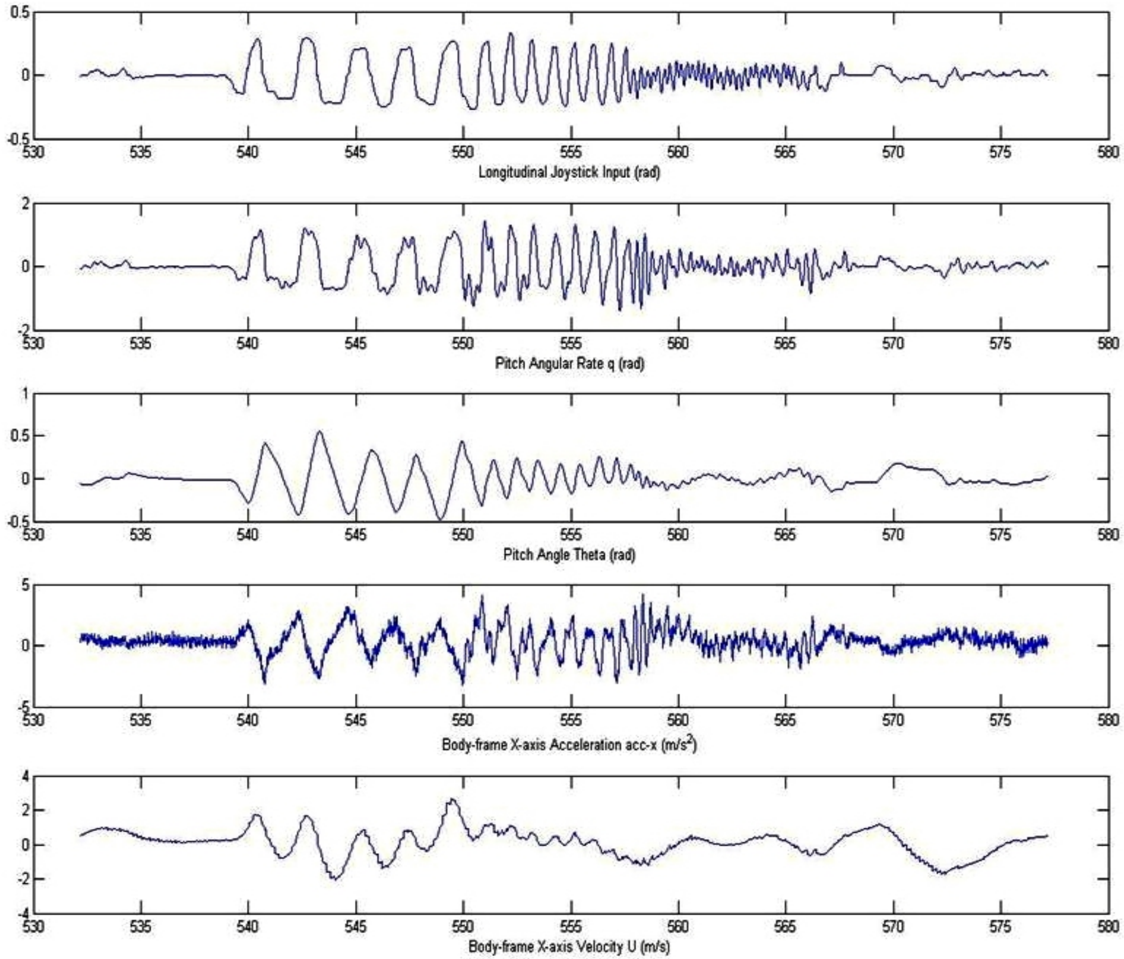
## Step 1: Data collection and preprocessing:



***Input signal selection***

***Data collection***

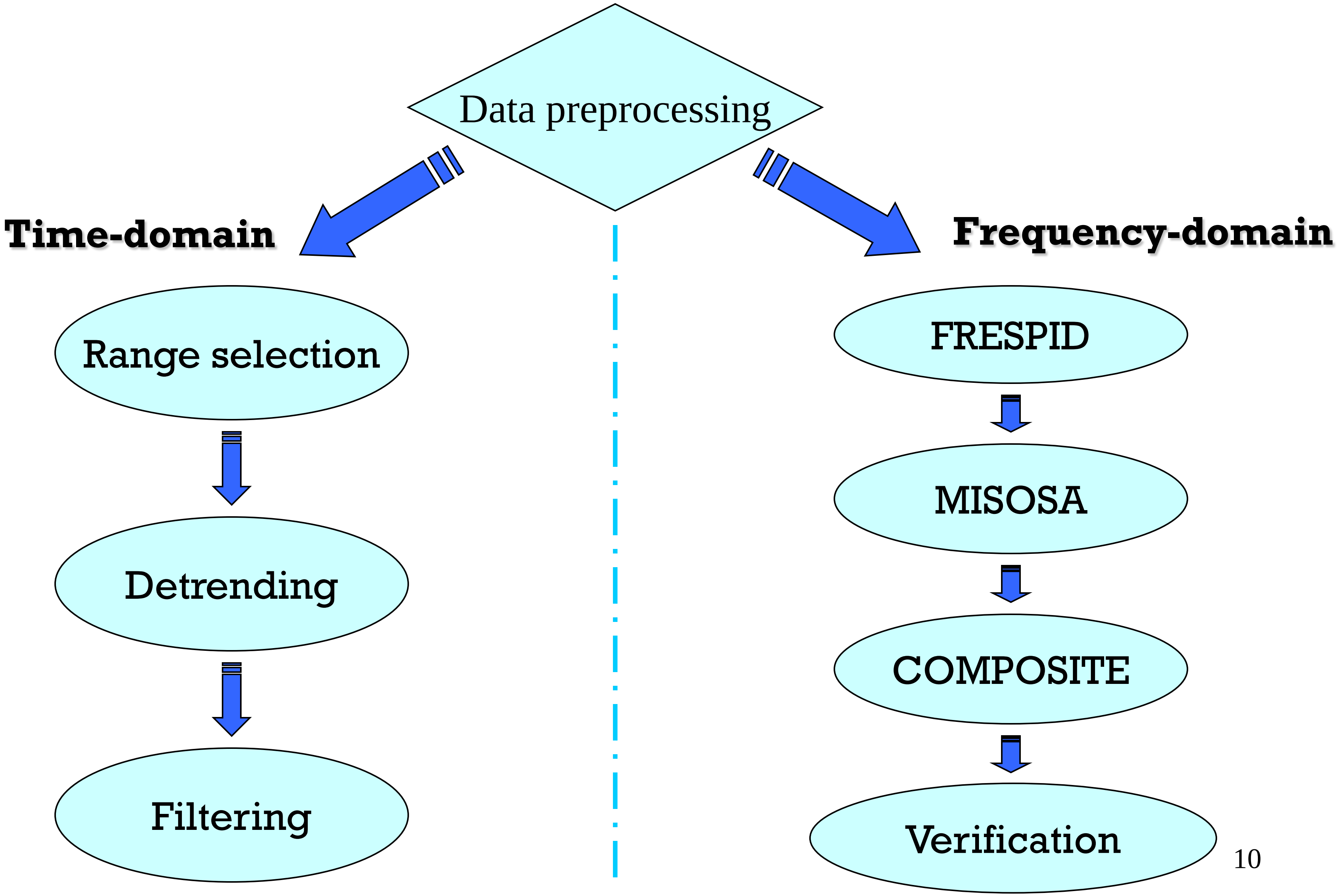
Data preprocessing




***Output responses***

# *System Identification Approach*

*Single-rotor helicopter*

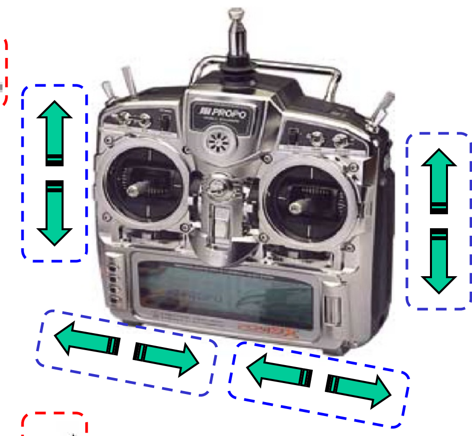


# System Identification Approach

## Single-rotor helicopter

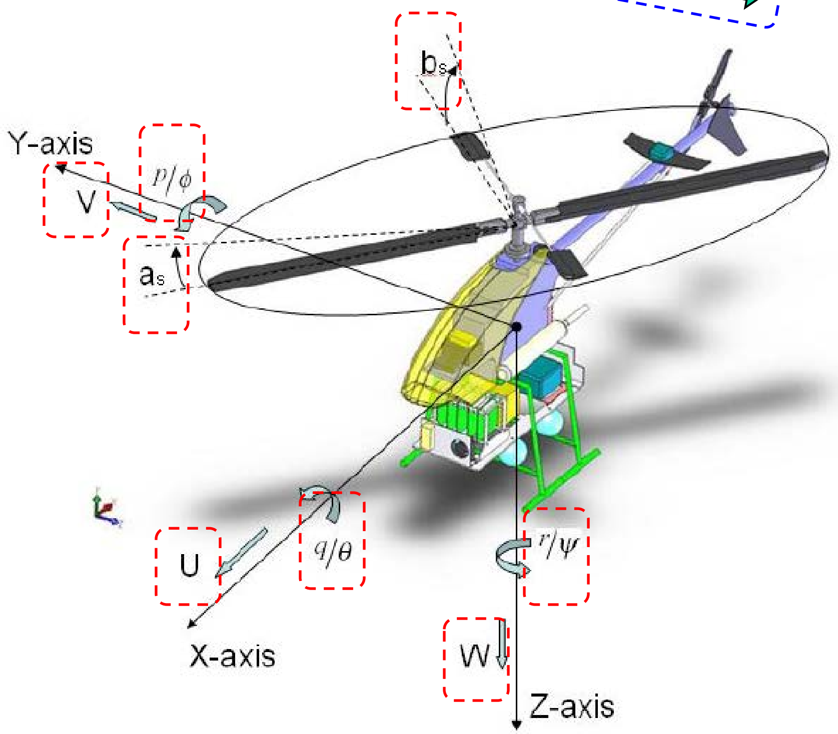
### Step 2: Model structure determination:

$$\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u} \quad \begin{cases} \mathbf{x} = [u \quad v \quad p \quad q \quad \phi \quad \theta \quad a_s \quad b_s \quad w \quad r \quad r_{fb}]', \\ \mathbf{u} = [\delta_{lat} \quad \delta_{lon} \quad \delta_{col} \quad \delta_{ped}]', \end{cases}$$



$$A = \begin{bmatrix} X_u & 0 & 0 & 0 & 0 & -g & X_{a_s} & 0 & 0 & 0 & 0 \\ 0 & Y_v & 0 & 0 & g & 0 & 0 & Y_{b_s} & 0 & 0 & 0 \\ L_u & L_v & 0 & 0 & 0 & 0 & L_{a_s} & L_{b_s} & 0 & 0 & 0 \\ M_u & M_v & 0 & 0 & 0 & 0 & M_{a_s} & M_{b_s} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & -1/\tau & A_{b_s} & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & B_{a_s} & -1/\tau & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & Z_w & Z_r & 0 \\ 0 & N_v & 0 & 0 & 0 & 0 & 0 & 0 & 0 & N_r & N_{rfb} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & K_r & K_{rfb} \end{bmatrix},$$

$$B = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ A_{lat} & A_{lon} & 0 & 0 \\ B_{lat} & B_{lon} & 0 & 0 \\ 0 & 0 & Z_{col} & 0 \\ 0 & 0 & 0 & N_{ped} \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$



## Step 3: Unknown parameter identification:

1). Angular rate dynamics; 2). Horizontal velocity dynamics;

3). Yaw dynamics; 4). Heave dynamics;

## a. Time-domain: Hover model of a single-rotor helicopter

$$A=\begin{bmatrix} -0.1778 & 0 & 0 & 0 & 0 & -9.7807 & -9.7807 & 0 & 0 & 0 & 0 \\ 0 & -0.3104 & 0 & 0 & 9.7807 & 0 & 0 & 9.7807 & 0 & 0 & 0 \\ -0.3326 & -0.5353 & 0 & 0 & 0 & 0 & 75.764 & 343.86 & 0 & 0 & 0 \\ -0.1903 & -0.2490 & 0 & 0 & 0 & 0 & 172.62 & -59.958 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & -8.1222 & 4.6535 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & -0.0921 & -8.1222 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.6821 & -0.0535 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.2892 & -5.5561 & -36.674 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2.7492 & -11.112 \end{bmatrix}, \quad B=\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0.0496 & 2.6224 & 0 & 0 \\ 2.4928 & 0.1741 & 0 & 0 \\ 0 & 0 & 7.8246 & 0 \\ 0 & 0 & 1.6349 & -58.4053 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

## b. Frequency-domain: Hovering model of a single-rotor helicopter

Extra metrics for parameter accuracy evaluation:

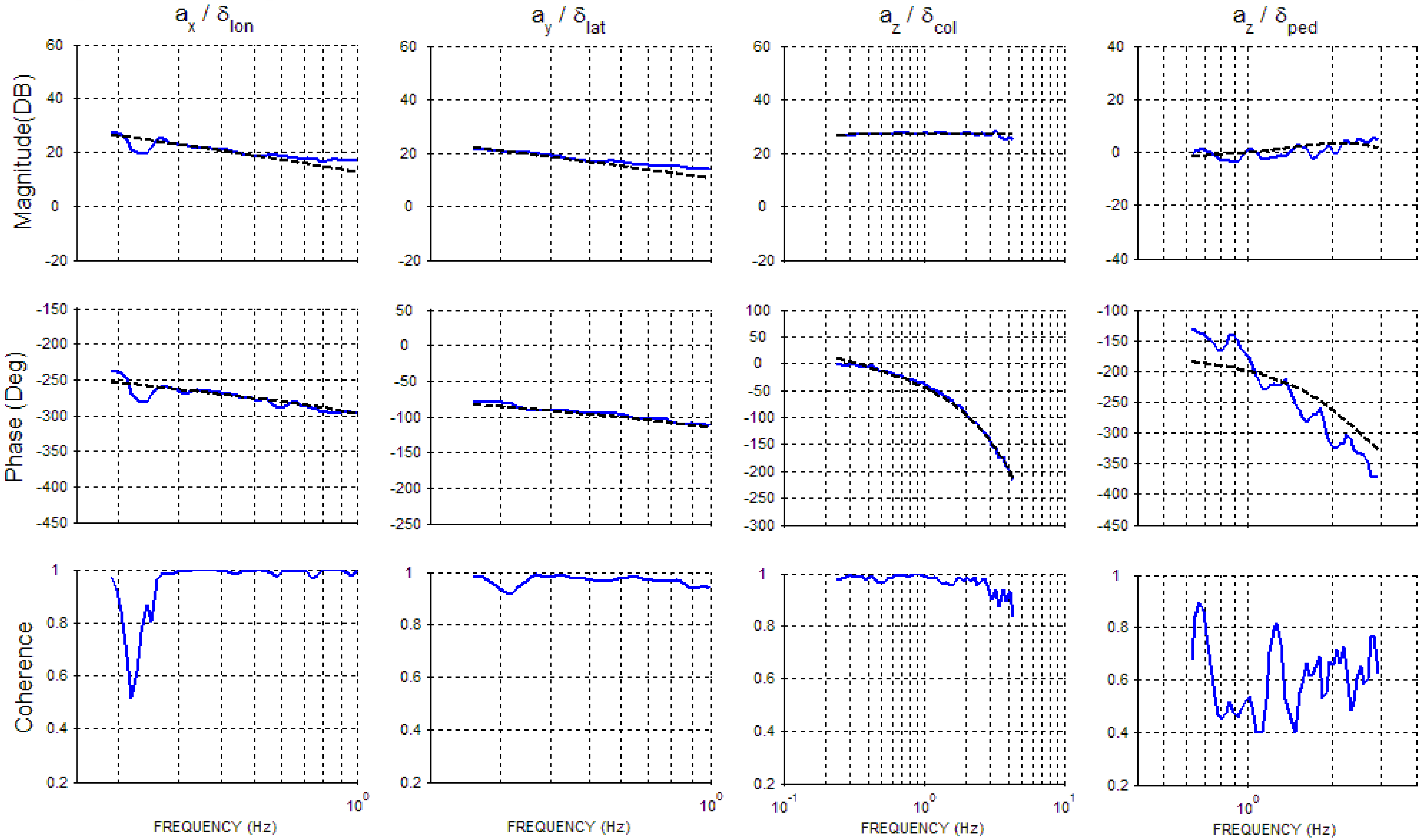
1). Cramer-Rao bound (CR% < 20%); 2). Insensitivity (I% < 10%)

| Parameter | Value | CR% | I% |
|---|---|---|---|
| $X_u$ | -0.5198 | 20.31 | 9.827 |
| $X_{a_s}$ | -9.7807 ! | —— | —— |
| $Y_v$ | -0.4225 | 19.51 | 9.231 |
| $Y_{b_s}$ | 9.7807 ! | —— | —— |
| $L_u$ | -0.3262 ! | —— | —— |
| $L_v$ | -0.0675 * | —— | —— |
| $L_{a_s}$ | 0 + | —— | —— |
| $L_{b_s}$ | 439.4000 | 2.489 | 1.020 |
| $M_u$ | -0.0814 * | —— | —— |
| $M_v$ | -2.1210 | 18.02 | 7.143 |
| $M_{a_s}$ | 275.1000 | 2.794 | 1.067 |
| $M_{b_s}$ | -42.1100 | 14.17 | 4.562 |
| $\tau$ | -4.7790 | 7.678 | 2.367 |
| $a_{b_s}$ | 2.3850 | 13.67 | 4.309 |
| $b_{a_s}$ | 0 | —— | —— |
| $Z_w$ | -0.5996 | 19.33 | 12.13 |
| $Z_r$ | 0.1103 | 6.369 | 2.514 |
| $N_r$ | -4.6150 | 7.515 | 2.441 |
| $N_{rfb}$ | 154.5000 * | —— | —— |
| $N_v$ | 0 | —— | —— |
| $K_r$ | -1.0830 | 5.805 | 1.929 |
| $K_{rfb}$ | -9.2300 * | —— | —— |
| $A_{lat}$ | 0.0610 | 16.62 | 8.979 |
| $A_{lon}$ | 2.8080 | 2.731 | 1.319 |
| $B_{lat}$ | 2.1810 | 2.687 | 1.202 |
| $B_{lon}$ | 0 + | —— | —— |
| $Z_{col}$ | 22.9400 | 3.776 | 1.856 |
| $N_{ped}$ | -154.5000 | 6.663 | 1.833 |

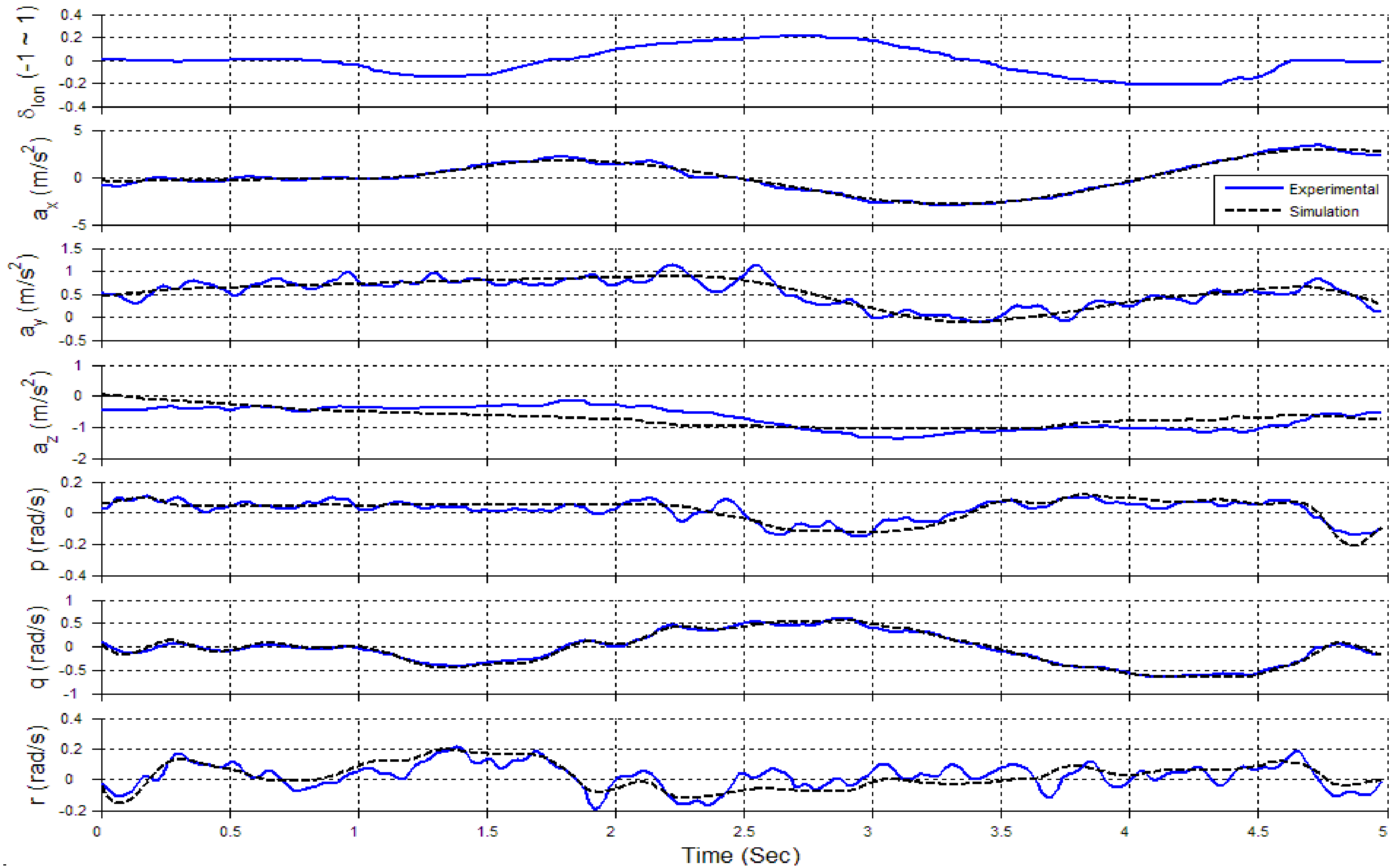
+ : Eliminated in model structure determination
! : Fixed empirically
* : Tied to other parameters

# Frequency-domain: Hovering model of a single-rotor helicopter

## Frequency responses matching



## Step 4: Model validation



## Rigid-body dynamics

$$\dot{u} = vr - wq - g\sin(\theta) + (X_{mr} + X_{fus})/m$$

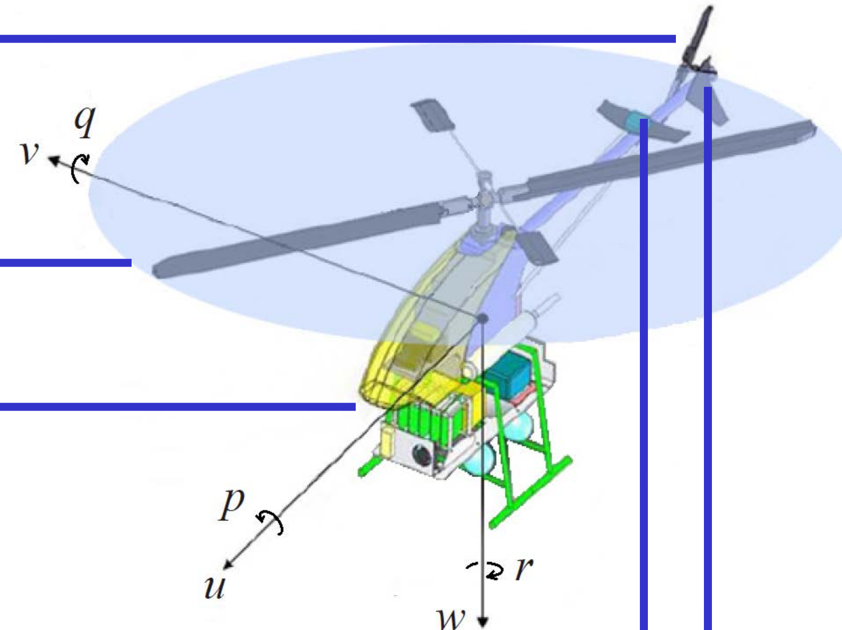
$$\dot{v} = wp - ur + g\sin(\phi)\cos(\theta) + (Y_{mr} + Y_{fus} + Y_{tr} + Y_{vf})/m$$

$$\dot{w} = uq - vp + g\cos(\phi)\cos(\theta) + (Z_{mr} + Z_{fus} + Z_{hf})/m$$

$$\dot{p} = qr(I_{yy} - I_{zz})/I_{xx} + (L_{mr} + L_{vf} + L_{tr})/I_{xx}$$

$$\dot{q} = pr(I_{zz} - I_{xx})/I_{yy} + (M_{mr} + M_{hf})/I_{yy}$$

$$\dot{r} = pq(I_{xx} - I_{yy})/I_{zz} + (N_{mr} + N_{vf} + N_{tr})/I_{zz}$$



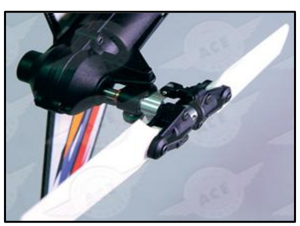

$()_{mr}$ for the main rotor

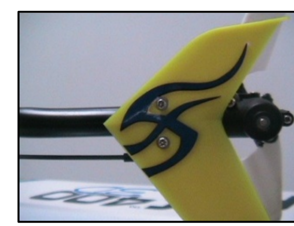
$()_{tr}$ for the tail rotor

$()_{fus}$ for the fuselage

$()_{hf}$ for the horizontal fin

$()_{vf}$ for the vertical fin

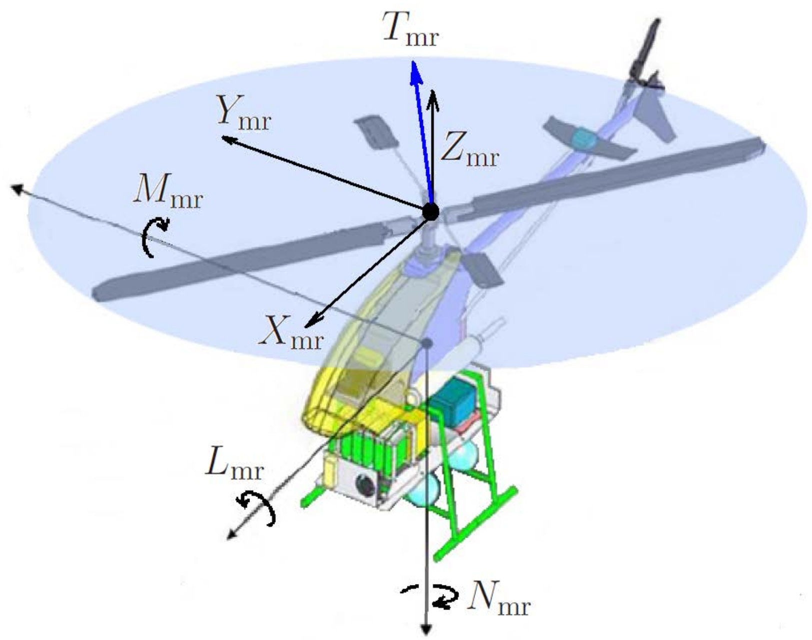




G. Cai, B. M. Chen, and T. H. Lee, *Unmanned Rotorcraft Systems*, New York: Springer, 2011.

## Main rotor forces and moments

$$\begin{cases} X_{\rm mr} = -T\sin(a_{\rm s}) \\ Y_{\rm mr} = T\sin(b_{\rm s}) \\ Z_{\rm mr} = -T\cos(a_{\rm s})\cos(b_{\rm s}) \\ L_{\rm mr} = (K_\beta + TH_{\rm mr})\sin(b_{\rm s}) \\ M_{\rm mr} = (K_\beta + TH_{\rm mr})\sin(a_{\rm s}) \\ N_{\rm mr} = -(P_{\rm pr} + P_{\rm i} + P_{\rm pa} + P_{\rm c})/\Omega \end{cases}$$




$$\begin{cases} T_{\rm mr} = \dfrac{\rho\,\Omega_{\rm mr}\,R_{\rm mr}^2\,C_{\rm l\alpha,mr}\,b_{\rm mr}\,c_{\rm mr}}{4}(w_{\rm bl,mr} - v_{\rm i,mr}) \\ v_{\rm i,mr}^2 = \sqrt{\left(\dfrac{\hat{v}_{\rm mr}^2}{2}\right)^2 + \left(\dfrac{T_{\rm mr}}{2\,\rho\,\pi\,R_{\rm mr}^2}\right)^2} - \dfrac{\hat{v}_{\rm mr}^2}{2} \\ \hat{v}_{\rm mr}^2 = u_{\rm a}^2 + v_{\rm a}^2 + w_{\rm r,mr}(w_{\rm r,mr} - 2\,v_{\rm i,mr}) \\ w_{\rm r,mr} = w_{\rm a} + a_{\rm s}\,u_{\rm a} - b_{\rm s}\,v_{\rm a} \\ w_{\rm bl,mr} = w_{\rm r,mr} + \dfrac{2}{3}\Omega_{\rm mr}\,R_{\rm mr}\,\theta_{\rm col} \end{cases}$$

Initial values: $T(0) = mg, \quad v_{\rm i}(0) = \dfrac{mg}{2\rho\pi R^2}, \quad \hat{v}(0) = 0.$

$$\begin{cases} P_{\rm pr} = \dfrac{\rho\Omega R^2 C_{\rm D0} b_{\rm mr} c_{\rm mr}}{8}\left[(\Omega R)^2 + 4.6(u_{\rm a}^2 + v_{\rm a}^2)\right] \\ P_{\rm i} = T v_{\rm i} \\ P_{\rm c} = -mgw_{\rm a} \\ P_{\rm pa} = |X_{\rm fus}u_{\rm a}| + |Y_{\rm fus}v_{\rm a}| + |Z_{\rm fus}(w_{\rm a} - v_{\rm i})|. \end{cases}$$

## Tail rotor forces and moments

$$T_{\rm tr} = \frac{\rho\,\Omega_{\rm tr}\,{R_{\rm tr}}^2\,C_{{\rm l}\alpha,{\rm tr}}\,b_{\rm tr}\,c_{\rm tr}}{4}(w_{\rm bl,tr} - v_{\rm i,tr})$$

$$v_{\rm i,tr}^2 = \sqrt{\left(\frac{\hat{v}_{\rm tr}^2}{2}\right)^2 + \left(\frac{T_{\rm tr}}{2\rho\,\pi\,R_{\rm tr}^2}\right)^2} - \frac{\hat{v}_{\rm tr}^2}{2}$$

$$\hat{v}_{\rm tr}^2 = (w_{\rm a} + qD_{\rm tr})^2 + u_{\rm a}^2 + w_{\rm r,tr}(w_{\rm r,tr} - 2v_{\rm i,tr})$$

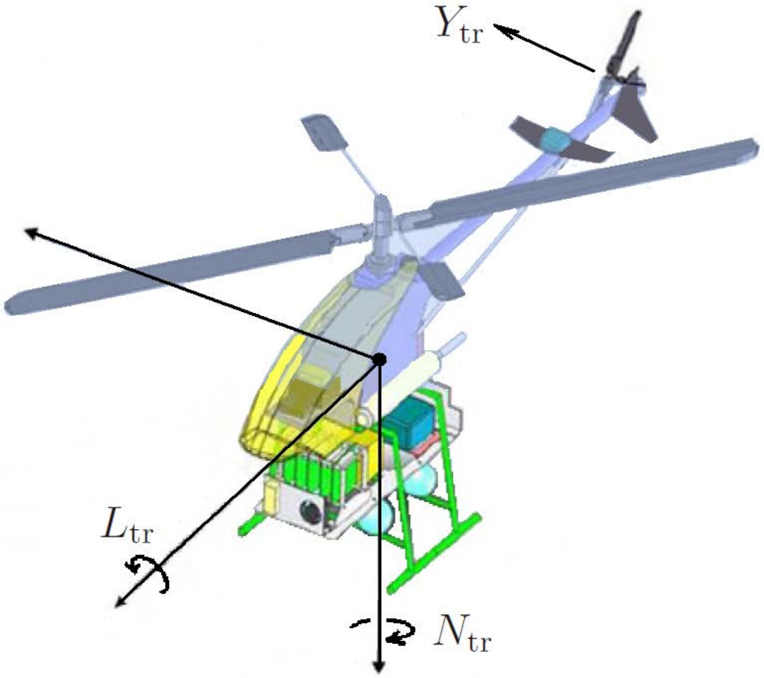
$$w_{\rm r,tr} = v_{\rm a} - rD_{\rm tr} + pH_{\rm tr}$$

$$w_{\rm bl,tr} = w_{\rm r,tr} + \frac{2}{3}\Omega_{\rm tr}R_{\rm tr}\theta_{\rm ped}$$

$$\theta_{\rm ped} = K_{\rm ped}\bar{\delta}_{\rm ped} + \theta_{\rm ped,0}$$

Initial values: $Y_{\rm tr}(0) = \dfrac{1}{\Omega D_{\rm tr}}\left(\dfrac{\rho\Omega^3R^4C_{\rm D0}b_{\rm mr}c_{\rm mr}}{8} + \dfrac{m^2g^2}{2\rho\pi R^2}\right)$, $v_{\rm i,tr}(0) = \dfrac{Y_{\rm tr}(0)}{2\rho\pi R_{\rm tr}^2}$, $\hat{v}_{\rm tr}(0) = 0$

$$Y_{\rm tr} = -T_{\rm tr} \quad L_{\rm tr} = -Y_{\rm tr}H_{\rm tr}, \quad N_{\rm tr} = Y_{\rm tr}D_{\rm tr}$$

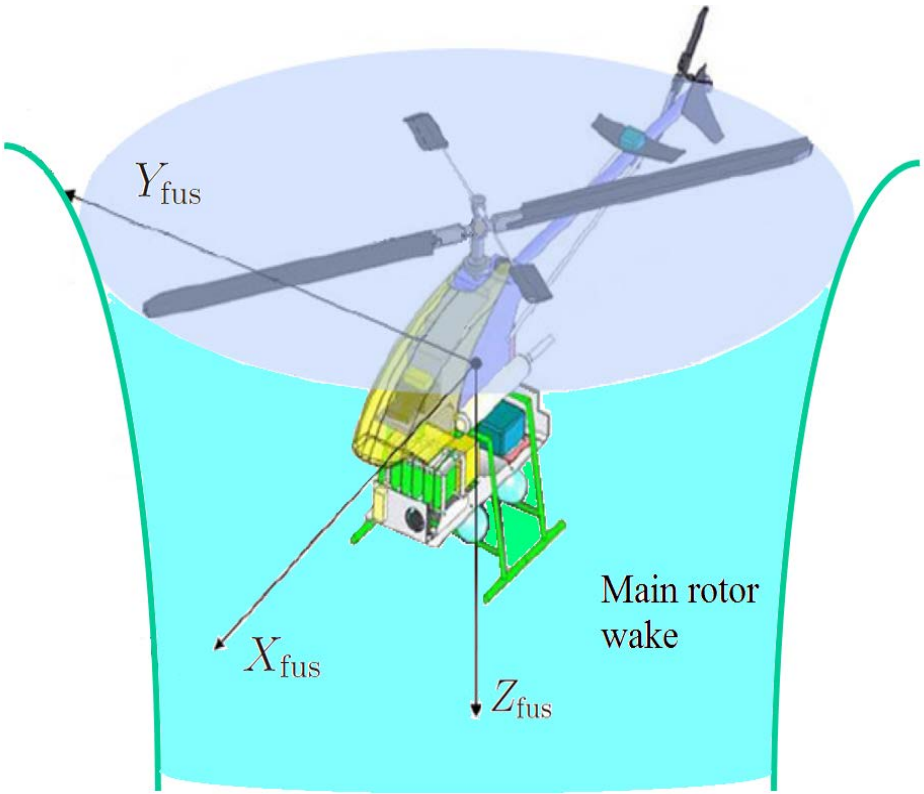


## Fuselage forces

$$X_{\text{fus}} = \begin{cases} -\frac{\rho}{2} S_{\text{fx}}\, u_{\text{a}}\, v_{\text{i,mr}}, & \text{if } |u_{\text{a}}| \le v_{\text{i,mr}} \\ -\frac{\rho}{2} S_{\text{fx}}\, u_{\text{a}}\, |u_{\text{a}}|, & \text{if } |u_{\text{a}}| > v_{\text{i,mr}} \end{cases}$$

$$Y_{\text{fus}} = \begin{cases} -\frac{\rho}{2} S_{\text{fy}}\, v_{\text{a}}\, v_{\text{i,mr}}, & \text{if } |v_{\text{a}}| \le v_{\text{i,mr}} \\ -\frac{\rho}{2} S_{\text{fy}}\, v_{\text{a}}\, |v_{\text{a}}|, & \text{if } |v_{\text{a}}| > v_{\text{i,mr}} \end{cases}$$

$$Z_{\text{fus}} = -\frac{\rho}{2} S_{\text{fz}} \left(w_{\text{a}} - v_{\text{i,mr}}\right) \left|w_{\text{a}} - v_{\text{i,mr}}\right|$$



# First-principles Modeling Approach

## Single-rotor helicopter

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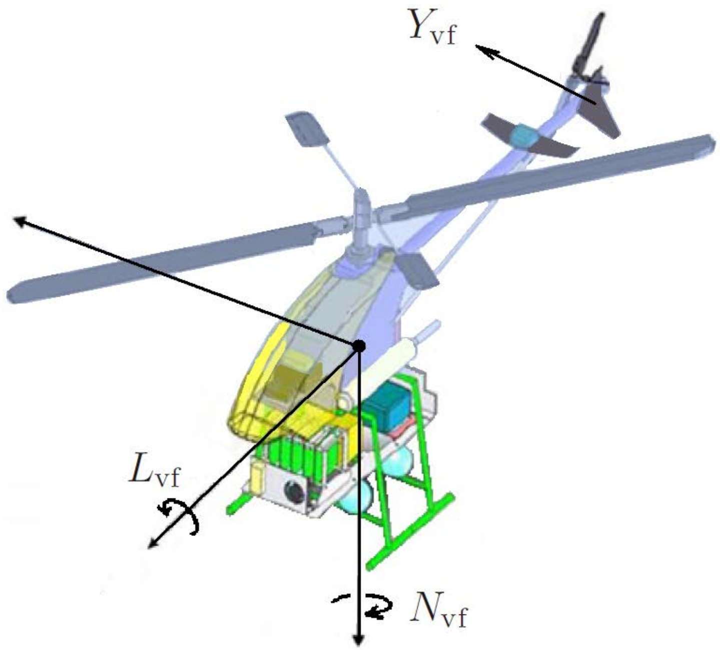
### Vertical fin forces and moments

$$Y_{\rm vf} = \begin{cases} -\dfrac{\rho}{2} C_{{\rm l}\alpha,{\rm vf}}\, S_{\rm vf}\, v_{\rm vf}\, |u_{\rm a}|, & \text{if } \left|\dfrac{v_{\rm vf}}{u_{\rm a}}\right| \le \tan(\alpha_{\rm st}) \\ -\dfrac{\rho}{2} S_{\rm vf}\, v_{\rm vf}\, |v_{\rm vf}|\,, & \text{if } \left|\dfrac{v_{\rm vf}}{u_{\rm a}}\right| > \tan(\alpha_{\rm st}) \ \text{(stalled)} \end{cases}$$

where $v_{\rm vf} = v_{\rm a} + v_{\rm i,tr} - r D_{\rm vf}$

$$L_{\rm vf} = Y_{\rm vf} H_{\rm vf}, \quad N_{\rm vf} = Y_{\rm vf} D_{\rm vf}$$

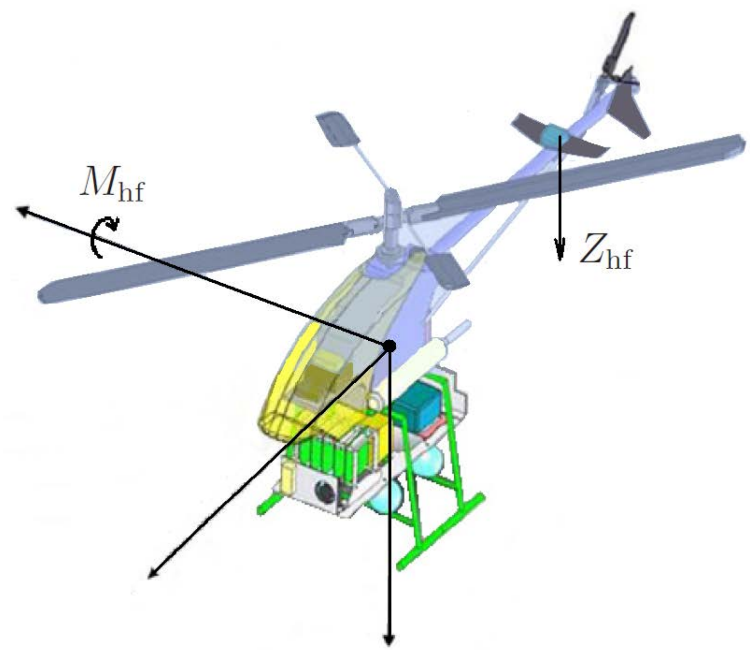



### Horizontal fin forces and moments

$$Z_{\rm hf} = \begin{cases} -\dfrac{\rho}{2} C_{{\rm l}\alpha,{\rm hf}}\, S_{\rm hf}\, w_{\rm hf}\, |u_{\rm a}|\,, & \text{if } \left|\dfrac{w_{\rm hf}}{u_{\rm a}}\right| \le \tan(\alpha_{\rm st}) \\ -\dfrac{\rho}{2} S_{\rm hf}\, w_{\rm hf}\, |w_{\rm hf}|\,, & \text{if } \left|\dfrac{w_{\rm hf}}{u_{\rm a}}\right| > \tan(\alpha_{\rm st}) \ \text{(stalled)} \end{cases}$$

where $w_{\rm hf} = w_{\rm a} + q\, D_{\rm hf} - v_{\rm i,mr}$

$$M_{\rm hf} = Z_{\rm hf} H_{\rm hf}$$



# First-principles Modeling Approach

*Single-rotor helicopter*

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## Direct measurement

| Parameter | Physical meaning |
|---|---|
| $I_{\text{mr}} = 0.055$ kg·m$^2$ | Main blade inertia w.r.t. rotor hub |
| $I_{\text{sb}} = 0.004$ kg·m$^2$ | Stabilizer bar inertia w.r.t. rotor hub |
| $R = 0.7$ m | Main rotor radius |
| $R_{\text{sb,in}} = 0.231$ m | Stabilizer bar inner radius |
| $R_{\text{sb,out}} = 0.312$ m | Stabilizer bar outer radius |
| $R_{\text{tr}} = 0.128$ m | Tail rotor radius |
| $S_{\text{fx}} = 0.103$ m$^2$ | Effective longitudinal fuselage drag area |
| $S_{\text{fy}} = 0.900$ m$^2$ | Effective lateral fuselage drag area |
| $S_{\text{fz}} = 0.084$ m$^2$ | Effective vertical fuselage drag area |
| $S_{\text{hf}} = 0.011$ m$^2$ | Horizontal fin area |
| $S_{\text{vf}} = 0.007$ m$^2$ | Vertical fin area |
| $b = 2$ | Main blade number |
| $b_{\text{tr}} = 2$ | Tail blade number |
| $c = 0.062$ m | Main blade chord length |
| $c_{\text{sb}} = 0.059$ m | Stabilizer bar chord length |
| $c_{\text{tr}} = 0.029$ m | Tail rotor chord length |
| $g = 9.781$ N·kg$^{-1}$ | Acceleration of gravity |
| $m = 9.750$ kg | Helicopter mass |
| $n_{\text{tr}} = 4.650$ | Gear ratio of the tail rotor to the main rotor |
| $\rho = 1.290$ kg/m$^3$ | Air density |

## Ground Tests

| Parameter | Physical meaning | |
|---|---|---|
| $D_{\text{hf}} = 0.751$ m | Horizontal fin location behind the CG | CG Location |
| $D_{\text{tr}} = 1.035$ m | Tail rotor hub location behind the CG | |
| $D_{\text{vf}} = 0.984$ m | Vertical fin location behind the CG | |
| $H_{\text{mr}} = 0.337$ m | Main rotor hub location above the CG | |
| $H_{\text{tr}} = 0.172$ m | Tail rotor hub location above the CG | |
| $H_{\text{vf}} = 0.184$ m | Vertical fin location above the CG | |
| $A_{\text{lon}} = 0.210$ rad | Direct linkage gain from $\delta_{\text{lon}}$ to main blade deflection | Airfoil deflection |
| $B_{\text{lat}} = 0.200$ rad | Direct linkage gain form $\delta_{\text{lat}}$ to main blade deflection | |
| $C_{\text{lon}} = 0.560$ rad | Linkage gain from $\delta_{\text{lon}}$ to stabilizer bar deflection | |
| $D_{\text{lat}} = 0.570$ rad | Linkage gain from $\delta_{\text{lat}}$ to stabilizer bar deflection | |
| $K_{\text{sb}} = 1$ | Ratio of main blade deflection to stabilizer bar TPP titling angle | |
| $I_{\text{xx}} = 0.249$ kg·m$^2$ | Rolling moment of inertia | Inertia |
| $I_{\text{yy}} = 0.548$ kg·m$^2$ | Pitching moment of inertia | |
| $I_{\text{zz}} = 0.787$ kg·m$^2$ | Yawing moment of inertia | |
| $K_{\text{I}}\,K_{\text{a}} = 8.499$ rad | Product of integral gain $K_{\text{I}}$ and scaling factor $K_{\text{a}}$ | Collective Pitch curve |
| $K_{\text{P}}\,K_{\text{a}} = 1.608$ rad | Product of proportional gain $K_{\text{P}}$ and scaling factor $K_{\text{a}}$ | |
| $K_{\text{col}} = -0.165$ rad | Ratio of $\theta_{\text{col}}$ to $\delta_{\text{col}}$ | |
| $K_{\text{ped}} = 1$ | Ratio of $\theta_{\text{ped}}$ to $\bar{\delta}_{\text{ped}}$ | |
| $\theta_{\text{col},0} = 0.075$ rad | $\theta_{\text{col}}$ value when $\delta_{\text{col}}$ is zero | |
| $\theta_{\text{ped},0} = 0.143$ rad | $\theta_{\text{ped}}$ value when $\bar{\delta}_{\text{col}}$ is zero | |

# First-principles Modeling Approach

## Single-rotor helicopter

NUS National University of Singapore

### Wind Tunnel Data

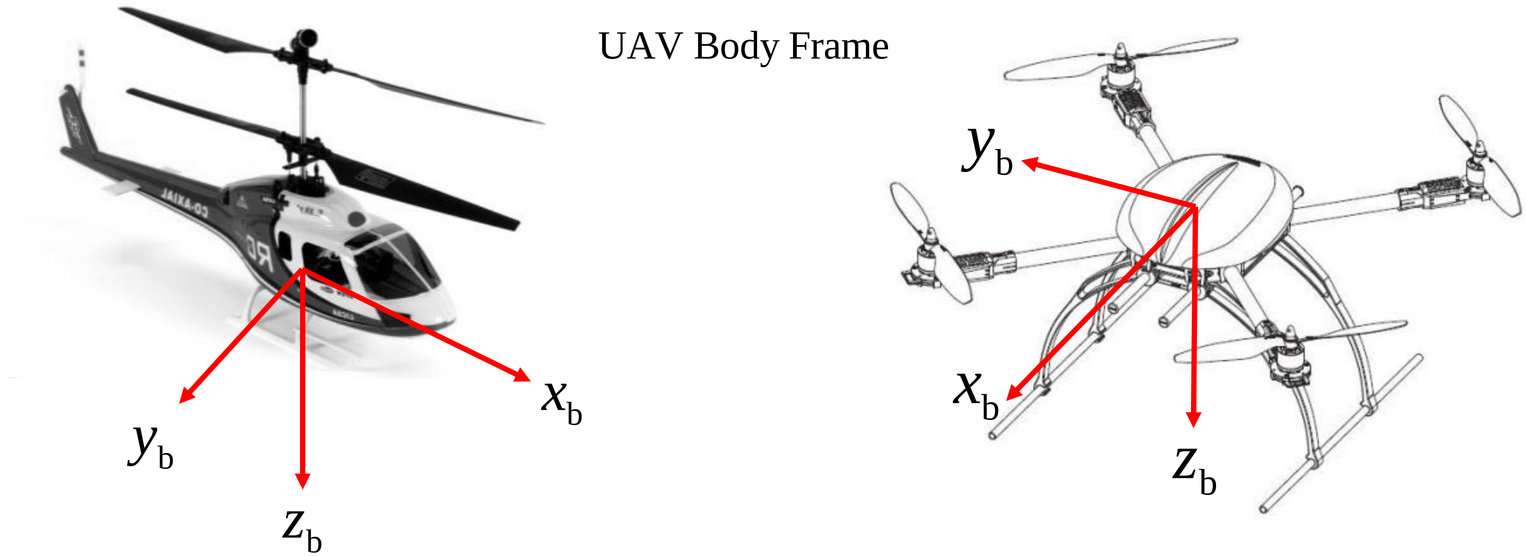
| Parameter | Physical meaning |
|---|---|
| $C_{l\alpha,\mathrm{hf}} = 2.85\ \mathrm{rad}^{-1}$ | Horizontal fin lift curve slope |
| $\hat{C}_{l\alpha,\mathrm{mr}} = 5.71\ \mathrm{rad}^{-1}$ | Main blade lift curve slope |
| $\hat{C}_{l\alpha,\mathrm{sb}} = 2.23\ \mathrm{rad}^{-1}$ | Stabilizer bar lift curve slope |
| $\hat{C}_{l\alpha,\mathrm{tr}} = 2.23\ \mathrm{rad}^{-1}$ | Tail blade lift curve slope |
| $C_{l\alpha,\mathrm{vf}} = 2.85\ \mathrm{rad}^{-1}$ | Vertical fin lift curve slope |

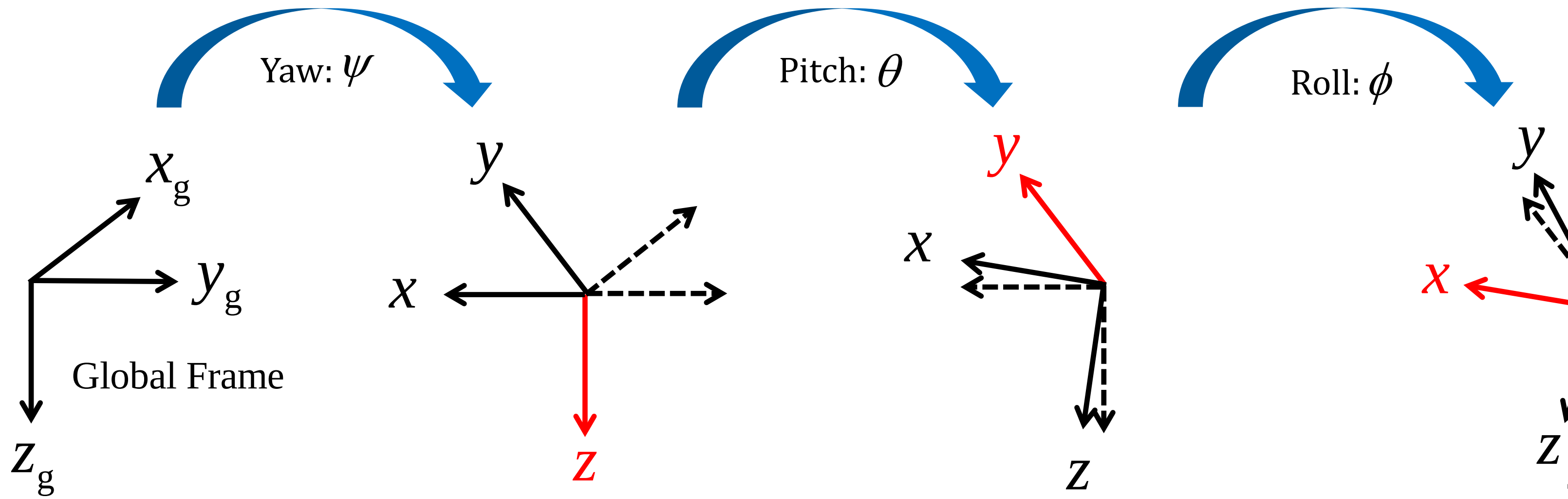
### Actual Flight Tests

| Parameters | Physical meaning |
|---|---|
| $A_{b_s} = 9.720\ \mathrm{sec}^{-1}$ | Coupling effect |
| $B_{a_s} = 10.704\ \mathrm{sec}^{-1}$ | Coupling effect |
| $K_{\mathrm{I}} = 2.2076$ | Integral gain of the embedded PI controller |
| $K_{\mathrm{P}} = 0.4177$ | Proportional gain of the embedded PI controller |
| $K_{\mathrm{a}} = -3.85\ \mathrm{rad}$ | Scaling factor of the amplifier circuit |
| $K_{\beta} = 112.84\ \mathrm{N{\cdot}m}$ | Main rotor spring constant |
| $C_{l\alpha,\mathrm{mr}} = 5.73\ \mathrm{rad}^{-1}$ | Main blade lift curve slope |
| $C_{l\alpha,\mathrm{sb}} = 2.13\ \mathrm{rad}^{-1}$ | Stabilizer bar lift curve slope |
| $C_{l\alpha,\mathrm{tr}} = 2.81\ \mathrm{rad}^{-1}$ | Tail blade lift curve slope |
| $\Omega_{\mathrm{mr}} = 193.73\ \mathrm{rad}$ | Main rotor rotating speed |
| $\Omega_{\mathrm{tr}} = 900.85\ \mathrm{rad}$ | Main rotor rotating speed |

# First-principles Modeling Approach

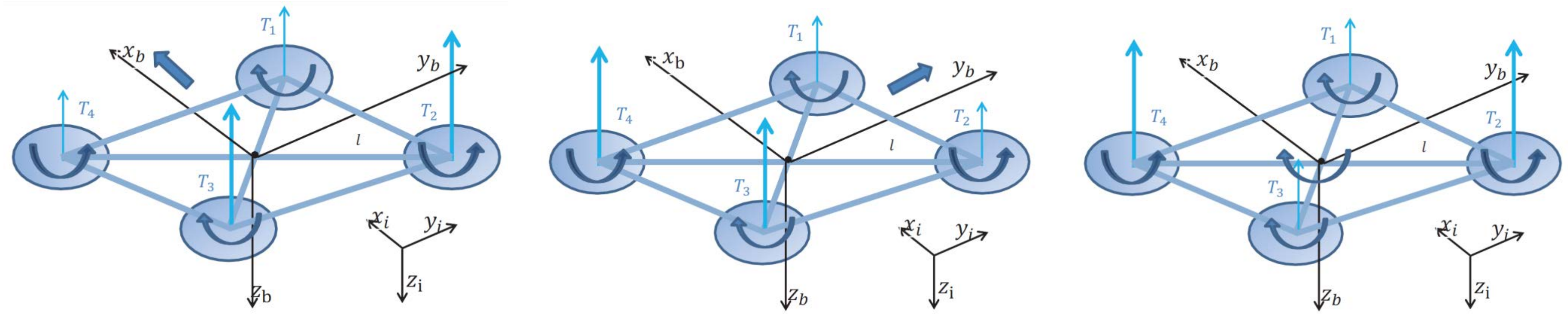
## Coaxial & quadrotor helicopters



# First-principles Modeling Approach

## Quadrotor working principles

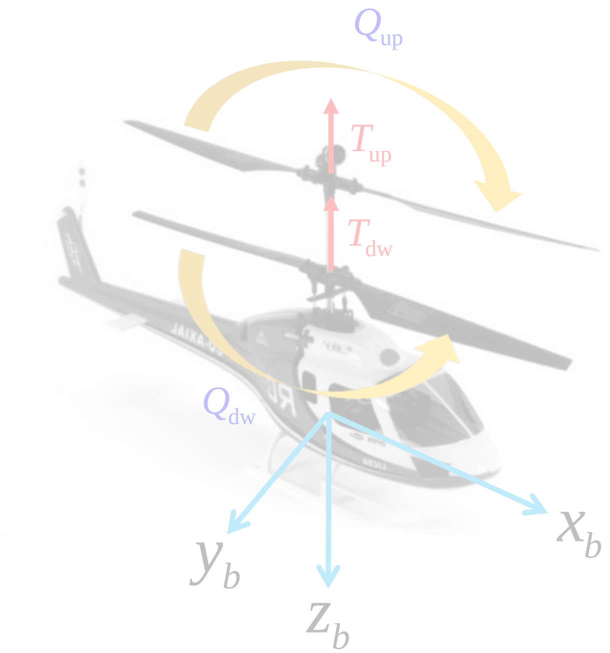


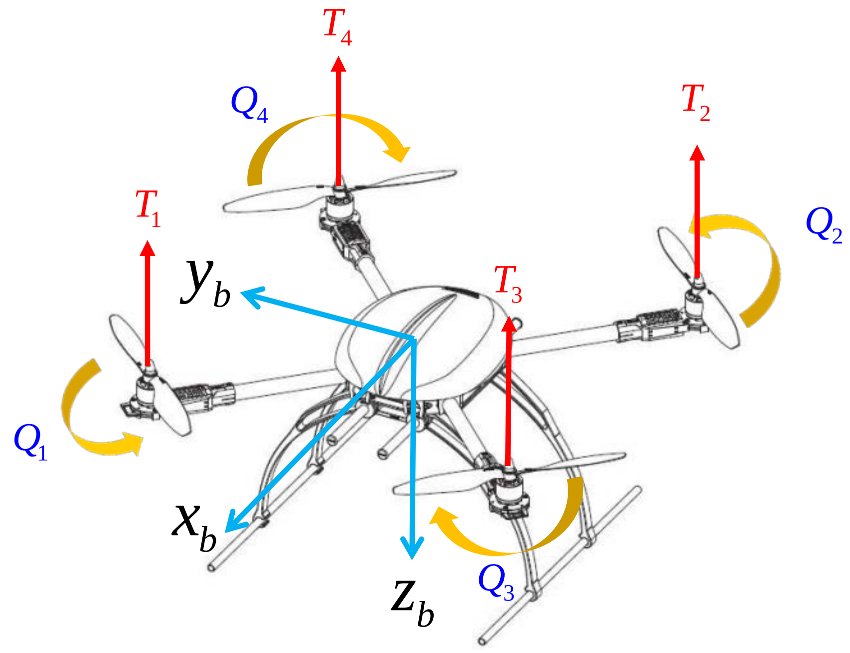

(a) Pitching (b) Rolling (c) Yawing

- Symmetric structure
- All types of motion controlled by adjusting motor speeds
- All thrust forces in the UAV body frame z-axis direction
- Open-loop dynamics fast and unstable

# First-principles Modeling Approach

## Quadrotor force & torque decomposition



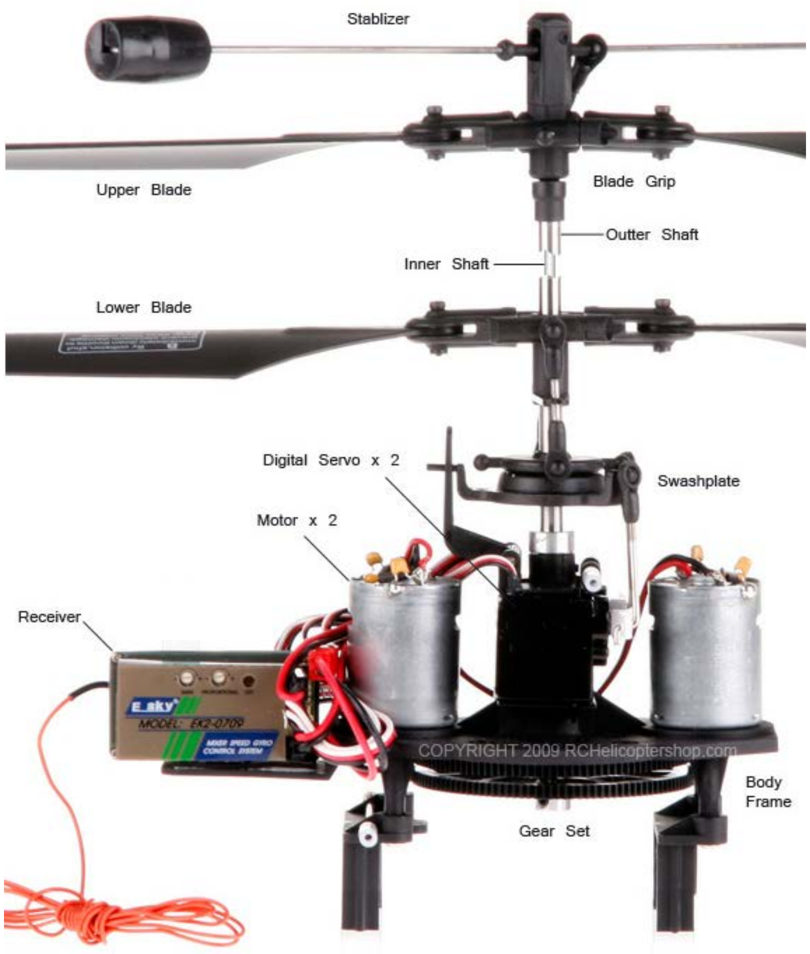

$$\begin{pmatrix} F_x \\ F_y \\ F_z \end{pmatrix} = -\begin{pmatrix} 0 \\ 0 \\ |T_1|+|T_2|+|T_3|+|T_4| \end{pmatrix} + mg\begin{pmatrix} -\sin\theta \\ \sin\phi\cos\theta \\ \cos\phi\cos\theta \end{pmatrix}$$

$$\begin{pmatrix} M_x \\ M_y \\ M_z \end{pmatrix} = \begin{pmatrix} (|T_2|+|T_3|-|T_1|-|T_4|)\times l/\sqrt{2} \\ (|T_1|+|T_3|-|T_2|-|T_4|)\times l/\sqrt{2} \\ |Q_1|+|Q_2|-|Q_3|-|Q_4| \end{pmatrix}$$

# *First-principles Modeling Approach*

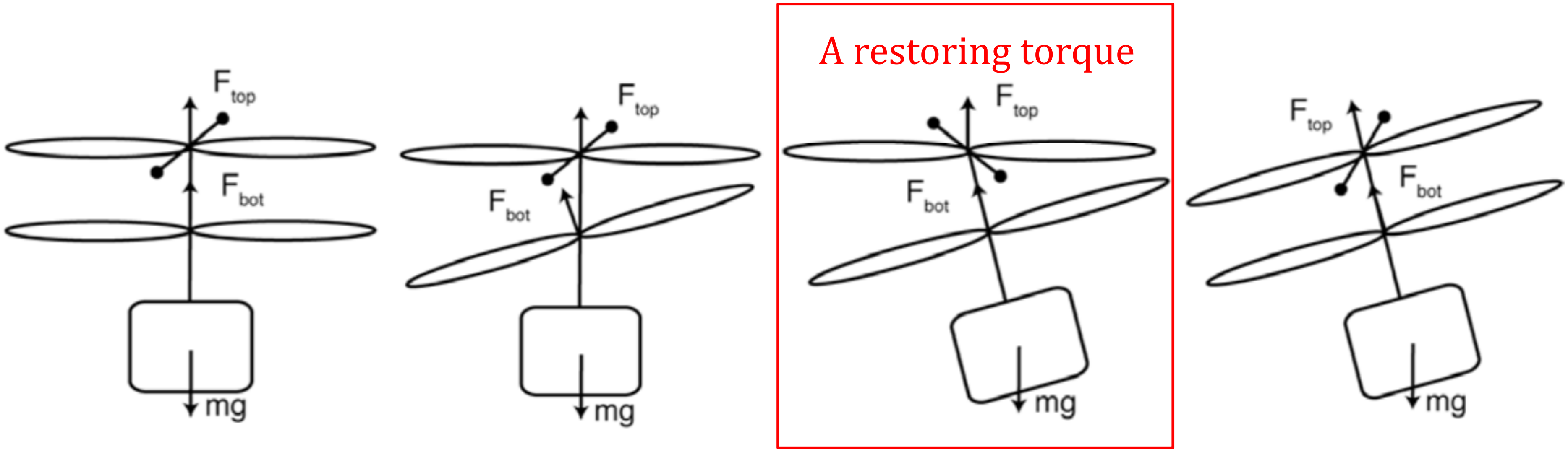
## *Coaxial helicopter working principles*

- Dual main rotors spin in opposite directions
- Rotational speed can be controlled
  - Sum → Heave motion
  - Difference → Yaw motion
- Bottom rotor attached to swashplate
  - Servo 1 → Roll motion
  - Servo 2 → Pitch motion
- Top rotor attached to stabilizer bar
  - Roll & Pitch motion damped



# *First-principles Modeling Approach*

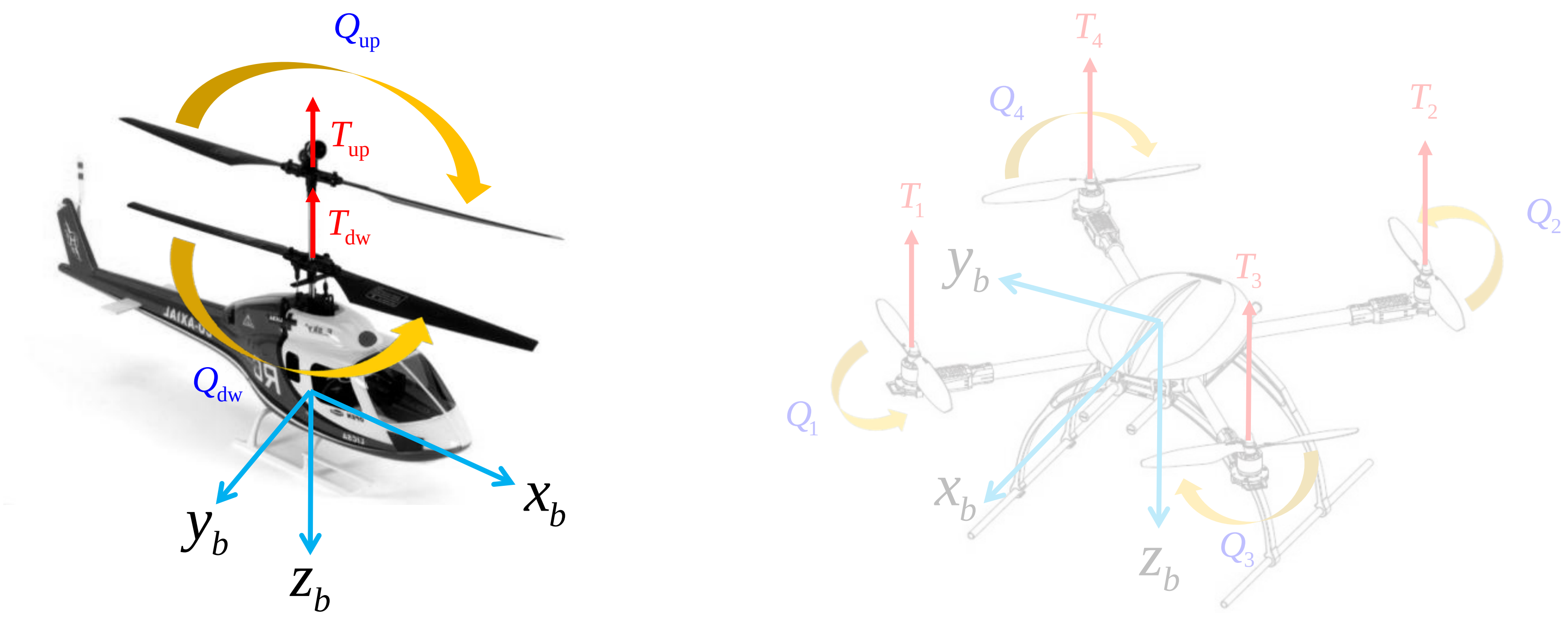
## *Coaxial helicopter working principles (stabilizer bar)*




- Roll and pitch dynamics slowed down
- Stability increased (inherently stable)
- Maneuverability decreased

# First-principles Modeling Approach

## Coaxial helicopter force & torque decomposition

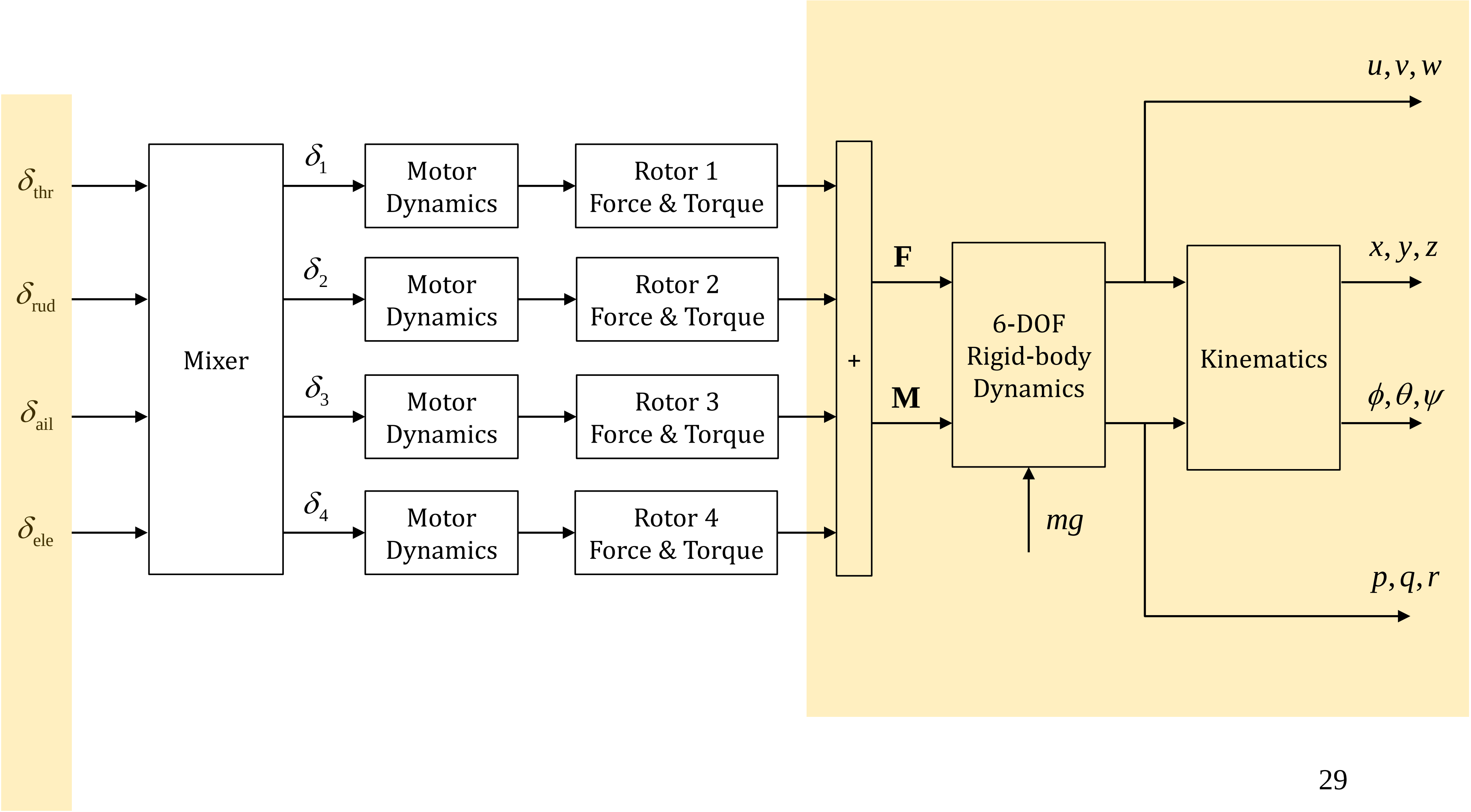



$$\begin{pmatrix} F_x \\ F_y \\ F_z \end{pmatrix} = |T_{up}| \begin{pmatrix} -\sin a_{up} \\ \sin b_{up} \\ -\cos a_{up} \cos b_{up} \end{pmatrix} + |T_{dw}| \begin{pmatrix} -\sin a_{dw} \\ \sin b_{dw} \\ -\cos a_{dw} \cos b_{dw} \end{pmatrix} + mg \begin{pmatrix} -\sin\theta \\ \sin\phi\cos\theta \\ \cos\phi\cos\theta \end{pmatrix}$$

$$\begin{pmatrix} M_x \\ M_y \\ M_z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -l_{up} \end{pmatrix} \times |T_{up}| \begin{pmatrix} -\sin a_{up} \\ \sin b_{up} \\ -\cos a_{up} \cos b_{up} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ -l_{dw} \end{pmatrix} \times |T_{dw}| \begin{pmatrix} -\sin a_{dw} \\ \sin b_{dw} \\ -\cos a_{dw} \cos b_{dw} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ |Q_{up}| - |Q_{dw}| \end{pmatrix}$$

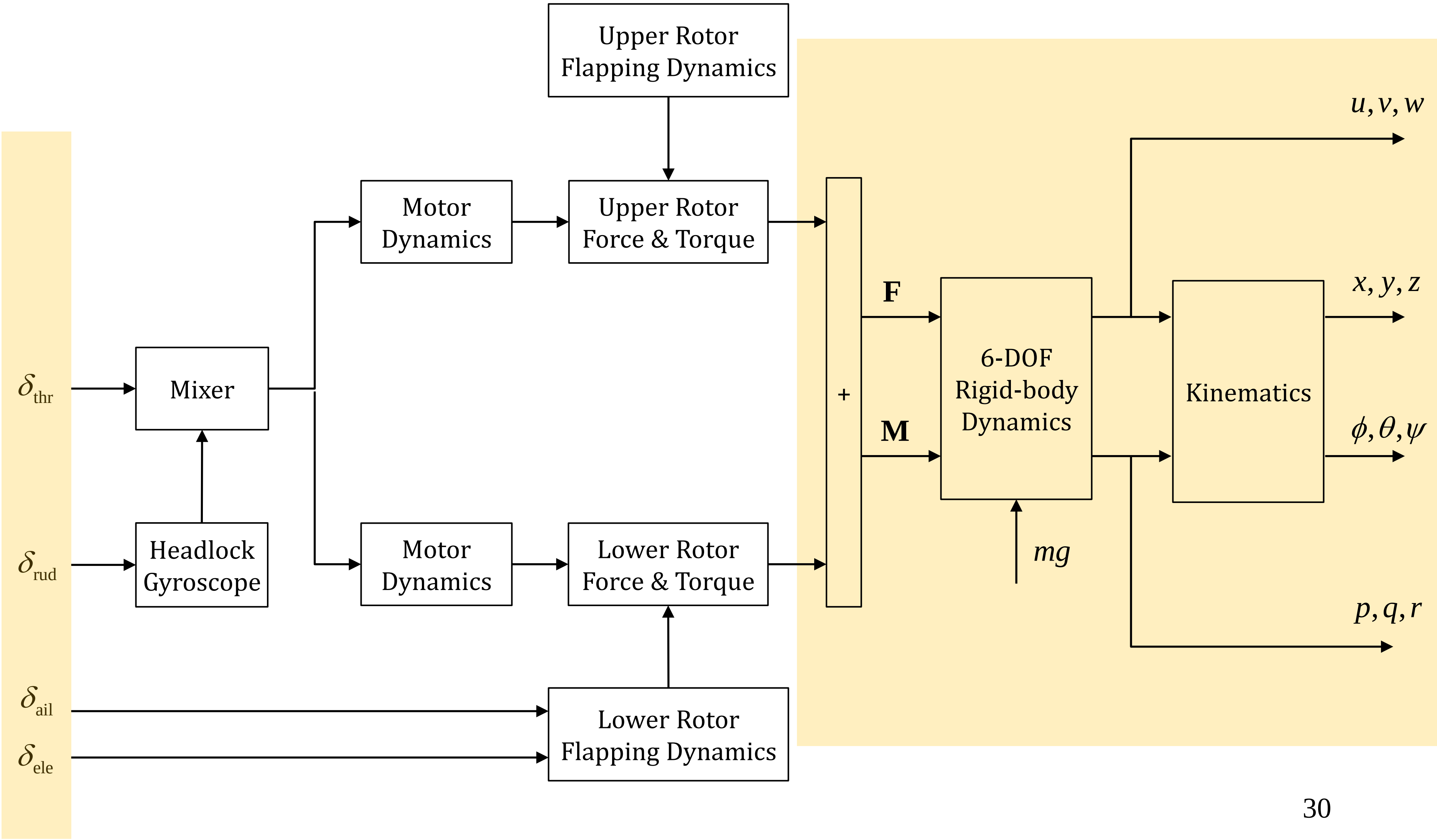
# First-principles Modeling Approach

## Quadrotor model structure



# First-principles Modeling Approach

## Coaxial helicopter model structure



# First-principles Modeling Approach

## Kinematics

Derivative of global-frame position

$\mathbf{R}_{\text{n/b}}$

Body-frame velocity

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{bmatrix} c_\psi c_\theta & c_\psi s_\theta s_\phi - s_\psi c_\phi & c_\psi s_\theta c_\phi + s_\psi s_\phi \\ s_\psi c_\theta & s_\psi s_\theta s_\phi + c_\psi c_\phi & s_\psi s_\theta c_\phi - c_\psi s_\phi \\ -s_\theta & c_\theta s_\phi & c_\theta c_\phi \end{bmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix}$$

$$\begin{pmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{pmatrix} = \begin{bmatrix} 1 & s_\phi s_\theta / c_\theta & c_\phi s_\theta / c_\theta \\ 0 & c_\phi & -s_\phi \\ 0 & s_\phi / c_\theta & c_\phi / c_\theta \end{bmatrix} \begin{pmatrix} p \\ q \\ r \end{pmatrix}$$

Derivative of Euler angles

$\mathbf{S}^{-1}$

Body-frame angular velocity

# *First-principles Modeling Approach*

## *Rigid-body dynamics*

Body-frame acceleration

Body-frame resultant force

$$\begin{pmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{pmatrix} = \frac{1}{m}\begin{pmatrix} F_x \\ F_y \\ F_z \end{pmatrix} - \begin{pmatrix} p \\ q \\ r \end{pmatrix} \times \begin{pmatrix} u \\ v \\ w \end{pmatrix}$$

Cross-coupling between axes

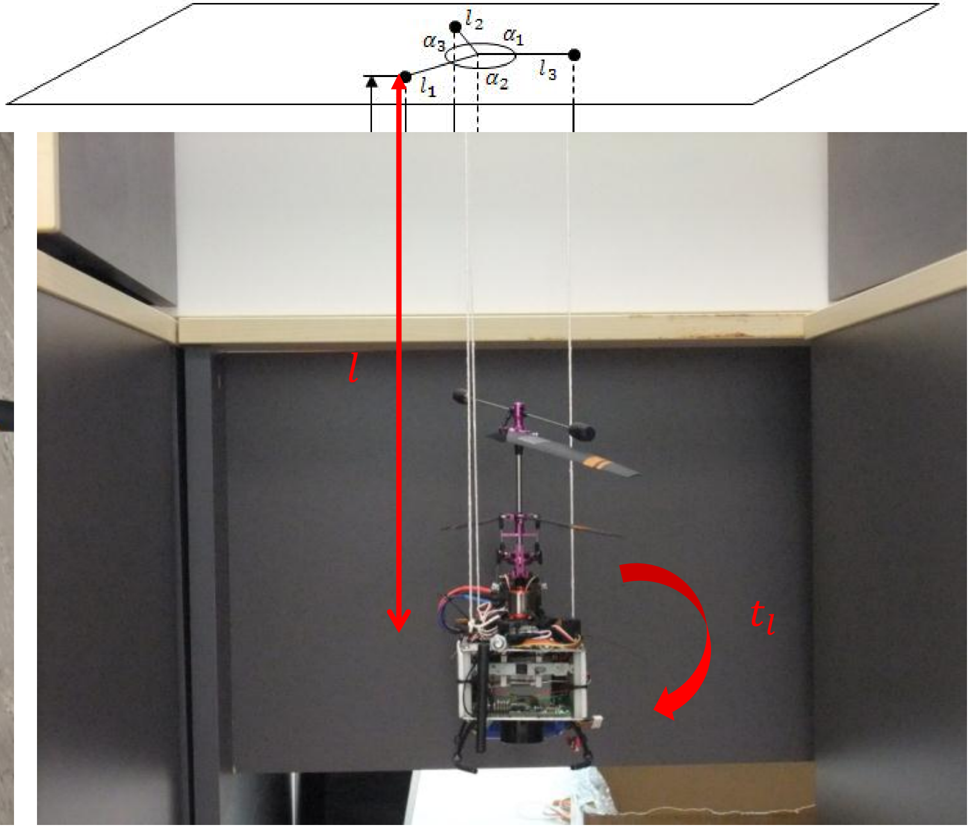
$$\begin{pmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{pmatrix} = \mathbf{J}^{-1}\left\{\begin{pmatrix} M_x \\ M_y \\ M_z \end{pmatrix} - \begin{pmatrix} p \\ q \\ r \end{pmatrix} \times \mathbf{J}\begin{pmatrix} p \\ q \\ r \end{pmatrix}\right\}$$

Body-frame angular acceleration

Body-frame resultant torque

# *First-principles Modeling Approach*

## *Mass & moment of inertia*

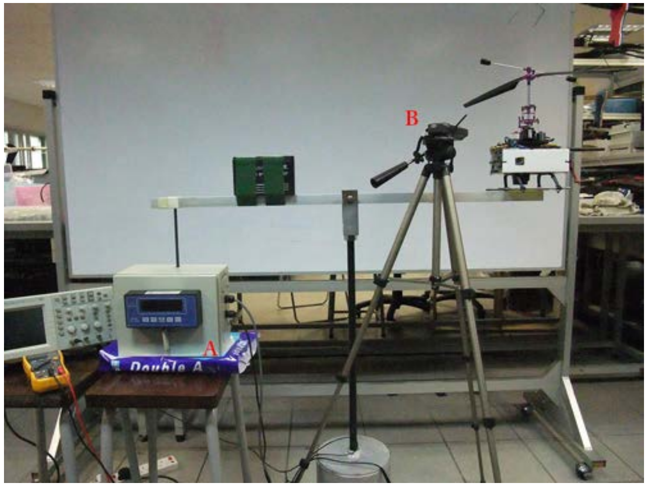



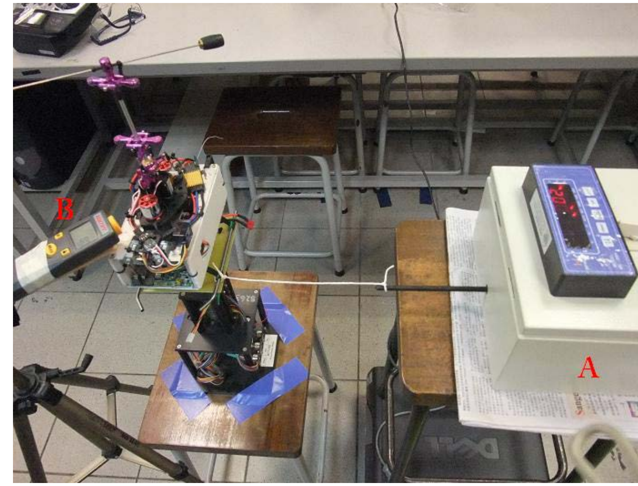

$$m = 0.934\ \text{kg}$$

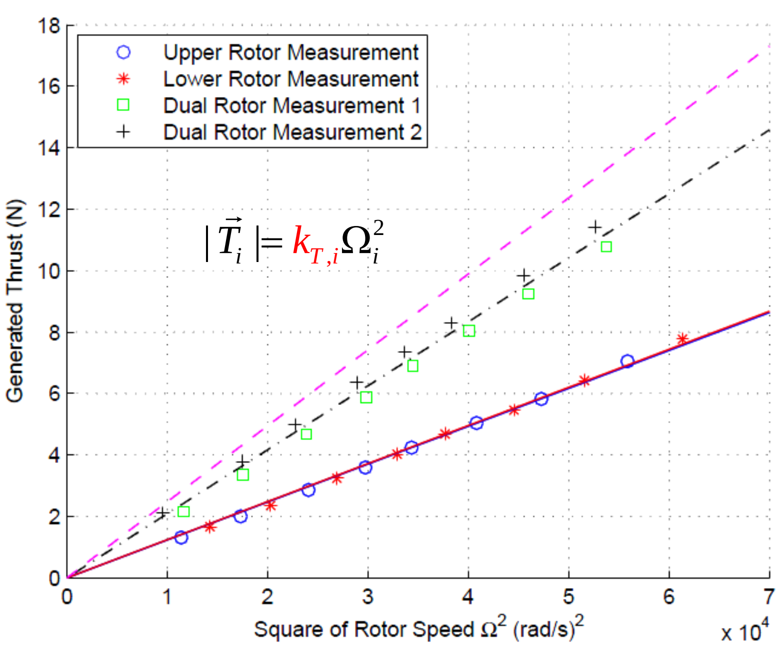
$$J_{zz} = \frac{mgl_1 l_2 l_3 t_l^2}{4\pi^2 l} \cdot \frac{l_1 \sin\alpha_1 + l_2 \sin\alpha_2 + l_3 \sin\alpha_3}{l_2 l_3 \sin\alpha_1 + l_1 l_3 \sin\alpha_2 + l_1 l_2 \sin\alpha_3}$$

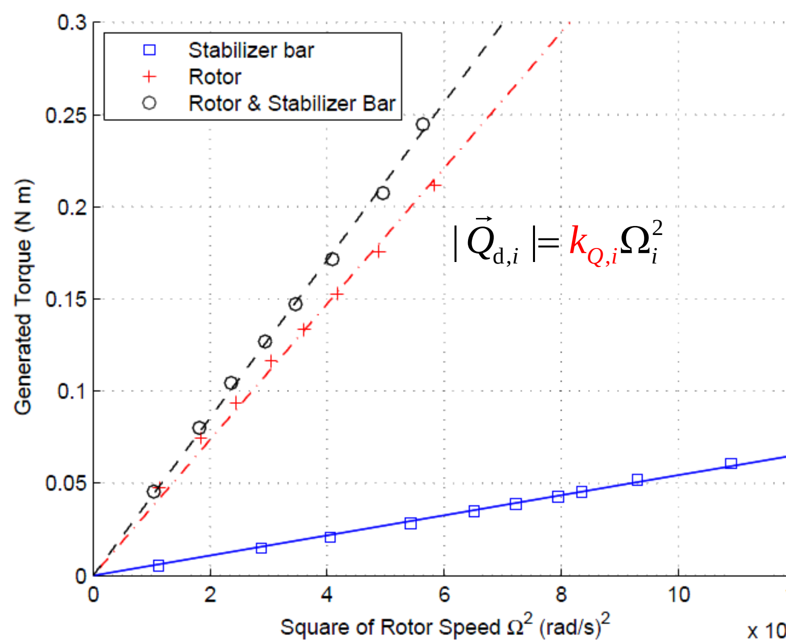
# *First-principles Modeling Approach*

## *Rotor thrust and torque generation*



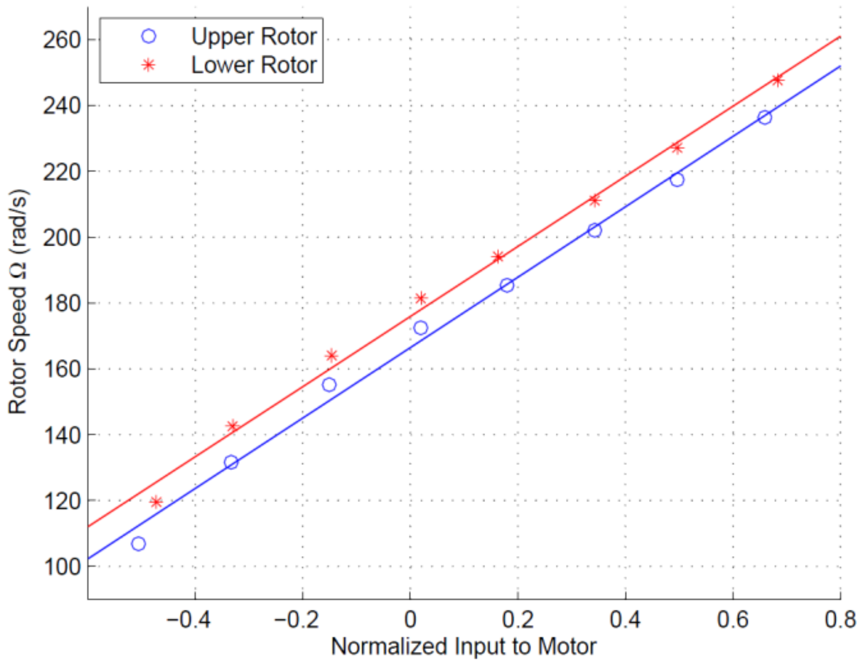



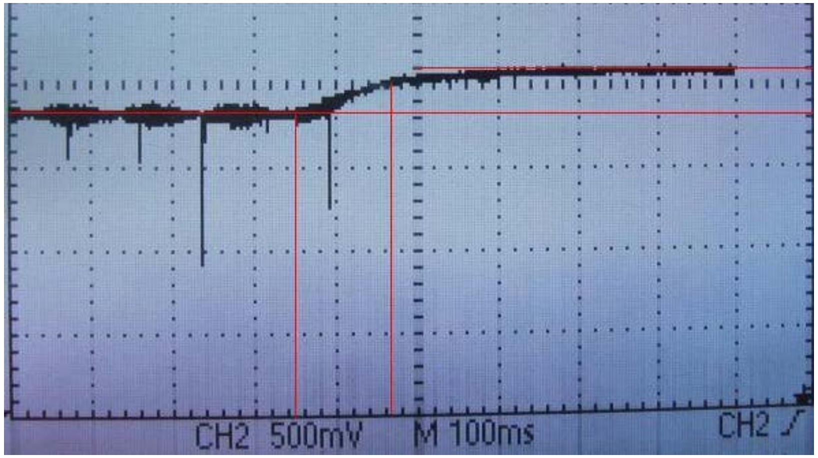



# First-principles Modeling Approach

## Motor dynamics



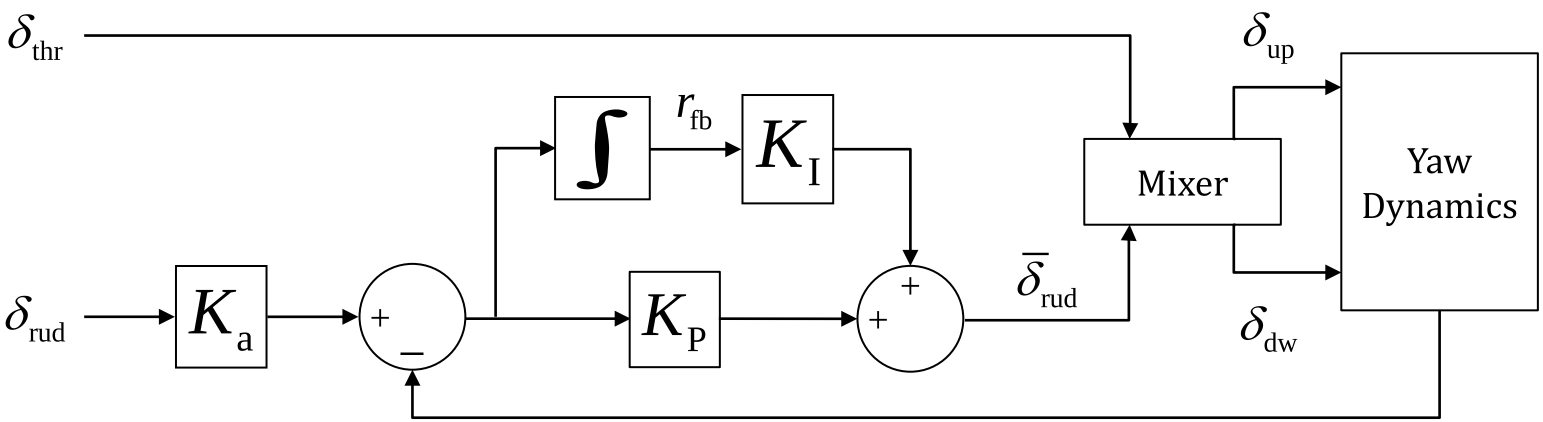




$$\dot{\Omega}_{\text{up}} = \frac{1}{\tau_{\text{mt}}}(m_{\text{up}}\delta_{\text{up}} + \Omega^*_{\text{up}} - \Omega_{\text{up}})$$

$$\dot{\Omega}_{\text{dw}} = \frac{1}{\tau_{\text{mt}}}(m_{\text{dw}}\delta_{\text{dw}} + \Omega^*_{\text{dw}} - \Omega_{\text{dw}})$$

# First-principles Modeling Approach

## Mixer & headlock gyro dynamics




$$\bar{\delta}_{\text{rud}} = K_P(K_a\delta_{\text{rud}} - r) + K_I r_{\text{fb}}$$

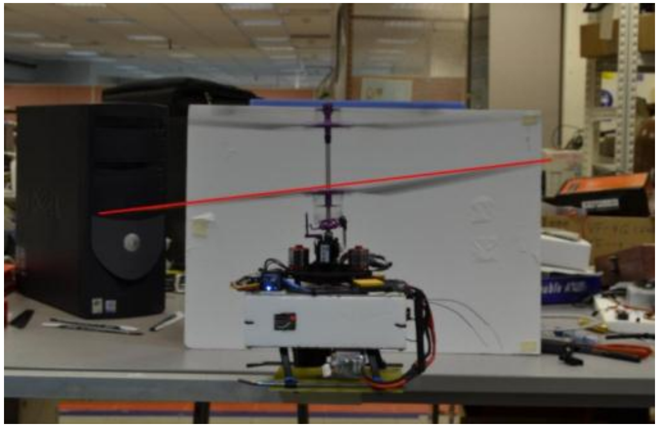
$$\dot{r}_{\text{fb}} = K_a\delta_{\text{rud}} - r$$

$$\delta_{\text{up}} = \delta_{\text{thr}} + \bar{\delta}_{\text{rud}}$$

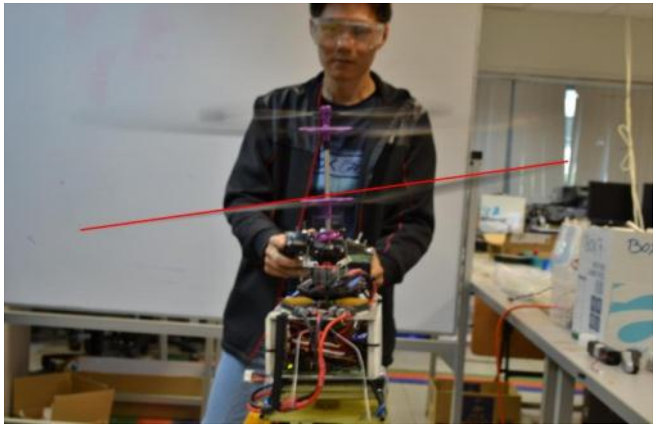
$$\delta_{\text{dw}} = \delta_{\text{thr}} - \bar{\delta}_{\text{rud}}$$

# First-principles Modeling Approach

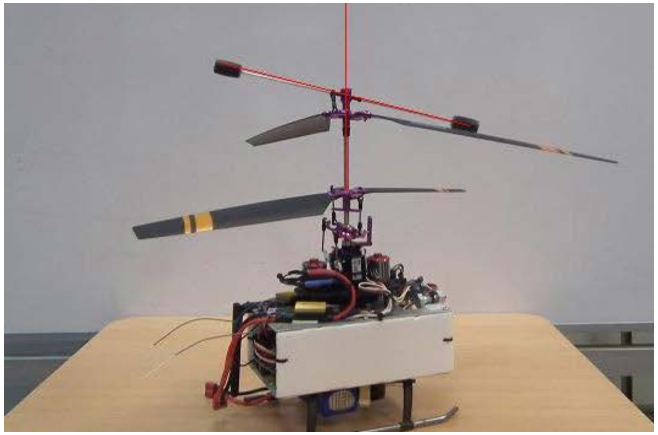
## Tip-path-plane dynamics



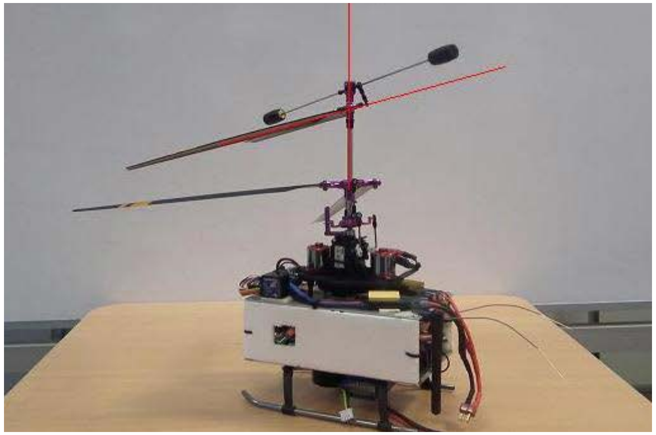
$$a_{\mathrm{dw}} = A_{a,\mathrm{dw}}\,\delta_{\mathrm{ele}} + A_{b,\mathrm{dw}}\,\delta_{\mathrm{ail}}$$



$$b_{\mathrm{dw}} = B_{b,\mathrm{dw}}\,\delta_{\mathrm{ail}} + B_{a,\mathrm{dw}}\,\delta_{\mathrm{ele}}$$



$$\dot{a}_{\mathrm{up}} = -\frac{1}{\tau_{\mathrm{up}}} a_{\mathrm{up}} - A_{a,\mathrm{up}} q - A_{b,\mathrm{up}} p$$



$$\dot{b}_{\mathrm{up}} = -\frac{1}{\tau_{\mathrm{up}}} b_{\mathrm{up}} - B_{b,\mathrm{up}} p - B_{a,\mathrm{up}} q$$

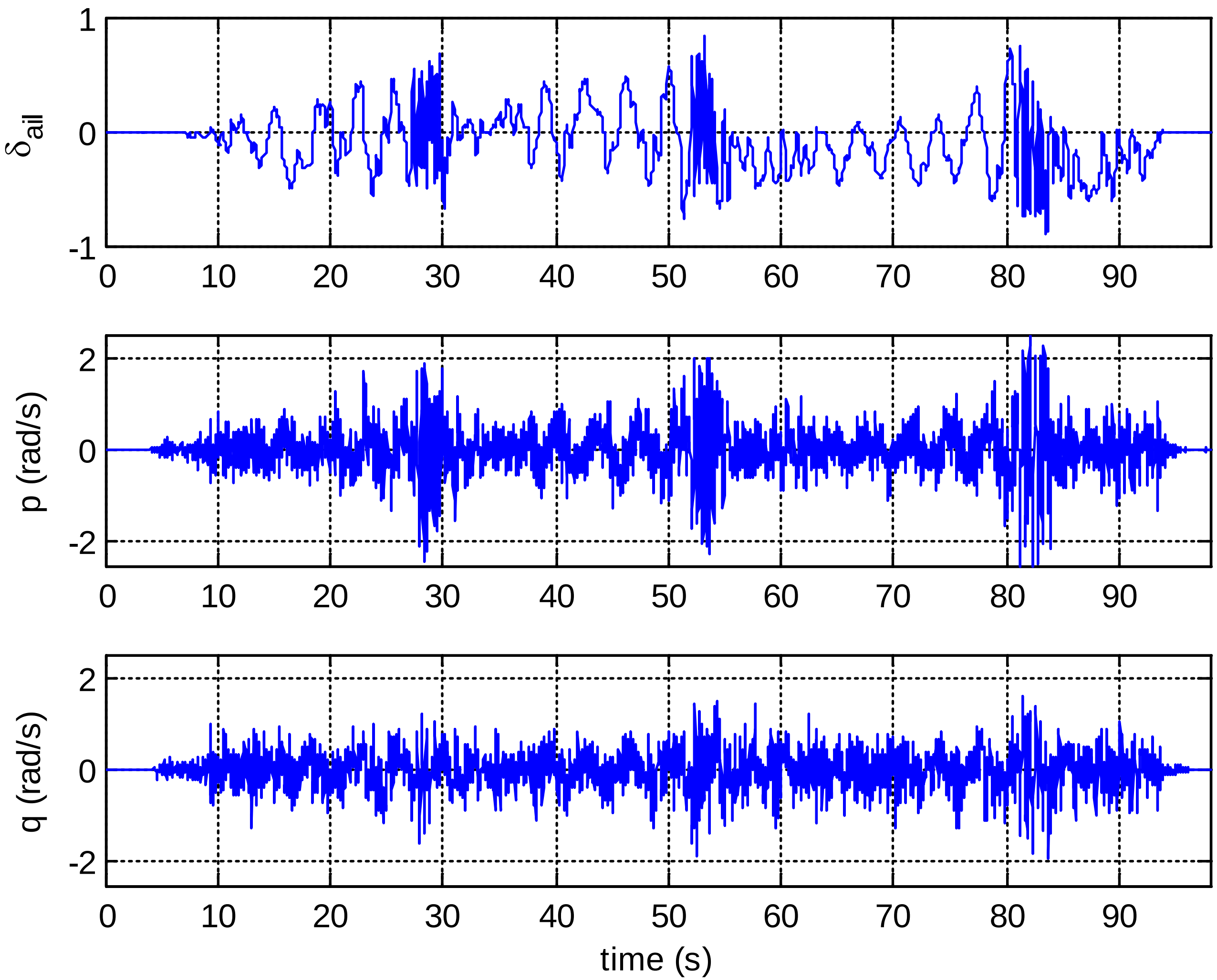
# First-principles Modeling Approach

## Tip-path-plane dynamics

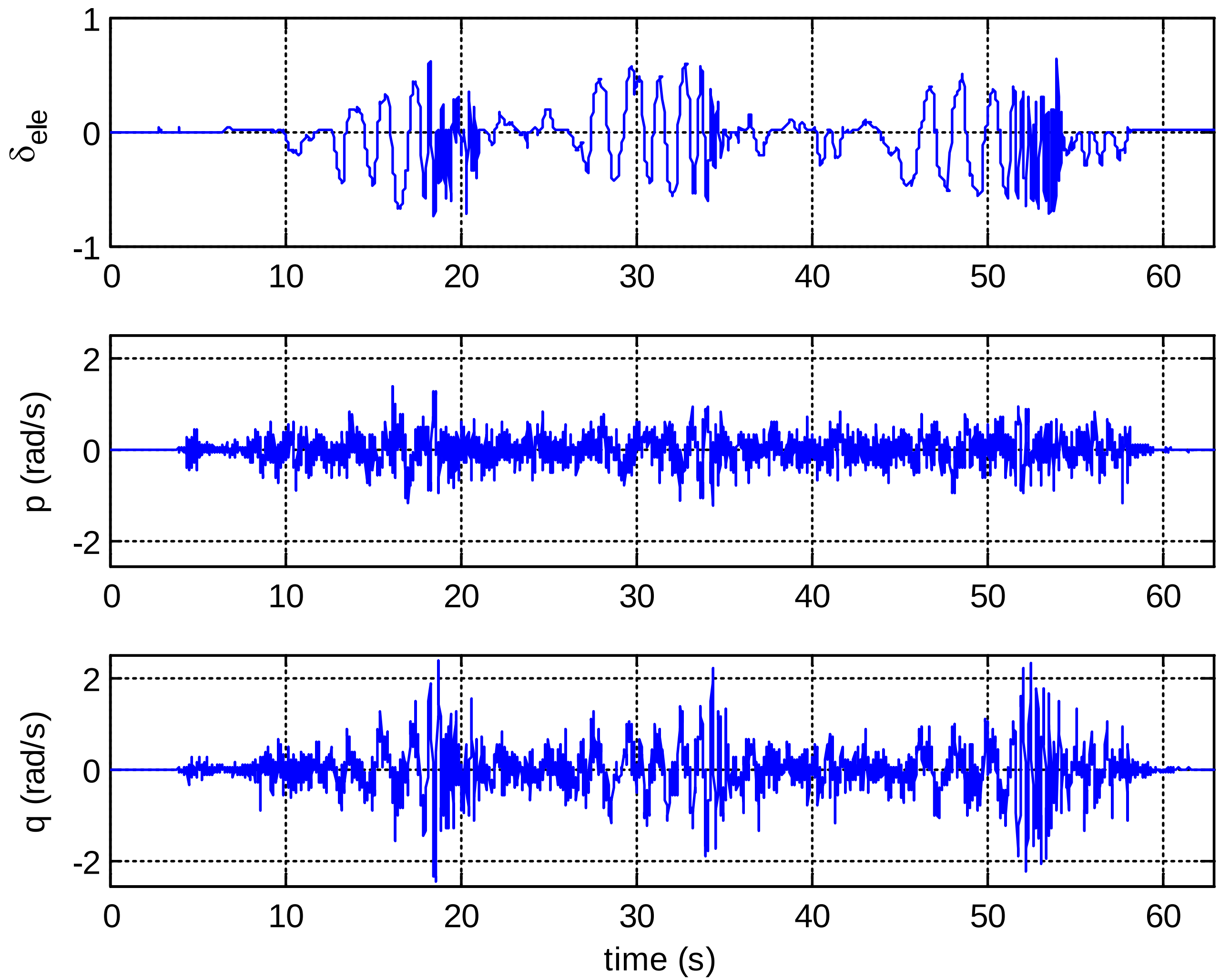
$$\begin{pmatrix} \dot{p} \\ q \\ a_{\text{up}} \\ b_{\text{up}} \end{pmatrix} = \begin{bmatrix} \frac{-X_{\text{dw}} B_{p,\text{dw}}}{J_{xx}} & 0 & 0 & \frac{X_{\text{up}}}{J_{xx}} \\ 0 & \frac{-X_{\text{dw}} A_{q,\text{dw}}}{J_{yy}} & \frac{X_{\text{up}}}{J_{yy}} & 0 \\ -A_{b,\text{up}} & -A_{a,\text{up}} & -\frac{1}{\tau_{\text{sb}}} & 0 \\ -B_{b,\text{up}} & -B_{a,\text{up}} & 0 & -\frac{1}{\tau_{\text{sb}}} \end{bmatrix} \begin{pmatrix} p \\ q \\ a_{\text{up}} \\ b_{\text{up}} \end{pmatrix} + \begin{bmatrix} \frac{X_{\text{dw}} B_{b,\text{dw}}}{J_{xx}} & \frac{X_{\text{dw}} B_{a,\text{dw}}}{J_{xx}} \\ \frac{X_{\text{dw}} A_{b,\text{dw}}}{J_{yy}} & \frac{X_{\text{dw}} A_{a,\text{dw}}}{J_{yy}} \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} \delta_{\text{ail}} \\ \delta_{\text{ele}} \end{pmatrix}$$

$$\begin{bmatrix} \dot{p} \\ q \\ a_{\text{up}} \\ b_{\text{up}} \end{bmatrix} = \begin{bmatrix} -17.19 & 0 & 0 & 934.1 \\ 0 & -5.360 & 291.3 & 0 \\ 0.2745 & -0.49 & -5 & 0 \\ -0.49 & -0.2745 & 0 & -5 \end{bmatrix} \begin{bmatrix} p \\ q \\ a_{\text{up}} \\ b_{\text{up}} \end{bmatrix} + \begin{bmatrix} -102.48 & -38.08 \\ -11.73 & 31.95 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \delta_{\text{ail}} \\ \delta_{\text{ele}} \end{bmatrix}$$

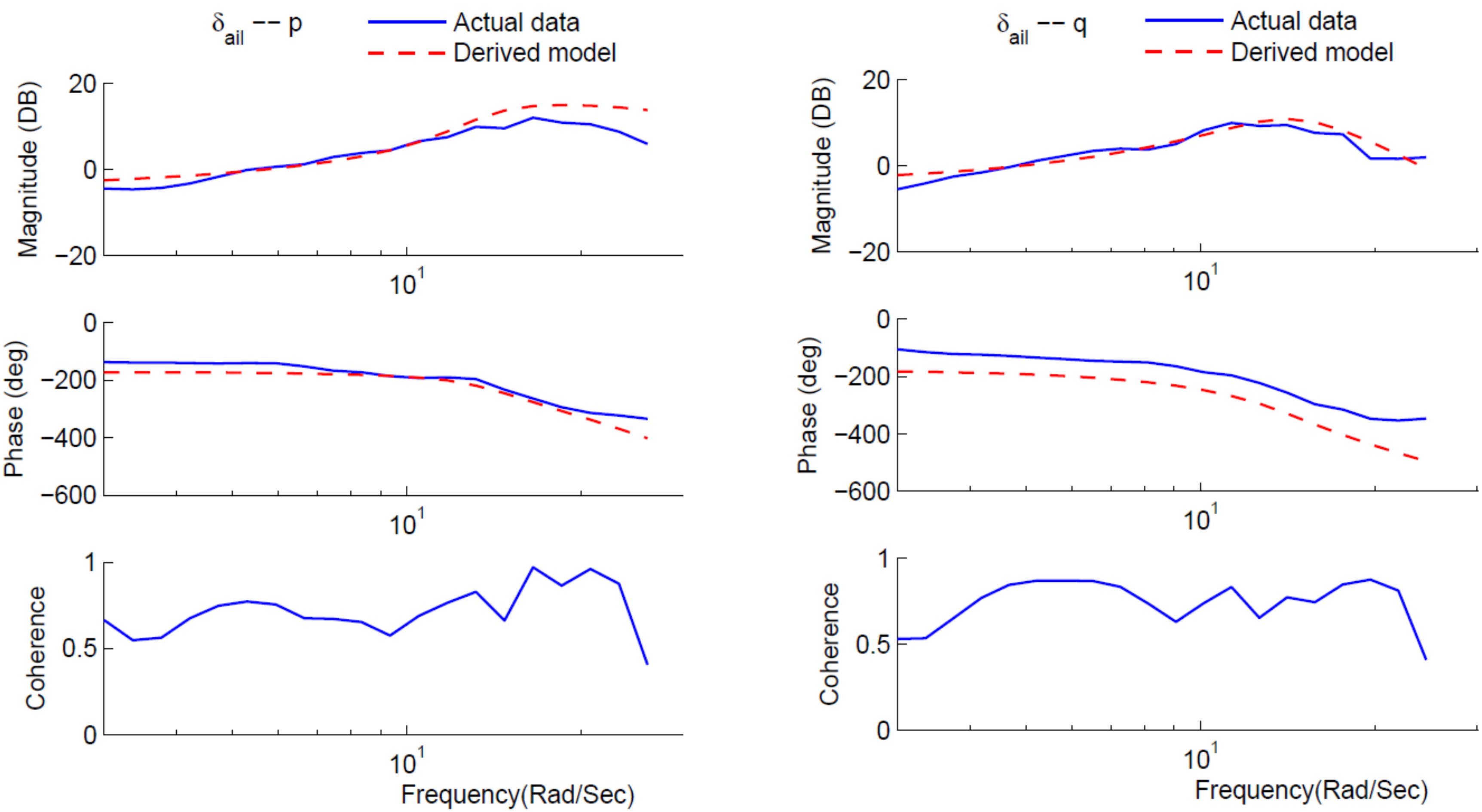
Aileron input frequency sweep



Elevator input frequency sweep



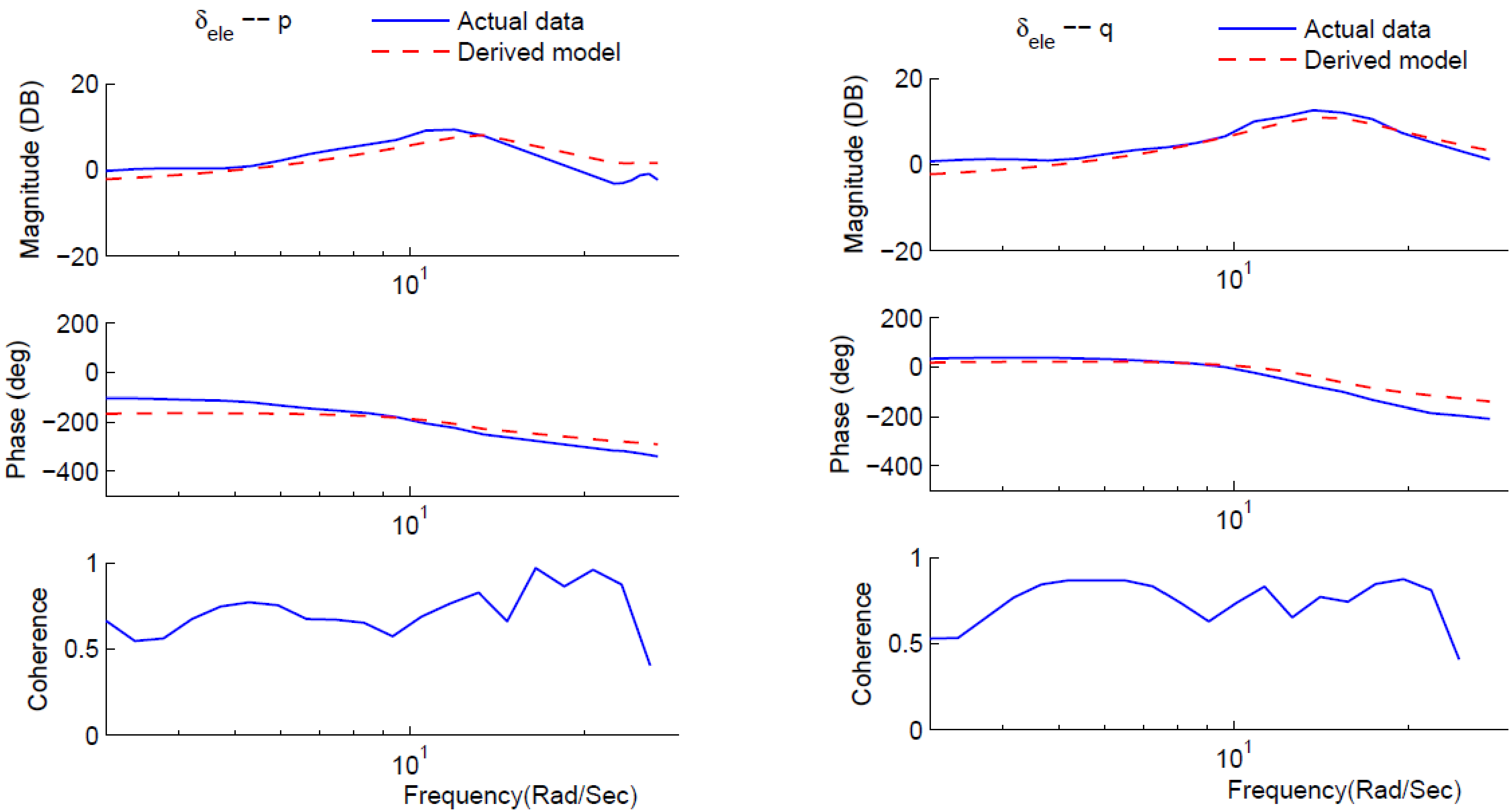
Parameter identification via aileron frequency sweep



# *First-principles Modeling Approach*

## *Tip-path-plane dynamics*

Parameter identification via elevator frequency sweep



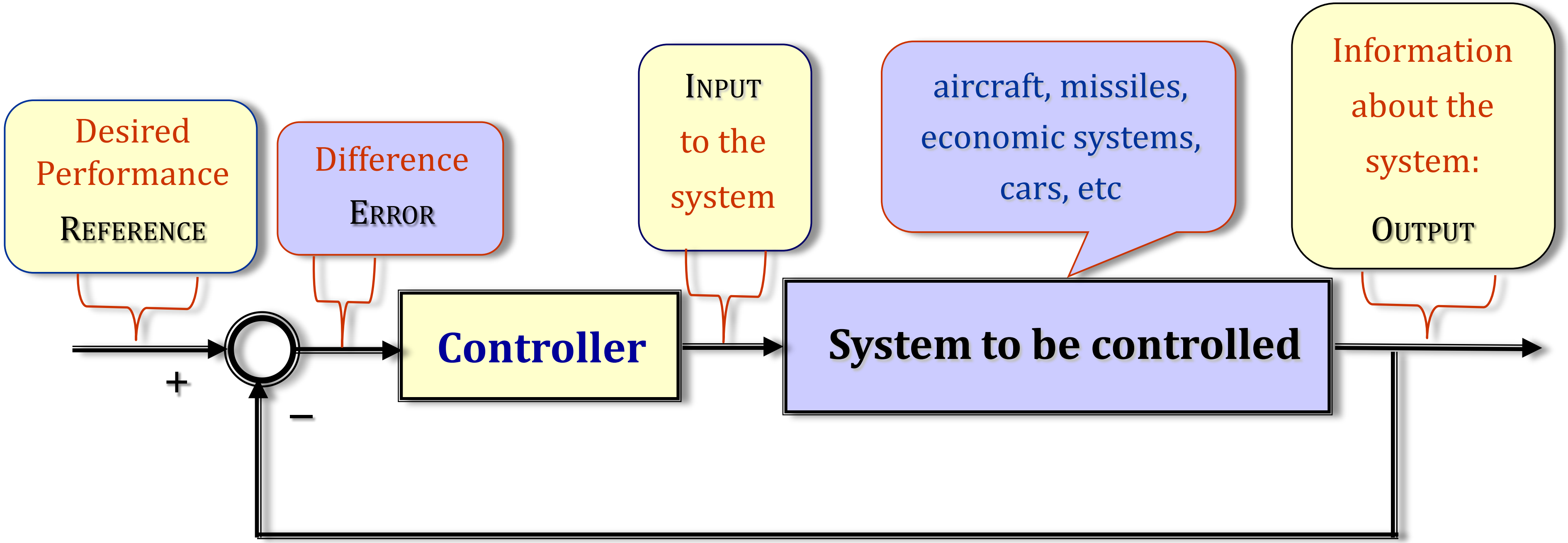
# First-principles Modeling Approach

## Identified parameters of coaxial helicopter

1. Direct measurement
2. Test-bench experiment
3. Flight test data

| | | |
|---|---|---|
| $\rho = 1.204$<br>$m = 0.977$<br>$R = 0.250$<br>$g = 9.781$ | $m_{\text{up}} = 106.90$<br>$m_{\text{dw}} = 106.45$<br>$\tau_{\text{sb}} = 0.2$<br>$\tau_{\text{mt}} = 0.12$ | $\Omega^*_{\text{up}} = 203.38$<br>$\Omega^*_{\text{dw}} = 217.88$<br>$K_\beta = 4.377$ |
| $S_{fx} = 0.00835$<br>$S_{fy} = 0.01310$<br>$S_{fz} = 0.01700$ | $K_{\text{a}} = 6.4267$<br>$K_{\text{P}} = 0.667/K_{\text{a}}$<br>$K_{\text{I}} = 0.713/K_{\text{a}}$ | $J_{xx} = 0.0059$<br>$J_{yy} = 0.0187$<br>$J_{zz} = 0.0030$ |
| $l_{up} = 0.195$<br>$l_{dw} = 0.120$ | $J_{\text{up}} = 6.8613 \cdot 10^{-4}$<br>$J_{\text{dw}} = 3.2906 \cdot 10^{-4}$ | $A_q = 0.0204$<br>$B_p = 0.0204$ |
| $k_{T,\text{up}} = 1.23 \cdot 10^{-4}$<br>$k_{T,\text{dw}} = 8.50 \cdot 10^{-5}$<br>$k_{Q,\text{up}} = 4.23 \cdot 10^{-6}$<br>$k_{Q,\text{dw}} = 3.68 \cdot 10^{-6}$ | $A_{a,\text{up}} = 0.4900$<br>$A_{b,\text{up}} = -0.2745$<br>$B_{a,\text{up}} = 0.2745$<br>$B_{b,\text{up}} = 0.4900$ | $A_{a,\text{dw}} = 0.1217$<br>$A_{b,\text{dw}} = -0.0450$<br>$B_{a,\text{dw}} = -0.0450$<br>$B_{b,\text{dw}} = -0.1217$ |

## *What is a control system?*

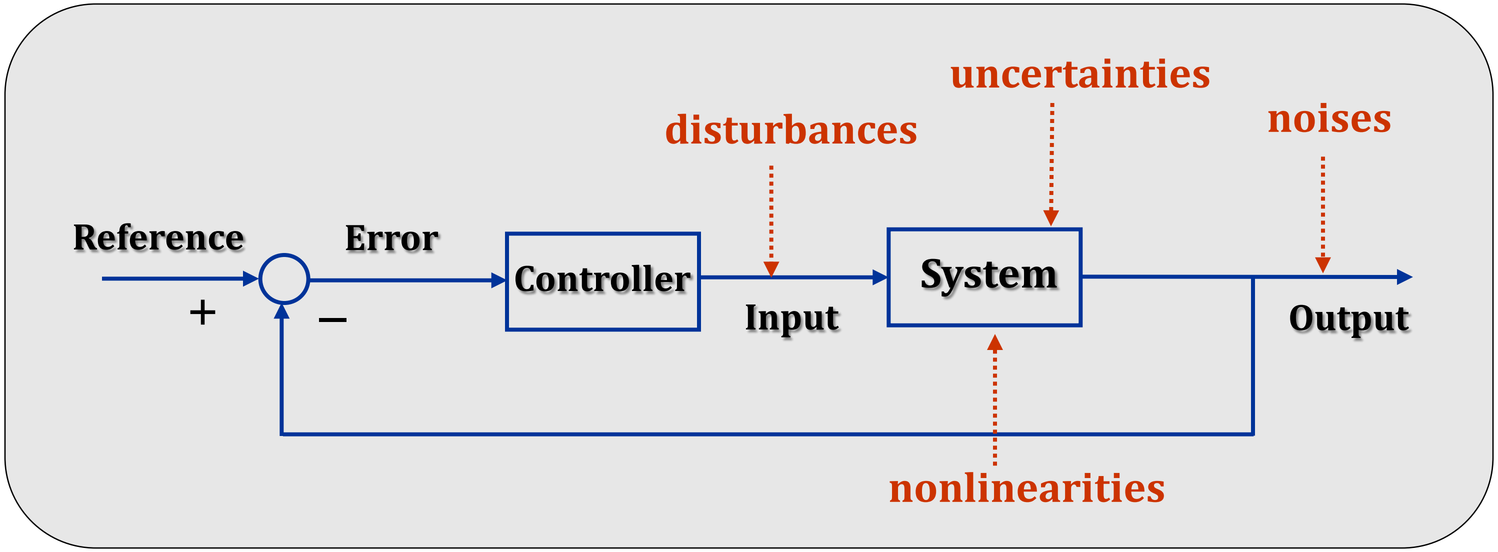



**Objective:** To make the system **OUTPUT** and the desired **REFERENCE** as close as possible, i.e., to make the **ERROR** as small as possible.

**Key Issues:** (1) How to describe the system to be controlled? **(Modeling)**

(2) How to design the controller? **(Control)**

There are many other factors of life have to be carefully considered when designing a control system for real-life problems. These factors include:



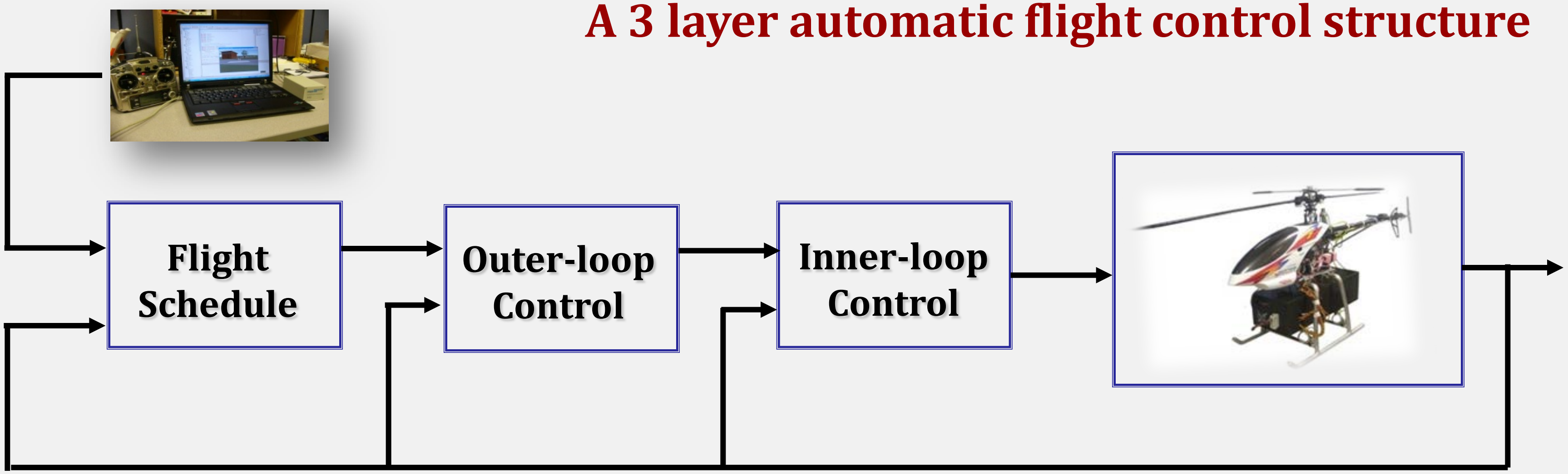

These real-life factors, i.e., disturbances, uncertainties, nonlinearities, give birth of many modern control techniques…

The following is my personal view on the clarification of control techniques:

- Classical control: PID control, developed in 1940s and utilized heavily for in industrial processes. Examples: everywhere in life…
- Optimal control: Linear quadratic regulator (LQR), $H_2$ optimal control, Kalman filter, developed in 1960s to achieve certain optimal performance and boomed by *NASA Apollo Project*.
- Robust control: $H_\infty$ control, developed in 1980s and 90s to handle systems with uncertainties and disturbances and with high performances.
- Nonlinear control: Still on-going research topics, developed to handle nonlinear systems with high performances.
- Intelligent control: Knowledge-based control, adaptive control, neural and fuzzy control, etc., researched heavily in 1990s, developed to handle systems with unknown models. Examples: economic systems, social systems...

## *Flight control system structure*

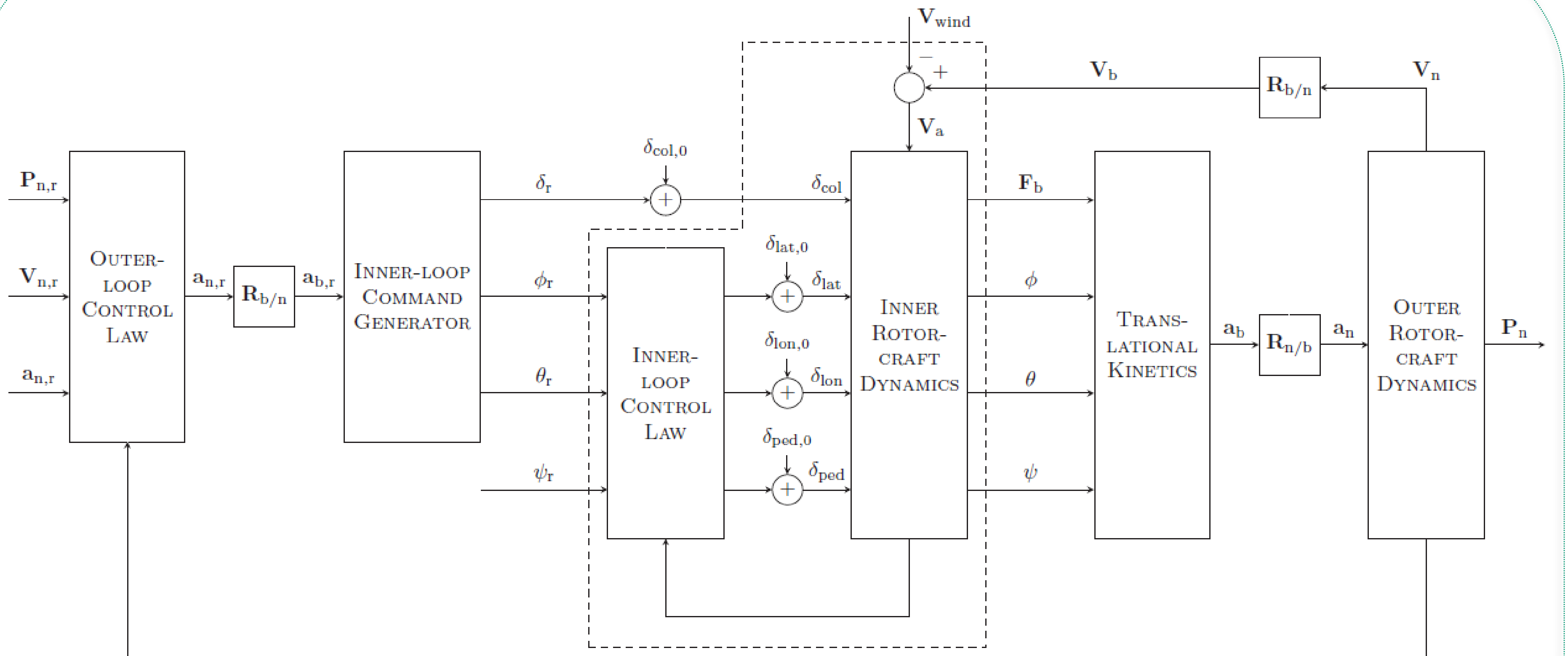
### A 3 layer automatic flight control structure




- Inner loop control is to guarantee the stability of the aircraft attitude
- Outer loop control is to control the aircraft position
- Flight schedule is to generate references for flight missions

# *UAV Control System Design*

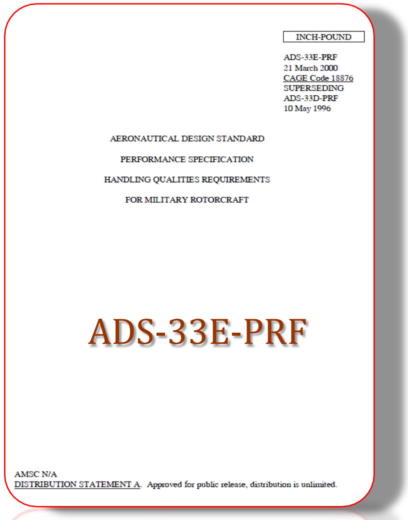
## *Detailed structure of inner- and outer-loop control*



Inner-loop control is to stabilize the overall aircraft and to control its attitude

We adopt the design specifications set by the USA military organization for military rotorcraft, which place strict requirements on:

- closed-loop stability
- bandwidth of pitch and roll attitude response
- coupling effect between roll and pitch channels
- coupling effect from heave control to yaw response
- disturbance rejection
- quickness of pitch, roll and yaw responses
- attitude hold for spike disturbance input



INCH-POUND

ADS-33E-PRF
21 March 2000
CAGE Code 18876
SUPERSEDING
ADS-33D-PRF
10 May 1996

AERONAUTICAL DESIGN STANDARD

PERFORMANCE SPECIFICATION

HANDLING QUALITIES REQUIREMENTS

FOR MILITARY ROTORCRAFT

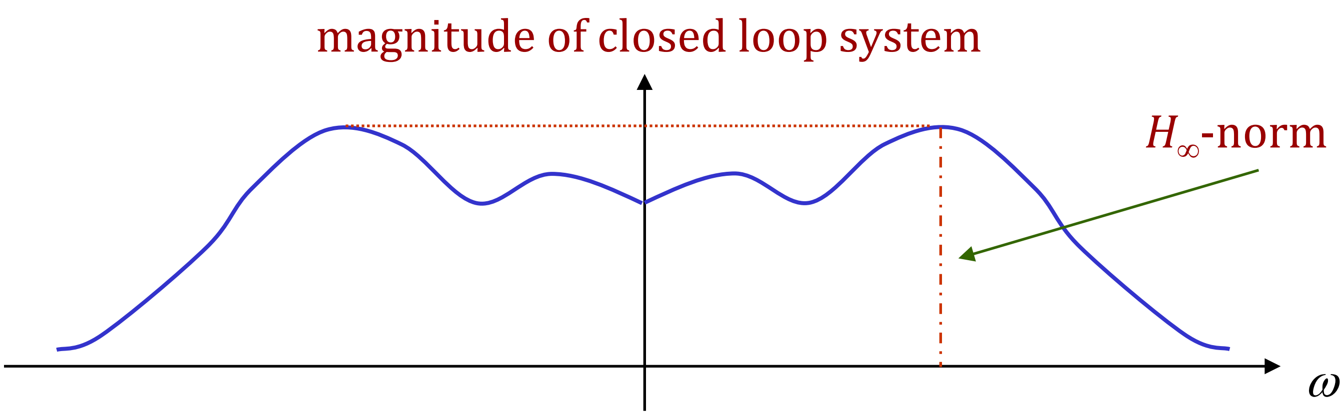
ADS-33E-PRF

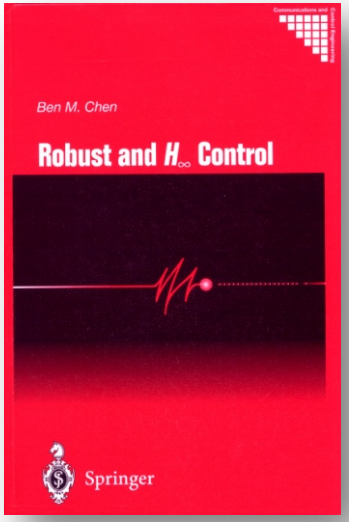
AMSC N/A
DISTRIBUTION STATEMENT A. Approved for public release, distribution is unlimited.

We aim to achieve **Level 1 performance** in all specifications in accordance with the military standards set by U.S. Army Aviation (ADS-33E-PRF).

## Control technique adopted for the inner loop

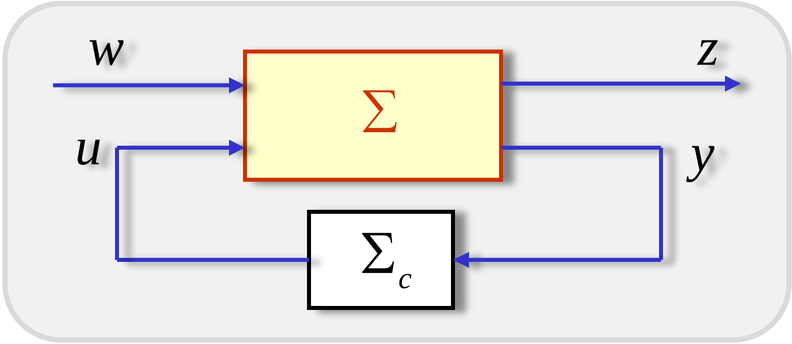
For the inner loop, we treat wind gusts as disturbance entering the system and then formulate the overall problem as an $H_\infty$ control problem, which is to design a control law such that when it is applied to the given system, the resulting closed-loop system is stable and the effect of the disturbance to the output to be controlled (the position of the aircraft in our case) is minimized in $H_\infty$ sense:




***$H_\infty$* optimization** is closely and commonly related to ***robust control***…

Consider a stabilizable and detectable linear time-invariant system Σ with a proper controller $\Sigma_c$




where

$$\Sigma: \begin{cases} \dot{x} = A\,x + B\,u + E\,w \\ y = C_1\,x + \qquad + D_1\,w \\ z = C_2\,x + D_2\,u \end{cases} \qquad \Sigma_c: \begin{cases} \dot{v} = A_{\mathrm{cmp}}\,v + B_{\mathrm{cmp}}\,y \\ u = C_{\mathrm{cmp}}\,v + D_{\mathrm{cmp}}\,y \end{cases}$$

$$\begin{cases} x & \Leftrightarrow & \text{state variable} \\ y & \Leftrightarrow & \text{measurement} \\ z & \Leftrightarrow & \text{controlled output} \end{cases} \qquad \begin{matrix} u & \Leftrightarrow & \text{control input} \\ w & \Leftrightarrow & \text{disturbance} \\ v & \Leftrightarrow & \text{controller state} \end{matrix}$$

The problem of $H_\infty$ control is to design a control law $\Sigma_c$ such that when it is applied to the given plant with disturbance, i.e., $\Sigma$, we have

- The resulting closed loop system is internally stable (this is necessary for any control system design).
- The $H_\infty$-norm of the resulting closed-loop transfer function from the disturbance to the controlled output is as small as possible, i.e., the effect of the disturbance on the controlled output is minimized.

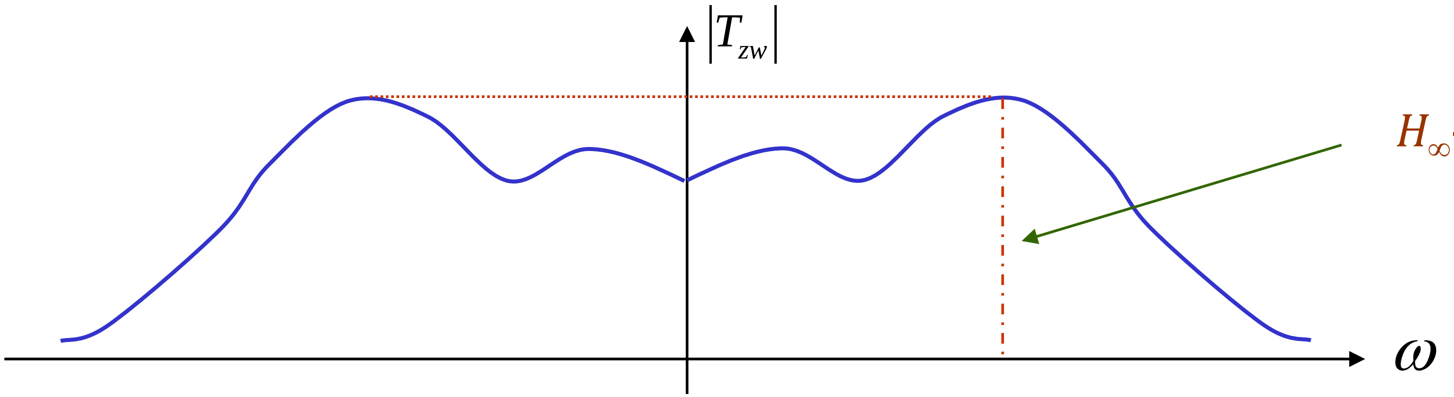
**Note:** A transfer function is a function of frequencies ranging from 0 to $\infty$. It is hard to tell if it is large or small. The common practice is to measure its norms instead. $H_2$-norm and $\boldsymbol{H_\infty}$**-norm** are two commonly used norms in measuring the sizes of a transfer function.

## Interpretation of $H_\infty$ norm

Given a stable and proper transfer function $T_{zw}(s)$, its $H_\infty$-norm is defined as

$$\|T_{zw}\|_\infty = \sup_{0\le\omega<\infty} \sigma_{\max}\left[T_{zw}(j\omega)\right]$$

where $\sigma_{\max}[T_{zw}(j\omega)]$ denotes the maximum singular value of $T_{zw}(j\omega)$. For a SISO transfer function $T_{zw}(s)$, it is equivalent to the magnitude of $T_{zw}(j\omega)$. Graphically,




**Note:** The $H_\infty$-norm is the worst case gain in $T_{zw}(s)$. Thus, minimization of the $H_\infty$-norm of $T_{zw}(s)$ is equivalent to the minimization of the worst case (gain) situation on the effect from the disturbance $w$ to the controlled output $z$.

Most results in $H_\infty$ control deal with a so-called a regular problem or regular case because it is simple. An $H_\infty$ control problem is said to be **regular** if the following conditions are satisfied,

1. $D_2$ is of maximal column rank, i.e., $D_2$ is a tall and full rank matrix
2. The subsystem ( $A,B,C_2,D_2$ ) has no invariant zeros on the imaginary axis;
3. $D_1$ is of maximal row rank, i.e., $D_1$ is a fat and full rank matrix
4. The subsystem ( $A,E,C_1,D_1$ ) has no invariant zeros on the imaginary axis.

An $H_\infty$ control problem is said to be **singular** if it is not regular, i.e., at least one of the above 4 conditions is not satisfied.

## Solution to regular $H_\infty$ state feedback problem

Given $\gamma > \gamma_\infty^*$ (see the note below), solve the following algebraic Riccati equation

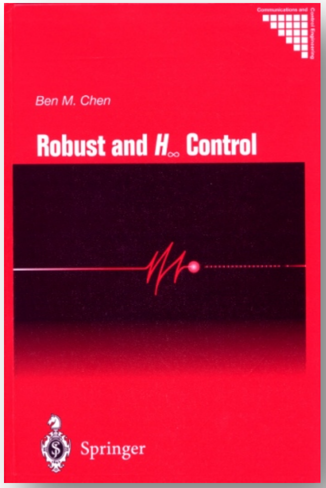
$$A^{\mathrm{T}}P + PA + C_2^{\mathrm{T}}C_2 + PEE^{\mathrm{T}}P/\gamma^2 - (PB + C_2^{\mathrm{T}}D_2)\left(D_2^{\mathrm{T}}D_2\right)^{-1}(D_2^{\mathrm{T}}C_2 + B^{\mathrm{T}}P) = 0$$

for a unique positive semi-definite solution $P \geq 0$. The $H_\infty$ state feedback law is then given by

$$u = F\,x = -(D_2^{\mathrm{T}}D_2)^{-1}\left(D_2^{\mathrm{T}}C_2 + B^{\mathrm{T}}P\right)x$$

The resulting closed-loop system $T_{zw}(s)$ has the following property: $\| T_{zw} \|_\infty < \gamma$.

**Note:** The computation of the best achievable $H_\infty$ attenuation level, $\gamma_\infty^*$, is very complicated. For certain cases, $\gamma_\infty^*$ can be computed exactly. Generally, $\gamma_\infty^*$ can only be obtained using some iterative algorithms. One method is to keep solving the Riccati equation for different values of $\gamma$ until it hits $\gamma_\infty^*$ for which and any $\gamma < \gamma_\infty^*$, the Riccati equation does not have a solution. See Chen (2000) for details.



## Solution to singular $H_\infty$ state feedback problem

$$\tilde{C}_2 := \begin{bmatrix} C_2 \\ \varepsilon I \\ 0 \end{bmatrix} \quad \text{and} \quad \tilde{D}_2 := \begin{bmatrix} D_2 \\ 0 \\ \varepsilon I \end{bmatrix}$$

**Step 1:** Given a $\gamma > \gamma_\infty^*$, choose $\varepsilon = 1$.

**Step 2:** Define the corresponding $\tilde{C}_2$ and $\tilde{D}_2$

**Step 3:** Solve the following Riccati equation for $\tilde{P}$:

$$A^{\mathrm{T}}\tilde{P} + \tilde{P}A + \tilde{C}_2^{\mathrm{T}}\tilde{C}_2 + \tilde{P}EE^{\mathrm{T}}\tilde{P}/\gamma^2 - (\tilde{P}B + \tilde{C}_2^{\mathrm{T}}\tilde{D}_2)\left(\tilde{D}_2^{\mathrm{T}}\tilde{D}_2\right)^{-1}(\tilde{D}_2^{\mathrm{T}}\tilde{C}_2 + B^{\mathrm{T}}\tilde{P}) = 0$$

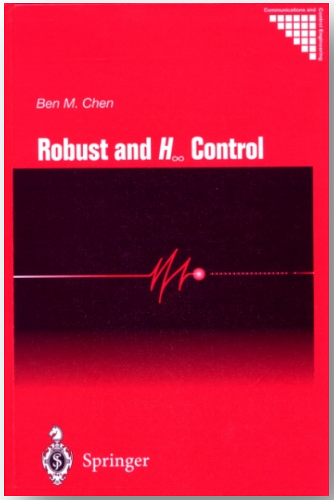
**Step 4:** If $\tilde{P} > 0$, go to Step 5. Otherwise, reduce the value of $\varepsilon$ and go to Step 2.

**Step 5:** Compute the required state feedback control law

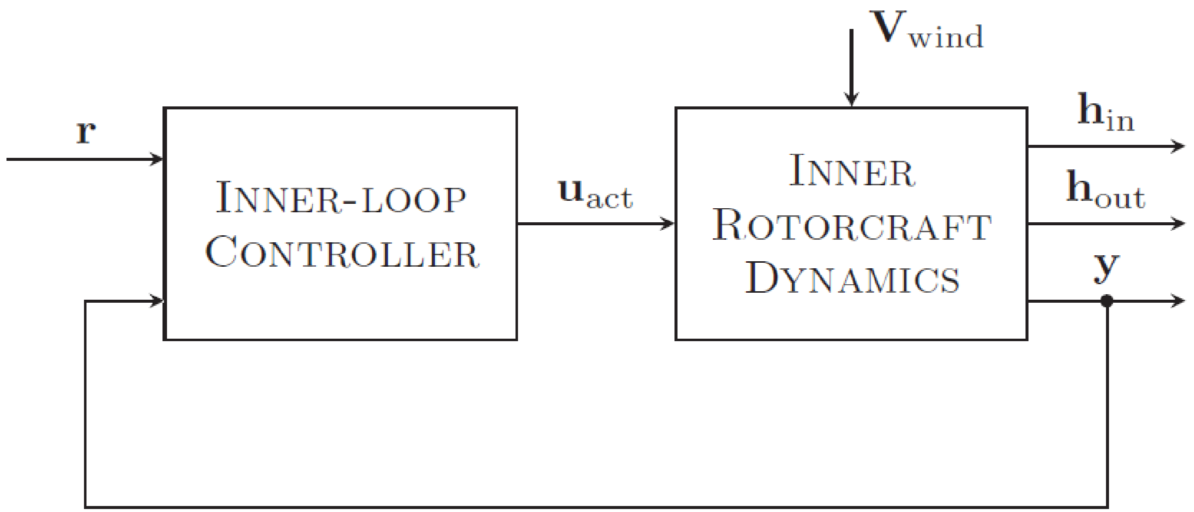
$$u = \tilde{F}\,x = -(\tilde{D}_2^{\mathrm{T}}\tilde{D}_2)^{-1}\left(\tilde{D}_2^{\mathrm{T}}\tilde{C}_2 + B^{\mathrm{T}}\tilde{P}\right)x$$

The resulting closed-loop system $T_{zw}(s)$ has: $\|T_{zw}\|_\infty < \gamma$.

More general results for the singular case can be found in Chen (2000).



## *Inner-loop control system design setup*




$$\mathbf{x} = [\phi \quad \theta \quad p \quad q \quad a_s \quad b_s \quad r \quad \delta_{ped,int} \quad \psi]^T$$

$$\mathbf{u}_{act} = [\delta_{lat} \quad \delta_{lon} \quad \delta_{ped}]^T$$

$$\mathbf{y} = [\phi \quad \theta \quad p \quad q \quad r \quad \psi]^T$$

$$\mathbf{h}_{out} := [\phi \quad \theta \quad \psi]^T$$

No gain scheduling is required in our flight control system!

## Inner-loop linearized model at hover

$$\begin{cases} \dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u} + E\mathbf{w} \\ \mathbf{y} = C_1\mathbf{x} \\ \mathbf{h}_{\text{out}} = C_{\text{out}}\mathbf{x} \end{cases}$$

$$\mathbf{y} = \begin{pmatrix} \phi \\ \theta \\ p \\ q \\ r \\ \psi \end{pmatrix}$$

$$\mathbf{h}_{\text{out}} = \begin{pmatrix} \phi \\ \theta \\ \psi \end{pmatrix}$$

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0.0009 & 0 & 0 \\ 0 & 0 & 0 & 0.9992 & 0 & 0 & -0.0389 & 0 & 0 \\ 0 & 0 & -0.0302 & -0.0056 & -0.0003 & 585.1165 & 11.4448 & -59.529 & 0 \\ 0 & 0 & 0 & -0.0707 & 267.7499 & -0.0003 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1.0000 & -3.3607 & 2.2223 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 2.4483 & -3.3607 & 0 & 0 & 0 \\ 0 & 0 & 0.0579 & 0.0108 & 0.0049 & 0.0037 & -21.9557 & 114.2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0.0389 & 0 & 0 & 0.9992 & 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 43.3635 \\ 0 & 0 & 0 \\ 0.2026 & 2.5878 & 0 \\ 2.5878 & -0.0663 & 0 \\ 0 & 0 & -83.1883 \\ 0 & 0 & -3.8500 \\ 0 & 0 & 0 \end{bmatrix} \qquad E = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -0.0001 & 0.1756 & -0.0395 \\ 0.0000 & 0.0003 & 0.0338 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ -0.0002 & -0.3396 & 0.6424 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

## $H_\infty$ state feedback control law...

$$\mathbf{u} = F\,\mathbf{x} + G\left(\mathbf{r} - \mathbf{h}_{\text{out,trim}}\right)$$

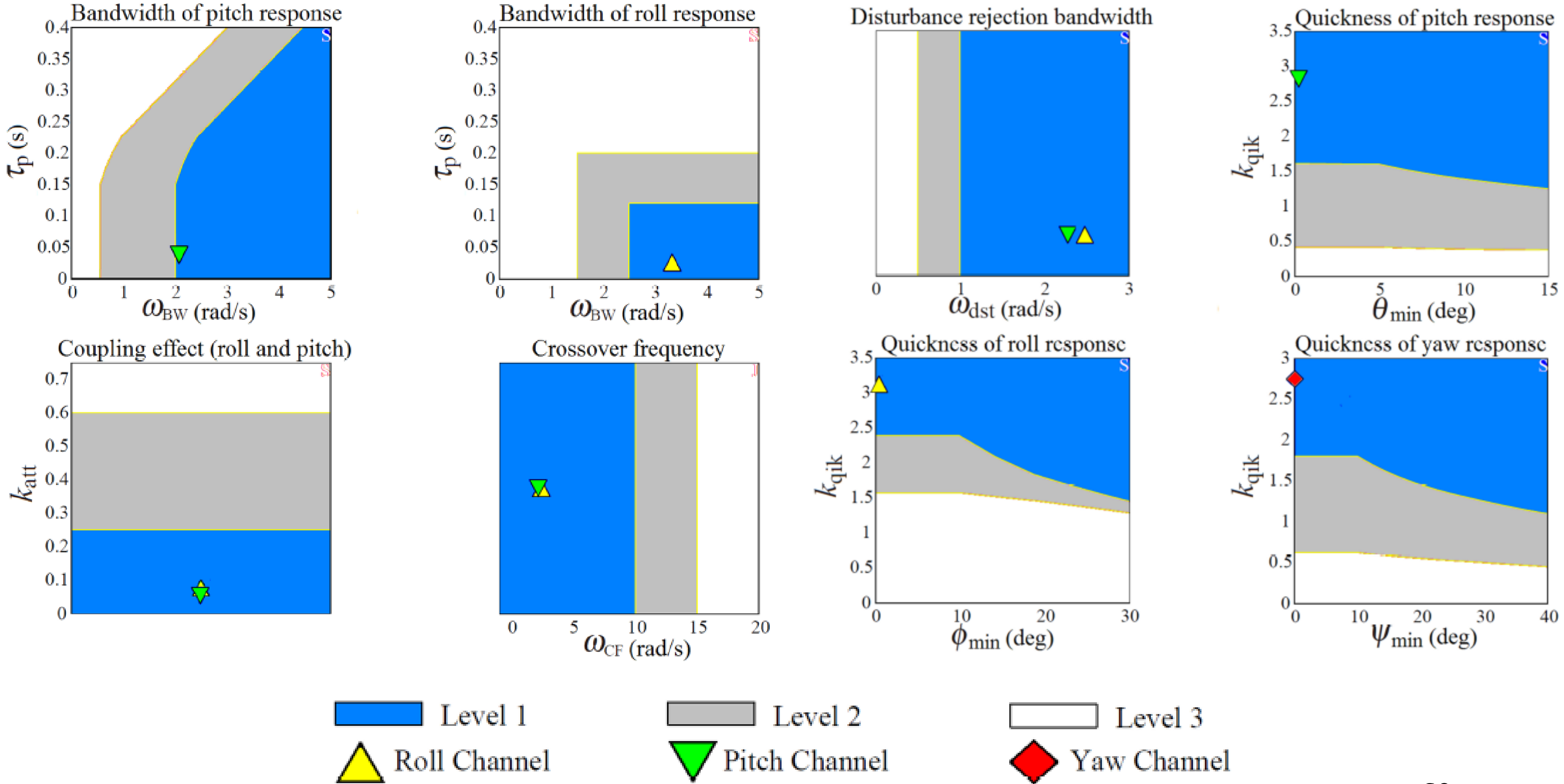
$$F = \begin{bmatrix} -1.0368 & -0.0604 & -0.0230 & -0.0083 & -0.2857 & -2.6165 & -0.0312 & 0.0499 & -0.0746 \\ 0.0760 & -0.9970 & 0.0174 & -0.0378 & -1.8340 & -0.1130 & 0.0026 & 0.0024 & -0.0169 \\ -0.0002 & -0.0185 & -0.0066 & 0.0004 & 0.0353 & 0.0990 & 0.0044 & 0.2295 & 0.2441 \end{bmatrix}$$

$$G = [C_{\text{out}}(A - BF)^{-1}B]^{-1} = \begin{bmatrix} 1.0368 & 0.0604 & 0.0746 \\ -0.0760 & 0.9970 & 0.0169 \\ 0.0002 & 0.0185 & -0.2441 \end{bmatrix}$$

The above state feedback control law is to be implemented together with a properly designed reduced-order observer for the state variables that cannot be measured...

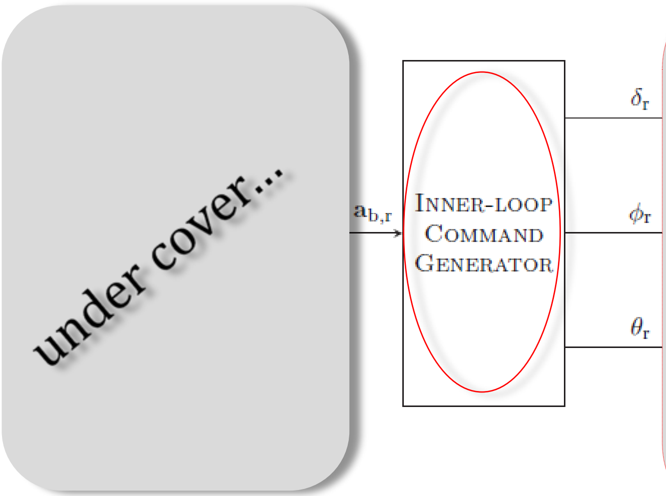
## Inner-loop control performance evaluation

Evaluation results with the standard set by U.S. Army Aviation…



# *UAV Control System Design*

## *Inner-loop command generator*

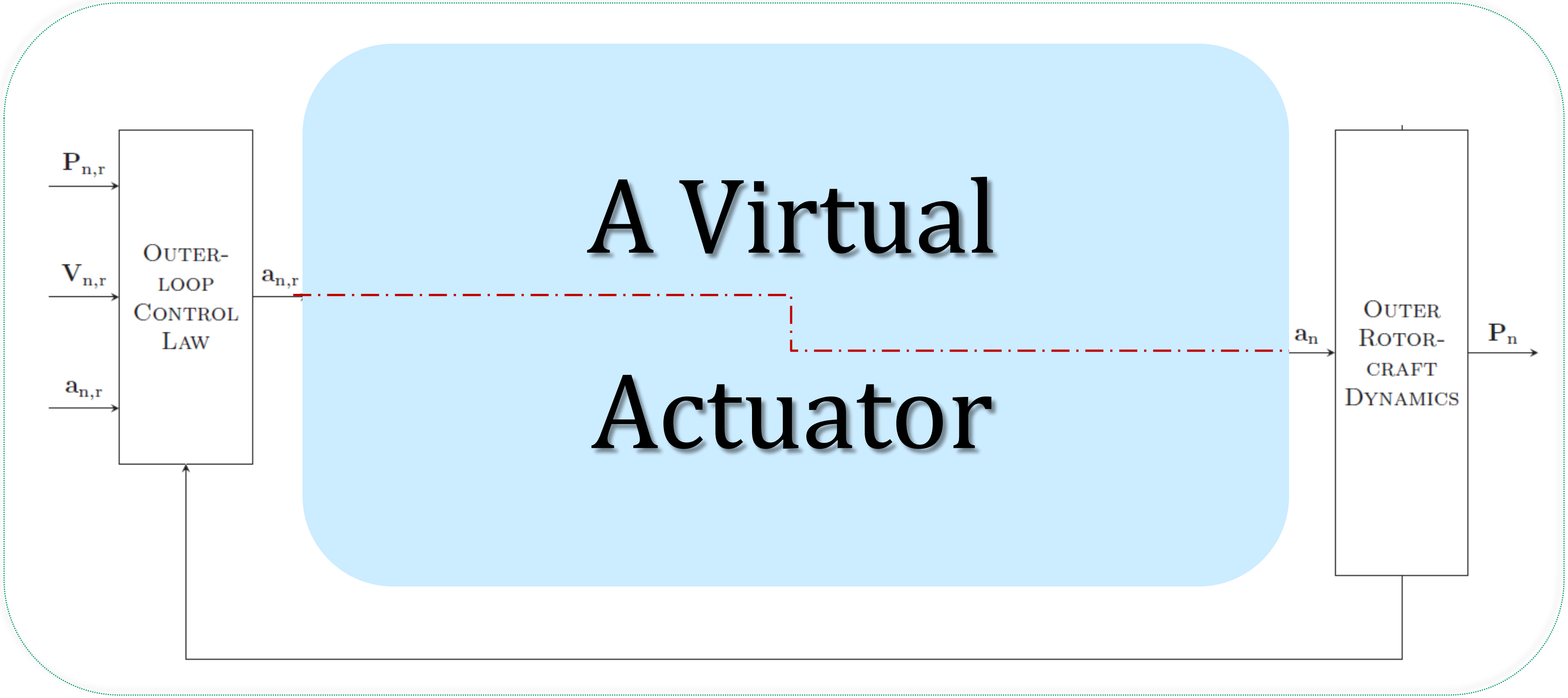



The inner-loop command generator is given as

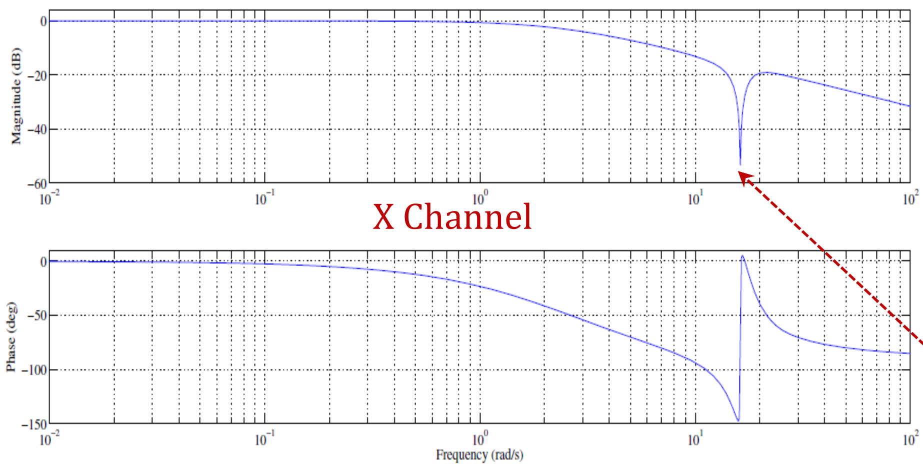
$$\begin{pmatrix} \delta_r \\ \phi_r \\ \theta_r \end{pmatrix} = \begin{bmatrix} -0.0001 & 0.0019 & 0.0478 \\ 0.0022 & -0.1031 & -0.0048 \\ 0.1022 & 0 & 0.0002 \end{bmatrix} \mathbf{a}_{b,r}$$

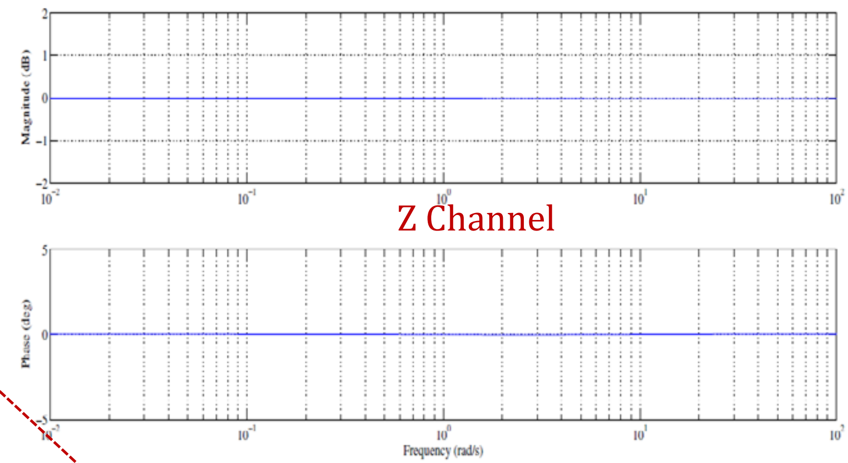
## *Outer-loop control system design setup*

The outer-loop control is to control the position of the aircraft and at the same time to generate necessary commands for the inner-loop control system…

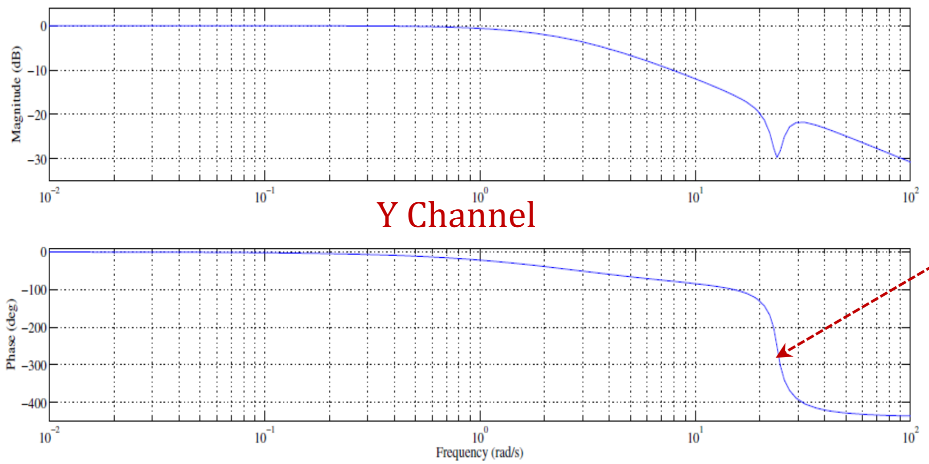


## Properties of the virtual actuator




*Frequency response of the virtual actuator…*




**Unstable Zeros!**

From practical point of view, it is safe to ignore them so long as the outer-loop bandwidth is within 1 rad/sec…

It can also be verified that coupling among each channel of the outer loop dynamics is very weak and thus can be ignored. As a result, all the x, y and z channels of the rotorcraft dynamics can be treated as decoupled and each channel can be characterized by

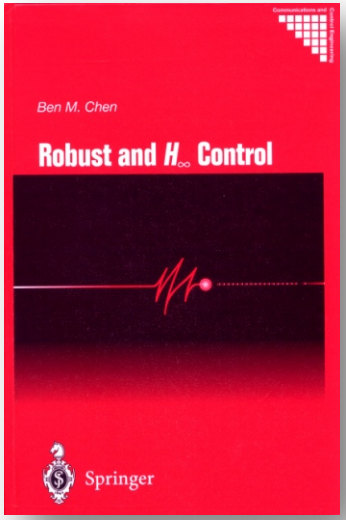
$$\begin{pmatrix} \dot{p}_* \\ \dot{v}_* \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} p_* \\ v_* \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} a_*$$

where $p_*$ is the position, $v_*$ is the velocity and $a_*$ is the acceleration, which is treated a control input in our formulation.

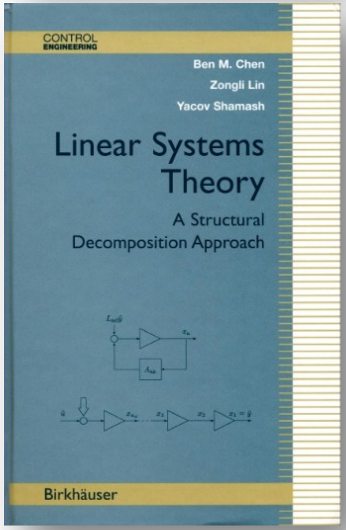
For such a simple system, it can be controlled by almost all the control techniques available in the literature, which include the most popular and the simplest one such as PID control…

## Control technique adopted for the outer loop

The outer-loop control system is designed using the so-called robust and perfect tracking (RPT) control technique developed by Chen and his co-workers. It is to design a controller such that the resulting closed-loop system is stable and the controlled output almost perfectly tracks a given reference signal in the presence of any initial conditions and external disturbances.




One of the most interesting features in the RPT control method is its capability of utilizing all possible information available in its controller structure. Such a feature is highly desirable for flight missions involving complicated maneuvers, in which not only the position reference is useful, but also its velocity and even acceleration information are important or even necessary to be used in order to achieve a good overall performance.




The RPT control renders flight formation of multiple UAVs a trivial task.

Consider the following continuous-time system:

$$\Sigma : \begin{cases} \dot{\boldsymbol{x}} = A\,\boldsymbol{x} + B\,\boldsymbol{u} + E\,\boldsymbol{w}, & \boldsymbol{x}(0) = \boldsymbol{x}_0, \\ \boldsymbol{y} = C_1\,\boldsymbol{x} + D_1\,\boldsymbol{w} \\ \boldsymbol{h} = C_2\,\boldsymbol{x} + D_2\,\boldsymbol{u} + D_{22}\,\boldsymbol{w} \end{cases} \tag{8.1}$$

where $\boldsymbol{x} \in \mathbb{R}^n$ is the state, $\boldsymbol{u} \in \mathbb{R}^m$ is the control input, $\boldsymbol{w} \in \mathbb{R}^q$ is the external disturbance, $\boldsymbol{y} \in \mathbb{R}^p$ is the measurement output, and $\boldsymbol{h} \in \mathbb{R}^\ell$ is the output to be controlled. Given the external disturbance $\boldsymbol{w} \in L_{\boldsymbol{p}}$, $\boldsymbol{p} \in [1, \infty)$, and any reference signal vector $\boldsymbol{r} \in \mathbb{R}^\ell$ with $r, \dot{\boldsymbol{r}}, \cdots, \boldsymbol{r}^{(\kappa-1)}$, $\kappa \geq 1$, being available, and $\boldsymbol{r}^{(\kappa)}$ being either a vector of delta functions or in $L_{\boldsymbol{p}}$, the RPT problem for the system in (8.1) is to find a parameterized dynamic measurement control law of the following form:

$$\begin{cases} \dot{\boldsymbol{v}} = A_{\text{cmp}}(\varepsilon)\boldsymbol{v} + B_{\text{cmp}}(\varepsilon)\boldsymbol{y} + G_0(\varepsilon)\boldsymbol{r} + \cdots + G_{\kappa-1}(\varepsilon)\boldsymbol{r}^{(\kappa-1)} \\ \boldsymbol{u} = C_{\text{cmp}}(\varepsilon)\boldsymbol{v} + D_{\text{cmp}}(\varepsilon)\boldsymbol{y} + H_0(\varepsilon)\boldsymbol{r} + \cdots + H_{\kappa-1}(\varepsilon)\boldsymbol{r}^{(\kappa-1)} \end{cases} \tag{8.2}$$

such that when the controller of (8.2) is applied to the system of (8.1), we have the following

1. There exists an $\varepsilon^* > 0$ such that the resulting closed-loop system with $\boldsymbol{r} = 0$ and $\boldsymbol{w} = 0$ is asymptotically stable for all $\varepsilon \in (0, \varepsilon^*]$.
2. Let $\boldsymbol{h}(t, \varepsilon)$ be the closed-loop controlled output response and let $\boldsymbol{e}(t, \varepsilon)$ be the resulting tracking error, i.e., $\boldsymbol{e}(t, \varepsilon) := \boldsymbol{h}(t, \varepsilon) - \boldsymbol{r}(t)$. Then, for any initial condition of the state, $\boldsymbol{x}_0 \in \mathbb{R}^n$,

$$\|\boldsymbol{e}\|_{\boldsymbol{p}} = \left( \int_0^\infty |\boldsymbol{e}(t)|^{\boldsymbol{p}}\, \mathrm{d}t \right)^{1/\boldsymbol{p}} \to 0 \text{ as } \varepsilon \to 0. \tag{8.3}$$

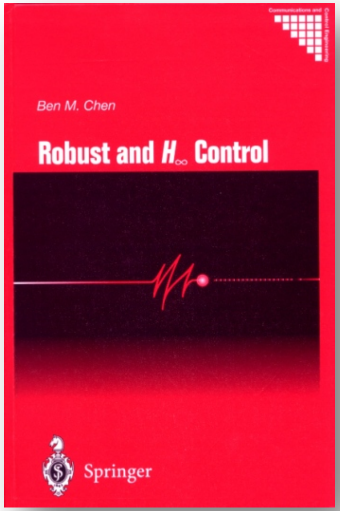
**R**obust to disturbance

**RPT Control**

**P**erfect in **T**racking

## RPT control solvability conditions

The necessary and sufficient condition for the solvability of the general RTP control problem can be found in Chen (200). For the case when $D_1 = 0$, the conditions are relatively simple and are given as




1. $(A, B)$ is stabilizable and $(A, C_1)$ is detectable.
2. $D_{22} = 0$.
3. $(A, B, C_2, D_2)$ is right invertible and of minimum phase.
4. $\mathrm{Ker}\,(C_2) \supset \mathrm{Ker}\,(C_1)$.

where Ker($X$) represents the null space of a constant matrix $X$.

We note that the last condition is automatically satisfied if the control output ***h*** of the given system is part of its measurement output ***y***.

## Solution to state feedback RPT control problem

**Step 1.** For a sufficiently small scalar $\varepsilon_0$, we define

$$\tilde{\boldsymbol{C}}_2 = \begin{bmatrix} \boldsymbol{C}_2 \\ \varepsilon I_{\kappa\ell+n} \\ 0 \end{bmatrix}, \quad \tilde{\boldsymbol{D}}_2 = \begin{bmatrix} \boldsymbol{D}_2 \\ 0 \\ \varepsilon I_m \end{bmatrix},$$

$$\tilde{\boldsymbol{A}} = \begin{bmatrix} \tilde{A}_0 & 0 \\ 0 & A \end{bmatrix}, \quad \tilde{A}_0 = -\varepsilon_0 I_{\kappa\ell} + \begin{bmatrix} 0 & I_\ell & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I_\ell \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

**Step 2.** Then, solve for positive-definite solution for the following Riccati equation

$$\boldsymbol{P}\tilde{\boldsymbol{A}} + \tilde{\boldsymbol{A}}^{\mathrm{T}}\boldsymbol{P} + \tilde{\boldsymbol{C}}_2^{\mathrm{T}}\tilde{\boldsymbol{C}}_2 - \left(\boldsymbol{P}\boldsymbol{B} + \tilde{\boldsymbol{C}}_2^{\mathrm{T}}\tilde{\boldsymbol{D}}_2\right)(\tilde{\boldsymbol{D}}_2^{\mathrm{T}}\tilde{\boldsymbol{D}}_2)^{-1}\left(\boldsymbol{P}\boldsymbol{B} + \tilde{\boldsymbol{C}}_2^{\mathrm{T}}\tilde{\boldsymbol{D}}_2\right)^{\mathrm{T}} = 0$$

**Step 3.** The required state feedback control law that solves the RPT problem is

$$\boldsymbol{u} = F(\varepsilon)\boldsymbol{x} + H_0(\varepsilon)\boldsymbol{r} + \cdots + H_{\kappa-1}(\varepsilon)\boldsymbol{r}^{(\kappa-1)}$$

where $\tilde{\boldsymbol{F}}(\varepsilon) = -(\tilde{\boldsymbol{D}}_2^{\mathrm{T}}\tilde{\boldsymbol{D}}_2)^{-1}\left(\boldsymbol{P}\boldsymbol{B} + \tilde{\boldsymbol{C}}_2^{\mathrm{T}}\tilde{\boldsymbol{D}}_2\right)^{\mathrm{T}} = [H_0(\varepsilon) \quad \cdots \quad H_{\kappa-1}(\varepsilon) \quad F(\varepsilon)]$.

## Outer-loop control law for the UAV

$$a_{x,n} = -\begin{bmatrix} \frac{\omega_{n,x}^2}{\varepsilon_x^2} & \frac{2\zeta_x\omega_{n,x}}{\varepsilon_x} \end{bmatrix} \begin{pmatrix} x_n \\ u_n \end{pmatrix} + \left(\frac{\omega_{n,x}^2}{\varepsilon_x^2}\right) x_{n,r} + \left(\frac{2\zeta_x\omega_{n,x}}{\varepsilon_x}\right) u_{n,r} + a_{x,n,r}$$

$$a_{y,n} = -\begin{bmatrix} \frac{\omega_{n,y}^2}{\varepsilon^2} & \frac{2\zeta_y\omega_{n,y}}{\varepsilon} \end{bmatrix} \begin{pmatrix} y_n \\ v_n \end{pmatrix} + \left(\frac{\omega_{n,y}^2}{\varepsilon^2}\right) y_{n,r} + \left(\frac{2\zeta_y\omega_{n,y}}{\varepsilon}\right) v_{n,r} + a_{y,n,r}$$

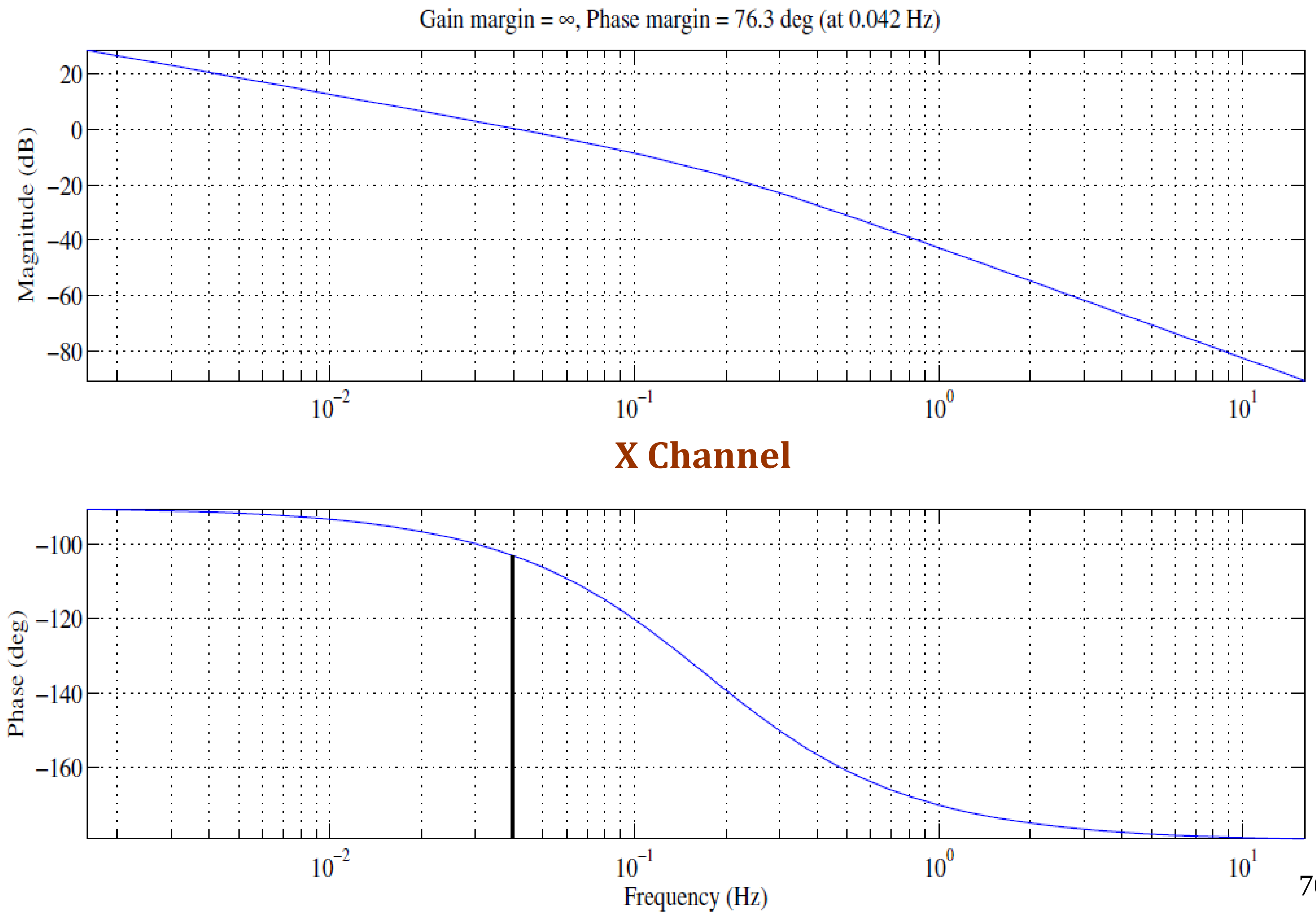
$$a_{z,n} = -\begin{bmatrix} \frac{\omega_{n,z}^2}{\varepsilon^2} & \frac{2\zeta_z\omega_{n,z}}{\varepsilon} \end{bmatrix} \begin{pmatrix} z_n \\ w_n \end{pmatrix} + \left(\frac{\omega_{n,z}^2}{\varepsilon^2}\right) z_{n,r} + \left(\frac{2\zeta_z\omega_{n,z}}{\varepsilon}\right) w_{n,r} + a_{z,n,r}$$

$$\varepsilon_x = \varepsilon_y = \varepsilon_z = 1$$

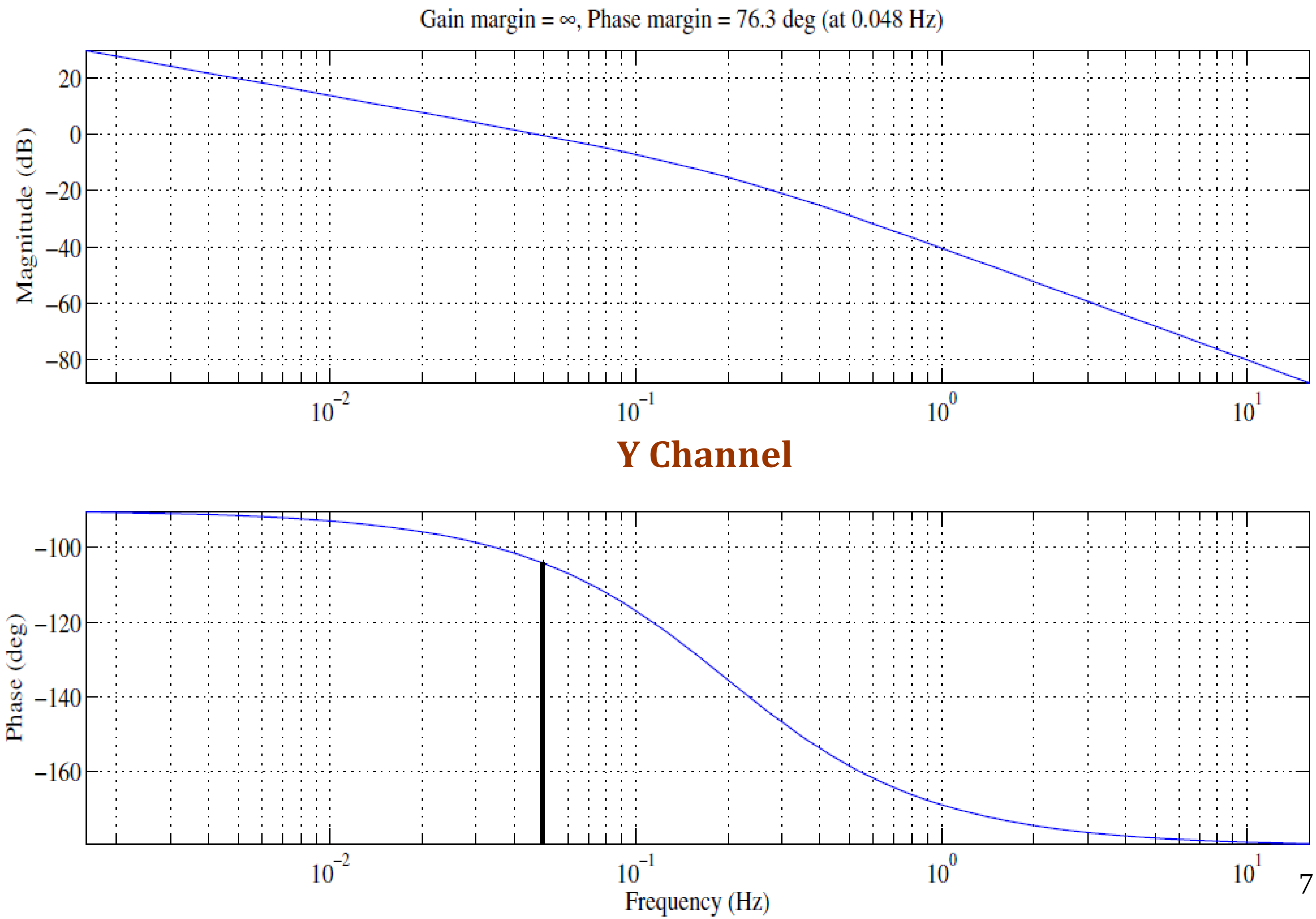
$$\zeta_x = 1, \quad \zeta_y = 1, \quad \zeta_z = 1.1$$

$$\omega_{n,x} = 0.54, \quad \omega_{n,y} = 0.62, \quad \omega_{n,z} = 0.78$$

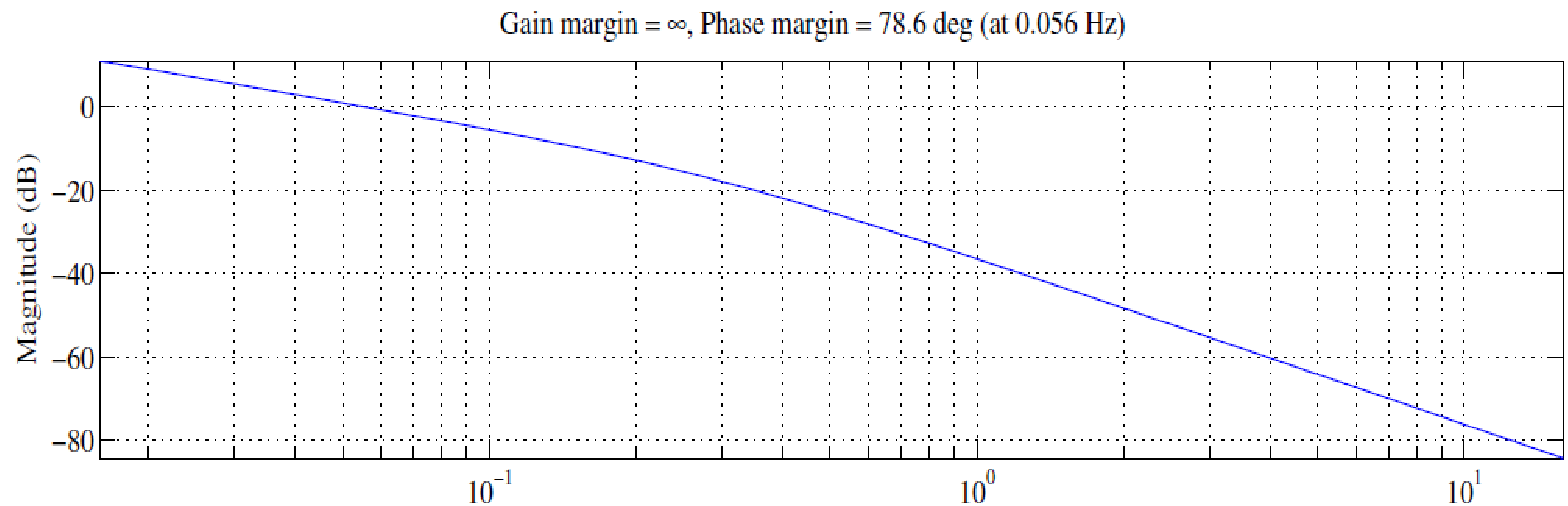
## *Outer-loop control system gain and phase margins*



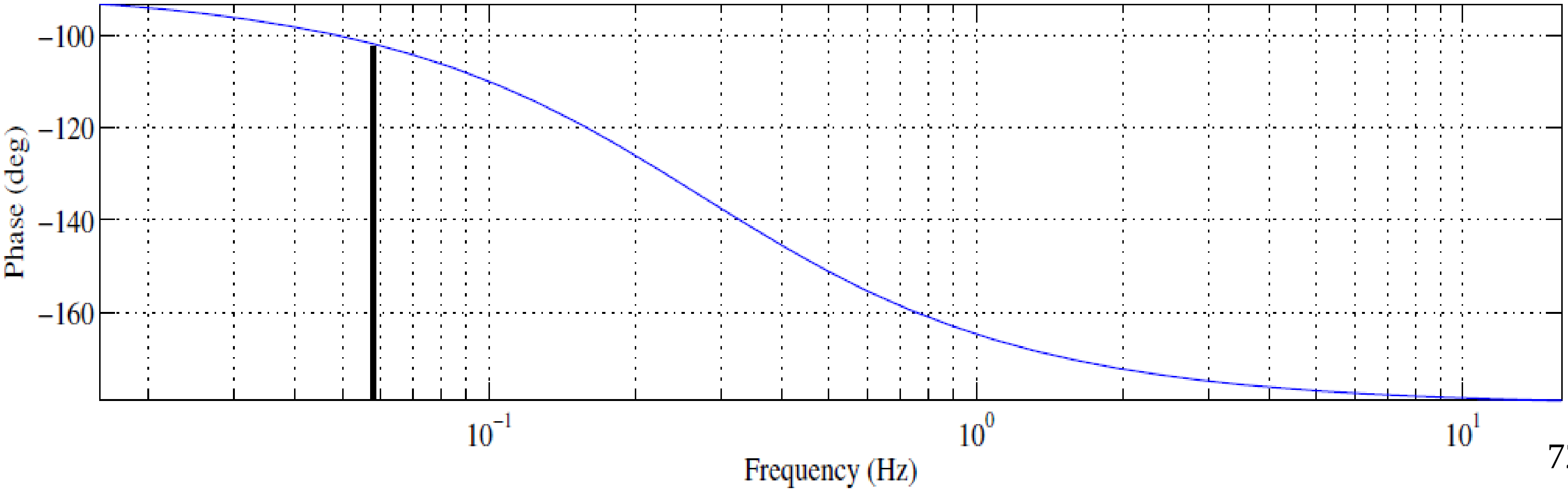
## *Outer-loop control system gain and phase margins*



## *Outer-loop control system gain and phase margins*

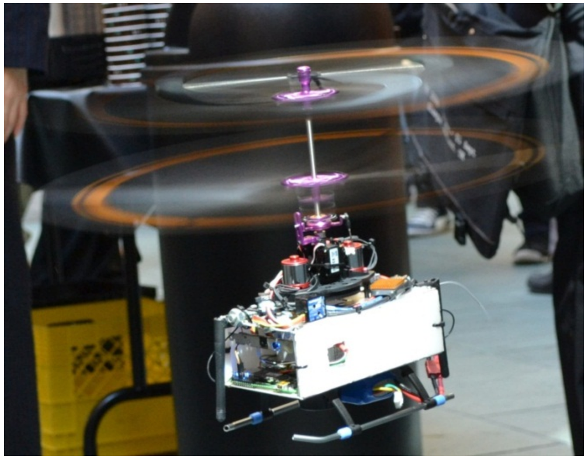



**Z Channel**



## *Control performance*

**Single-rotor Helicopter**



**Coaxial Helicopter**

**Quadrotor**

# Questions & Answers …