EE5/6102: Multivariable Control Systems

Part 2: Time Domain Approaches

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1. Introduction to Part 2

1.1 Course Outline

- <u>Revision:</u> General introduction to control systems; demonstration of actual control example; review of basic linear system theory.

- Properties of linear quadratic regulation (LQR) control; returned differences; guaranteed gain and phase margins; Kalman filter; linear quadratic Gaussian (LQG) design technique.

- Introduction to modern control system design; H_2 and H_∞ optimal control; solutions to regular and singular H_2 and H_∞ optimal control problems; solutions to some robust control problems.

- Robust & perfect tracking (RPT) control technique.

- Loop transfer recovery (LTR) design technique.

1.2 Reference Textbooks

- B. M. Chen, Z. Lin, Y. Shamash, *Linear Systems Theory: A Structural Decomposition Approach*, Birkhauser, Boston, 2004.

- F. L. Lewis, L. Xie and D. Popa, Optimal and Robust Estimation, CRC Press, 2007.

- A. Saberi, P. Sannuti, B. M. Chen, H_2 *Optimal Control,* Prentice Hall, London, 1995.

- B. M. Chen, *Robust and H_∞ Control,* Springer, New York, 2000.

- A. Saberi, B. M. Chen, P. Sannuti, *Loop Transfer Recovery: Analysis and Design,* Springer, London, 1993.

- J. M. Maciejowski, *Multivariable Feedback Design*, Addison Wesley, New York, 1989.

- G. Cai, B. M. Chen, T. H. Lee, *Unmanned Rotorcraft Systems*, Springer, New York, 2011.

- B. M. Chen, T. H. Lee, K. Peng, V. Venkataramanan, *Hard Disk Drive Servo Systems*, 2nd Edn., Springer, New York, 2006.

1.3 Homework Assignments

There will be three (3) homework assignments (for EE5102, it is to design control systems for an HDD servo system, whereas for EE6102, it is to design a flight control system for an unmanned helicopter), which require computer simulations. All students are expected to have knowledge in MATLAB™ (Control Toolbox and Robust Control Toolbox) and SIMULINK™ after completing these assignments. Homework assignments are to be marked and counted as a certain percentage in your final grade.

❖ EE5102 students can choose to do the EE6102 assignments instead!

❖ You are welcome to do both assignments if you like ☺

1.4 Final Grades for Part 2

Final Grade = 70% × Final exam marks for Part 2 (max = 50) + …

30% × Homework assignments marks (max = 50)

2. Revision: Basic Control Concepts

2.1 What is a control system?

Desired Performance REFERENCE

Difference ERROR

INPUT to the system

aircraft, missiles, economic systems, cars, etc

Information about the system: OUTPUT

+ −

Controller

System to be controlled
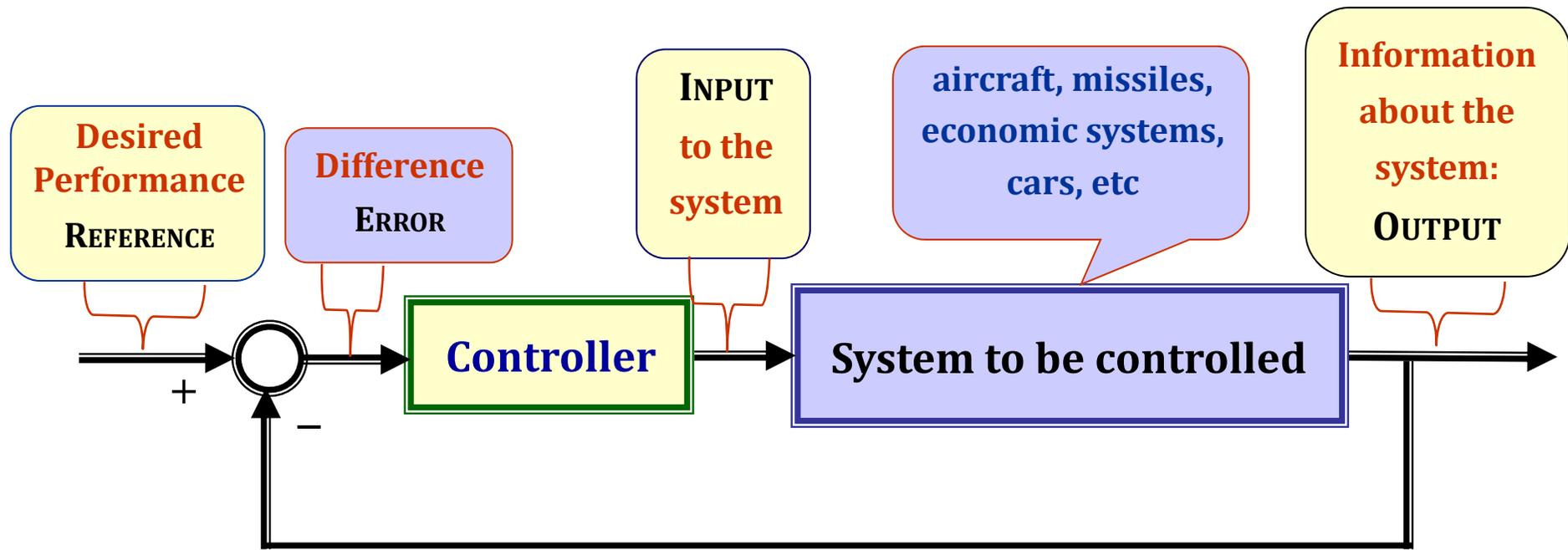
Objective: To make the system **OUTPUT** and the desired **REFERENCE** as close as possible, i.e., to make the **ERROR** as small as possible.

Key Issues: (1) How to describe the system to be controlled? **(Modeling)**

(2) How to design the controller? **(Control)**

2.2 Some Control Systems Examples

REFERENCE

INPUT

OUTPUT

Controller

+

−

Desired Performance

Government Policies

Economic System
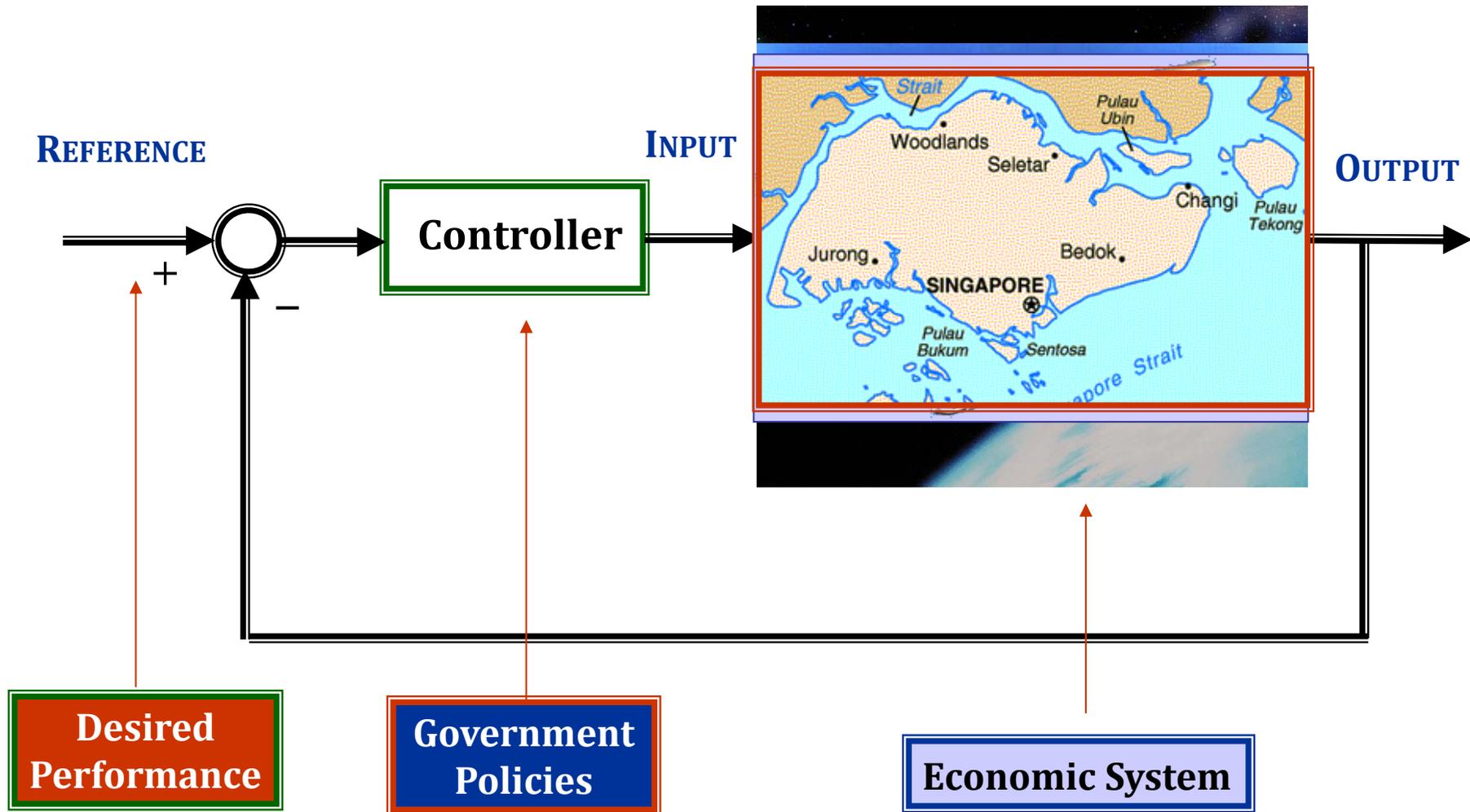
2.3 Uncertainties, Nonlinearities and Disturbances

There are many other factors of life have to be carefully considered when dealing with real-life problems. These factors include:

?
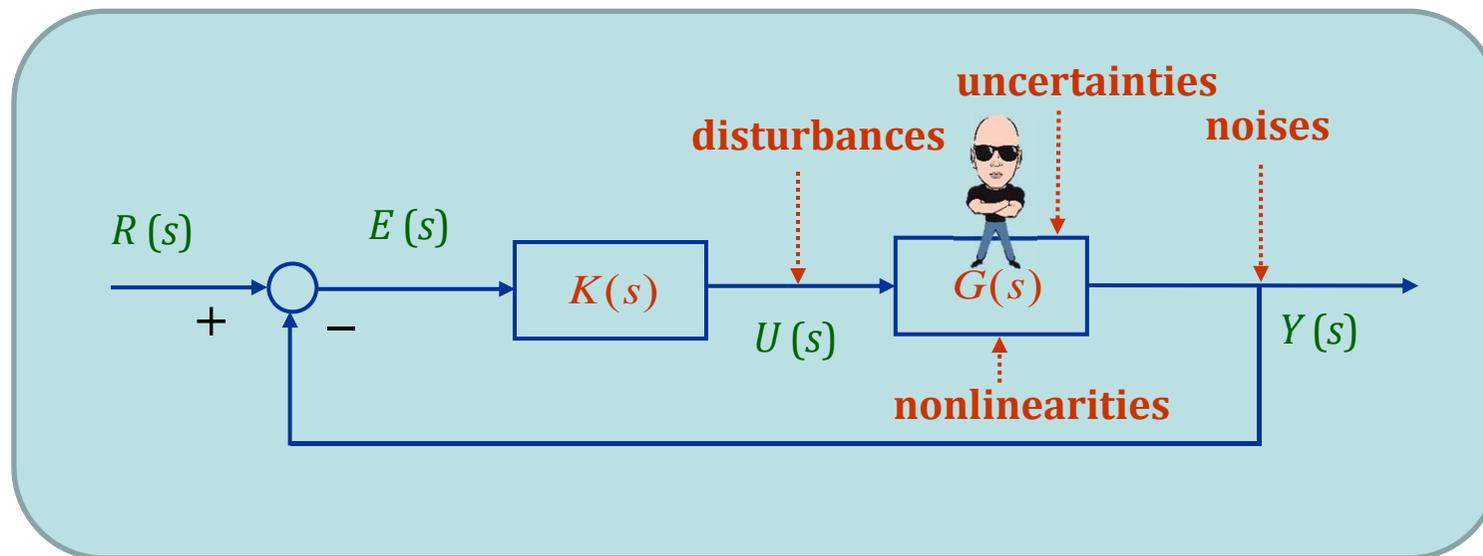
2.4 Control Techniques – A Brief View

The following is my personal view on the clarification of control techniques…

➢ **Classical control**

  PID control, developed in 1940s and used heavily for in industrial processes.

➢ **Optimal control**

  Linear quadratic regulator control, Kalman filter, H_2 control, developed in 1960s to achieve certain optimal performance.

➢ **Robust control**

  H_∞ control, developed in 1980s & 90s to handle systems with uncertainties and disturbances and with high performances.

➢ **Nonlinear control**

  Developed to handle nonlinear systems with high performances.

➢ **Multi-agent systems & cooperative control**

  It is a hot topic at moment.

➢ **Intelligent control (with a link to deep learning…)**

  Knowledge-based control, adaptive control, neural and fuzzy control, etc., developed to handle systems with unknown models.

2.5 An Accident due to Control Failure

Respect the **Unstable**

Gunter Stein

The Chernobyl Story

Finally, to make this point most dramatically, let me recount the story of the nuclear accident at Chernobyl (Figure 2). On 28 April 1986, news came out of Ukraine that a nuclear power plant had destroyed itself two days earlier and had released significant amounts of radioactive contaminants over a wide area. Short of nuclear war or impending long-term climate changes, this kind of accident certainly looms large in the public mind as one of the more serious threats to our well being.

Whether we choose to recognize it or not, control played a major role in that accident. The plant's hardware did not fail. No valve hung up, no electronic box went dead, and no metallurgical flaw caused a critical part to break. Instead, the reactor control system systematically drove the plant into an operating condition from which there was no safe way to recover. This is true, at least, if we count the control system's hardware, its human operators, and its operating policies as part of the system.

Figure 2. *Chernobyl nuclear power plant shortly after the accident on 26 April 1986.*

We will consider unstable to be synonymous with dangerous.

2.6 An Actual Control System Design Example

– *HeLion*

Bare Helicopter

Command

Control Signal

Real-time Data

Avionic System

Measurement

Manual Operation

Ground Station
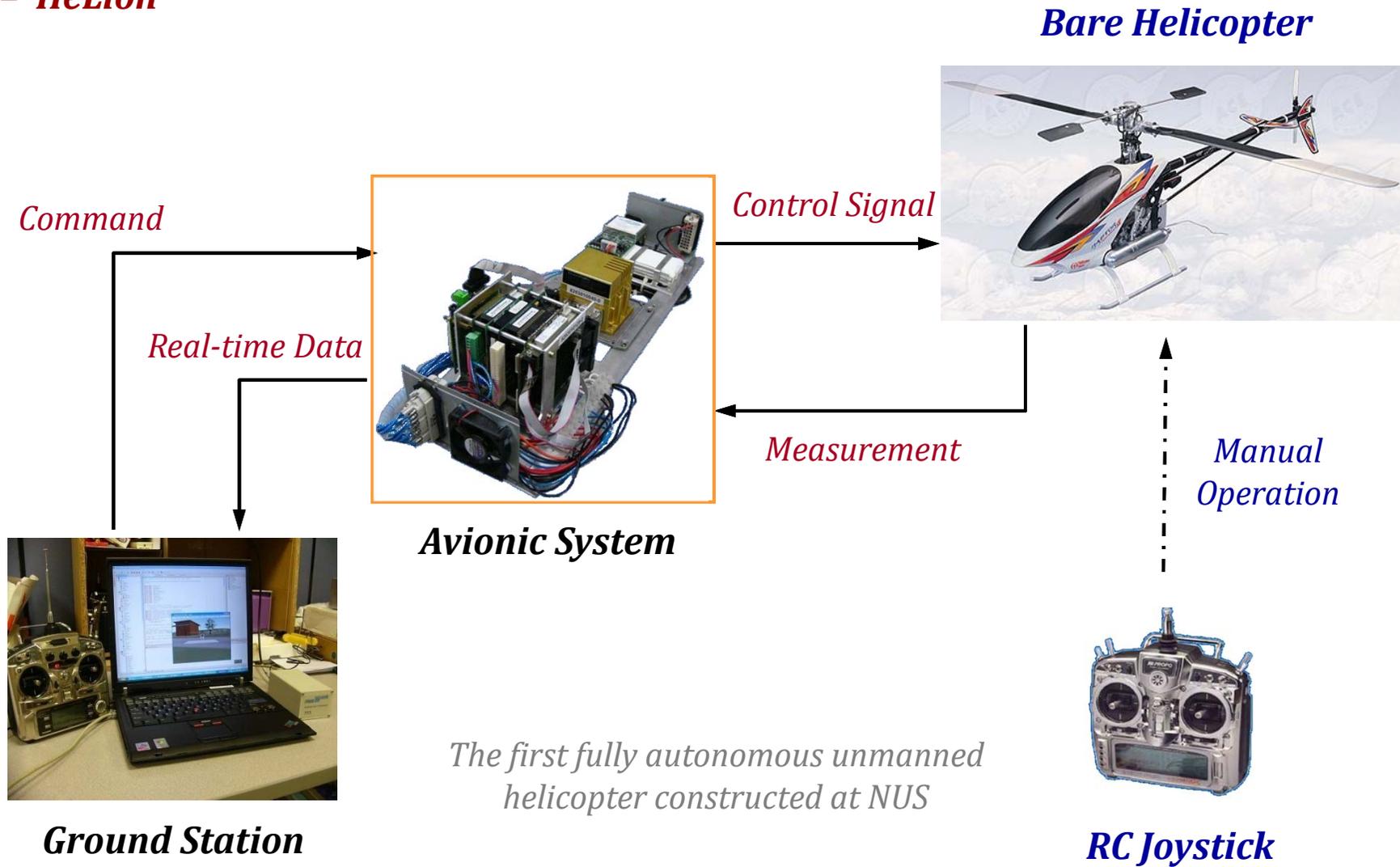
The first fully autonomous unmanned helicopter constructed at NUS

RC Joystick

Modeling – Data Collection

Data Collection Procedure:

1). Chirp-like signal issued

   in single channel;

2). Chirp-like signal issued

   in multi-channels;

3). Step-like and random

   signals issued for validation.
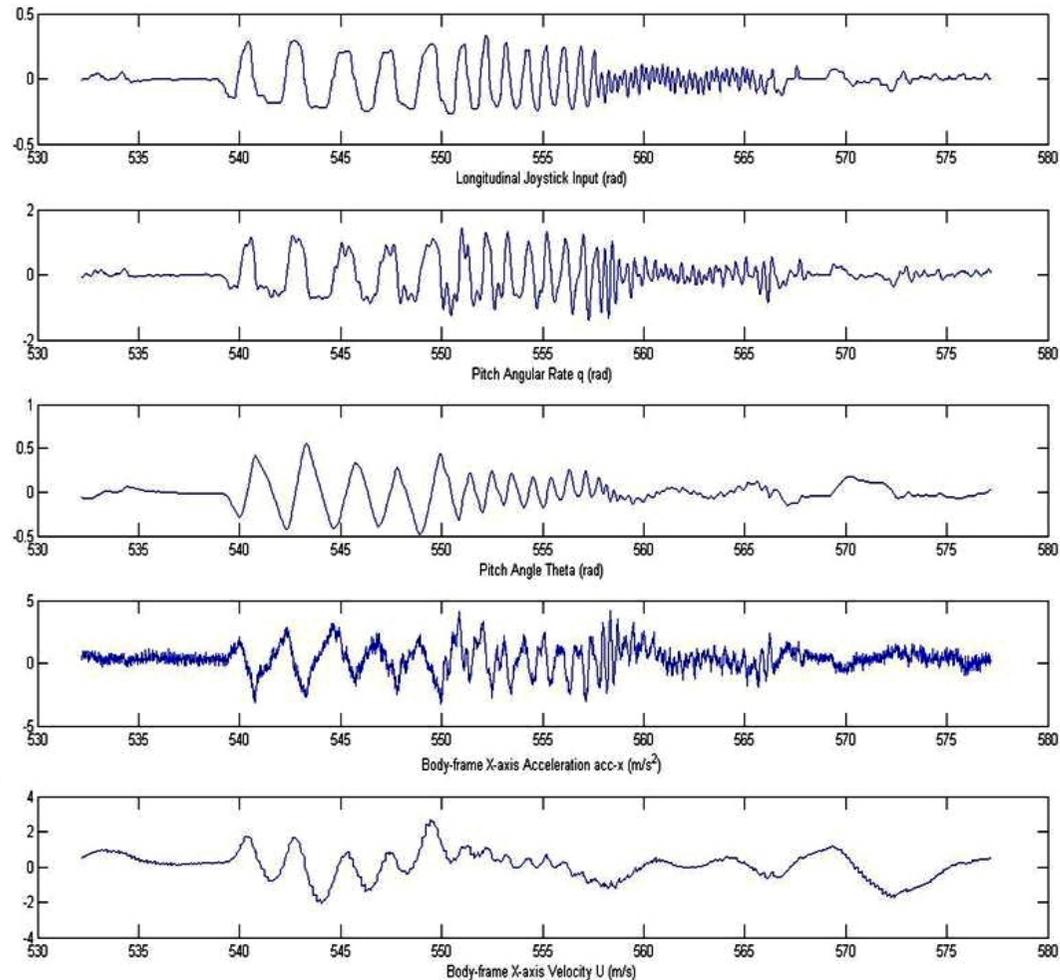
Chirp-like signal and corresponding responses

Modeling – Test Flights

Flight testing for modeling purpose

Modeling – Model Structure

Hover Model of HeLion

$$\dot{x} = Ax + Bu$$

$$
= \begin{bmatrix}
X_U & 0 & 0 & 0 & 0 & -g & X_{a_s} & 0 & 0 & 0 & 0 \\
0 & Y_V & 0 & 0 & g & 0 & 0 & Y_{b_s} & 0 & 0 & 0 \\
L_U & L_V & 0 & 0 & 0 & 0 & L_{a_s} & L_{b_s} & 0 & 0 & 0 \\
M_U & M_V & 0 & 0 & 0 & 0 & M_{a_s} & M_{b_s} & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & -1/\tau_f & A_{b_s} & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & B_{a_s} & -1/\tau_f & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & Z_{a_s} & Z_{b_s} & Z_w & Z_r & 0 \\
0 & 0 & N_p & 0 & 0 & 0 & 0 & 0 & N_w & N_r & -N_{ped} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & K_r & -K_{r_{fb}}
\end{bmatrix}
\begin{bmatrix}
U \\ V \\ p \\ q \\ \phi \\ \theta \\ a_s \\ b_s \\ W \\ r \\ r_{fb}
\end{bmatrix}
$$

$$
+ \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
A_{lati} & A_{longi} & 0 & 0 \\
B_{lati} & B_{longi} & 0 & 0 \\
0 & 0 & Z_{col} & 0 \\
0 & 0 & N_{col} & N_{ped} \\
0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\delta_{lat} \\ \delta_{lon} \\ \delta_{col} \\ \delta_{ped}
\end{bmatrix}
$$
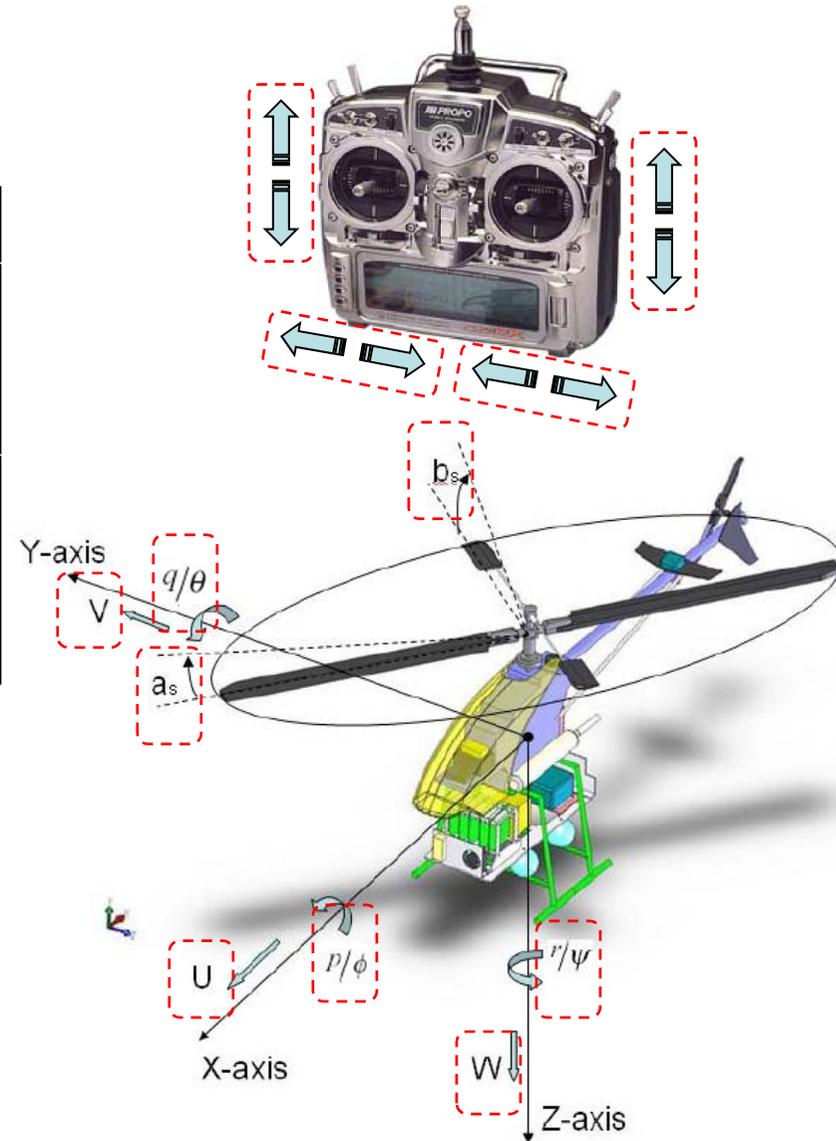
Physical meanings of the plant parameters

Table 6.1 State and input variables of the flight dynamics model

Variable	Physical description	Unit
$\mathbf{P}_\text{n} = (x_\text{n} \quad y_\text{n} \quad z_\text{n})'$	Position vector in the local NED frame	m
$\mathbf{V}_\text{b} = (u \quad v \quad w)'$	Local NED velocity projected onto the body frame	m/s
$\boldsymbol{\omega}_{\text{b/n}}^{\text{b}} = (p \quad q \quad r)'$	Angular velocity of the local NED respect to the body frames	rad/s
ϕ, θ, ψ	Euler angles	rad
a_s, b_s	Tip-path-plane (TPP) flapping angles of the main rotor	rad
$\delta_{\text{ped,int}}$	Intermediate state of yaw rate feedback controller	NA
δ_{lat}	Normalized aileron servo input $(-1, 1)$	NA
δ_{lon}	Normalized elevator servo input $(-1, 1)$	NA
δ_{col}	Normalized collective pitch servo input $(-1, 1)$	NA
δ_{ped}	Normalized rudder servo input $(-1, 1)$	NA

Modeling – Parameter Identification

1). Angular rate dynamics;   2). Horizontal velocity dynamics;

3). Yaw dynamics;   4). Heave dynamics.

$$
A = \begin{bmatrix}
-0.489 & 0 & 0 & 0 & 0 & -9.78 & -9.78 & 0 & 0 & 0 & 0 \\
0 & -0.524 & 0 & 0 & 9.78 & 0 & 0 & 9.78 & 0 & 0 & 0 \\
-1.491 & -0.427 & 0 & 0 & 0 & 0 & 51.08 & 360.24 & 0 & 0 & 0 \\
-0.096 & -0.543 & 0 & 0 & 0 & 0 & 172.45 & -58.8 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & -7.31 & 4.902 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0.891 & -7.31 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1.052 & 9.779 & -0.541 & -0.131 & 0 \\
0 & 0 & -0.283 & 0 & 0 & 0 & 0 & 0 & -0.144 & -5.556 & -36.674 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2.749 & -11.112
\end{bmatrix}
$$

$$
B = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
-0.022 & 3.314 & 0 & 0 \\
3.121 & 0.365 & 0 & 0 \\
0 & 0 & 20.033 & 0 \\
0 & 0 & 2.081 & -74.364 \\
0 & 0 & 0 & 0
\end{bmatrix}
$$

Flight Control System Structure
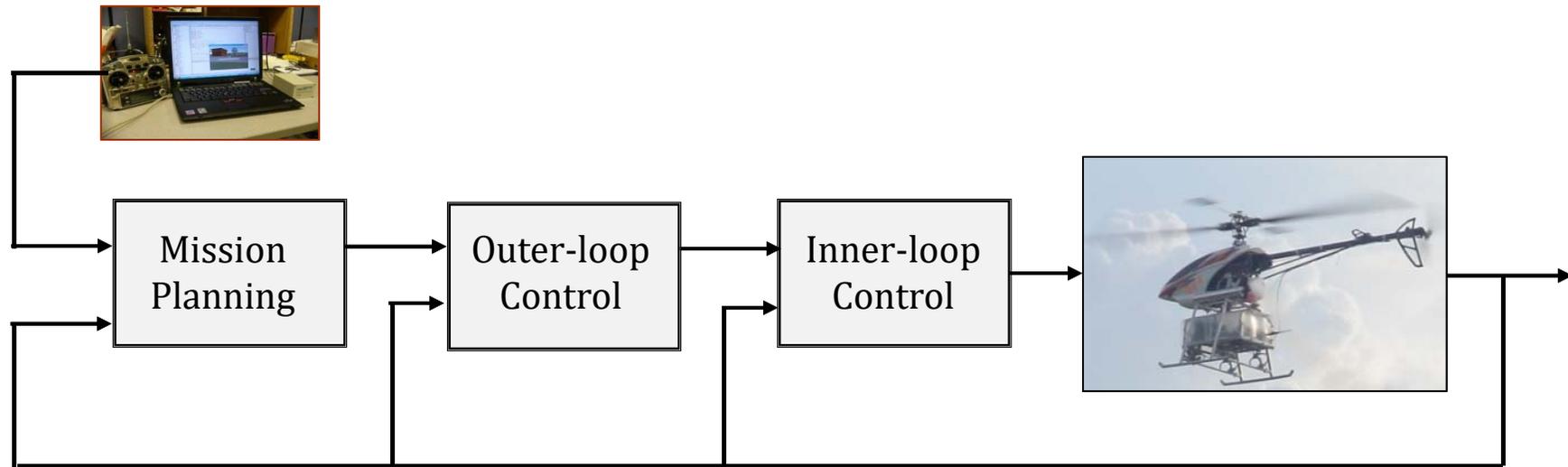
Inner-loop control is to stabilize the overall aircraft and to properly control its attitude.

The outer-loop control is to control the position of the aircraft and at the same time to generate necessary commands for the inner-loop control system…

NUS research team & unmanned systems platforms...
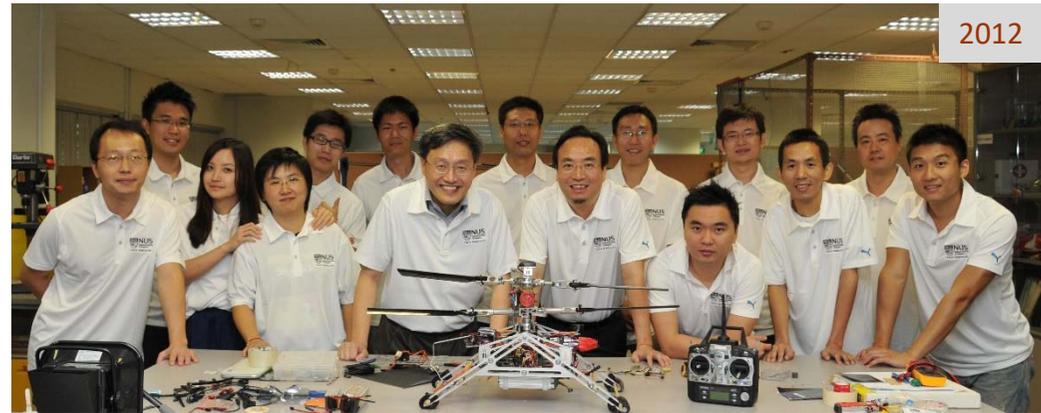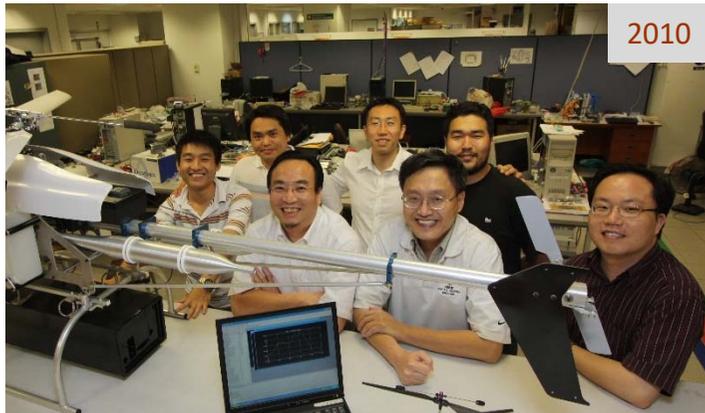
Video demo of a fully automatic UAV flight control systems

Indoor Navigation & Control

Firefighting

Inside Forest Navigation

Micro UAV

Hit-and-Run UAVs

Flight Formation

Hybrid UAVs

Cargo Transportation
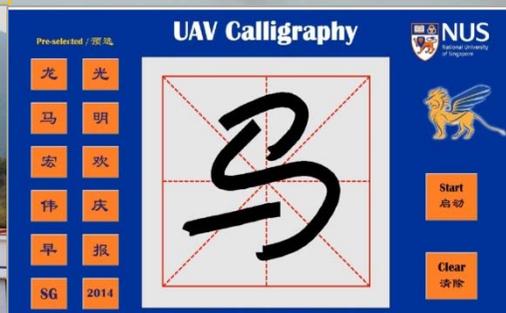
UAV Calligraphy
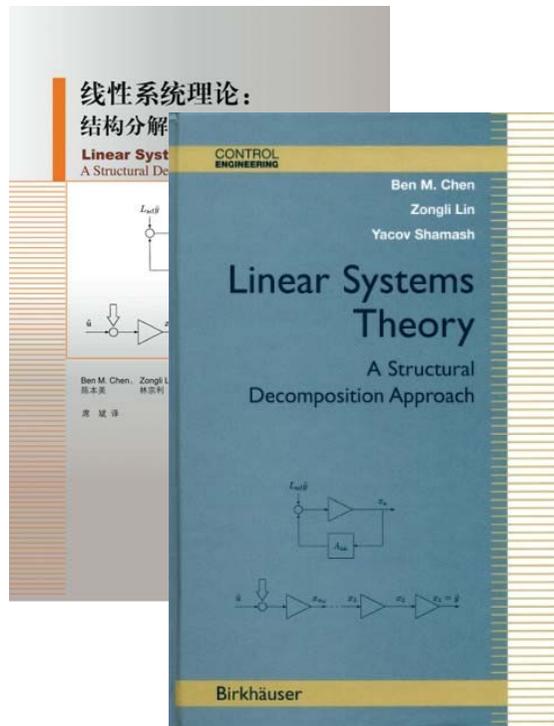
3. Brief Review of Basic Linear Systems Theory

...Chapter 3...

3.2 Dynamical Responses

Given a linear time-invariant system

$$\Sigma \; : \; \begin{cases} \dot{x}(t) = A\,x(t) + B\,u(t), & x(t) \in \mathbb{R}^n, \, u(t) \in \mathbb{R}^m \\ y(t) = C\,x(t) + D\,u(t), & y(t) \in \mathbb{R}^p \end{cases} \qquad (3.1.1)$$

The solution of the state variable or the state response, $x(t)$, of Σ with an initial condition $x_0 = x(0)$ can be uniquely expressed as

$$x(t) = \mathrm{e}^{At}x_0 + \int_0^t \mathrm{e}^{A(t-\tau)}Bu(\tau)d\tau, \quad t \geq 0, \qquad (3.2.1)$$

where the first term is the response due to the initial state, x_0, and the second term is the response excited by the external control force, $u(t)$.

Lastly, it is simple to see that the corresponding output response of the system (3.1.1) is given as:

$$y(t) = C\mathrm{e}^{At}x_0 + \int_0^t C\mathrm{e}^{A(t-\tau)}Bu(\tau)d\tau + Du(t), \quad t \geq 0. \qquad (3.2.13)$$

3.3 System Stability

A linear time-invariant system is said to be asymptotically stable if all its closed-loop poles are located on the left-half complex plane (LHP), unstable if at least of its poles are on the right-half plane (RHP)...
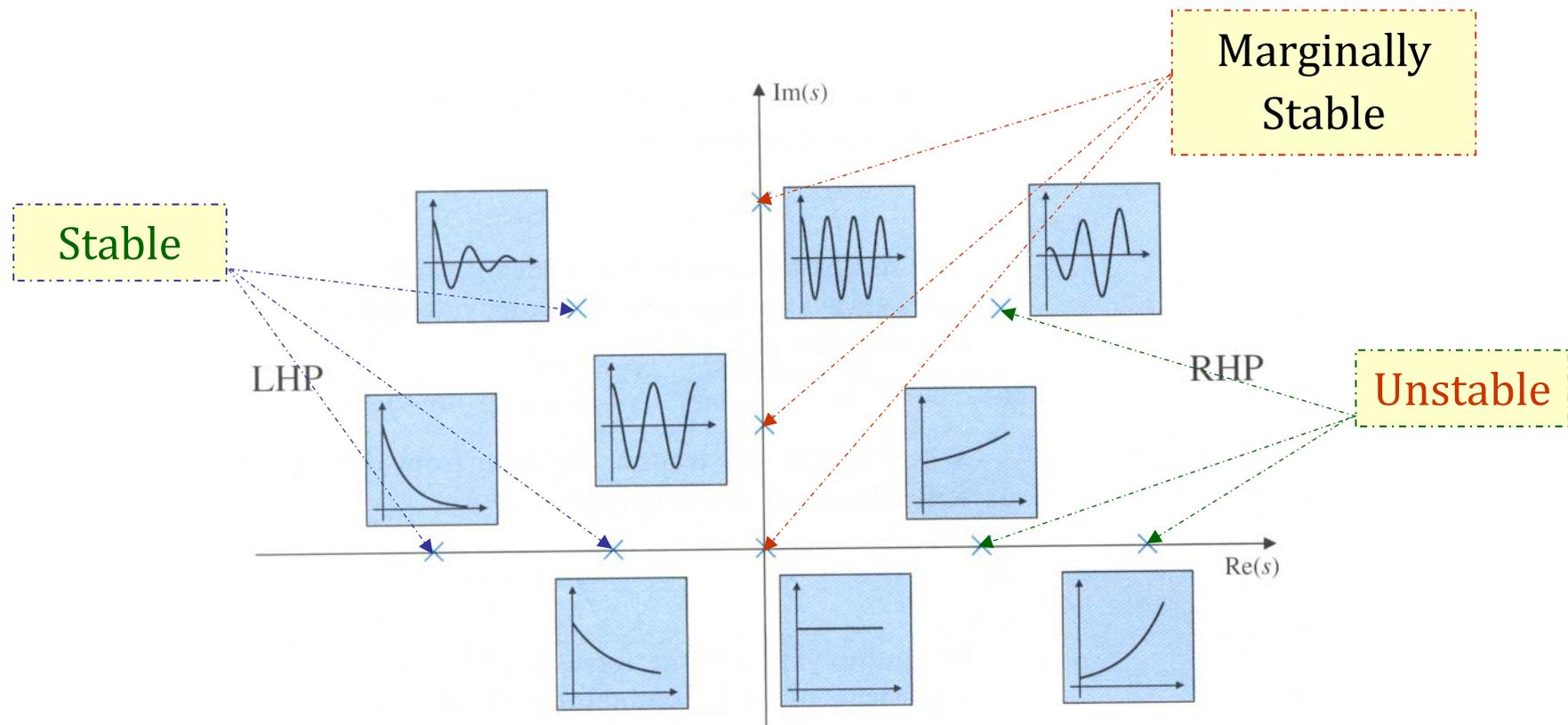
Lyapunov Stability

Consider a general dynamic system, $\dot{x} = f(x)$ with $f(0) = 0$.

If there exists a so-called Lyapunov function $V(x)$, which satisfies the following conditions:

Aleksandr Lyapunov
1857–1918

1. $V(x)$ is continuous in x and $V(0) = 0$;

2. $V(x) > 0$ (positive definite);

3. $\dot{V}(x) = \dfrac{\partial V}{\partial x} f(x) < 0$ (negative definite),

then we can say that the system is asymptotically stable at $x = 0$. If in addition,

$$V(x) \to \infty, \quad \text{as} \quad \|x\| \to \infty$$

then we can say that the system is globally asymptotically stable at $x = 0$. In this case, the stability is independent of the initial condition $x(0)$.

Lyapunov Stability for Linear Systems

Consider a linear system, $\dot{x} = A\,x$. The system is asymptotically stable (i.e., the eigenvalues of matrix A are all in the open LHP) if for any given appropriate dimensional real positive definite matrix $Q = Q^{\mathrm{T}} > 0$, there exists a real positive definite solution $P = P^{\mathrm{T}} > 0$ for the following Lyapunov equation:

$$A^{\mathrm{T}}P + PA = -Q$$

Proof. Define a Lyapunov function $V(x) = x^{\mathrm{T}}P\,x$. Obviously, the first and second conditions on the previous page are satisfied. Now consider

$$\dot{V}(x) = \dot{x}^{\mathrm{T}}P\,x + x^{\mathrm{T}}P\,\dot{x} = (A\,x)^{\mathrm{T}}P\,x + x^{\mathrm{T}}P\,A\,x = x^{\mathrm{T}}\left(A^{\mathrm{T}}P + PA\right)x = -x^{\mathrm{T}}Qx < 0$$

Hence, the third condition is also satisfied. The result follows.

Note that the condition, $Q = Q^{\mathrm{T}} > 0$, can be replaced by $Q = Q^{\mathrm{T}} \geq 0$ and $\left(A,\, Q^{1/2}\right)$ being detectable.

3.4 Controllability and Observability

Theorem 3.4.2. *The given system Σ of (3.1.1) is controllable if and only if*

$$\text{rank}\,(Q_c) = n, \qquad (3.4.11)$$

where

$$Q_c := \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} \qquad (3.4.12)$$

is called the controllability matrix *of* Σ.

Theorem 3.4.3. *The given system Σ of (3.1.1) is controllable if and only if, for every eigenvalue of A, λ_i, $i = 1, 2, \ldots, n$,*

$$\text{rank}\,\begin{bmatrix} \lambda_i I - A & B \end{bmatrix} = n. \qquad (3.4.21)$$

Definition 3.4.2. *The given system Σ of (3.1.1) is said to be* stabilizable *if all its uncontrollable modes are asymptotically stable. Otherwise, Σ is said to be* unstabilizable.

Theorem 3.4.6. *The given system Σ of (3.1.1) is observable if and only if either one of the following statements is true:*

1. *The observability matrix of Σ,*

$$Q_o := \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \qquad (3.4.27)$$

is of full rank, i.e., rank $(Q_o) = n$.

2. *For every eigenvalue of A, λ_i, $i = 1, 2, \ldots, n$,*

$$\text{rank} \begin{bmatrix} \lambda_i I - A \\ C \end{bmatrix} = n. \qquad (3.4.28)$$

Definition 3.4.4. *The given system Σ of (3.1.1) is said to be* detectable *if all its unobservable modes are asymptotically stable. Otherwise, Σ is said to be* undetectable.

3.5 System Invertibility

Recall the given system (3.1.1), which has a transfer function

$$H(s) = C(sI - A)^{-1}B + D. \tag{3.5.1}$$

Definition 3.5.1. *Consider the linear time-invariant system Σ of (3.1.1). Then,*

1. *Σ is said to be* left invertible *if there exists a rational matrix function of s, say $L(s)$, such that*
$$L(s)H(s) = I_m. \tag{3.5.2}$$

2. *Σ is said to be* right invertible *if there exists a rational matrix function of s, say $R(s)$, such that*
$$H(s)R(s) = I_p. \tag{3.5.3}$$

3. *Σ is said to be* invertible *if it is both left and right invertible.*

4. *Σ is said to be* degenerate *if it is neither left nor right invertible.*

3.6 Normal Rank and Invariant Zeros

Definition 3.6.1. *Consider the given system Σ of (3.1.1). The normal rank of its transfer function $H(s) = C(sI - A)^{-1}B + D$, or in short, normrank$\{H(s)\}$, is defined as*

$$\text{normrank}\{H(s)\} = \max\{\text{rank}[H(\lambda)] \mid \lambda \in \mathbb{C}\}. \qquad (3.6.2)$$

Definition 3.6.2. *Consider the given system Σ of (3.1.1). A scalar $\beta \in \mathbb{C}$ is said to be an invariant zero of Σ if*

$$\text{rank}\{P_\Sigma(\beta)\} < n + \text{normrank}\{H(s)\}. \qquad (3.6.4)$$

Here

$$P_\Sigma(s) := \begin{bmatrix} sI - A & -B \\ C & D \end{bmatrix}$$

which is known as the so-called Rosenbrock system matrix.

Howard H. Rosenbrock
1920–2010

3.7 Frequency Responses

Consider the following feedback control system,
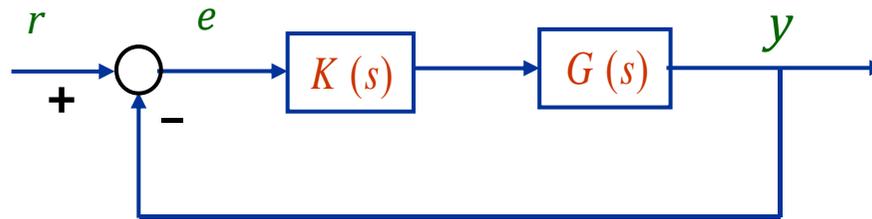
Hendrik Wade Bode
1905–1982

Bode Plots are the magnitude and phase responses of the open-loop transfer function, i.e., $K(s)G(s)$, with s being replaced by $j\omega$. For example, for the ball and beam system with a PD controller, which has an open-loop transfer function

$$K(s)G(s)\big|_{s=j\omega} = \left(0.37 + 0.23s\right)\frac{10}{s^2}\bigg|_{s=j\omega} = \frac{3.7 + 2.3s}{s^2}\bigg|_{s=j\omega} = \frac{3.7 + j2.3\omega}{-\omega^2}$$

$$\left|K(j\omega)G(j\omega)\right| = \frac{\sqrt{3.7^2 + (2.3\omega)^2}}{\omega^2}, \quad \angle K(j\omega)G(j\omega) = \tan^{-1}\left(\frac{2.3\omega}{3.7}\right) - 180^\circ$$

Gain and phase margins
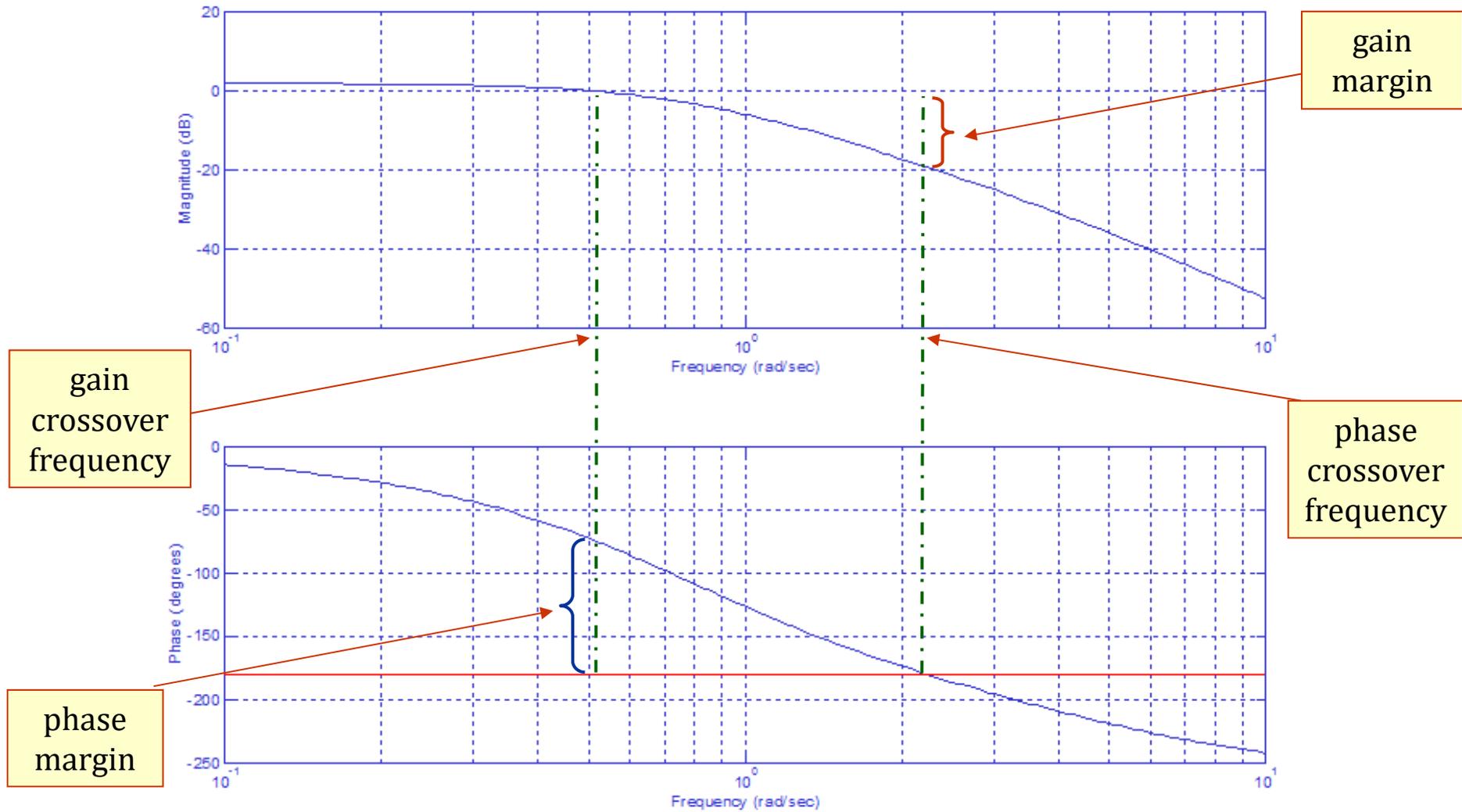
Nyquist Plot

Instead of separating into magnitude and phase diagrams as in Bode plots, Nyquist plot maps the open-loop transfer function $K(s)G(s)$ directly onto a complex plane, e.g.,
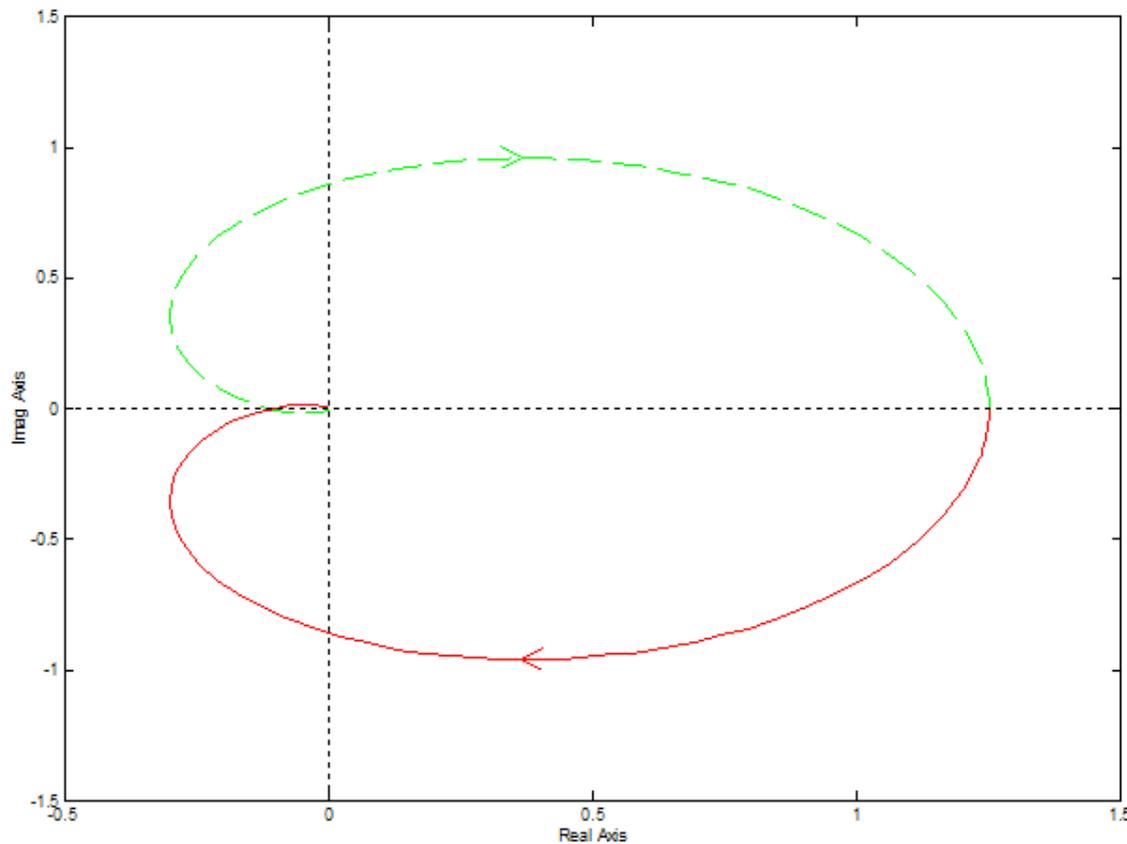
Harry Nyquist
1889–1976

Gain and phase margins

The gain margin and phase margin can also be found from the Nyquist plot by zooming in the region in the neighbourhood of the origin.
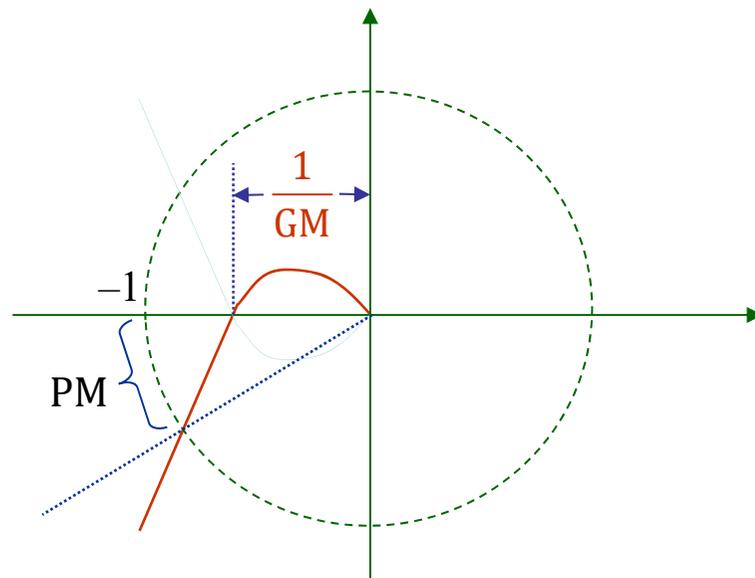
Remark: Gain margin is the maximum additional gain you can apply to the closed-loop system such that it will still remain stable. Similarly, phase margin is the maximum phase you can tolerate to the closed-loop system such that it will still remain stable.

Mathematically,

$$\text{GM} = \frac{1}{\left| K(j\omega_p)G(j\omega_p) \right|}, \quad \text{where } \omega_p \text{ is such that } \angle K(j\omega_p)G(j\omega_p) = 180°$$

$$\text{PM} = \angle K(j\omega_g)G(j\omega_g) + 180°, \quad \text{where } \omega_g \text{ is such that } \left| K(j\omega_g)G(j\omega_g) \right| = 1$$

Sensitivity functions

Consider the typical feedback control scheme
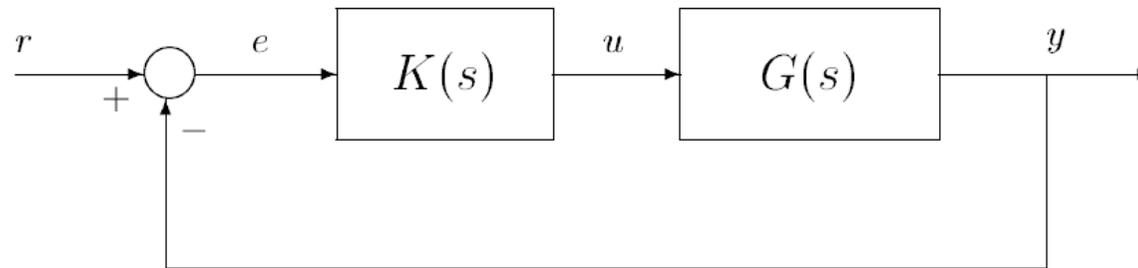
The sensitivity function is defined as the closed-loop transfer function from the reference signal, *r*, to the tracking error, *e*, and given by

$$S(s) = \frac{1}{1 + K(s)G(s)}$$

The complimentary sensitivity function is defined as the closed-loop transfer function between the reference, *r,* and the system output, *y*, and is given as

$$T(s) = \frac{K(s)G(s)}{1 + K(s)G(s)}$$

Clearly, $S(s) + T(s) \equiv 1$.

Figure 3. *Sensitivity reduction at low frequency unavoidably leads to sensitivity increase at higher frequencies.*

A good control system design should have a sensitivity function that is small at low frequencies for good tracking performance and disturbance rejection and is equal to unity at high frequencies. On the other hand, the complementary sensitivity function should be made unity at low frequencies. It must roll off at high frequencies to possess good attenuation of high-frequency noise.

Gunter Stein

4. Properties of LQR Control

Linear Quadratic Regulator (LQR)

Consider a linear system characterized by

$$\dot{x} = A\,x + B\,u$$

where (A, B) is stabilizable. We define a cost index

Jacopo Francesco Riccati
1676–1754

$$J(x, u, Q, R) = \int_0^\infty (x^\top Q x + u^\top R u)\,dt, \qquad Q \geq 0, \ R > 0$$

and $(A, Q^{1/2})$ is detectable. The linear quadratic regulation problem is to find

a control law $u = -F\,x$ such that $(A - B\,F)$ is stable and J is minimized. The

solution is given by $F = R^{-1} B^\top P$, with P being a positive semi-definite solution

of the following Riccati equation:

$$PA + A^\top P - PBR^{-1}B^\top P + Q = 0$$

(See the reference by Saberi et al, 1995, for the methods on how to solve Riccati equations)

H_2 OPTIMAL
CONTROL

ALI SABERI
PEDDAPULLAIAH SANNUTI
BEN M. CHEN

If we arrange the LQR control in the following block diagram,
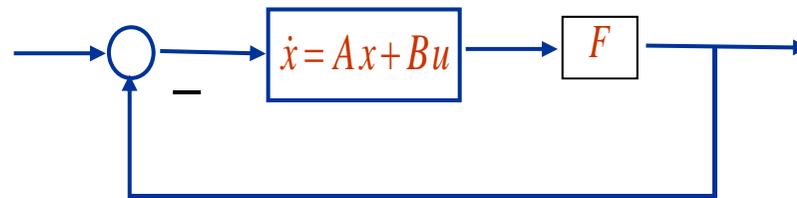
we can find its gain margin and phase margin as we have done in classical control. It is clear that the open-loop transfer function,

$$\text{Open loop transfer function} = F(sI-A)^{-1}B = R^{-1}B^{\mathrm{T}}P(sI-A)^{-1}B$$

The block diagram can be re-drawn as follows,
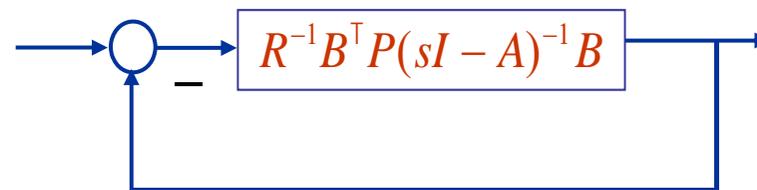
Return Difference Equality and Inequality

Consider the LQR control law. The following so-called return difference equality holds:

$$R + B^{\mathrm{T}}(-j\omega I - A^{\mathrm{T}})^{-1} Q(j\omega I - A)^{-1} B = [I + B^{\mathrm{T}}(-j\omega I - A^{\mathrm{T}})^{-1} F^{\mathrm{T}}] R [I + F(j\omega I - A)^{-1} B]$$

The following is called the return difference inequality:

$$[I + B^{\mathrm{T}}(-j\omega I - A^{\mathrm{T}})^{-1} F^{\mathrm{T}}] R [I + F(j\omega I - A)^{-1} B] \geq R$$

Proof. Recall that

$$F = R^{-1} B^{\mathrm{T}} P \qquad\qquad PA + A^{\mathrm{T}} P - PBR^{-1} B^{\mathrm{T}} P + Q = 0$$

Then we have

$$-P j\omega I + PA + P j\omega I + A^{\mathrm{T}} P - (PBR^{-1}) R (R^{-1} B^{\mathrm{T}} P) + Q = 0$$

$$\Downarrow$$

$$P(j\omega I - A) + (-j\omega I - A^{\mathrm{T}}) P + F^{\mathrm{T}} R F = Q$$

Multiplying it on the left by $B^\top(-j\omega I - A^\top)^{-1}$ and on the right by $(j\omega I - A)^{-1}B$,

$$B^\top(-j\omega I - A^\top)^{-1}P(j\omega I - A)(j\omega I - A)^{-1}B + B^\top(-j\omega I - A^\top)^{-1}(-j\omega I - A^\top)P(j\omega I - A)^{-1}B$$
$$+ B^\top(-j\omega I - A^\top)^{-1}F^\top RF(j\omega I - A)^{-1}B = B^\top(-j\omega I - A^\top)^{-1}Q(j\omega I - A)^{-1}B$$

$\Downarrow$

$$B^\top(-j\omega I - A^\top)^{-1}PB + B^\top P(j\omega I - A)^{-1}B + B^\top(-j\omega I - A^\top)^{-1}F^\top RF(j\omega I - A)^{-1}B$$
$$= B^\top(-j\omega I - A^\top)^{-1}Q(j\omega I - A)^{-1}B$$

Noting the fact that

$$F = R^{-1}B^\top P \quad \Rightarrow \quad B^\top P = RF \quad \& \quad PB = F^\top R$$

we have

$$R + B^\top(-j\omega I - A^\top)^{-1}F^\top R + RF(j\omega I - A)^{-1}B + B^\top(-j\omega I - A^\top)^{-1}F^\top RF(j\omega I - A)^{-1}B$$
$$= B^\top(-j\omega I - A^\top)^{-1}Q(j\omega I - A)^{-1}B + R$$

$\Downarrow$

$$[I + B^\top(-j\omega I - A^\top)^{-1}F^\top]R[I + F(j\omega I - A)^{-1}B] = R + B^\top(-j\omega I - A^\top)^{-1}Q(j\omega I - A)^{-1}B$$

Single Input Case

In the single input case, the transfer function

$$\text{Open loop transfer function} = f(sI - A)^{-1}b$$

is a scalar function. Then, the return difference equation is reduced to

$$r + b^{\mathrm{T}}(-j\omega I - A^{\mathrm{T}})^{-1}Q(j\omega I - A)^{-1}b = r\,[1 + b^{\mathrm{T}}(-j\omega I - A^{\mathrm{T}})^{-1}f^{\mathrm{T}}][1 + f(j\omega I - A)^{-1}b]$$

$$r + \alpha = r\left| 1 + f(j\omega I - A)^{-1}b \right|^2 \quad \text{where } \alpha \geq 0$$

$$r\left| 1 + f(j\omega I - A)^{-1}b \right|^2 \geq r$$

$$\left| 1 + f(j\omega I - A)^{-1}b \right|^2 \geq 1 \quad \text{Return Difference Inequality...}$$

Graphically, $\left|1+ f\left(j\omega I - A\right)^{-1}b\right|^2 \geq 1 \implies \left| f\left(j\omega I - A\right)^{-1}b -\left(-1+ j0\right)\right| \geq 1$ implies

−1

$60°$

PM $\geq 60°$

− 2   −1
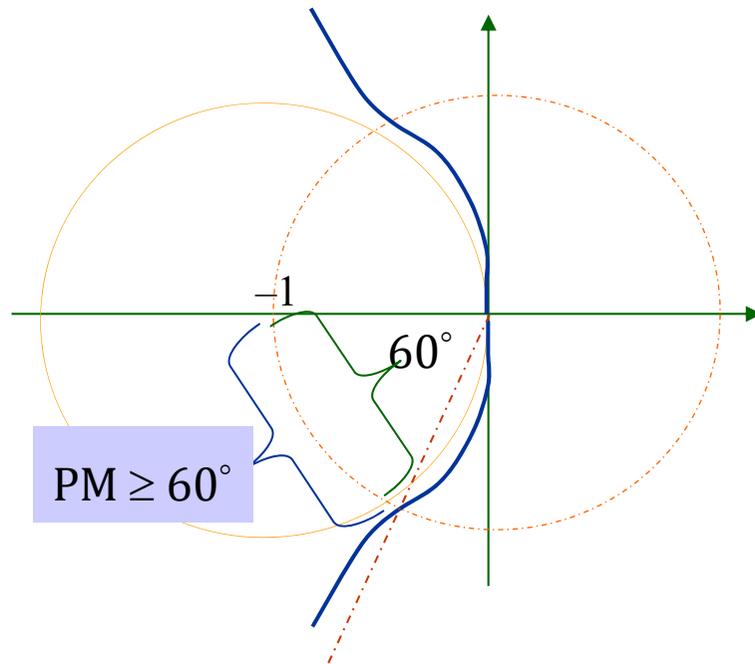
Clearly, the phase margin resulting from the LQR design is at least 60 deg.
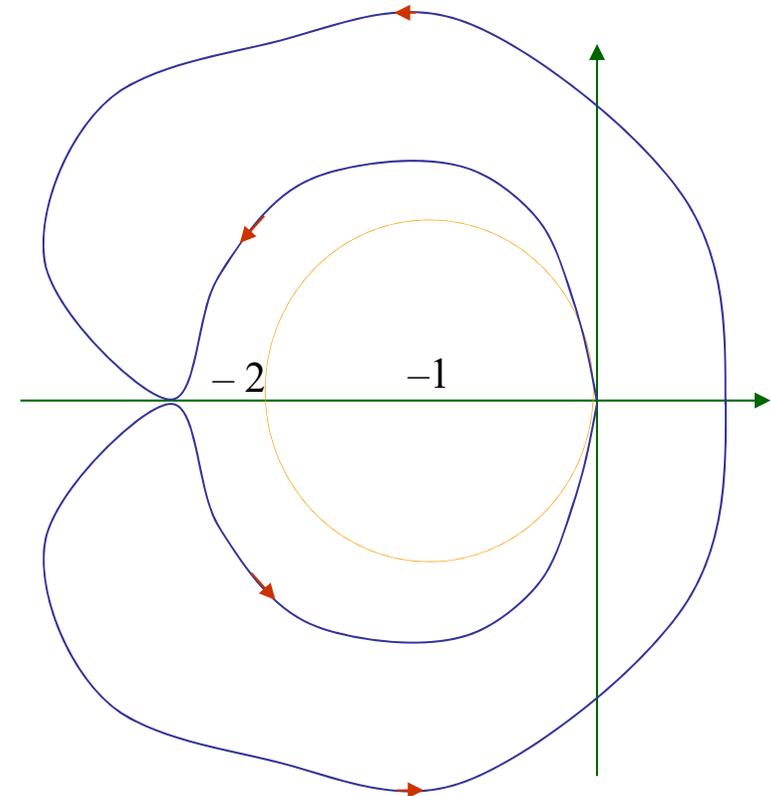
The gain margin is from [0.5, ∞).

*   *R. E. Kalman, When is a linear control system optimal? Journal of Basic Engineering, Transactions of the ASME, Series D, Vol. 86, No. 1, pp. 51–60, 1964.*

Example: Consider a given plant characterized by

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

Solving the LQR problem which minimizes the following cost function

$$J(x,u,Q,R) = \int_0^\infty (x^{\mathrm{T}}Qx + u^{\mathrm{T}}Ru)dt, \qquad \text{with} \quad Q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad R = 0.1$$

we obtain

$$P = \begin{bmatrix} 0.6872 & 0.2317 \\ 0.2317 & 0.1373 \end{bmatrix} \quad \text{and} \quad F = \begin{bmatrix} 2.3166 & 1.3734 \end{bmatrix}$$

which results the closed-loop eigenvalues at $-1.1867 \pm j1.3814$. Clearly, the closed-loop system is asymptotically stable.

Bode Diagrams

From: U(1)
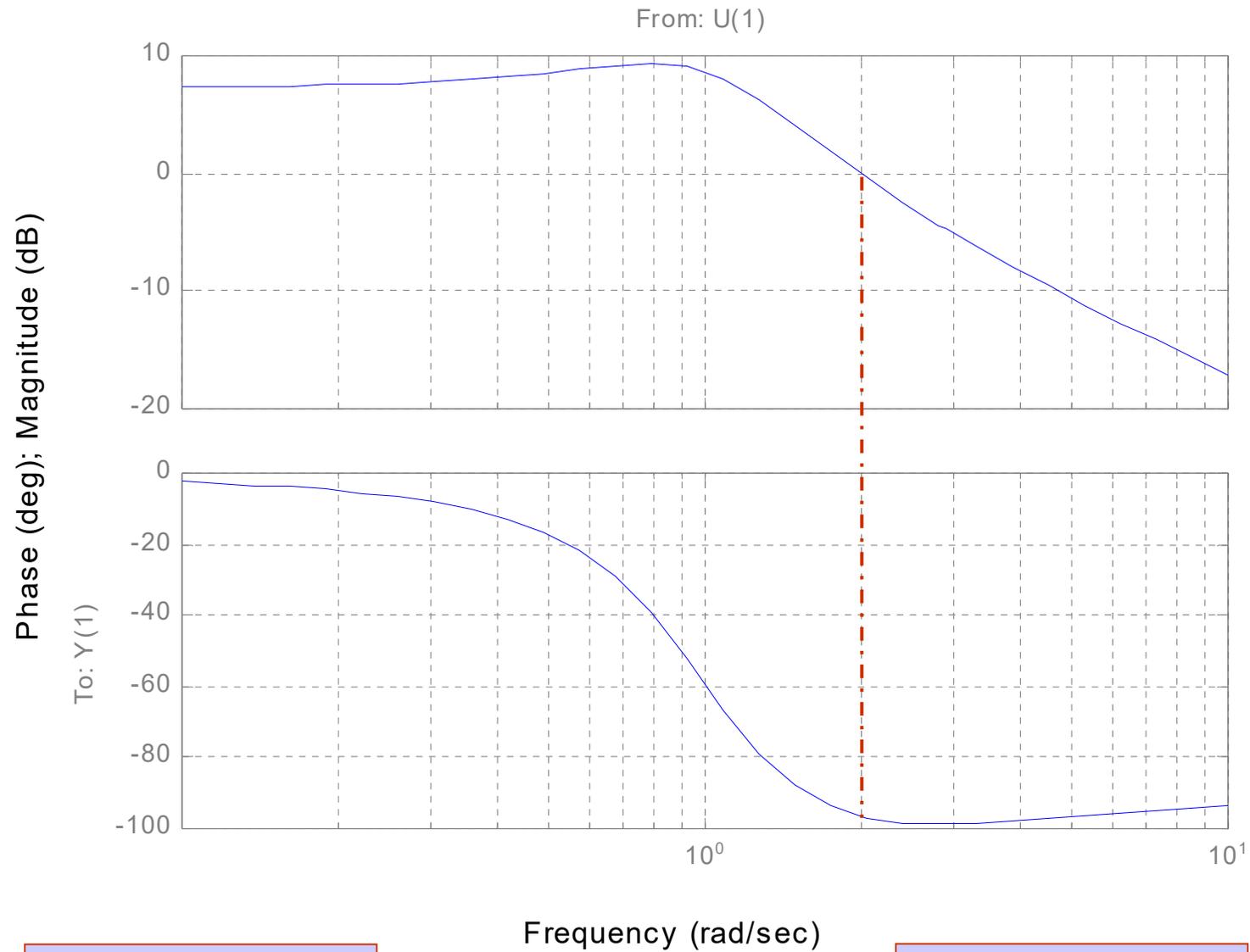
PM = 84°

GM = ∞

5. Kalman Filter
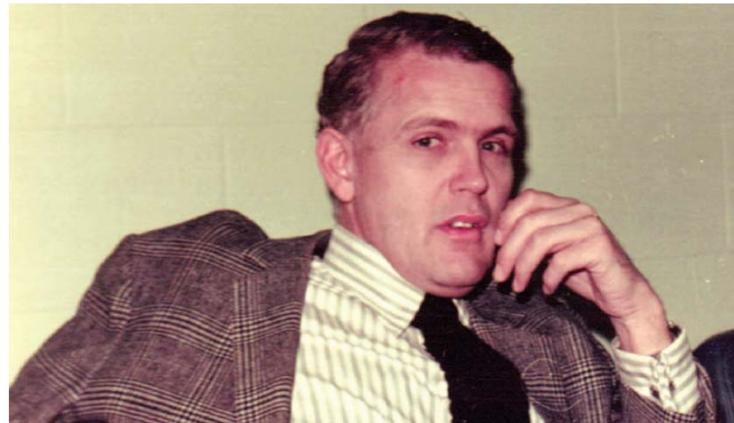
Rudolf E. Kalman, 1930–2016

Kalman-Bucy Filter?
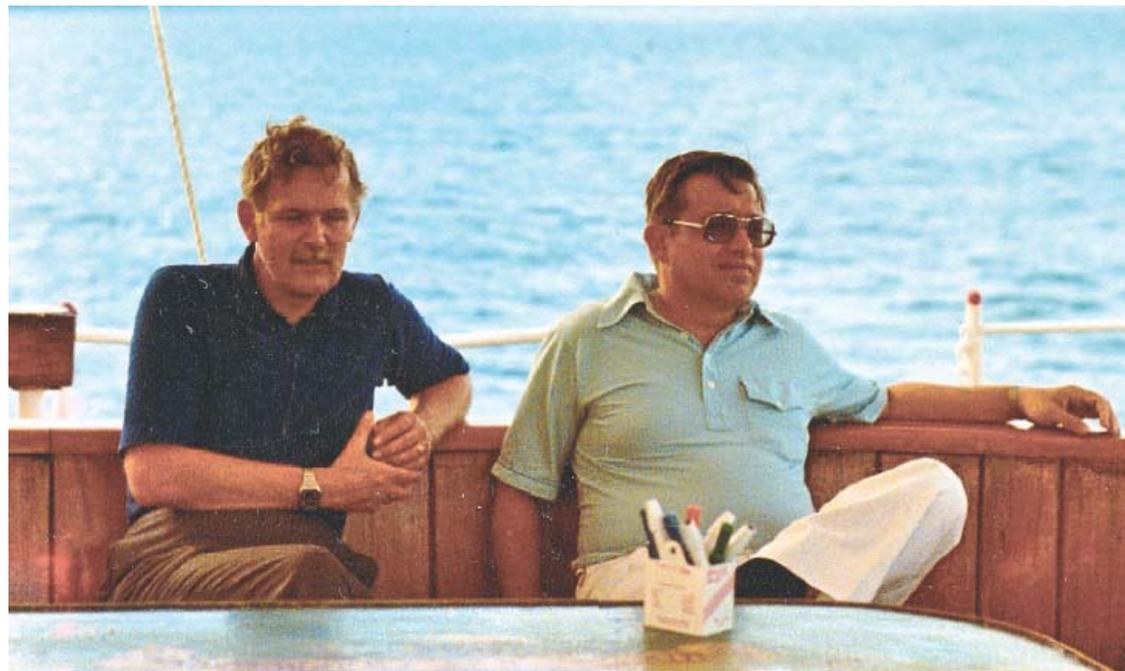
Kalman and Bucy (1977)

Review: Random Process

A **random variable** X is a mapping between the sample space and the real numbers.

A **random process** (a.k.a **stochastic process**) is a mapping from the sample space into an ensemble of time functions (known as sample functions). To every member in the sample space, there corresponds a function of time (a sample function) $X(t)$.
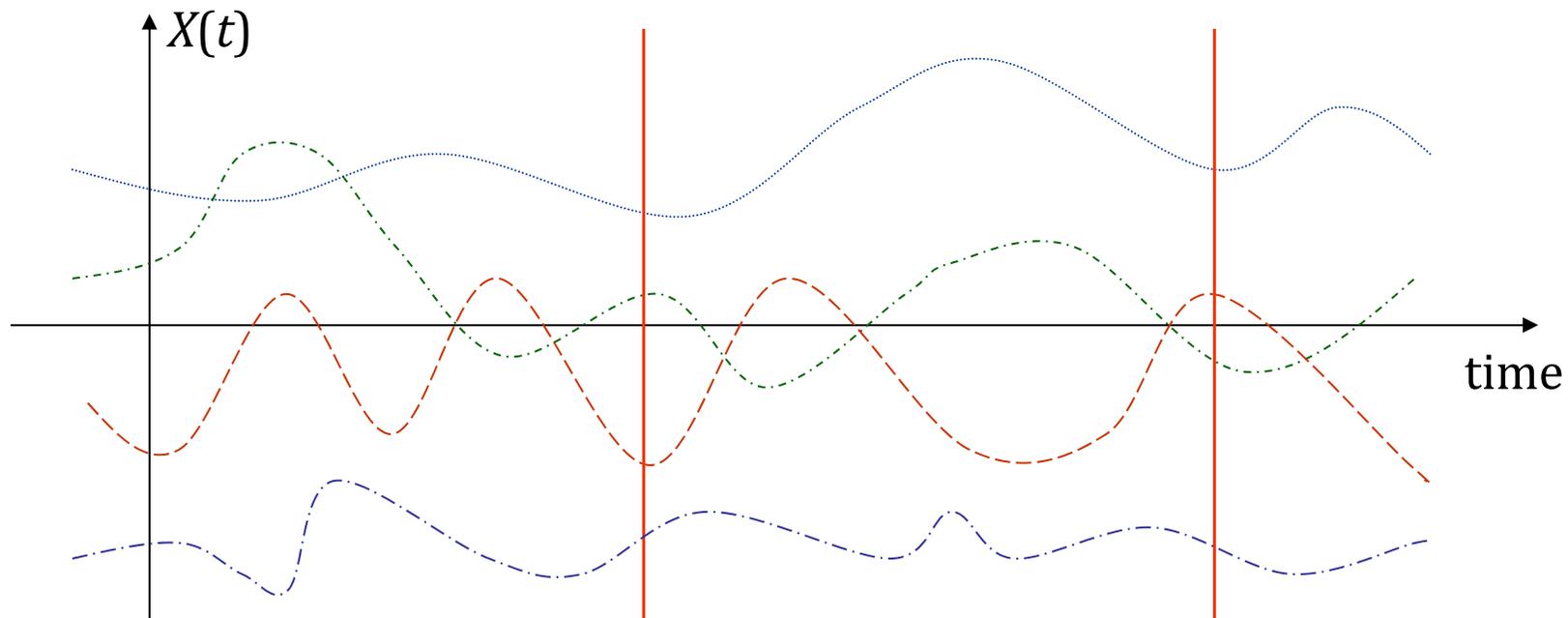
Mean, Moment, Variance, Covariance of Random Process

Let $f(x,t)$ be the **probability density function** (p.d.f.) associated with a random process $X(t)$. If the p.d.f. is independent of time t, i.e., $f(x,t) = f(x)$, then the corresponding random process is said to be **stationary**. We will focus our attention only on this class of random processes in this course. For this type of random processes (RP), we define:

1) **mean** (or **expectation**):

$$m = E[X] = \int_{-\infty}^{\infty} x \cdot f(x)dx$$

2) **moment** (j-th order moment)

$$E[X^j] = \int_{-\infty}^{\infty} x^j \cdot f(x)dx$$

3) **variance**

$$\sigma^2 = E[(x-m)^2] = \int_{-\infty}^{\infty} (x-m)^2 f(x)dx$$

4) **covariance of two random processes**

$$con(v,w) = E[(v-E[v])(w-E[w])]$$

Two RPs v and w are said to be **independent** if their joint p.d.f. $f(v,w) = f(v) \cdot f(w)$

$$\Rightarrow E[vw] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} vwf(v,w)dvdw = \int_{-\infty}^{\infty} vf(v)dv \int_{-\infty}^{\infty} wf(w)dw = E[v]E[w]$$

Autocorrelation Function and Power Spectrum

Autocorrelation function is used to describe the time domain property of a random process. Given a random process v, its **autocorrelation function** is defined as follows:

$$R_x(t_1, t_2) = E\big[v(t_1)v(t_2)\big]$$

If v is a wide sense stationary (WSS) process,

$$R_x(t_1, t_2) = R_x(t_2 - t_1) = R_x(\tau) = R_x(t, t+\tau) = E\big[v(t)v(t+\tau)\big]$$

Note that $R_x(0)$ is the time average of the power or energy of the random process.

Power spectrum of a random process is the Fourier transform of its autocorrelation function. It is a frequency domain property of the random process. To be more specific, it is defined as

$$S_x(\omega) = \int_{-\infty}^{\infty} R_x(\tau) e^{-j\omega\tau} d\tau$$

White Noise, Colored Noise and Gaussian Random Process

White Noise is a random process with a constant power spectrum, and an autocorrelation function

$$R_x(\tau) = q \cdot \delta(\tau)$$

which implies that a white noise has an infinite power and thus it is non existent in real life. However, many noises (or the so-called colored noises, or noises with finite energy and finite frequency components) can be modeled as the outputs of linear systems with an injection of white noise into their inputs, i.e., a **colored noise** can be generated by a white noise

white noise $\longrightarrow$ | Linear System | $\longrightarrow$ colored noise

Gaussian Process v is also known as normal process has a p.d.f.

$$f(v) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(v-\mu)^2/2\sigma^2}, \qquad \mu = \text{mean}, \quad \sigma^2 = \text{variance}$$

Kalman Filter for a Linear Time Invariant (LTI) System

Consider an LTI system characterized by

$$
\begin{cases}
\dot{x} = Ax + Bu + v(t) & v \text{ is the input noise} \\
y = Cx + w(t) & w \text{ is the measurement noise}
\end{cases}
$$

Assume: (1) (A,C) is observable

(2) $v(t)$ and $w(t)$ are independent white noises with the following properties

$$
E[v(t)] = 0, \quad E[v(t)v^{\mathrm{T}}(\tau)] = Q\delta(t-\tau), \ Q = Q^{\mathrm{T}} \geq 0,
$$

$$
E[w(t)] = 0, \ E[w(t)w^{\mathrm{T}}(\tau)] = R\delta(t-\tau), \ R = R^{\mathrm{T}} > 0
$$

(3) $\left(A, Q^{1/2} \right)$ is stabilizable (to guarantee closed-loop stability).

The problem of **Kalman Filter** is to design a state estimator to estimate the state $x(t)$ by $\hat{x}(t)$ such that the estimation error covariance is minimized, i.e., the following index is minimized:

$$
J_e = E\left[\{x(t) - \hat{x}(t)\}^{\mathrm{T}} \{x(t) - \hat{x}(t)\} \right]
$$

Construction of Steady State Kalman Filter

Kalman filter is a state observer with a specially selected observer gain (or Kalman filter gain). It has the dynamic equation:

$$\dot{\hat{x}} = A\hat{x} + Bu + K_e(y - \hat{y}), \qquad \hat{x}(0) \text{ is given}$$

$$\hat{y} = C\hat{x}$$

with the Kalman filter gain K_e being given as

$$K_e = P_e C^{\mathrm{T}} R^{-1}$$

where P_e is the positive definite solution of the following Riccati equation,

$$P_e A^{\mathrm{T}} + A P_e - P_e C^{\mathrm{T}} R^{-1} C P_e + Q = 0$$

Let $e = x - \hat{x}$. We can show (see next) that such a Kalman filter has the following properties:

$$\lim_{t\to\infty} E\big[\, e(t) \big] = \lim_{t\to\infty} E\big[\, x(t) - \hat{x}(t) \big] = 0, \quad \lim_{t\to\infty} J_e = \lim_{t\to\infty} E\big[\, e^{\mathrm{T}}(t)\, e(t) \big] = \mathrm{trace}\, P_e$$

Kalman Filter and LQR – They Are Dual

Recall the optimal regulator problem,

$$\dot{x} = Ax + Bu \qquad x(0) = x_0 \;\; \text{given}$$

$$J = \int_0^\infty \left(x^{\mathrm{T}} Q x + u^{\mathrm{T}} R u \right) dt, \quad Q = Q^{\mathrm{T}} \geq 0 \;\; \text{and} \;\; R = R^{\mathrm{T}} > 0$$

The LQR problem is to find a state feedback law $u = -Fx$ such that J is minimized. It was shown that the solution to the above problem is given by

$$F = R^{-1} B^{\mathrm{T}} P \qquad PA + A^{\mathrm{T}} P - PBR^{-1} B^{\mathrm{T}} P + Q = 0, \quad P = P^{\mathrm{T}} > 0$$

and the optimal value of J is given by $J = x_0^{\mathrm{T}} P x_0$. Note that x_0 is arbitrary. Let us consider a special case when x_0 is a random vector with   .

$$E[x_0] = 0, \;\; E[x_0 x_0^{\mathrm{T}}] = I$$

Then, we have

$$E[J] = E[x_0^{\mathrm{T}} P x_0] = E\left[\sum_{i=1}^n \sum_{j=1}^n p_{ij} x_{0i} x_{0j} \right] = \sum_{i=1}^n \sum_{j=1}^n p_{ij} E[x_{0i} x_{0j}] = \sum_{i=1}^n p_{ii} = \text{trace } P$$

The Duality

♣ **Linear Quadratic Regulator** ♠ **Kalman Filter**

$$F = R^{-1}B^{\mathrm{T}}P$$

$$PA + A^{\mathrm{T}}P - PBR^{-1}B^{\mathrm{T}}P + Q = 0$$

$$J_{\mathrm{optimal}} = \mathrm{trace}\ P$$

$$K_e = P_e C^{\mathrm{T}} R^{-1}$$

$$P_e A^{\mathrm{T}} + AP_e - P_e C^{\mathrm{T}} R^{-1} CP_e + Q = 0$$

$$J_{\mathrm{optimal}} = \mathrm{trace}\ P_e$$

These two problems are equivalent (or dual) if we let

$$A^{\mathrm{T}} \longleftrightarrow A$$
$$B^{\mathrm{T}} \longleftrightarrow C$$
$$F^{\mathrm{T}} \longleftrightarrow K_e$$
$$P \qquad P_e$$

Proof of Properties of Kalman Filter

Recall that the dynamics of the given plant and Kalman filter, i.e.,

$$\dot{x} = Ax + Bu + v(t)$$
$$y = Cx + w(t)$$

&

$$\dot{\hat{x}} = A\hat{x} + Bu + K_e(y - \hat{y})$$
$$\hat{y} = C\hat{x}$$

We have

$$\dot{e} = \dot{x} - \dot{\hat{x}} = Ax + Bu + v(t) - A\hat{x} - Bu - K_e[Cx + w(t) - C\hat{x}]$$

$$= (A - K_eC)(x - \hat{x}) + v(t) - K_ew(t)$$

$$= (A - K_eC)e + \begin{bmatrix} I & -K_e \end{bmatrix}\begin{pmatrix} v \\ w \end{pmatrix} = \bar{A}e + d(t)$$

with

$$E[d(t)] = E\left[\begin{bmatrix} I & -K_e \end{bmatrix}\begin{pmatrix} v(t) \\ w(t) \end{pmatrix}\right] = \begin{bmatrix} I & -K_e \end{bmatrix}\begin{pmatrix} E[v(t)] \\ E[w(t)] \end{pmatrix} = \begin{bmatrix} I & -K_e \end{bmatrix}\begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0$$

Next, it is reasonable to assume that initial error $e(0)$ and $d(t)$ are independent, i.e.,

$$E[e(0)\, d^{\mathrm{T}}(t)] = E[e(0)] \cdot E[d^{\mathrm{T}}(t)] = 0$$

Furthermore,

$$E[d(t)d^\mathrm{T}(\tau)] = \begin{bmatrix} I & -K_e \end{bmatrix} \begin{pmatrix} E[v(t)v^\mathrm{T}(\tau)] & E[v(t)w^\mathrm{T}(\tau)] \\ E[w(t)v^\mathrm{T}(\tau)] & E[w(t)w^\mathrm{T}(\tau)] \end{pmatrix} \begin{bmatrix} I \\ -K_e^\mathrm{T} \end{bmatrix}$$

$$= \begin{bmatrix} I & -K_e \end{bmatrix} \begin{pmatrix} Q\delta(t-\tau) & 0 \\ 0 & R\delta(t-\tau) \end{pmatrix} \begin{bmatrix} I \\ -K_e^\mathrm{T} \end{bmatrix}$$

$$= \left(Q + K_e R K_e^\mathrm{T} \right) \delta(t-\tau)$$

$$= \nabla \delta(t-\tau)$$

where $\nabla = Q + K_e R K_e^\mathrm{T} \geq 0$.

We will next show $\overline{A}$ is asymptotically stable and

$$\lim_{t\to\infty} E[\, e(t)\, e^\mathrm{T}(t)\,] = P_e$$

Recall that $K_e = P_e C^T R^{-1}$ and

$$P_e A^T + A P_e - P_e C^T R^{-1} C P_e + Q = 0$$

We have

$$P_e A^T - P_e C^T R^{-1} C P_e + A P_e - P_e C^T R^{-1} C P_e + P_e C^T R^{-1} C P_e + Q = 0$$

$$\implies P_e \left(A^T - C^T R^{-1} C P_e \right) + \left(A - P_e C^T R^{-1} C \right) P_e + P_e C^T R^{-1} C P_e + Q = 0$$

$$\implies P_e \left(A^T - C^T K_e^T \right) + \left(A - K_e C \right) P_e + P_e C^T R^{-1} R R^{-1} C P_e + Q = 0$$

$$\implies P_e \overline{A}^T + \overline{A} P_e = -K_e R K_e^T - Q = -\nabla \leq 0$$

Since $Q = Q^T \geq 0$ and $\left(A, \, Q^{\frac{1}{2}} \right)$ is assumed to be stabilizable, it follows from Lyapunov stability theory that matrix $\overline{A} = \left(A - K_e C \right)$ is asymptotically stable.

Recall also the solution to $\dot{e} = \overline{A} e + d(t)$, i.e.,

$$e(t) = e^{\overline{A} t} \cdot e(0) + \int_0^t e^{\overline{A}(t-\tau)} d(\tau) \cdot d\tau$$

Noting that $e^{\bar{A}t}$ is deterministic, we have

$$P(t) = E[e(t)e^{\mathrm{T}}(t)] = E\left[\left(e^{\bar{A}t}\cdot e(0) + \int_0^t e^{\bar{A}(t-\tau)}d(\tau)\cdot d\tau\right)\cdot\left(e^{\bar{A}t}\cdot e(0) + \int_0^t e^{\bar{A}(t-\tau)}d(\tau)\cdot d\tau\right)^{\mathrm{T}}\right]$$

$$= e^{\bar{A}t}E[e(0)e^{\mathrm{T}}(0)]e^{\bar{A}^{\mathrm{T}}t} + \int_0^t e^{\bar{A}(t-\tau)}E[d(\tau)e^{\mathrm{T}}(0)]e^{\bar{A}^{\mathrm{T}}t}\cdot d\tau$$

0

$$+ \int_0^t e^{\bar{A}t}E[e(0)d^{\mathrm{T}}(\tau)]e^{\bar{A}^{\mathrm{T}}(t-\tau)}\cdot d\tau + \int_0^t e^{\bar{A}(t-\tau)}d\tau\int_0^t E[d(\tau)d^{\mathrm{T}}(\sigma)]e^{\bar{A}^{\mathrm{T}}(t-\sigma)}\cdot d\sigma$$

0

$$= e^{\bar{A}t}E[e(0)e^{\mathrm{T}}(0)]e^{\bar{A}^{\mathrm{T}}t} + \int_0^t e^{\bar{A}(t-\tau)}d\tau\int_0^t \nabla\delta(\tau-\sigma)e^{\bar{A}^{\mathrm{T}}(t-\sigma)}\cdot d\sigma$$

$$= e^{\bar{A}t}E[e(0)e^{\mathrm{T}}(0)]e^{\bar{A}^{\mathrm{T}}t} + \int_0^t e^{\bar{A}(t-\tau)}\nabla e^{\bar{A}^{\mathrm{T}}(t-\tau)}\cdot d\tau = e^{\bar{A}t}E[e(0)e^{\mathrm{T}}(0)]e^{\bar{A}^{\mathrm{T}}t} + \int_0^t e^{\bar{A}\eta}\nabla e^{\bar{A}^{\mathrm{T}}\eta}\cdot d\eta$$

Since $\bar{A}$ is stable, we have $e^{\bar{A}t} \to 0,$ as $t \to \infty$. Thus,

$$P(\infty) = \int_0^{\infty} e^{\bar{A}\eta}\nabla e^{\bar{A}^{\mathrm{T}}\eta}\cdot d\eta$$

We next show that $P(\infty) = P_e$, i.e., the solution to the Kalman filter ARE. Let

$$\dot{z} = \overline{A}^\mathrm{T} z, \quad z(0) \text{ given} \quad \Rightarrow \quad z(t) = e^{\overline{A}^\mathrm{T} t} z(0), \quad z(\infty) = 0$$

In view of $P_e \overline{A}^\mathrm{T} + \overline{A} P_e = -\nabla$, we have

$$z^\mathrm{T} [P_e \overline{A}^\mathrm{T} + \overline{A} P_e] z = -z^\mathrm{T} \nabla z \quad \Rightarrow \quad z^\mathrm{T} P_e \overline{A}^\mathrm{T} z + z^\mathrm{T} \overline{A} P_e z = -z^\mathrm{T} \nabla z$$

$$\Rightarrow \quad z^\mathrm{T} P_e \dot{z} + \dot{z}^\mathrm{T} P_e z = -z^\mathrm{T} \nabla z \quad \Rightarrow \quad \frac{d}{dt}(z^\mathrm{T} P_e z) = -z^\mathrm{T} \nabla z$$

Next, we have

$$-\int_0^\infty z^\mathrm{T} \nabla z \, dt = -\int_0^\infty z^\mathrm{T}(0) e^{\overline{A} t} \nabla e^{\overline{A}^\mathrm{T} t} z(0) \, dt = -z^\mathrm{T}(0) \left[\int_0^\infty e^{\overline{A} t} \nabla e^{\overline{A}^\mathrm{T} t} dt \right] z(0) = -z^\mathrm{T}(0) P(\infty) z(0)$$

$$\int_0^\infty \frac{d}{dt}(z^\mathrm{T} P_e z) \, dt = z^\mathrm{T}(t) P_e z(t) \Big|_0^\infty = z^\mathrm{T}(\infty) P_e z(\infty) - z^\mathrm{T}(0) P_e z(0) = 0 - z^\mathrm{T}(0) P_e z(0)$$

Thus, we have for every given $z(0)$,

$$z^\mathrm{T}(0) P_e z(0) = z^\mathrm{T}(0) P(\infty) z(0) \quad \Rightarrow \quad P_e = P(\infty) = \int_0^\infty e^{\overline{A}^\mathrm{T} \eta} \nabla e^{\overline{A} \eta} d\eta$$

It is now simple to see that

$$\lim_{t\to\infty} E[e(t)e^{\mathrm{T}}(t)] = P(\infty) = P_e \quad \Rightarrow \quad \lim_{t\to\infty} E[e^{\mathrm{T}}(t)e(t)] = \text{trace } P_e$$

Finally, we have

$$\lim_{t\to\infty} E[e(t)] = \lim_{t\to\infty}\left[e^{\bar{A}t} \cdot E[e(0)] + \int_0^t e^{\bar{A}(t-\tau)} E[d(\tau)] \cdot d\tau \right] = 0$$

Example: Consider a given plant characterized by the following state space model,

$$\begin{cases} \dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + v(t), \quad E[v(t)v^{\mathrm{T}}(\tau)] = Q\delta(t-\tau) = \begin{bmatrix} 0.1 & 0 \\ 0 & 0 \end{bmatrix}\delta(t-\tau) \\ \\ y = \begin{bmatrix} 1 & 0 \end{bmatrix} x + w(t), \quad E[w(t)w^{\mathrm{T}}(\tau)] = R\delta(t-\tau) = 0.2\delta(t-\tau) \end{cases}$$

Solving the Kalman filter ARE, we obtain

$$P_e = \begin{bmatrix} 0.0792 & -0.0343 \\ -0.0343 & 0.0314 \end{bmatrix}, \quad K_e = \begin{bmatrix} 0.3962 \\ -0.1715 \end{bmatrix} \qquad \begin{cases} \dot{\hat{x}} = A\hat{x} + Bu + K_e(y - \hat{y}) \\ \hat{y} = C\hat{x} \end{cases}$$

Related Topics:

Extended Kalman Filter (EKF): In estimation theory, the EKF is the nonlinear version of the Kalman filter, which linearizes about the current mean and covariance. The EKF has been considered the *de facto* standard in the theory of nonlinear state estimation, navigation systems and GPS.

Unscented Kalman filter (UKF): When the state transition and observation models, the predict and update functions are highly nonlinear, the extended Kalman filter can give particularly poor performance. This is because the covariance is propagated through linearization of the underlying non-linear model. UKF uses a deterministic sampling technique known as the unscented transform to pick a minimal set of sample points (called sigma points) around the mean. These sigma points are then propagated through the non-linear functions, from which the mean and covariance of the estimate are then recovered. The result is a filter which more accurately captures the true mean and covariance.

6. Linear Quadratic Gaussian (LQG)

Problem Statement

It is very often in control system design for a real life problem that one cannot measure all the state variables of the given plant. Thus, the linear quadratic regulator, although it has very impressive gain and phase margins (GM = ∞ and PM > 60 degrees), is impractical as it utilizes all state variables in the feedback, i.e., $u = -Fx$. In most of practical situations, only partial information of the state of the given plant is accessible or can be measured for feedback. The natural questions one would ask:

- Can we recover or estimate the state variables of the plant through the partially measurable information? The answer is yes. The solution is Kalman filter.
- Can we replace x the control law in LQR, i.e., $u = -Fx$, by the estimated state to carry out a meaningful control system design? The answer is yes. The solution is called LQG.
- Do we still have impressive properties associated with LQG? The answer is no. Any solution? Yes. It is called a **loop transfer recovery** (LTR) technique.

Linear Quadratic Gaussian Design

Consider a given plant characterized by

$$\begin{cases} \dot{x} = Ax + Bu + v(t) & \quad v \text{ is the input noise} \quad\quad\quad\quad (A, B) \text{ stabilizable} \\ y = Cx + w(t) & \quad w \text{ is the measurement noise} \quad (A, C) \text{ detectable} \end{cases}$$

where $v(t)$ and $w(t)$ are white with zero means. $v(t)$, $w(t)$ and $x(0)$ are independent, and

$$E[v(t)v^{\mathrm{T}}(\tau)] = Q_e \delta(t - \tau), \ \ Q_e \geq 0, \ \ E[w(t)w^{\mathrm{T}}(\tau)] = R_e \delta(t - \tau), \ \ R_e > 0, \ \ \ E[x(0)] = x_0$$

The performance index has to be modified as follows:

$$J = \lim_{T \to \infty} \frac{1}{T} E\left[\int_0^T (x^{\mathrm{T}}Qx + u^{\mathrm{T}}Ru)dt \right], \quad\quad Q \geq 0, \ \ R > 0$$

The **Linear Quadratic Gaussian** (LQG) control is to design a control law that only requires the measurable information such that when it is applied to the given plant, the overall system is stable and the performance index is minimized.

Solution to LQG Problem – Separation Principle

Step 1. Design an LQR control law $u = -Fx$ which solves the following problem,

$$\dot{x} = A\,x + B\,u \qquad J(x,u,Q,R) = \int_0^\infty (x^\mathsf{T}Qx + u^\mathsf{T}Ru)dt, \qquad Q \geq 0, \; R > 0$$

i.e., compute

$$PA + A^\mathsf{T}P - PBR^{-1}B^\mathsf{T}P + Q = 0, \quad P > 0, \qquad\qquad F = R^{-1}B^\mathsf{T}P.$$

Step 2. Design a Kalman filter for the given plant, i.e.,

$$\dot{\hat{x}} = A\hat{x} + Bu + K_e(y - \hat{y}), \quad \hat{y} = C\hat{x}$$

where $K_e = P_e C^\mathsf{T} R^{-1}, \;\; P_e A^\mathsf{T} + A P_e - P_e C^\mathsf{T} R_e^{-1} C P_e + Q_e = 0, \quad P_e > 0.$

Step 3. The LQG control law is given by $u = -F\,\hat{x}$, i.e.,

$$\begin{cases} \dot{\hat{x}} = A\,\hat{x} + B\,u + K_e(y - C\,\hat{x}) \\ u = -F\,\hat{x} \end{cases} \quad\Longrightarrow\quad \begin{cases} \dot{\hat{x}} = (A - BF - K_eC)\,\hat{x} + K_e\,y \\ u = -F\,\hat{x} \end{cases}$$

Block Diagram Implementation of LQG Control Law

Reference

r → G → ⊖ → u → **PLANT** → y

$$G = [C_2 (A - BF)^{-1} B]^{-1}$$

LQR CONTROL ← $\hat{x}$ ← **KALMAN FILTER**
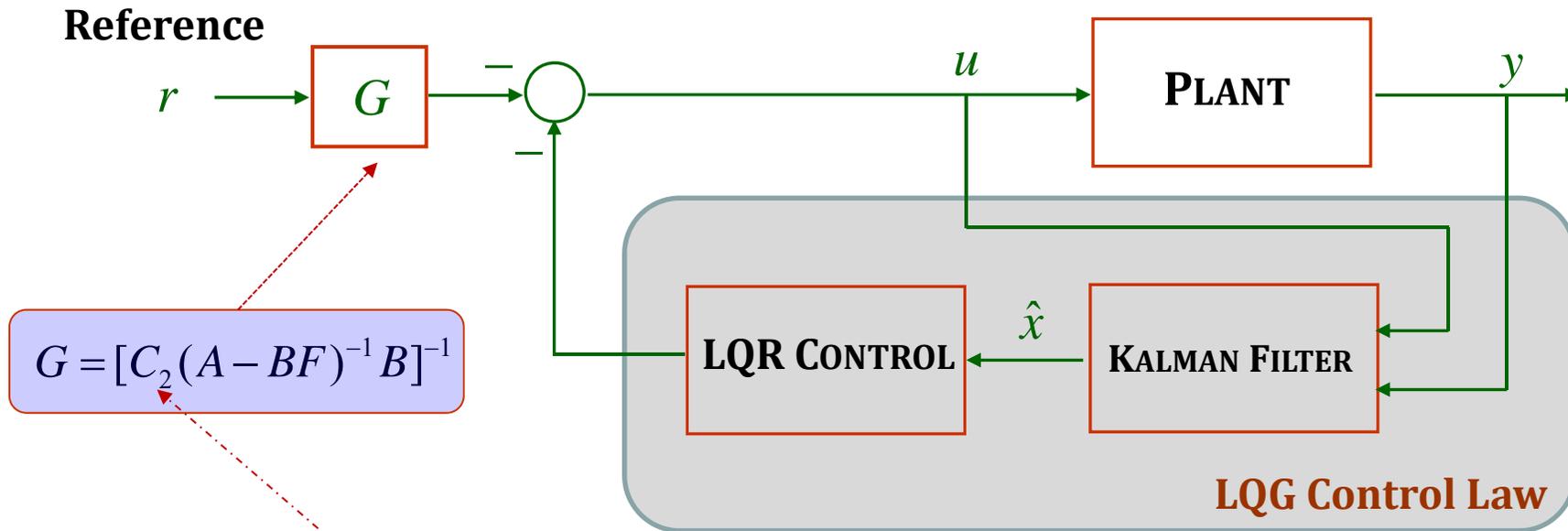
LQG Control Law

Matrix C_2 is related to output variables of interest, say

$$z = C_2 x$$

where z is to track the reference r.

Closed-Loop Dynamics of Given Plant together with LQG Controller

Recall the plant: $\begin{cases} \dot{x} = Ax + Bu + v(t) \\ y = Cx + w(t) \end{cases}$ and controller $\begin{cases} \dot{\hat{x}} = (A - BF - K_e C)\,\hat{x} - BGr + K_e\,y \\ u = -F\,\hat{x} - Gr \end{cases}$

We define a new variable $e = x - \hat{x}$ and thus

$$\dot{e} = \dot{x} - \dot{\hat{x}} = Ax - BF\hat{x} - BGr + v(t) - A\hat{x} + BF\hat{x} + K_e C\hat{x} + BGr - K_e Cx - K_e w(t)$$

$$= A(x - \hat{x}) - K_e C(x - \hat{x}) + v(t) - K_e w(t) = (A - K_e C)e + v(t) - K_e w(t)$$

and

$$\dot{x} = Ax + Bu + v(t) = Ax - BF\hat{x} - BGr + v(t) = Ax - BF(x - e) - BGr + v(t)$$

$$= (A - BF)x + BFe - BGr + v(t)$$

Clearly, the closed-loop system is characterized by the following state space equation,

$$\begin{cases} \begin{pmatrix} \dot{x} \\ \dot{e} \end{pmatrix} = \begin{bmatrix} A - BF & BF \\ 0 & A - K_e C \end{bmatrix} \begin{pmatrix} x \\ e \end{pmatrix} - \begin{bmatrix} BG \\ 0 \end{bmatrix} r + \tilde{v}, \quad \tilde{v} = \begin{pmatrix} v \\ v - K_e w \end{pmatrix} \\ \\ y = \begin{bmatrix} C & 0 \end{bmatrix} \begin{pmatrix} x \\ e \end{pmatrix} + w \end{cases}$$

The closed-loop poles are given by $\lambda(A - BF) \cup \lambda(A - K_e C)$, which are stable.

Homework Assignment 1:

Using the LQR design, the Kalman filter design and their combination, i.e., the LQG control method to design an appropriate measurement feedback control law that meets all the design specification specified in the (HDD or helicopter) problem.

Show all the detailed calculation and simulate your design using MATLAB and Simulink. Give all the necessary plots that show the evidence of your design.

7. Introduction to Robust Control

George Zames
1934–1997

A Real Control Problem

disturbances → **Plant** → response

sensor noise →

control input

measurements

Controller ← commands
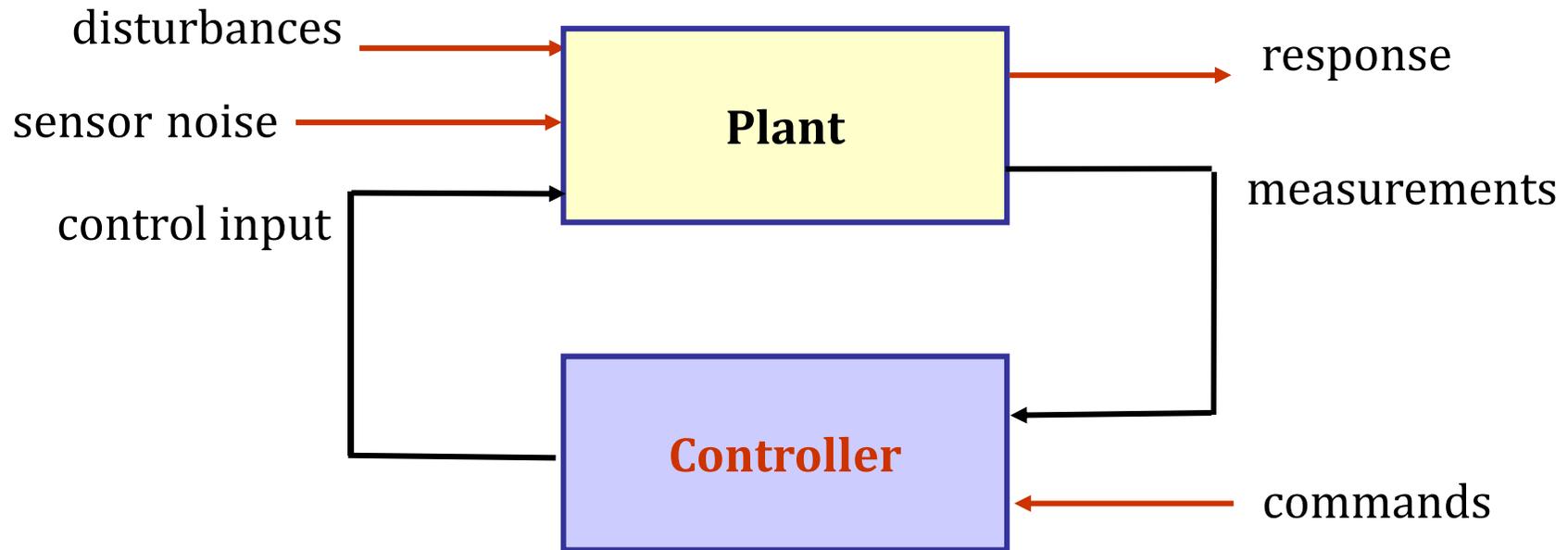
Controller Objective: To provide desired responses in face of

- Uncertain plant dynamics + External inputs ⎰ disturbances
  sensor noise
  control input

Representation of Uncertain Plant Dynamics
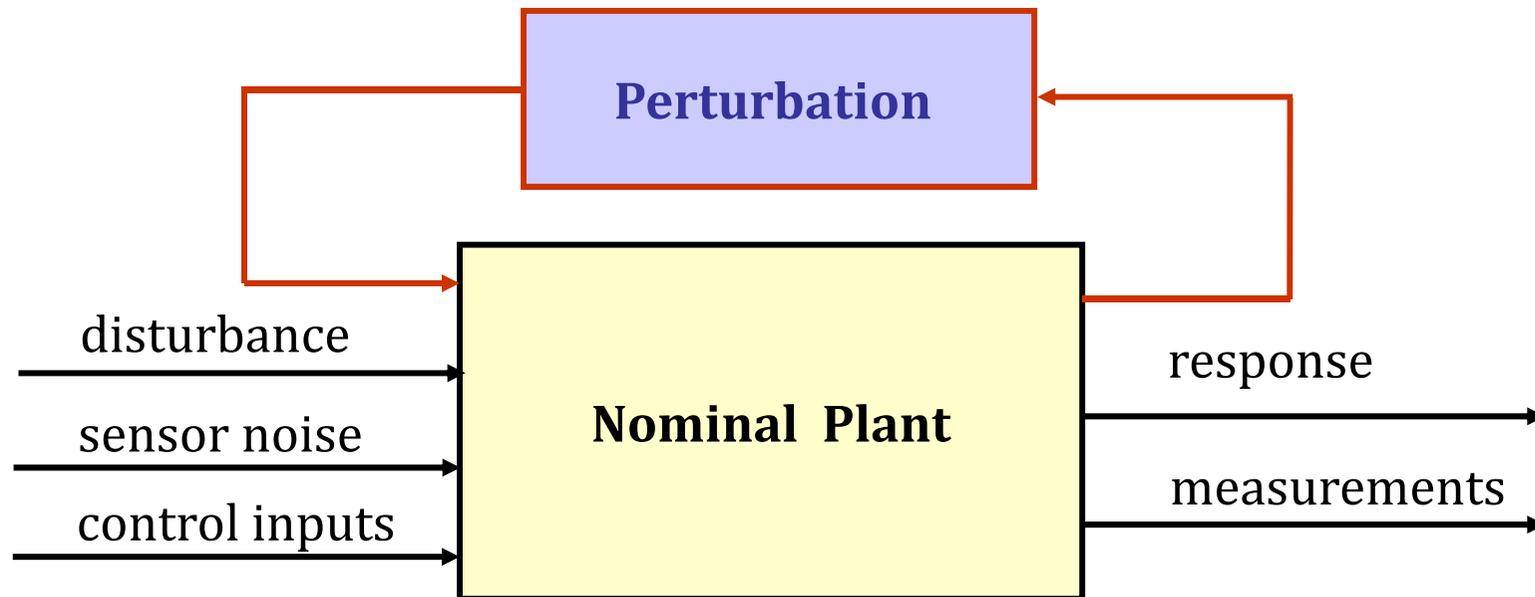
- Nominal Plant is an FDLTI System

- Perturbation is Member of Set of Possible Perturbations

Analysis Objectives

- **Nominal Performance Question (H_2 Optimal Control):**

  Are closed loop responses acceptable for disturbances? sensor noise?

- **Robust Stability Question (H_∞ Optimal Control):**

  Is closed loop system stable for nominal plant? for all possible perturbations?

- **Robust Performance Question (Mixed H_2 / H_∞ Optimal Control):**

  Are closed loop responses acceptable for all possible perturbations and all external inputs? Simultaneously?

Complete Picture of Robust Control Problem
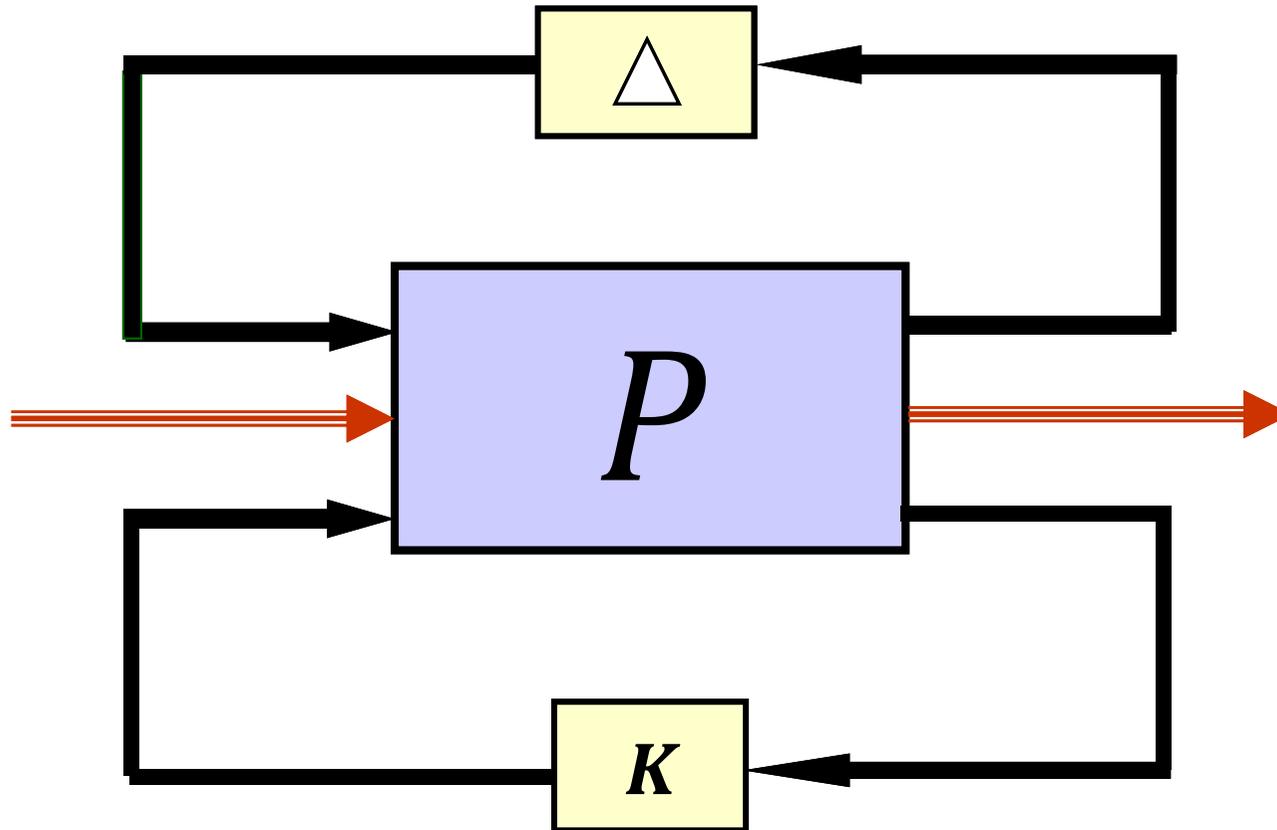
Standard Feedback Loops in terms of General Interconnection Structure
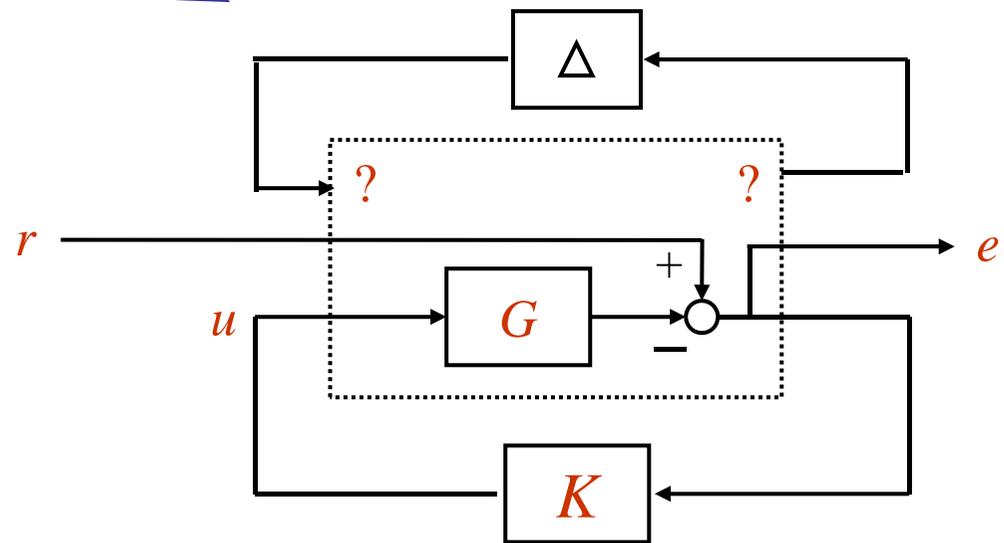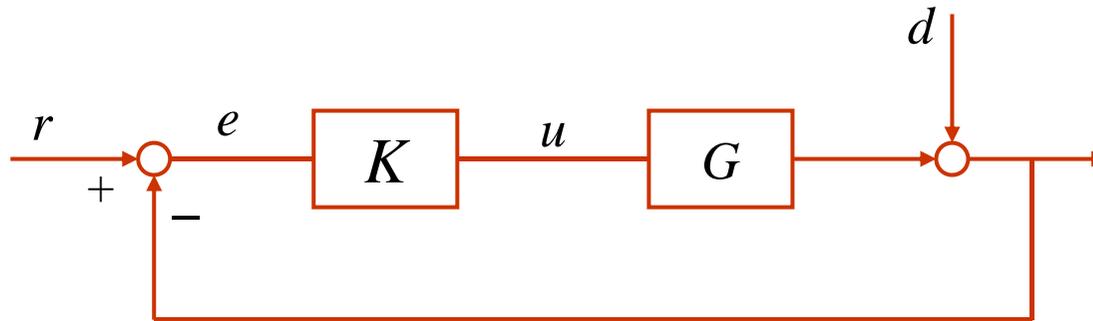
8. H_2 Optimal Control and H_∞ Control
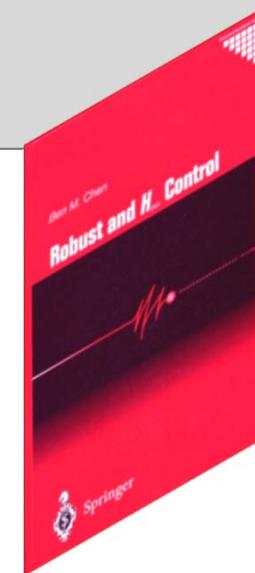
Introduction to the Problems

Consider a stabilizable and detectable linear time-invariant system Σ with a proper controller Σ_c
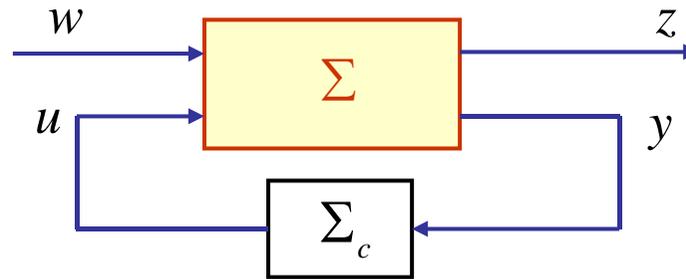
where

$$\Sigma : \begin{cases} \dot{x} = A\,x + B\,u + E\,w \\ y = C_1\,x + 0\,u + D_1\,w \\ z = C_2\,x + D_2\,u + 0\,w \end{cases} \qquad \Sigma_c : \begin{cases} \dot{v} = A_c\,v + B_c\,y \\ u = C_c\,v + D_c\,y \end{cases}$$

$$\begin{cases} x \in R^n \iff \text{state variable} \\ y \in R^p \iff \text{measurement} \\ z \in R^q \iff \text{controlled output} \end{cases} \quad \& \quad \begin{matrix} u \in R^m \iff \text{control input} \\ w \in R^l \iff \text{disturbance} \\ v \in R^k \iff \text{controller state} \end{matrix}$$

The problems of H_2 and H_∞ optimal control are to design a proper control law Σ_c such that when it is applied to the given plant with disturbance, i.e., Σ, we have

- The resulting closed loop system is internally stable (this is necessary for any control system design).

- The resulting closed-loop transfer function from the disturbance w to the controlled output z, say, $T_{zw}(s)$, is as small as possible, i.e., the effect of the disturbance on the controlled output is minimized.

  - H_2 optimal control: the H_2-norm of $T_{zw}(s)$ is minimized.

  - H_∞ optimal control: the H_∞-norm of $T_{zw}(s)$ is minimized.

Note: A transfer function is a function of frequencies ranging from 0 to ∞. It is hard to tell if it is large or small. The common practice is to measure its norms instead. H_2-norm and H_∞-norm are two commonly used norms in measuring the sizes of a transfer function.

Closed Loop Transfer Function from Disturbance to Controlled Output

Recall that

$$\Sigma: \begin{cases} \dot{x} = A\,x + B\,u + E\,w \\ y = C_1\,x + 0\,u + D_1\,w \\ z = C_2\,x + D_2\,u + 0\,w \end{cases}$$

and

$$\Sigma_c: \begin{cases} \dot{v} = A_c\,v + B_c\,y \\ u = C_c\,v + D_c\,y \end{cases}$$

$$\Rightarrow \quad \dot{v} = A_c v + B_c(C_1 x + D_1 w)$$
$$= A_c v + B_c C_1 x + B_c D_1 w$$

$$\Rightarrow \begin{cases} \dot{x} = A\,x + B(C_c v + D_c y) + E\,w \\ y = C_1\,x \qquad\qquad\quad + D_1\,w \\ z = C_2\,x + D_2(C_c v + D_c y) \end{cases} \Rightarrow \begin{cases} \dot{x} = Ax + BC_c v + BD_c(C_1 x + D_1 w) + Ew \\ z = C_2 x + D_2 C_c v + D_2 D_c(C_1 x + D_1 w) \end{cases}$$

$$\Rightarrow \begin{cases} \dot{x} = Ax + BC_c v + BD_c y + Ew \\ z = C_2 x + D_2 C_c v + D_2 D_c y \end{cases} \Rightarrow \begin{cases} \dot{x} = (A + BD_c C_1)x + BC_c v + (E + BD_c D_1)w \\ z = (C_2 + D_2 D_c C_1)x + D_2 C_c v + D_2 D_c D_1 w \end{cases}$$

$$\Rightarrow \begin{cases} \begin{pmatrix} \dot{x} \\ \dot{v} \end{pmatrix} = \begin{bmatrix} A + BD_c C_1 & BC_c \\ B_c C_1 & A_c \end{bmatrix}\begin{pmatrix} x \\ v \end{pmatrix} + \begin{bmatrix} E + BD_c D_1 \\ B_c D_1 \end{bmatrix} w = A_{cl}\,\tilde{x} + B_{cl}\,w \\ z = \begin{bmatrix} C_2 + D_2 D_c C_1 & D_2 C_c \end{bmatrix}\begin{pmatrix} x \\ v \end{pmatrix} + \quad D_2 D_c D_1 \quad w = C_{cl}\,\tilde{x} + D_{cl}\,w \end{cases}$$

Thus, the closed-loop transfer function from w to z is given by

$$T_{zw}(s) = C_{cl}\left(sI - A_{cl}\right)^{-1} B_{cl} + D_{cl}$$

The resulting closed-loop system is internally stable if and only if the eigenvalues of

$$A_{cl} = \begin{bmatrix} A + BD_cC_1 & BC_c \\ B_cC_1 & A_c \end{bmatrix}$$

are all in open left half complex plane.

Remark: For the state feedback case, $C_1 = I$ and $D_1 = 0$, i.e., all the states of the given system can be measured, Σ_c can then be reduced to $u = F\,x$ and the corresponding closed-loop transfer function is reduced to

$$T_{zw}(s) = \left(\, C_2 + D_2F \,\right)\left(sI - A - BF\right)^{-1} E$$

The closed-loop stability implies and is implied that $A + B\,F$ has stable eigenvalues.

H_2-norm and H_∞-norm of a Transfer Function

Definition: (H_2-norm) Given a stable and proper transfer function $T_{zw}(s)$, its H_2-norm is defined as

$$\left\| T_{zw} \right\|_2 = \left(\frac{1}{2\pi} \text{trace}\left[\int_{-\infty}^{+\infty} T_{zw}(j\omega)T_{zw}(j\omega)^{\mathsf{H}} d\omega \right] \right)^{\frac{1}{2}}$$

Graphically,
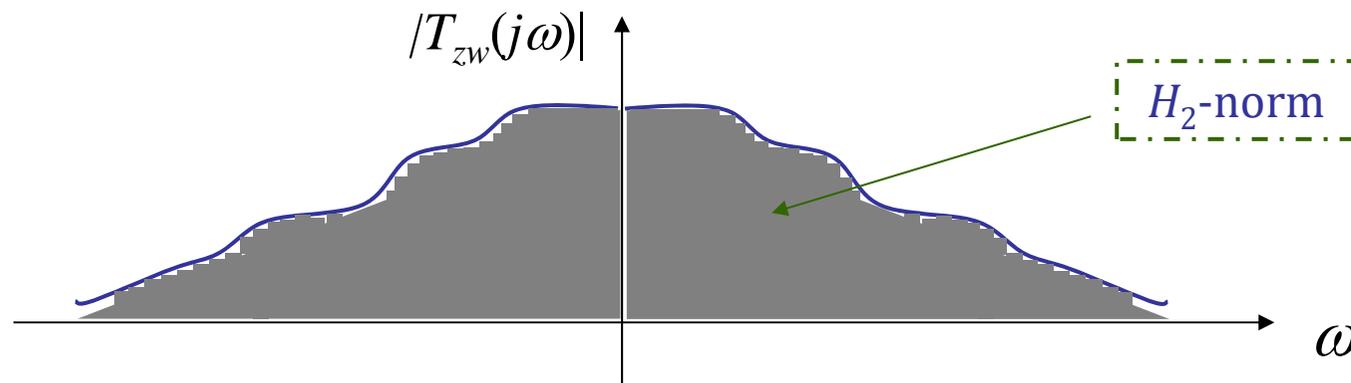
H_2-norm

Note: The H_2-norm is the total energy corresponding to the impulse response of $T_{zw}(s)$. Thus, minimization of the H_2-norm of $T_{zw}(s)$ is equivalent to the minimization of the total energy from the disturbance w to the controlled output z.

Definition: (H_∞-norm) Given a stable and proper transfer function $T_{zw}(s)$, its H_∞-norm is defined as

$$\left\| T_{zw} \right\|_\infty = \sup_{0 \le \omega < \infty} \sigma_{\max}\left[T_{zw}(j\omega) \right]$$

where $\sigma_{\max}\left[T_{zw}(j\omega) \right]$ denotes the maximum singular value of $T_{zw}(j\omega)$. For a single-input-single-output transfer function $T_{zw}(s)$, it is equivalent to the magnitude of $T_{zw}(j\omega)$. Graphically,
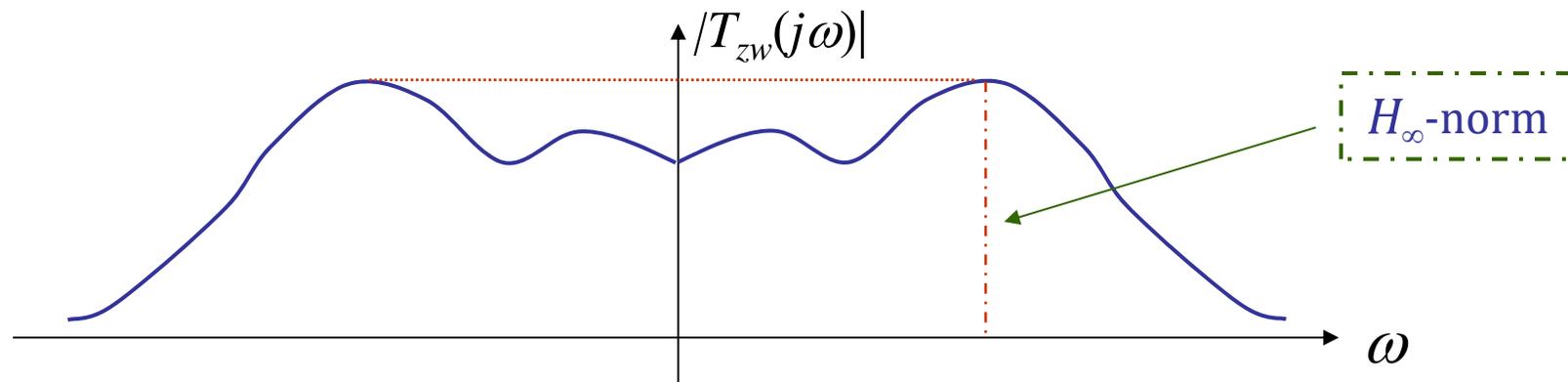
Note: The H_∞-norm is the worst case gain in $T_{zw}(s)$. Thus, minimization of the H_∞-norm of $T_{zw}(s)$ is equivalent to the minimization of the worst case (gain) situation on the effect from the disturbance w to the controlled output z.

Infima and Optimal Controllers

Definition: (The infimum of H_2 optimization) The infimum of the H_2 norm of the closed-loop transfer matrix $T_{zw}(s)$ over all stabilizing proper controllers is denoted by γ_2^*, that is

$$\gamma_2^* := \inf\left\{\ \left\| T_{zw} \right\|_2\ \middle|\ \Sigma_c \text{ internally stabilizes } \Sigma \ \right\}.$$

Definition: (The H_2 optimal controller) A proper controller Σ_c is said to be an H_2 optimal controller if it internally stabilizes Σ and $\left\| T_{zw} \right\|_2 = \gamma_2^*$.

Definition: (The infimum of H_∞ optimization) The infimum of the H_∞-norm of the closed-loop transfer matrix $T_{zw}(s)$ over all stabilizing proper controllers is denoted by γ_∞^*, that is

$$\gamma_\infty^* := \inf\left\{\ \left\| T_{zw} \right\|_\infty\ \middle|\ \Sigma_c \text{ internally stabilizes } \Sigma \ \right\}.$$

Definition: (The H_∞ γ-suboptimal controller) A proper controller Σ_c is said to be an H_∞ γ- suboptimal controller if it internally stabilizes Σ and $\left\| T_{zw} \right\|_\infty < \gamma \left(> \gamma_\infty^*\right)$.

Critical Assumptions: Regular Case vs Singular Case

Most results in H_2 and H_∞ optimal control deal with a so-called a regular problem or regular case because it is simple. An H_2 or H_∞ control problem is said to be **regular** if the following conditions are satisfied,

1. D_2 is of maximal column rank, i.e., D_2 is a tall and full rank matrix $\begin{bmatrix} \\ \\ \end{bmatrix}$

2. The subsystem (A,B,C_2,D_2) has no invariant zeros on the imaginary axis;

3. D_1 is of maximal row rank, i.e., D_1 is a fat and full rank matrix $\begin{bmatrix} \end{bmatrix}$

4. The subsystem (A,E,C_1,D_1) has no invariant zeros on the imaginary axis;

An H_2 or H_∞ control problem is said to be **singular** if it is not regular, i.e., at least one of the above 4 conditions is not satisfied.

Note: For state feedback control, Conditions 1 and 2 are sufficient for the regular case.

Solutions to the State Feedback Problems: the Regular Case

The state feedback H_2 and H_∞ control problems are referred to the problems in which all the states of the given plant Σ are available for feedback. That is the given system is

$$\Sigma: \begin{cases} \dot{x} = A\,x + B\,u + E\,w \\ y = \quad x \\ z = C_2\,x + D_2\,u \end{cases}$$

where (A, B) is stabilizable, D_2 is of maximal column rank and (A, B, C_2, D_2) has no invariant zeros on the imaginary axis.

In the state feedback case, we are looking for a static control law

$$u = F\,x$$

Solution to the Regular H_2 State Feedback Problem

Solve the following algebraic Riccati equation (H_2-ARE)

$$A^\mathsf{T} P + PA + C_2^\mathsf{T} C_2 - \left(PB + C_2^\mathsf{T} D_2 \right) \left(D_2^\mathsf{T} D_2 \right)^{-1} \left(D_2^\mathsf{T} C_2 + B^\mathsf{T} P \right) = 0$$

for a unique positive semi-definite stabilizing solution $P \geq 0$. The H_2 optimal state feedback law is then given by

$$u = F\, x = -\left(D_2^\mathsf{T} D_2 \right)^{-1} \left(D_2^\mathsf{T} C_2 + B^\mathsf{T} P \right) x$$

It can be showed that the resulting closed-loop system $T_{zw}(s)$ has the following property:

$$\| T_{zw} \|_2 = \gamma_2^*.$$

It can also be showed that $\gamma_2^* = \left[\mathrm{trace}(E^\mathsf{T} P E) \right]^{1/2}$. Note that the **trace** of a matrix is defined as the **sum of all its diagonal elements**.

Example: Consider a system characterized by

$$\Sigma : \begin{cases} \dot{x} = \overset{A}{\begin{bmatrix} 5 & 2 \\ 3 & 4 \end{bmatrix}} x + \overset{B}{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} u + \overset{E}{\begin{bmatrix} 1 \\ 2 \end{bmatrix}} w \\[2mm] y = x \\[2mm] z = \underset{C_2}{\begin{bmatrix} 1 & 1 \end{bmatrix}} x + \underset{D_2}{1 \cdot u} \end{cases}$$

Solving the following H_2-ARE using MATLAB, we obtain a positive definite solution

$$P = \begin{bmatrix} 144 & 40 \\ 40 & 16 \end{bmatrix}$$

and

$$F = \begin{bmatrix} -41 & -17 \end{bmatrix}$$

The closed-loop magnitude response from the disturbance to the controlled output:
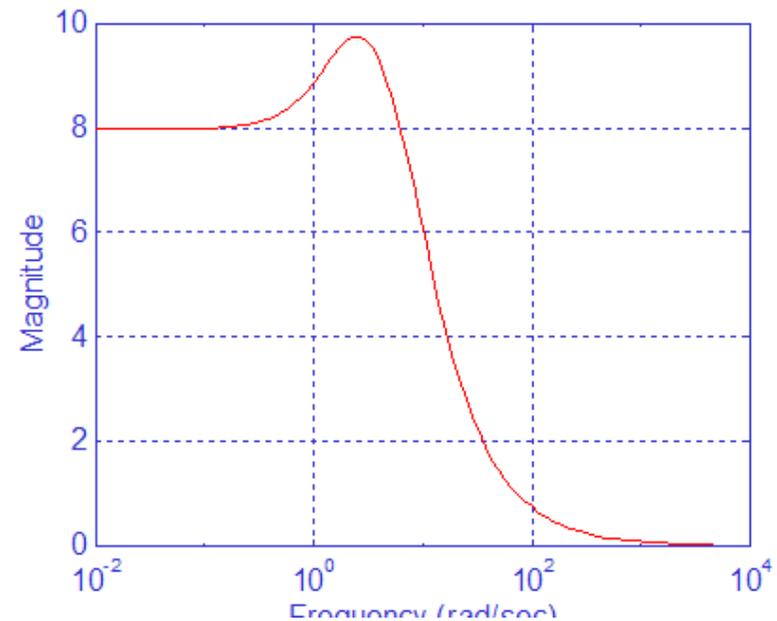
The optimal performance or infimum is given by

$$\gamma_2^* = 19.1833$$

Classical LQR Problem is a Special Case of H_2 Control

It can be shown that the well-known LQR problem can be re-formulated as an H_2 optimal control problem. Consider a linear system,

$$\dot{x} = A\,x + B\,u, \quad x(0) = X_0$$

The LQR problem is to find a control law $u = F\,x$ such that the following index is minimized:

$$J = \int_0^{\infty} \left(x^{\mathrm{T}} Q\,x + u^{\mathrm{T}} R u \right) dt$$

where $Q \geq 0$ is a positive semi-definite matrix and $R > 0$ is a positive definite matrix. The problem is equivalent to finding a static state feedback H_2 optimal control law $u = F\,x$ for

$$\left\{ \begin{aligned} \dot{x} &= A\,x + B\,u + X_0 w \\ y &= x \\ z &= \begin{bmatrix} 0 \\ Q^{1/2} \end{bmatrix} x + \begin{bmatrix} R^{1/2} \\ 0 \end{bmatrix} u \end{aligned} \right.$$

Solution to the Regular H_∞ State Feedback Problem

Given $\gamma > \gamma_\infty^*$, solve the following algebraic Riccati equation (H_∞-ARE)

$$A^\top P + PA + C_2^\top C_2 + PEE^\top P/\gamma^2 - (PB + C_2^\top D_2)(D_2^\top D_2)^{-1}(D_2^\top C_2 + B^\top P) = 0$$

for a unique positive semi-definite stabilizing solution $P \geq 0$. The H_∞ γ-suboptimal state feedback law is then given by

$$u = F\,x = -(D_2^\top D_2)^{-1}\left(D_2^\top C_2 + B^\top P\right)x$$

The resulting closed-loop system $T_{zw}(s)$ has the following property: $\left\| T_{zw} \right\|_\infty < \gamma$.

Remark: The computation of the best achievable H_∞ attenuation level, γ_∞^*, is in general quite complicated. For certain cases, γ_∞^* can be computed exactly. There are cases in which γ_∞^* can only be obtained using some iterative algorithms. One method is to keep solving the H_∞-ARE for different values of γ until it hits γ_∞^* for which and any $\gamma < \gamma_\infty^*$, the H_∞-ARE does not have a solution. Please see the reference by Chen (2000) for details.

Example: Again, consider the following system

$$
\Sigma : \begin{cases} \dot{x} = \overset{A}{\begin{bmatrix} 5 & 2 \\ 3 & 4 \end{bmatrix}} x + \overset{B}{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} u + \overset{E}{\begin{bmatrix} 1 \\ 2 \end{bmatrix}} w \\[4mm] y = x \\[2mm] z = \underset{C_2}{\begin{bmatrix} 1 & 1 \end{bmatrix}} x + \underset{D_2}{1 \cdot u} \end{cases}
$$

It can be showed that the best achievable H_∞ performance for this system is $\boxed{\gamma_\infty^* = 5}$.

Solving the following H_∞-ARE using MATLAB with $\gamma = 5.001$, we obtain a positive definite solution

$$
P = \begin{bmatrix} 330111.5 & 110028.8 \\ 110028.8 & 26679.1 \end{bmatrix}
$$

$$
F = \begin{bmatrix} -110029.8 & -36680.1 \end{bmatrix}
$$

The closed-loop magnitude response from the disturbance to the controlled output:
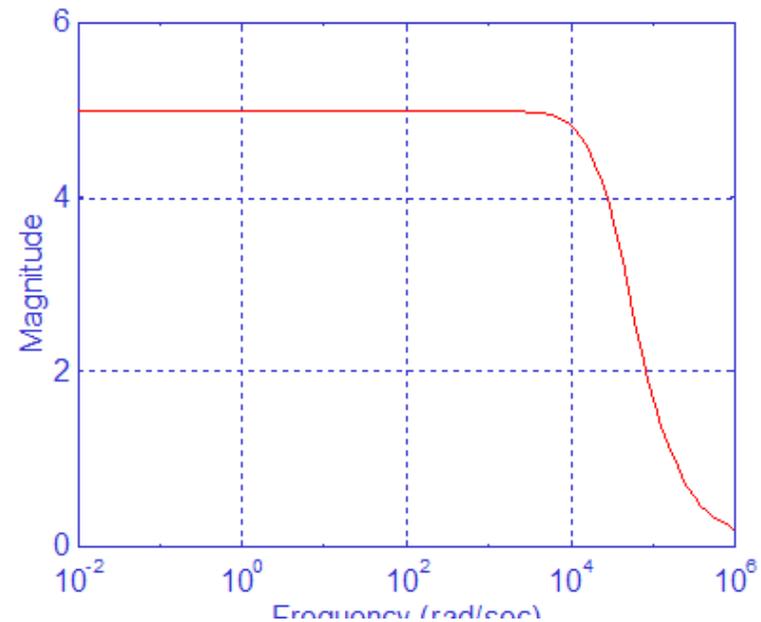
Clearly, the worse case gain, occurred at the low frequency is roughly equal to 5 (actually between 5 and 5.001)

Why are zeros more interesting?...

Consider the system

$$\Sigma: \begin{cases} \dot{x} = A\,x + B\,u + E\,w \\ y = x \\ z = C_2\,x + D_2\,u \end{cases}$$

with D_2 being square and full rank, i.e., it is nonsingular. We can then apply a pre-feedback $u = -D_2^{-1}C_2 x + D_2^{-1} v$ to the given system, which yields

$$\begin{cases} \dot{x} = (A - BD_2^{-1}C_2)\,x + BD_2^{-1}\,v + E\,w \\ y = x \\ z = 0 \quad x + I \quad v \end{cases} \implies \begin{cases} \dot{x} = \bar{A}\,x + \bar{B}\,v + E\,w \\ y = x \\ z = 0\,x + I\,v \end{cases}$$

and the Rosenbrock system matrix of the subsystem from v to z is given by

$$P_\Sigma(s) = \begin{bmatrix} sI - \bar{A} & -\bar{B} \\ 0 & I \end{bmatrix} \implies$$

> All the eigenvalues of $\bar{A}$ are the invariant zeros of the system!

If (A, B, C_2, D_2) is of minimum phase, i.e., all its invariant zeros are stable, or equivalently $\overline{A}$ is a stable matrix, the its corresponding H_2-ARE, i.e.,

$$A^\mathsf{T}P + PA + C_2^\mathsf{T}C_2 - \left(PB + C_2^\mathsf{T}D_2\right)\left(D_2^\mathsf{T}D_2\right)^{-1}\left(D_2^\mathsf{T}C_2 + B^\mathsf{T}P\right) = 0$$

can be simplified as

$$\overline{A}^\mathsf{T}P + P\overline{A} - P\overline{B}\,\overline{B}^\mathsf{T}P = 0$$

Then, it can be seen that P = 0 is the required solution! The optimal solution is given by

$$v = F\,x = -\left(I^\mathsf{T}I\right)^{-1}\left(I^\mathsf{T}\cdot 0 + \overline{B}^\mathsf{T}\cdot 0\right)x = 0$$

and the solution in terms of the original control input is given by

$$u = -D_2^{-1}C_2 x + D_2^{-1}v = -D_2^{-1}C_2 x$$

Similarly, the corresponding H_∞-ARE, i.e.,

$$A^\mathsf{T}P + PA + C_2^\mathsf{T}C_2 + PEE^\mathsf{T}P/\gamma^2 - (PB + C_2^\mathsf{T}D_2)\left(D_2^\mathsf{T}D_2\right)^{-1}(D_2^\mathsf{T}C_2 + B^\mathsf{T}P) = 0$$

can be simplified as

$$\overline{A}^\mathsf{T}P + P\overline{A} + PEE^\mathsf{T}P/\gamma^2 - P\overline{B}\overline{B}^\mathsf{T}P = 0$$

Again, $P = 0$ is the required solution. The optimal solution (for this special situation, the H_∞ control has an optimal solution) is given by

$$v = \overline{F}\, x = -(I^\mathsf{T} I)^{-1}\left(I^\mathsf{T} \cdot 0 + \overline{B}^\mathsf{T} \cdot 0 \right) x = 0$$

and the solution in terms of the original control input is given by

$$u = -D_2^{-1}C_2 x + D_2^{-1}v = -D_2^{-1}C_2 x$$

In both cases, the closed-loop transfer function matrix from w to z is

$$T_{zw}(s) = 0 \cdot (sI - \overline{A})^{-1} E = 0 !$$

and all the stable invariant zeros of (A, B, C_2, D_2) are the closed-loop system poles!

If (A, B, C_2, D_2) has all its invariant zeros to be unstable, or equivalently $\overline{A}$ is an anti-stable matrix, the its corresponding H_2-ARE, i.e.,

$$\overline{A}^{\mathrm{T}} P + P\overline{A} - P\overline{B}\,\overline{B}^{\mathrm{T}} P = 0$$

has a solution $P = 0$ too. But, it does not give a stabilizing control law (why?). However, it can be converted into a Lyapunov equation

$$P^{-1}\left(-\overline{A}\right)^{\mathrm{T}} + \left(-\overline{A}\right)P^{-1} = -\overline{B}\,\overline{B}^{\mathrm{T}} \qquad \left[\quad \Rightarrow \quad P^{-1}\left(-\overline{A}\right)^{\mathrm{T}} P + \left(-\overline{A}\right) = -\overline{B}\,\overline{B}^{\mathrm{T}} P \quad \right]$$

From the Lyapunov stability theorem, it has a unique positive definite solution. The optimal solution is given by

$$v = \overline{F} x = -(I^{\mathrm{T}} I)^{-1}\left(I^{\mathrm{T}} \cdot 0 + \overline{B}^{\mathrm{T}} \cdot P \right) x = -\overline{B}^{\mathrm{T}} P x \quad \Rightarrow \quad u = -\left(D_2^{-1} C_2 + \left(D_2^{\mathrm{T}} D_2 \right)^{-1} BP \right) x$$

and the resulting closed-loop system matrix

$$\overline{A} + \overline{B}\overline{F} = \overline{A} - \overline{B}\overline{B}^{\mathrm{T}} P = \overline{A} + P^{-1}\left(-\overline{A}\right)^{\mathrm{T}} P - \overline{A} = P^{-1}\left(-\overline{A}\right)^{\mathrm{T}} P$$

The mirror images of the unstable invariant zeros of (A, B, C_2, D_2), i.e., $\lambda\left(-\overline{A}\right)$ are the closed-loop system poles!

Similarly, the corresponding H_∞-ARE, i.e.,

$$\bar{A}^T P + P\bar{A} + PEE^T P / \gamma^2 - P\bar{B}\,\bar{B}^T P = 0$$

can be re-written as

$$P^{-1}\bar{A}^T + \bar{A}P^{-1} = \bar{B}\,\bar{B}^T - EE^T / \gamma^2$$

which can be solved by solving two Lyapunov equations:

Ian Petersen

$$S\bar{A}^T + \bar{A}S = \bar{B}\,\bar{B}^T \qquad \text{and} \qquad T\bar{A}^T + \bar{A}T = EE^T$$

It can be showed that

$$\gamma_\infty^* = \sqrt{\lambda_{\max}(TS^{-1})} \qquad \text{and} \qquad P = \left(S - \frac{1}{\gamma^2}T\right)^{-1} > 0, \quad \forall\, \gamma > \gamma_\infty^*$$

The γ-suboptimal solution is given as

$$u = -\left(D_2^{-1}C_2 + \left(D_2^T D_2\right)^{-1} B^T P\right)x$$

Exact Computation of the Infimum in
H_∞-Optimization Via Output Feedback

More general results for the singular case can be found in Chen et al (1992).
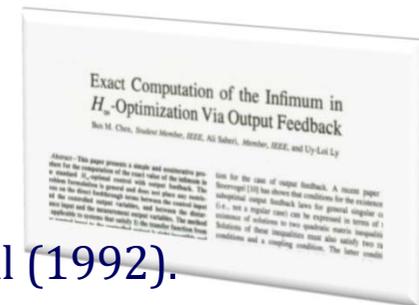
Q: What happens if the given system has both stable and unstable invariant zeros?

A: For the case when $\bar{A}$ has both stable and unstable eigenvalues, there exists a similarity transformation T such that

$$T^{-1}\bar{A}\,T = \begin{bmatrix} A_- & 0 \\ 0 & A_+ \end{bmatrix}, \quad A_- \text{ stable}, A_+ \text{ anti-stable}.$$
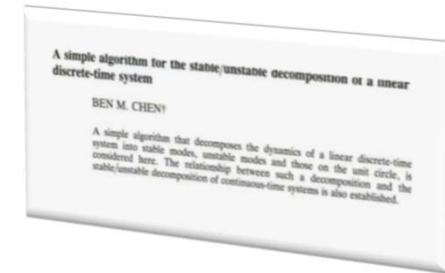
One can then deal with each part separately. The solution to the ARE corresponding to the stable part is 0 and the solution to the ARE corresponding to the unstable part can be calculated by solving Lyapunov equations as on the previous page.

Q: It can be seen that when the given system is of nonminimum phase, the overall performance of the closed-loop system is limited. Can we relocate the zeros as the way that we have changed the locations of poles?

A: Yes. It involves relocations of the sensors and/or actuators of the given system. It is called sensor/actuator placement.
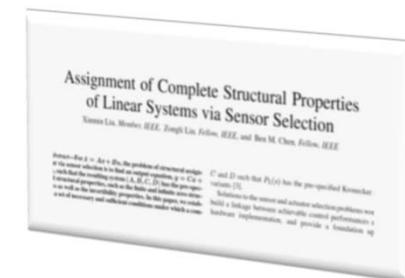
Solutions to the State Feedback Problems – the Singular Case

Consider the following system again,

$$\Sigma: \begin{cases} \dot{x} = A\,x + B\,u + E\,w \\ y = x \\ z = C_2\,x + D_2\,u \end{cases}$$

Kemin Zhou

where (A, B) is stabilizable, D_2 is not necessarily of maximal rank and (A, B, C_2, D_2) might have invariant zeros on the imaginary axis.

Solution to this kind of problems can be done using the following trick (or so-called a **perturbation approach**): Define a new controlled output

P Khargonekar

$$\tilde{z} = \begin{bmatrix} z \\ \varepsilon\,x \\ \varepsilon\,u \end{bmatrix} = \begin{bmatrix} C_2 \\ \varepsilon I \\ 0 \end{bmatrix} x + \begin{bmatrix} D_2 \\ 0 \\ \varepsilon I \end{bmatrix} u$$

small perturbations

Clearly, $z \propto \tilde{z}$ if $\varepsilon = 0$.

Now let us consider the perturbed system

$$
\widetilde{\Sigma} \quad : \quad \begin{cases} \dot{x} = A\,x + B\,u + E\,w \\ y = \quad x \\ \widetilde{z} = \widetilde{C}_2 x + \widetilde{D}_2 u \end{cases}
\qquad \text{where} \quad \widetilde{C}_2 := \begin{bmatrix} C_2 \\ \varepsilon I \\ 0 \end{bmatrix} \quad \text{and} \quad \widetilde{D}_2 := \begin{bmatrix} D_2 \\ 0 \\ \varepsilon I \end{bmatrix}
$$

Obviously, $\widetilde{D}_2$ is of maximal column rank and $(A, B, \widetilde{C}_2, \widetilde{D}_2)$ is free of invariant zeros for any $\varepsilon > 0$. Thus, $\widetilde{\Sigma}$ satisfies the conditions of the regular state feedback case, and hence we can apply the procedures for regular cases to the perturbed system to find the H_2 and H_∞ control laws.

Example:

$$
\Sigma : \begin{cases} \dot{x} = \begin{bmatrix} 5 & 2 \\ 3 & 4 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 1 \\ 2 \end{bmatrix} w \\[4pt] y = x \\[4pt] z = \begin{bmatrix} 1 & 1 \end{bmatrix} x + 0 \cdot u \end{cases}
\quad \Longrightarrow \quad
\widetilde{\Sigma} : \begin{cases} \dot{x} = \begin{bmatrix} 5 & 2 \\ 3 & 4 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 1 \\ 2 \end{bmatrix} w \\[4pt] y = \qquad x \\[4pt] z = \begin{bmatrix} 1 & 1 \\ \varepsilon & 0 \\ 0 & \varepsilon \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \varepsilon \end{bmatrix} u \end{cases}
$$

Solution to the General H_2 State Feedback Problem

Given a small $\varepsilon > 0$, Solve the following algebraic Riccati equation (H_2-ARE)

$$A^{\mathrm{T}}\tilde{P} + \tilde{P}A + \tilde{C}_2^{\mathrm{T}}\tilde{C}_2 - \left(\tilde{P}B + \tilde{C}_2^{\mathrm{T}}\tilde{D}_2\right)\left(\tilde{D}_2^{\mathrm{T}}\tilde{D}_2\right)^{-1}\left(\tilde{D}_2^{\mathrm{T}}\tilde{C}_2 + B^{\mathrm{T}}\tilde{P}\right) = 0$$

for a unique positive definite solution $\tilde{P} > 0$. Obviously, $\tilde{P}$ is a function of ε. The H_2 suboptimal state feedback law is then given by

$$u = \tilde{F}\,x = -\left(\tilde{D}_2^{\mathrm{T}}\tilde{D}_2\right)^{-1}\left(\tilde{D}_2^{\mathrm{T}}\tilde{C}_2 + B^{\mathrm{T}}\tilde{P}\right)x$$

It can be showed that the resulting closed-loop system $T_{zw}(s)$ has

$$\left\|\,T_{zw}\,\right\|_2 \to \gamma_2^* \quad \text{as } \varepsilon \to 0$$

It can also be showed that

$$\left[\,\mathrm{trace}(E^{\mathrm{T}}\tilde{P}E)\,\right]^{1/2} \to \gamma_2^* \quad \text{as } \varepsilon \to 0.$$

Example: Consider a system characterized by

$$\Sigma: \begin{cases} \dot{x} = \begin{bmatrix} 5 & 2 \\ 3 & 4 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 1 \\ 2 \end{bmatrix} w \\ y = x \\ z = \begin{bmatrix} 1 & 1 \end{bmatrix} x + 0 \cdot u \end{cases}$$

Solving the following H_2-ARE using MATLAB with $\varepsilon = 1$, we obtain

$$\tilde{P} = \begin{bmatrix} 186.1968 & 46.2778 \\ 46.2778 & 18.2517 \end{bmatrix}, \quad F = \begin{bmatrix} -46.2778 & -18.2517 \end{bmatrix}$$

- $\varepsilon = 0.1$

$$\tilde{P} = \begin{bmatrix} 21.2472 & 4.9311 \\ 4.9311 & 1.8975 \end{bmatrix}, \quad F = \begin{bmatrix} -49.3111 & -18.9748 \end{bmatrix}$$

- $\varepsilon = 0.0001$

$$\tilde{P} = \begin{bmatrix} 1.6701 & 0.0424 \\ 0.0424 & 0.0112 \end{bmatrix}, \quad F = \begin{bmatrix} -423.742 & -112.222 \end{bmatrix}$$

The closed-loop magnitude response from the disturbance to the controlled output:
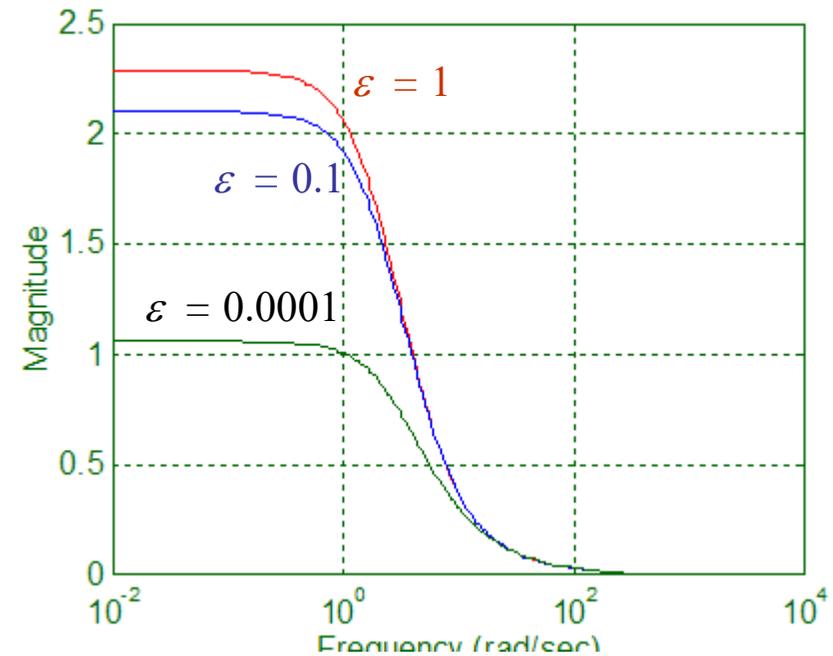
The optimal performance or infimum is given by

$$\boxed{\gamma_2^* = 1.225}$$

Solution to General H_∞ State Feedback Problem

Step 1: Given a $\gamma > \gamma_\infty{}^*$, choose $\varepsilon = 1$.

$$\tilde{C}_2 := \begin{bmatrix} C_2 \\ \varepsilon I \\ 0 \end{bmatrix} \quad \text{and} \quad \tilde{D}_2 := \begin{bmatrix} D_2 \\ 0 \\ \varepsilon I \end{bmatrix}$$

Step 2: Define the corresponding $\tilde{C}_2$ and $\tilde{D}_2$

Step 3: Solve the following algebraic Riccati equation (H_∞-ARE) for $\tilde{P}$:

$$A^{\mathrm{T}}\tilde{P} + \tilde{P}A + \tilde{C}_2^{\mathrm{T}}\tilde{C}_2 + \tilde{P}EE^{\mathrm{T}}\tilde{P}/\gamma^2 - (\tilde{P}B + \tilde{C}_2^{\mathrm{T}}\tilde{D}_2)(\tilde{D}_2^{\mathrm{T}}\tilde{D}_2)^{-1}(\tilde{D}_2^{\mathrm{T}}\tilde{C}_2 + B^{\mathrm{T}}\tilde{P}) = 0$$

Step 4: If $\tilde{P} > 0$, go to Step 5. Otherwise, reduce the value of ε and go to Step 2.

Step 5: Compute the required state feedback control law

$$u = \tilde{F} \, x = -(\tilde{D}_2^{\mathrm{T}} \tilde{D}_2)^{-1}(\tilde{D}_2^{\mathrm{T}} \tilde{C}_2 + B^{\mathrm{T}} \tilde{P}) x$$

It can be showed that the resulting closed-loop system $T_{zw}(s)$ has: $\left\| T_{zw} \right\|_\infty < \gamma$.

More general results for the singular case can be found in Chen (2000).

Example: Again, consider the following system

$$\Sigma: \begin{cases} \dot{x} = \begin{bmatrix} 5 & 2 \\ 3 & 4 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 1 \\ 2 \end{bmatrix} w \\ y = x \\ z = \begin{bmatrix} 1 & 1 \end{bmatrix} x + 0 \cdot u \end{cases}$$

It can be showed that the best achievable H_∞ performance for this system is $\boxed{\gamma_\infty^* = 0.5}$.

Solving the following H_∞-ARE using MATLAB with $\gamma = 0.6$ and $\varepsilon = 0.001$, we obtain a positive definite solution

$$P = \begin{bmatrix} 15.1677 & 0.9874 \\ 0.9874 & 0.0981 \end{bmatrix}$$

and

$$F = \begin{bmatrix} -987.363 & -98.1161 \end{bmatrix}$$

The closed-loop magnitude response from the disturbance to the controlled output:
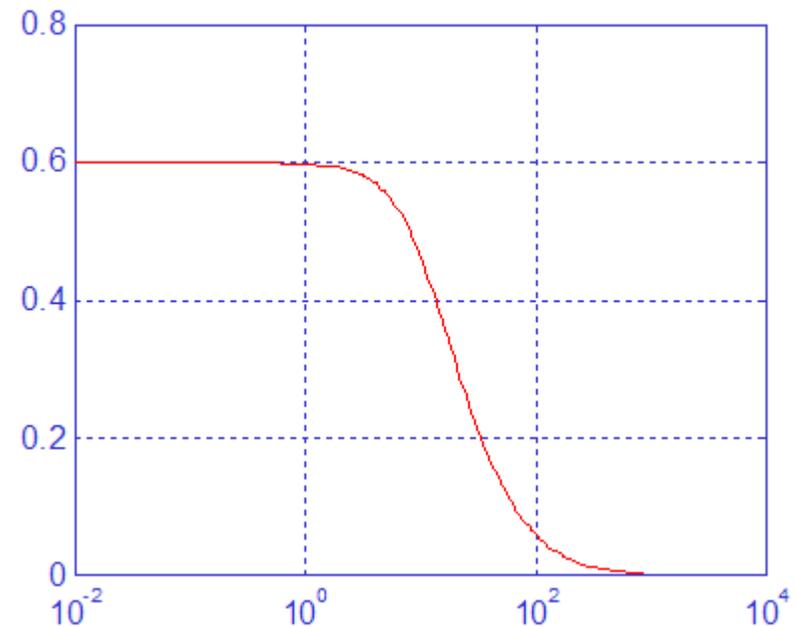
Clearly, the worse case gain, occurred at the low frequency is slightly less than 0.6. The design specification is achieved.

Solutions to Output Feedback Problems – Regular Case

Recall the system with measurement feedback, i.e.,

$$\Sigma: \begin{cases} \dot{x} = A\,x + B\,u + E\,w \\ y = C_1\,x \qquad\quad + D_1\,w \\ z = C_2\,x + D_2\,u \end{cases}$$

where (A, B) is stabilizable and (A, C_1) is detectable. Also, it satisfies the following regularity assumptions:

1. D_2 is of maximal column rank, i.e., D_2 is a tall and full rank matrix

2. The subsystem (A, B, C_2, D_2) has no invariant zeros on the imaginary axis

3. D_1 is of maximal row rank, i.e., D_1 is a fat and full rank matrix

4. The subsystem (A, E, C_1, D_1) has no invariant zeros on the imaginary axis

Solution to Regular H_2 Output Feedback Problem

Solve the following algebraic Riccati equation (H_2-ARE)

$$A^\mathrm{T} P + PA + C_2^\mathrm{T} C_2 - \left(PB + C_2^\mathrm{T} D_2 \right) \left(D_2^\mathrm{T} D_2 \right)^{-1} \left(D_2^\mathrm{T} C_2 + B^\mathrm{T} P \right) = 0$$

for a unique positive semi-definite stabilizing solution $P \geq 0$, and the following ARE

$$QA^\mathrm{T} + AQ + EE^\mathrm{T} - \left(QC_1^\mathrm{T} + ED_1^\mathrm{T} \right) \left(D_1 D_1^\mathrm{T} \right)^{-1} \left(D_1 E^\mathrm{T} + C_1 Q \right) = 0$$

for a unique positive semi-definite stabilizing solution $Q \geq 0$. The H_2 optimal output feedback law is then given by

$$\Sigma_c : \begin{cases} \dot{v} = \left(A + BF + KC_1 \right) v - K \, y \\ u = \qquad F \quad v \end{cases}$$

where $F = -\left(D_2^\mathrm{T} D_2 \right)^{-1} \left(D_2^\mathrm{T} C_2 + B^\mathrm{T} P \right)$ and $K = -\left(QC_1^\mathrm{T} + ED_1^\mathrm{T} \right) \left(D_1 D_1^\mathrm{T} \right)^{-1}$

Furthermore,

$$\gamma_2^* = \left\{ \mathrm{trace}(E^\mathrm{T} PE) + \mathrm{trace}\left[\left(A^\mathrm{T} P + PA + C_2^\mathrm{T} C_2 \right) Q \right] \right\}^{1/2}$$

Example: Consider a system characterized by

$$\Sigma: \begin{cases} \dot{x} = \begin{bmatrix} 5 & 2 \\ 3 & 4 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 1 \\ 2 \end{bmatrix} w \\ y = \begin{bmatrix} 0 & 1 \end{bmatrix} x \qquad + 1 \cdot w \\ z = \begin{bmatrix} 1 & 1 \end{bmatrix} x + 1 \cdot u \end{cases}$$

Solving the following H_2-AREs using MATLAB, we obtain

$$P = \begin{bmatrix} 144 & 40 \\ 40 & 16 \end{bmatrix} \quad F = \begin{bmatrix} -41 & -17 \end{bmatrix}$$

$$Q = \begin{bmatrix} 49.7778 & 23.3333 \\ 23.3333 & 14.0000 \end{bmatrix} \quad K = \begin{bmatrix} -24.3333 \\ -16.0000 \end{bmatrix}$$

and an output feedback control law,

$$\Sigma_c: \begin{cases} \dot{v} = \begin{bmatrix} 5 & -22.3333 \\ -38 & -29 \end{bmatrix} v + \begin{bmatrix} 24.3333 \\ 16 \end{bmatrix} y \\ u = \begin{bmatrix} -41 & -17 \end{bmatrix} v \end{cases}$$

The closed-loop magnitude response from the disturbance to the controlled output:
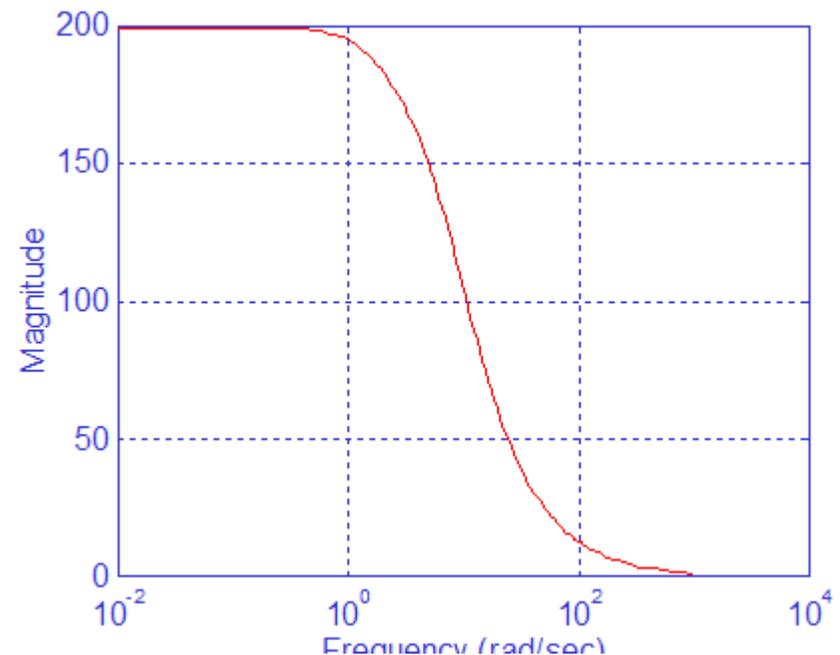
The optimal performance or infimum is given by

$$\gamma_2^* = 347.3$$

Solution to Regular H_∞ Output Feedback Problem

Given a $\gamma > \gamma_\infty{}^*$, solve the following algebraic Riccati equation (H_∞-ARE)

$$A^T P + PA + C_2^T C_2 + PEE^T P / \gamma^2 - \left(PB + C_2^T D_2 \right) \left(D_2^T D_2 \right)^{-1} \left(D_2^T C_2 + B^T P \right) = 0$$

John Doyle

for a positive semi-definite stabilizing solution $P \geq 0$, and the following ARE

$$QA^T + AQ + EE^T + QC_2^T C_2 Q / \gamma^2 - \left(QC_1^T + ED_1^T \right) \left(D_1 D_1^T \right)^{-1} \left(D_1 E^T + C_1 Q \right) = 0$$

for a positive semi-definite stabilizing solution $Q \geq 0$. In fact, these

Keith Glover

P and Q satisfy the so-called coupling condition: $\rho(PQ) < \gamma^2$. The

H_∞ γ-suboptimal output feedback law is then given by [DGKF] (1989)

$$\Sigma_{\text{cmp}} : \begin{cases} \dot{v} = A_{\text{cmp}}\, v + B_{\text{cmp}}\, y \\ u = C_{\text{cmp}}\, v \end{cases}$$

P Khargonekar

Gilead Tadmor

where $B_{\text{cmp}} = -\left(I - \gamma^{-2} QP \right)^{-1} K$, $\qquad C_{\text{cmp}} = F$

$$A_{\text{cmp}} = A + \gamma^{-2} EE^T P + BF + \left(I - \gamma^{-2} QP \right)^{-1} K \left(C_1 + \gamma^{-2} D_1 E^T P \right)$$

and where $F = -\left(D_2^T D_2 \right)^{-1} \left(D_2^T C_2 + B^T P \right)$, $K = -\left(QC_1^T + ED_1^T \right) \left(D_1 D_1^T \right)^{-1}$.

Bruce Francis

Example: Consider a system characterized by

$$\Sigma: \begin{cases} \dot{x} = \begin{bmatrix} 5 & 2 \\ 3 & 4 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 1 \\ 2 \end{bmatrix} w \\[2mm] y = \begin{bmatrix} 0 & 1 \end{bmatrix} x \qquad + 1 \cdot w \\[2mm] z = \begin{bmatrix} 1 & 1 \end{bmatrix} x + 1 \cdot u \end{cases}$$

It can be showed that the best achievable H_∞ performance for this system is $\boxed{\gamma_\infty^* = 96.32864}$.

Solving the following H_∞-AREs using MATLAB with $\gamma = 97$, we obtain

$$P = \begin{bmatrix} 144.353 & 40.1168 \\ 40.1168 & 16.0392 \end{bmatrix} \quad Q = \begin{bmatrix} 49.8205 & 23.3556 \\ 23.3556 & 14.0118 \end{bmatrix}$$

$$\Sigma_{cmp}: \begin{cases} \dot{v} = \begin{bmatrix} -38.814 & -1848.66 \\ -59.414 & -914.112 \end{bmatrix} v + \begin{bmatrix} 1836.58 \\ 894.227 \end{bmatrix} y \\[2mm] u = \begin{bmatrix} -41.116 & -17.039 \end{bmatrix} v \end{cases}$$

The closed-loop magnitude response from the disturbance to the controlled output:
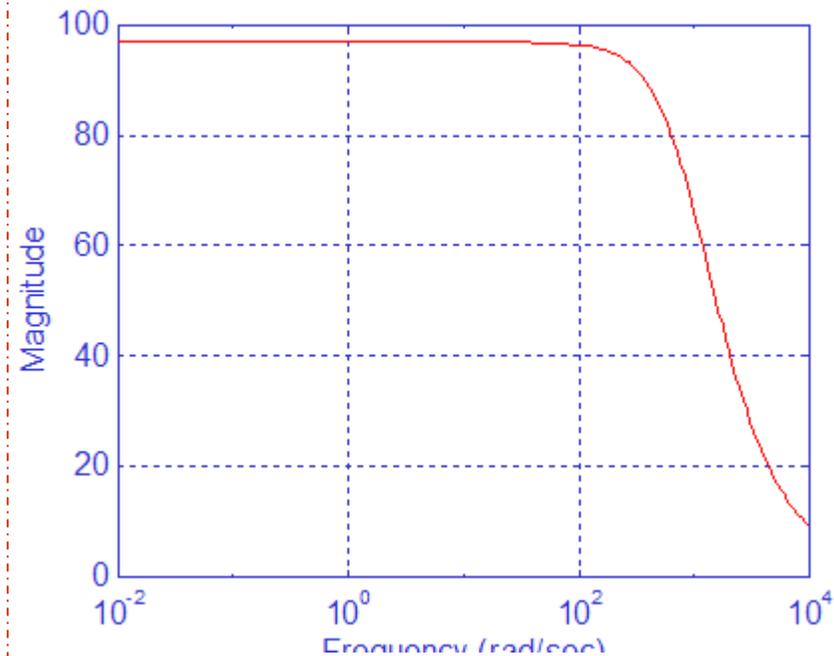
Clearly, the worse case gain, occurred at the low frequency is slightly less than 97. The design specification is achieved.

Solutions to Output Feedback Problems – Singular Case

For general systems for which the regularity conditions are not satisfied, it can be solved again using the perturbation approach. We define a new controlled output:

$$\tilde{z} = \begin{bmatrix} z \\ \varepsilon x \\ \varepsilon u \end{bmatrix} = \begin{bmatrix} C_2 \\ \varepsilon I \\ 0 \end{bmatrix} x + \begin{bmatrix} D_2 \\ 0 \\ \varepsilon I \end{bmatrix} u$$

and new matrices associated with the disturbance inputs:

$$\tilde{E} = \begin{bmatrix} E & \varepsilon I & 0 \end{bmatrix} \quad \text{and} \quad \tilde{D}_1 = \begin{bmatrix} D_1 & 0 & \varepsilon I \end{bmatrix}.$$

The H_2 and H_∞ control problems for singular output feedback case can be obtained by solving the following perturbed **regular** system with sufficiently small ε:

$$\tilde{\Sigma}: \begin{cases} \dot{x} = A\,x + B\,u + \tilde{E}\,\tilde{w} \\ y = C_1\,x \qquad\quad + \tilde{D}_1\,\tilde{w} \\ \tilde{z} = \tilde{C}_2\,x + \tilde{D}_2\,u \end{cases}$$

Remark: *Perturbation approach might have serious numerical problems!*

Side Notes on H_∞ Singular Case

1. D_2 is of maximal column rank, i.e., D_2 is a tall and full rank matrix

2. (A, B, C_2, D_2) has no invariant zeros on the imaginary axis

3. D_1 is of maximal row rank, i.e., D_1 is a fat and full rank matrix

4. (A, E, C_1, D_1) has no invariant zeros on the imaginary axis

Construction of closed-form solutions and computation of γ_∞^* etc...

Anton Stoorvogel & coworkers   *Carsten Scherer*   *Bugs Bunny & coworkers*

Side Notes on (Almost) Disturbance Decoupling

1. If $\gamma_2{}^* = 0$, then the corresponding H_2 optimal control problem is also called an H_2 (almost) disturbance decoupling problem. It can be showed that the H_2 almost disturbance decoupling problem is solvable if the following conditions are satisfied:

   - (A, B) is stabilizable and (A, C_1) is detectable
   - (A, B, C_2, D_2) is right invertible and has no invariant zeros on open RHP
   - (A, E, C_1, D_1) is left invertible and has no invariant zeros on open RHP

   Necessary and sufficient conditions for the solvability of the almost disturbance decoupling problem is available in the literature. However, they can only be expressed in terms of certain geometric subspaces on the given system…

2. If $\gamma_\infty^*= 0$, then the corresponding H_∞ optimal control problem is also called an H_∞ almost disturbance decoupling problem. It can be showed that the H_∞ almost disturbance decoupling problem is solvable if the following conditions are satisfied:

- (A, B) is stabilizable and (A, C_1) is detectable
- (A, B, C_2, D_2) is right invertible and of minimum phase
- (A, E, C_1, D_1) is left invertible and of minimum phase

Studies on disturbance decoupling problems led to the development of the geometric theory in linear systems...

W. M. Wonham   *Jan C. Willems*   *Carsten Scherer*   *Bugs Bunny*
1939–2013

Applications: Some Robust Control Problems

Robust Stabilization of Systems with Unstructured Uncertainties

Consider an uncertain plant with an unstructured perturbation,
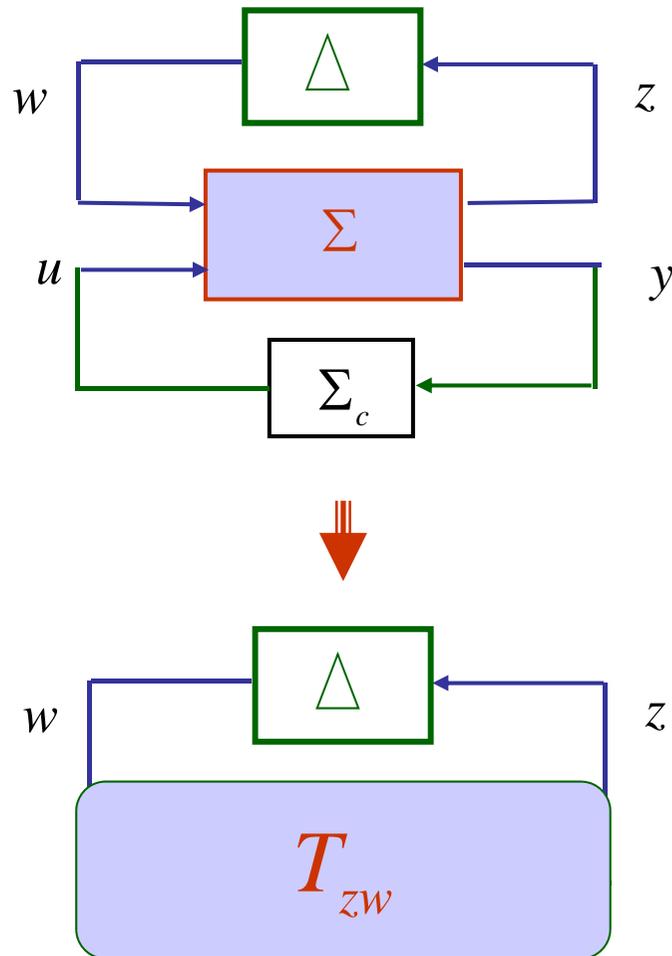
Small Gain Theory (!)

If $\triangle$ is stable and $\|\Delta\|_\infty \cdot \|M\|_\infty < 1,$ then the interconnected system is stable.
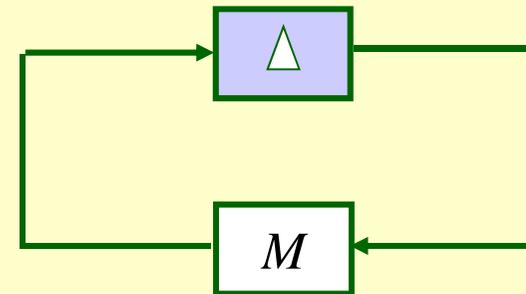
Assume $\|T_{zw}\|_\infty < \gamma$. Then the system with unstructured uncertainty if

$$\|T_{zw}\|_\infty \cdot \|\Delta\|_\infty < \gamma \cdot \|\Delta\|_\infty < 1 \implies \|\Delta\|_\infty < \frac{1}{\gamma}$$

Robust Stabilization with Additive Perturbation

Consider an uncertain plant with additive perturbations,
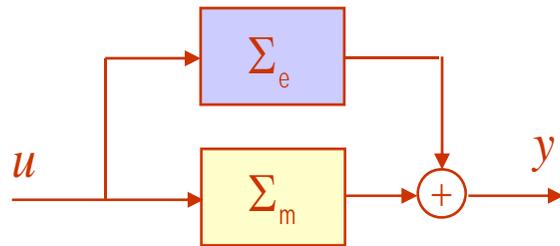
Σ_m has a transfer function $G_m(s) = C_m(sI - A_m)^{-1}B_m + D_m$

Σ_e is an unknown perturbation.

Σ_m and $\Sigma_m + \Sigma_e$ have same number of unstable poles.

Given a $\gamma_a > 0$, the problem of robust stabilization for plants with additive perturbations is to find a proper controller such that when it is applied to the uncertain plant, the resulting closed-loop system is stable for all possible perturbations with their L_∞-norm $\leq \gamma_a$. (The definition of L_∞-norm is the same as that of H_∞-norm except for L_∞-norm, the system need not be stable.) Such a problem is equivalent to find an H_∞ γ-suboptimal control law (with $\gamma = 1/\gamma_a$) for

$$\Sigma_{add} : \begin{cases} \dot{x} = A_m\, x + B_m\, u + 0w \\ y = C_m\, x + D_m\, u + I\, w \\ z = \ 0\ x + \ I\ u \end{cases}$$

Robust Stabilization with Multiplicative Perturbation

Consider an uncertain plant with multiplicative perturbations,
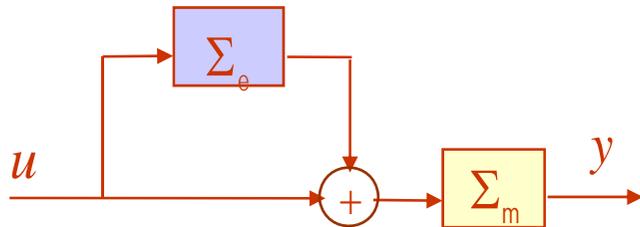
Σ_{m} has a transfer function $G_{\mathrm{m}}(s) = C_{\mathrm{m}}(sI - A_{\mathrm{m}})^{-1} B_{\mathrm{m}} + D_{\mathrm{m}}$

Σ_{e} is an unknown perturbation.

Σ_{m} and $\Sigma_{\mathrm{m}} \times \Sigma_{\mathrm{e}}$ have same number of unstable poles.

Given a $\gamma_{\mathrm{m}} > 0$, the problem of robust stabilization for plants with multiplicative

perturbations is to find a proper controller such that when it is applied to the uncertain

plant, the resulting closed-loop system is stable for all possible perturbations with their

L_∞-norm $\le \gamma_{\mathrm{m}}$. Again, such a problem is equivalent to find an H_∞ γ-suboptimal control law

(with $\gamma = 1/\gamma_{\mathrm{m}}$) for the following system,

$$\Sigma_{\mathrm{multi}} : \begin{cases} \dot{x} = A_{\mathrm{m}}\, x + B_{\mathrm{m}}\, u + B_{\mathrm{m}}\, w \\ y = C_{\mathrm{m}}\, x + D_{\mathrm{m}}\, u + D_{\mathrm{m}}\, w \\ z = \ 0\ x + \ I\ u \end{cases}$$

Homework Assignment 2:

Using both H_2 and H_∞ control techniques to design appropriate measurement feedback control laws that meet all the design specification specified in the (HDD or helicopter) problem.

Show all the detailed calculation and simulate your design using MATLAB and Simulink. Give all the necessary plots that show the evidence of your design.

9. Robust & Perfect Tracking
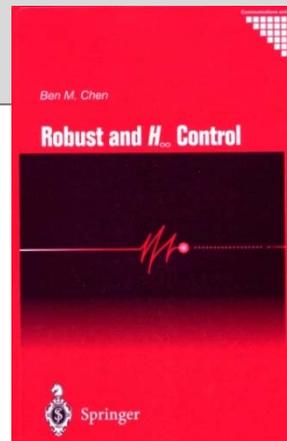
Robust and Perfect Tracking Control

The robust and perfect tracking (RPT) control technique developed by Chen and his co-workers is to design a controller such that the resulting closed-loop system is stable and the controlled output almost perfectly tracks a given reference signal in the presence of any initial conditions and external disturbances.

One of the most interesting features in the RPT control method is its capability of utilizing all possible information available in its controller structure. Such a feature is highly desirable for flight missions involving complicated maneuvers, in which not only the position reference is useful, but also its velocity and even acceleration information are important or even necessary to be used in order to achieve a good overall performance.

The RPT control renders flight formation of multiple UAVs a trivial task.
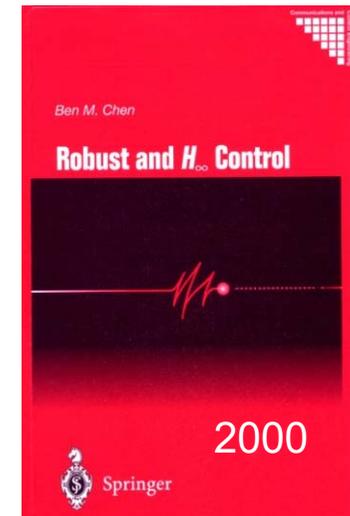
Bugs Bunny

Problem formulation

Consider the following continuous-time system:

$$\Sigma : \begin{cases} \dot{\boldsymbol{x}} = A\,\boldsymbol{x} + B\,\boldsymbol{u} + E\,\boldsymbol{w}, & \boldsymbol{x}(0) = \boldsymbol{x}_0, \\ \boldsymbol{y} = C_1\,\boldsymbol{x} \qquad\qquad + D_1\,\boldsymbol{w} \\ \boldsymbol{z} = C_2\,\boldsymbol{x} + D_2\,\boldsymbol{u} + D_{22}\,\boldsymbol{w} \end{cases} \qquad \begin{array}{l} (9.1.1) \\ (8.1) \end{array}$$

where $\boldsymbol{x} \in \mathbb{R}^n$ is the state, $\boldsymbol{u} \in \mathbb{R}^m$ is the control input, $\boldsymbol{w} \in \mathbb{R}^q$ is the external disturbance, $\boldsymbol{y} \in \mathbb{R}^p$ is the measurement output, and $\boldsymbol{z} \in \mathbb{R}^\ell$ is the output to be controlled. Given the external disturbance $\boldsymbol{w} \in L_p$, $\boldsymbol{p} \in [1, \infty)$, and any reference signal vector $\boldsymbol{r} \in \mathbb{R}^\ell$ with $r, \dot{r}, \cdots, r^{(\kappa-1)}$, $\kappa \geq 1$, being available, and $r^{(\kappa)}$ being either a vector of delta functions or in L_p, the RPT problem for the system in (8.1) is to find a parameterized dynamic measurement control law of the following form:

$$\begin{cases} \dot{\boldsymbol{v}} = A_{\text{cmp}}(\varepsilon)\boldsymbol{v} + B_{\text{cmp}}(\varepsilon)\boldsymbol{y} + G_0(\varepsilon)\boldsymbol{r} + \cdots + G_{\kappa-1}(\varepsilon)\boldsymbol{r}^{(\kappa-1)} \\ \boldsymbol{u} = C_{\text{cmp}}(\varepsilon)\boldsymbol{v} + D_{\text{cmp}}(\varepsilon)\boldsymbol{y} + H_0(\varepsilon)\boldsymbol{r} + \cdots + H_{\kappa-1}(\varepsilon)\boldsymbol{r}^{(\kappa-1)} \end{cases} \qquad \begin{array}{l} (8.2) \\ (9.1.2) \end{array}$$

such that when the controller of (8.2) is applied to the system of (8.1), we have the following

1. There exists an $\varepsilon^* > 0$ such that the resulting closed-loop system with $\boldsymbol{r} = 0$ and $\boldsymbol{w} = 0$ is asymptotically stable for all $\varepsilon \in (0, \varepsilon^*]$.
2. Let $\bar{\boldsymbol{z}}(t, \varepsilon)$ be the closed-loop controlled output response and let $\boldsymbol{e}(t, \varepsilon)$ be the resulting tracking error, i.e., $\boldsymbol{e}(t, \varepsilon) := \bar{\boldsymbol{z}}(t, \varepsilon) - \boldsymbol{r}(t)$. Then, for any initial condition of the state, $\boldsymbol{x}_0 \in \mathbb{R}^n$,

$$\|\boldsymbol{e}\|_{\boldsymbol{p}} = \left(\int_0^\infty |\boldsymbol{e}(t)|^{\boldsymbol{p}}\, \mathrm{d}t \right)^{1/\boldsymbol{p}} \longrightarrow 0 \quad \text{as } \varepsilon \to 0. \qquad (8.3)$$

Robust to disturbance & initial condition
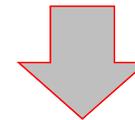
RPT Control

Perfect in Tracking

Solvability conditions:

Corollary 9.2.1. Consider the given system (9.1.1) with its external disturbance $w \in L_p$, $p \in [1, \infty)$, its initial condition $x(0) = x_0$. Assume that all its states are measured for feedback, i.e., $C_1 = I$ and $D_1 = 0$. Then, for any reference signal $r(t)$, which has all its i-th order derivatives, $i = 1, 2, \cdots, \kappa - 1$, $\kappa \geq 1$, being available and $r^{(\kappa)}(t)$ being either a vector of delta functions or in L_p, the proposed robust and perfect tracking (RPT) problem is solvable by the control law of (9.1.2) if and only if the following conditions are satisfied:

1. (A, B) is stabilizable;

2. $D_{22} = 0$;

3. $\Sigma_{\rm P}$, i.e., (A, B, C_2, D_2), is right invertible and of minimum phase. Ⓒ

The solvability condition for the general measurement feedback case is rather complicated. Please refer to the reference text for details (Theorem 9.2.1).

Solution:

Remark 9.2.1. Note that the required gain matrices for the state feedback RPT problem might be computed by solving the following Riccati equation,

$$P\tilde{A} + \tilde{A}'P + \tilde{C}_2'\tilde{C}_2 - \left(PB + \tilde{C}_2'\tilde{D}_2\right)\left(\tilde{D}_2'\tilde{D}_2\right)^{-1}\left(PB + \tilde{C}_2'\tilde{D}_2\right)' = 0, \quad (9.2.43)$$

for a positive definite solution $P > 0$, where

$$\tilde{C}_2 = \begin{bmatrix} C_2 \\ \varepsilon I_{\kappa\ell+n} \\ 0 \end{bmatrix}, \quad \tilde{D}_2 = \begin{bmatrix} D_2 \\ 0 \\ \varepsilon I_m \end{bmatrix}, \quad (9.2.44)$$

$$\tilde{A} = \begin{bmatrix} \tilde{A}_0 & 0 \\ 0 & A \end{bmatrix}, \quad \tilde{A}_0 = -\varepsilon I_{\kappa\ell} + \begin{bmatrix} 0 & I_\ell & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I_\ell \\ 0 & 0 & \cdots & 0 \end{bmatrix}, \quad (9.2.45)$$

and where B, C_2 and D_2 are as defined in (9.2.30) and (9.2.31). The required gain matrix is then given by

$$\tilde{F}(\varepsilon) = -\left(\tilde{D}_2'\tilde{D}_2\right)^{-1}\left(PB + \tilde{C}_2'\tilde{D}_2\right)' = [\,H_0(\varepsilon) \quad \cdots \quad H_{\kappa-1}(\varepsilon) \quad F(\varepsilon)\,], \quad (9.2.46)$$

where $H_i(\varepsilon) \in \mathbb{R}^{m\times\ell}$ and $F(\varepsilon) \in \mathbb{R}^{m\times n}$. Finally, we note that solutions to the Riccati equation (9.2.43) might have severe numerical problems as ε tends smaller and smaller. ®

Special Case...

For the special case when the given plant is of a double integrator, i.e.,

$$\begin{pmatrix} \dot{p} \\ \dot{v} \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} p \\ v \end{pmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + E\,w, \quad y = \begin{pmatrix} p \\ v \end{pmatrix}, \quad z = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{pmatrix} p \\ v \end{pmatrix}$$

where p is the position and v is the acceleration, assuming the reference position (p_r), velocity (v_r) and acceleration (a_r) are all available, it can be shown that the RPT control law can be calculated in the following closed-form

$$u = -\begin{bmatrix} \dfrac{\omega_n^2}{\varepsilon^2} & \dfrac{2\varsigma\omega_n}{\varepsilon} \end{bmatrix} \begin{pmatrix} p \\ v \end{pmatrix} + \left(\dfrac{\omega_n^2}{\varepsilon^2} \right) p_r + \left(\dfrac{2\varsigma\omega_n}{\varepsilon} \right) v_r + a_r$$

where ς is the damping ratio and ω_n is the natural frequency of the closed-loop system, and ε is the tuning parameter.

We note such a plant is very common in real applications including the outer loop flight control systems. In fact, the RPT control is very effective in improving flight performance for UAVs.

Case Study... Unmanned Helicopter Flight Control Systems

An unmanned system

| Mission Management | | | | | |
| Path Planning | Trajectory Generation | Outer Loop Control | Inner Loop Control | | Measured Signal |
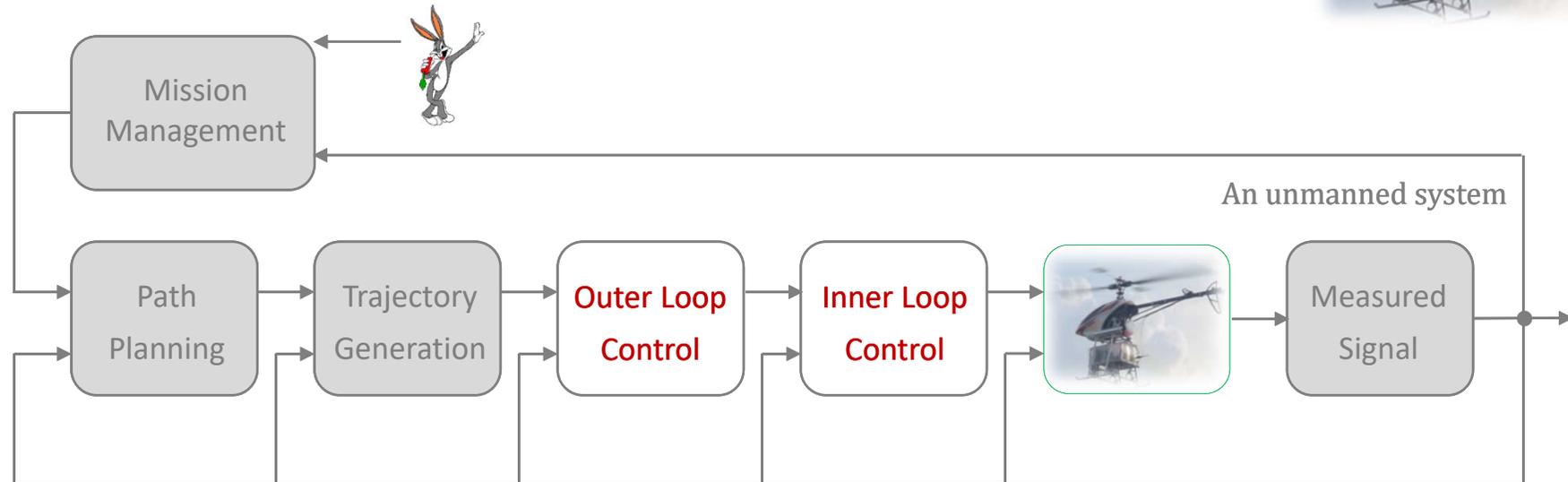
✓ **Inner Loop to stabilize UAV attitude**

- o PID Control (commonly used)
- o Optimal Control
- o Robust Control
- o Nonlinear Control
- o ......

✓ **Outer Loop to control position/velocity**

- o PID Control (commonly used)
- o Pole placement
- o RPT Control
- o Robust Control
- o ......

Detailed control structure
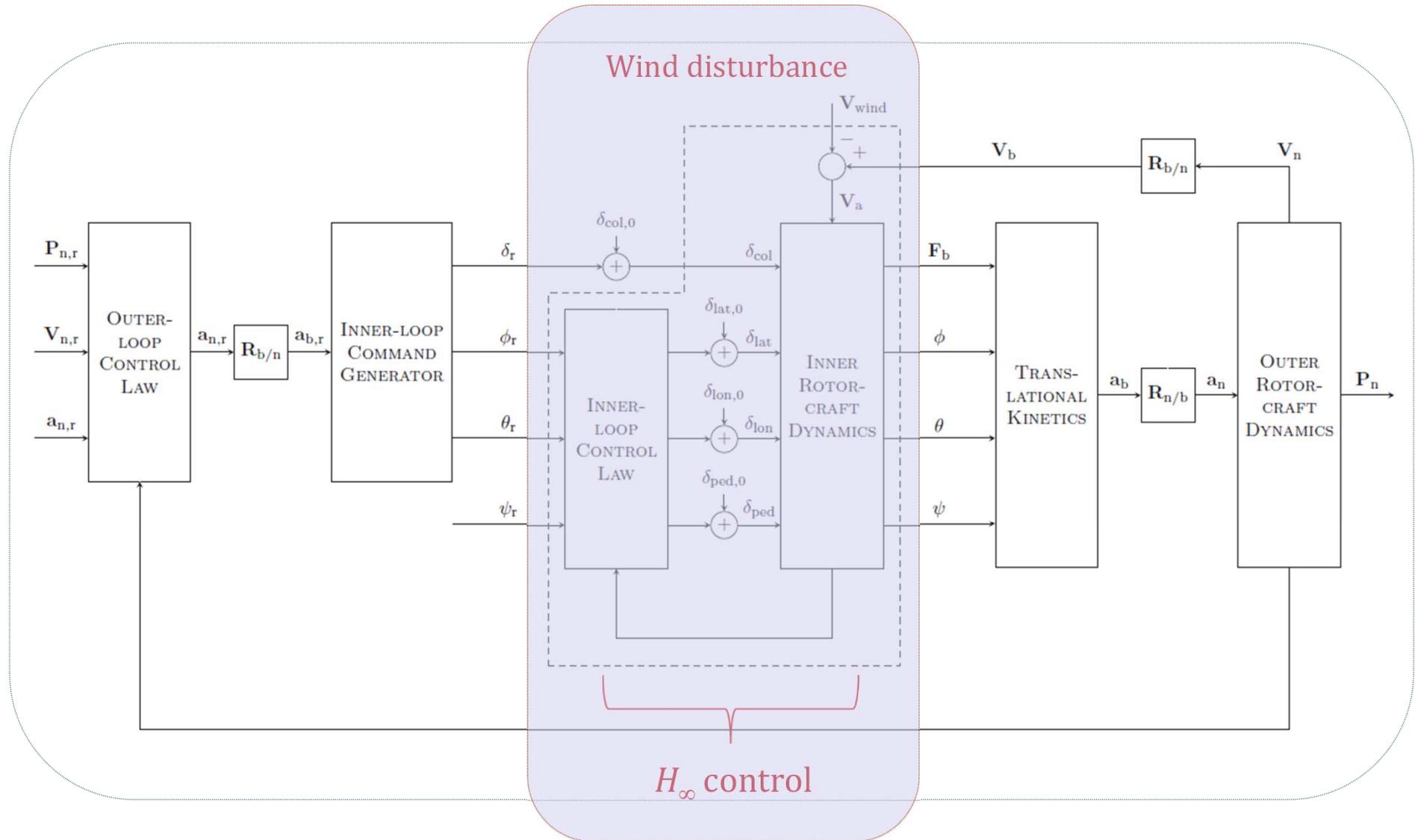
Inner-loop control system design setup

$$\mathbf{x} = \begin{bmatrix} \phi & \theta & p & q & a_s & b_s & r & \delta_{ped,int} & \psi \end{bmatrix}^{T}$$

$$\mathbf{u}_{act} = \begin{bmatrix} \delta_{lat} & \delta_{lon} & \delta_{ped} \end{bmatrix}^{T}$$

$$\mathbf{y} = \begin{bmatrix} \phi & \theta & p & q & r & \psi \end{bmatrix}^{T}$$
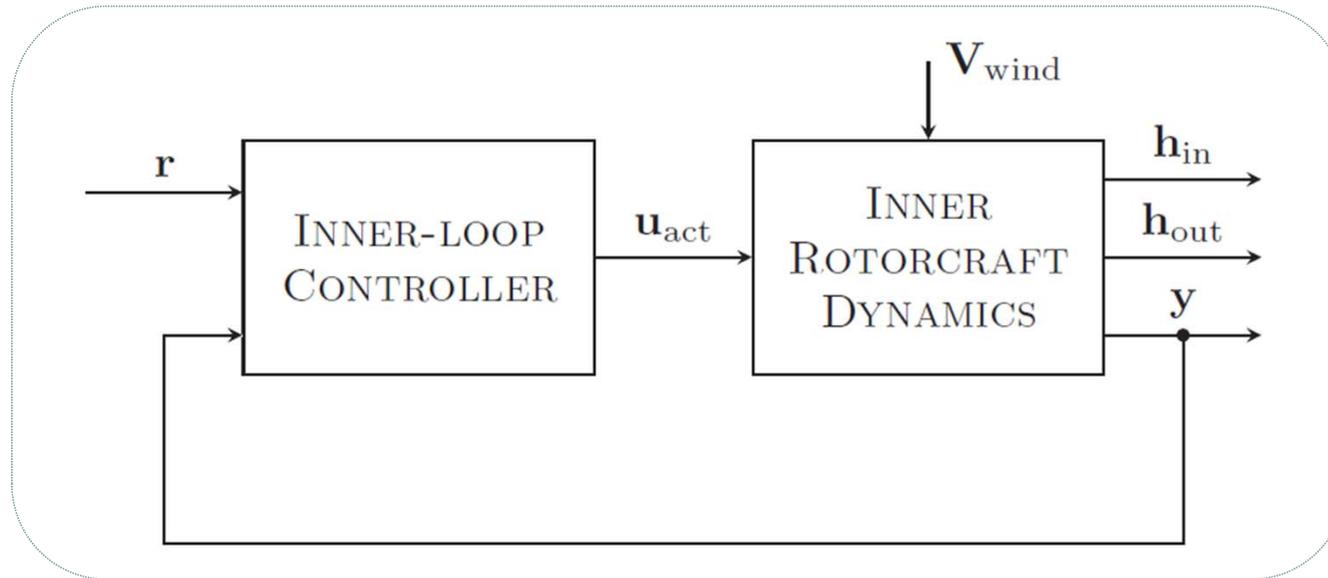
$$\mathbf{h}_{out} := \begin{bmatrix} \phi & \theta & \psi \end{bmatrix}^{T}$$

Inner-loop linearized model at hover

$$\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u} + E\mathbf{w}$$

$$\mathbf{y} = C_1\mathbf{x}$$

$$\mathbf{h}_{\text{out}} = C_{\text{out}}\mathbf{x}$$

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0.0009 & 0 & 0 \\ 0 & 0 & 0 & 0.9992 & 0 & 0 & -0.0389 & 0 & 0 \\ 0 & 0 & -0.0302 & -0.0056 & -0.0003 & 585.1165 & 11.4448 & -59.529 & 0 \\ 0 & 0 & 0 & -0.0707 & 267.7499 & -0.0003 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1.0000 & -3.3607 & 2.2223 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 2.4483 & -3.3607 & 0 & 0 & 0 \\ 0 & 0 & 0.0579 & 0.0108 & 0.0049 & 0.0037 & -21.9557 & 114.2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0.0389 & 0 & 0 & 0.9992 & 0 & 0 \end{bmatrix}$$

$$\mathbf{y} = \begin{pmatrix} \phi \\ \theta \\ p \\ q \\ r \\ \psi \end{pmatrix}$$

$$\mathbf{h}_{\text{out}} = \begin{pmatrix} \phi \\ \theta \\ \psi \end{pmatrix}$$

$$B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 43.3635 \\ 0 & 0 & 0 \\ 0.2026 & 2.5878 & 0 \\ 2.5878 & -0.0663 & 0 \\ 0 & 0 & -83.1883 \\ 0 & 0 & -3.8500 \\ 0 & 0 & 0 \end{bmatrix} \qquad E = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -0.0001 & 0.1756 & -0.0395 \\ 0.0000 & 0.0003 & 0.0338 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ -0.0002 & -0.3396 & 0.6424 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

One can use the techniques covered earlier, i.e., H_2 control, H_∞ control, or LQG to design an appropriate inner-loop controller for the above system.

Inner-loop command generator

INNER-LOOP
COMMAND
GENERATOR

under cover...

$\mathbf{a}_{b,r}$

δ_r

ϕ_r

θ_r

The inner-loop command generator is given as

$$\begin{pmatrix} \delta_r \\ \phi_r \\ \theta_r \end{pmatrix} = \begin{bmatrix} 0 & 0.0019 & 0.0477 \\ 0 & 0.1022 & -0.0037 \\ -0.1022 & 0 & 0.0001 \end{bmatrix} \mathbf{a}_{b,r}$$
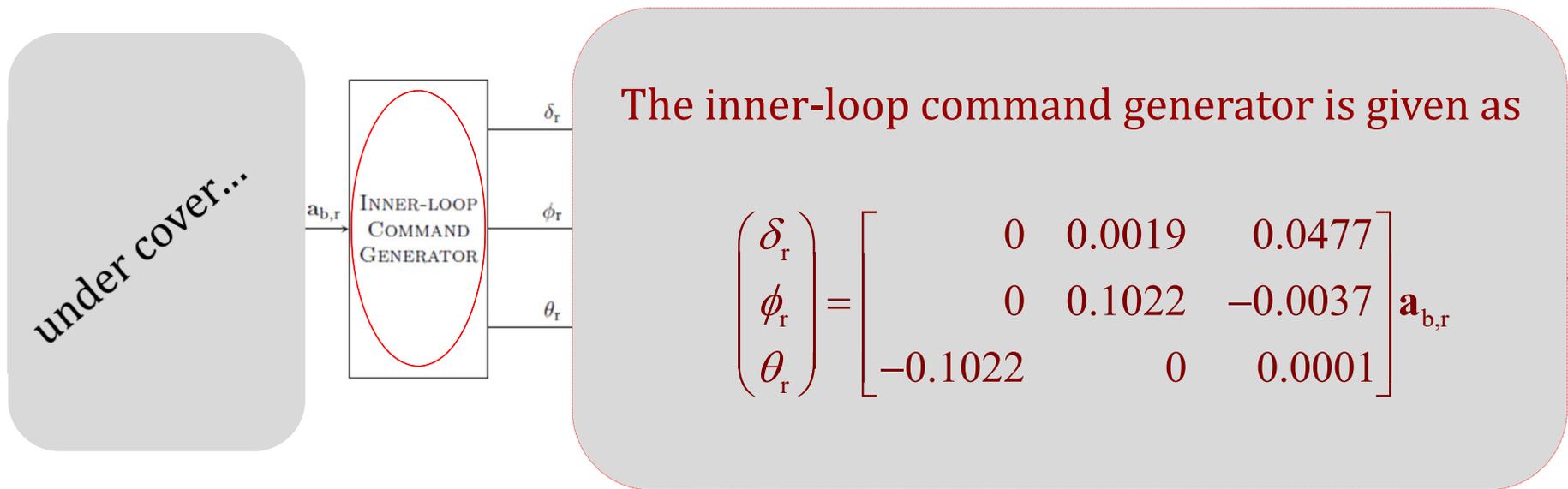
Outer-loop control system design setup
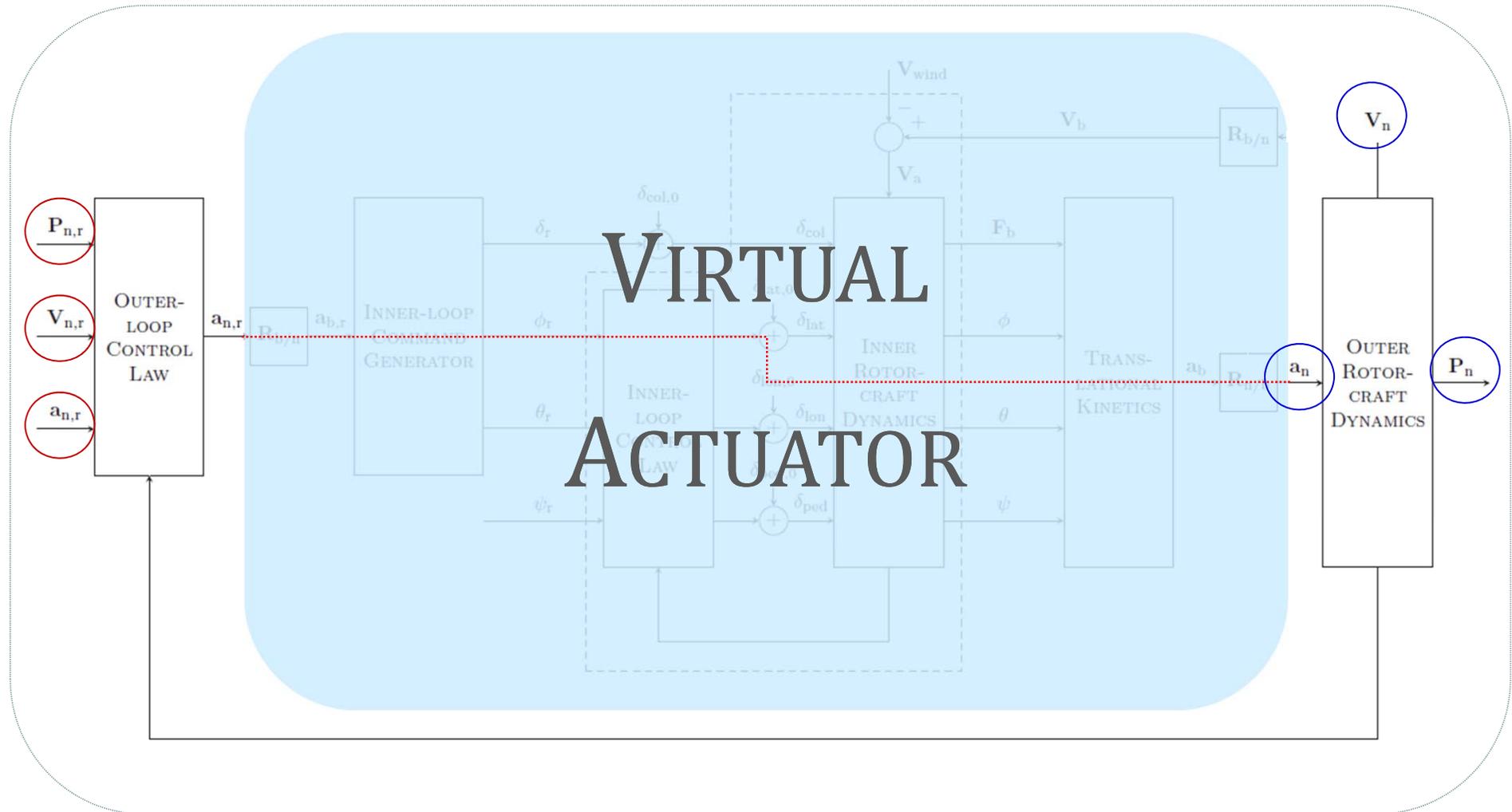
Properties of the virtual actuator

X Channel

Z Channel
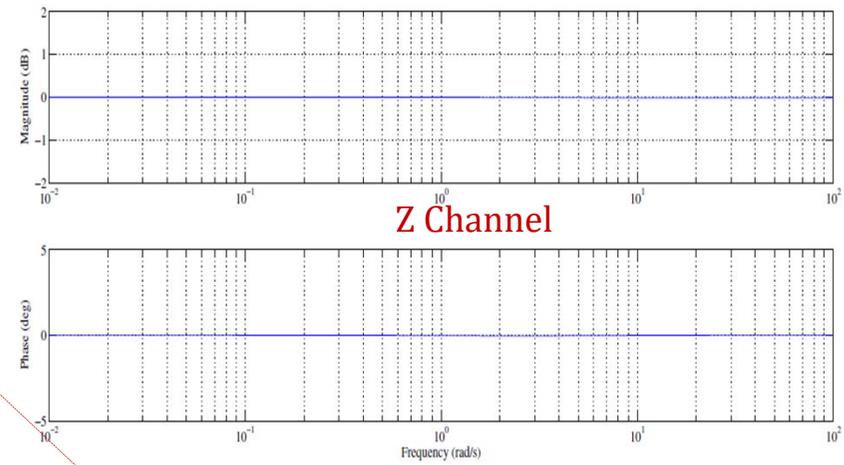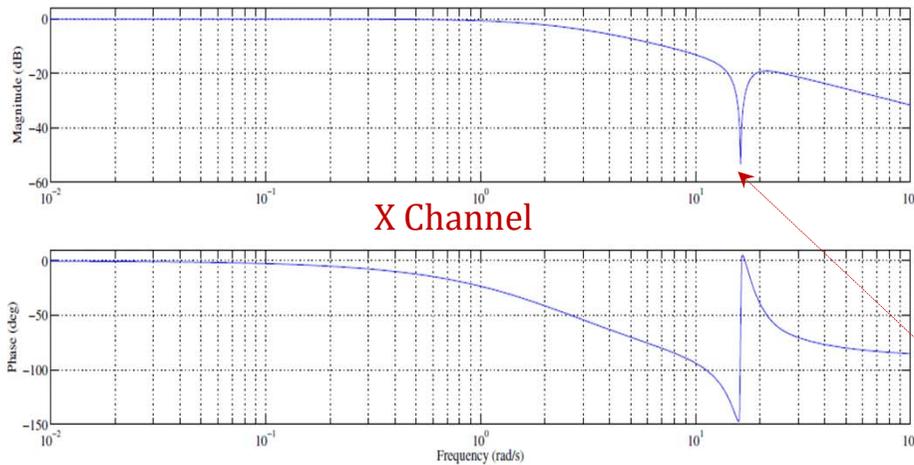
Frequency response of the virtual actuator…
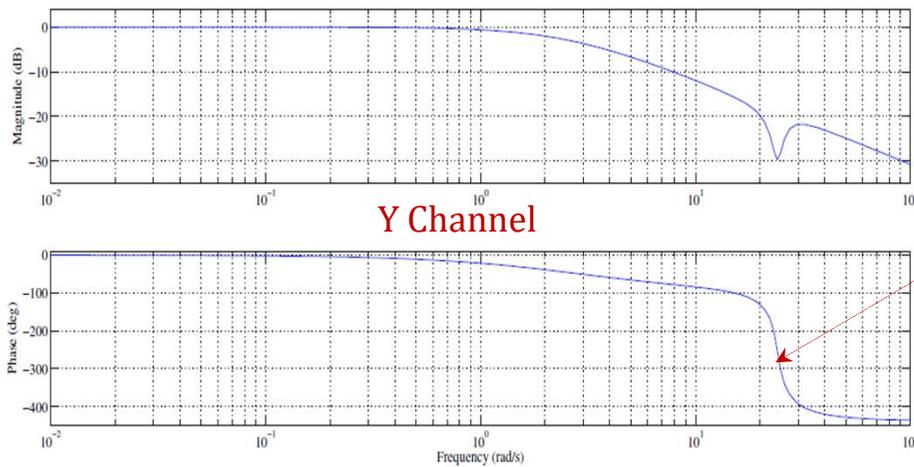
Y Channel

Unstable Zeros!

From practical point of view, it is safe to ignore them so long as the outer-loop bandwidth is within 1 rad/sec…

Properties of the outer-loop dynamics

It can also be verified that coupling among each channel of the outer loop dynamics is very weak and thus can be ignored. As a result, all the x, y and z channels of the rotorcraft dynamics can be treated as decoupled and each channel can be characterized by

$$\begin{pmatrix} \dot{p}_* \\ \dot{v}_* \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} p_* \\ v_* \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} a_*$$

where p_* is the position, v_* is the velocity and a_* is the acceleration, which is treated a control input in our formulation.

For such a simple system, it can be controlled by almost all the control techniques available in the literature, which include the most popular and the simplest one such as PID control...

Outer-loop RPT control law

$$a_{x,n} = -\begin{bmatrix} \dfrac{\omega_{n,x}^2}{\varepsilon_x^2} & \dfrac{2\zeta_x\omega_{n,x}}{\varepsilon_x} \end{bmatrix}\begin{pmatrix} x_n \\ u_n \end{pmatrix} + \left(\dfrac{\omega_{n,x}^2}{\varepsilon_x^2}\right)x_{n,r} + \left(\dfrac{2\zeta_x\omega_{n,x}}{\varepsilon_x}\right)u_{n,r} + a_{x,n,r}$$

$$a_{y,n} = -\begin{bmatrix} \dfrac{\omega_{n,y}^2}{\varepsilon^2} & \dfrac{2\zeta_y\omega_{n,y}}{\varepsilon} \end{bmatrix}\begin{pmatrix} y_n \\ v_n \end{pmatrix} + \left(\dfrac{\omega_{n,y}^2}{\varepsilon^2}\right)y_{n,r} + \left(\dfrac{2\zeta_y\omega_{n,y}}{\varepsilon}\right)v_{n,r} + a_{y,n,r}$$

$$a_{z,n} = -\begin{bmatrix} \dfrac{\omega_{n,z}^2}{\varepsilon^2} & \dfrac{2\zeta_z\omega_{n,z}}{\varepsilon} \end{bmatrix}\begin{pmatrix} z_n \\ w_n \end{pmatrix} + \left(\dfrac{\omega_{n,z}^2}{\varepsilon^2}\right)z_{n,r} + \left(\dfrac{2\zeta_z\omega_{n,z}}{\varepsilon}\right)w_{n,r} + a_{z,n,r}$$

$$\varepsilon_x = \varepsilon_y = \varepsilon_z = 1$$

$$\zeta_x = 1, \quad \zeta_y = 1, \quad \zeta_z = 1.1$$

$$\omega_{n,x} = 0.54, \quad \omega_{n,y} = 0.62, \quad \omega_{n,z} = 0.78$$

Simulation of RPT control with $\zeta = 0.7$ & $\omega_n = 1$...
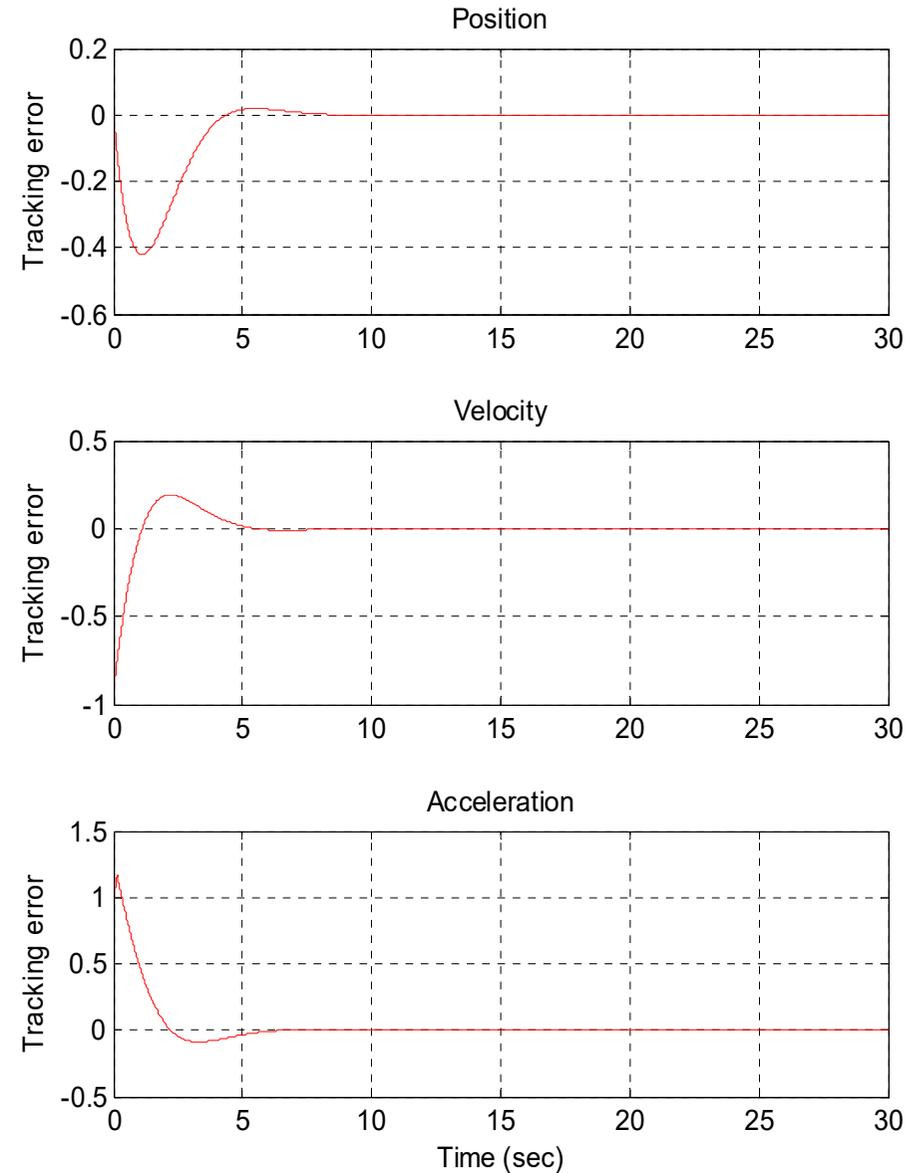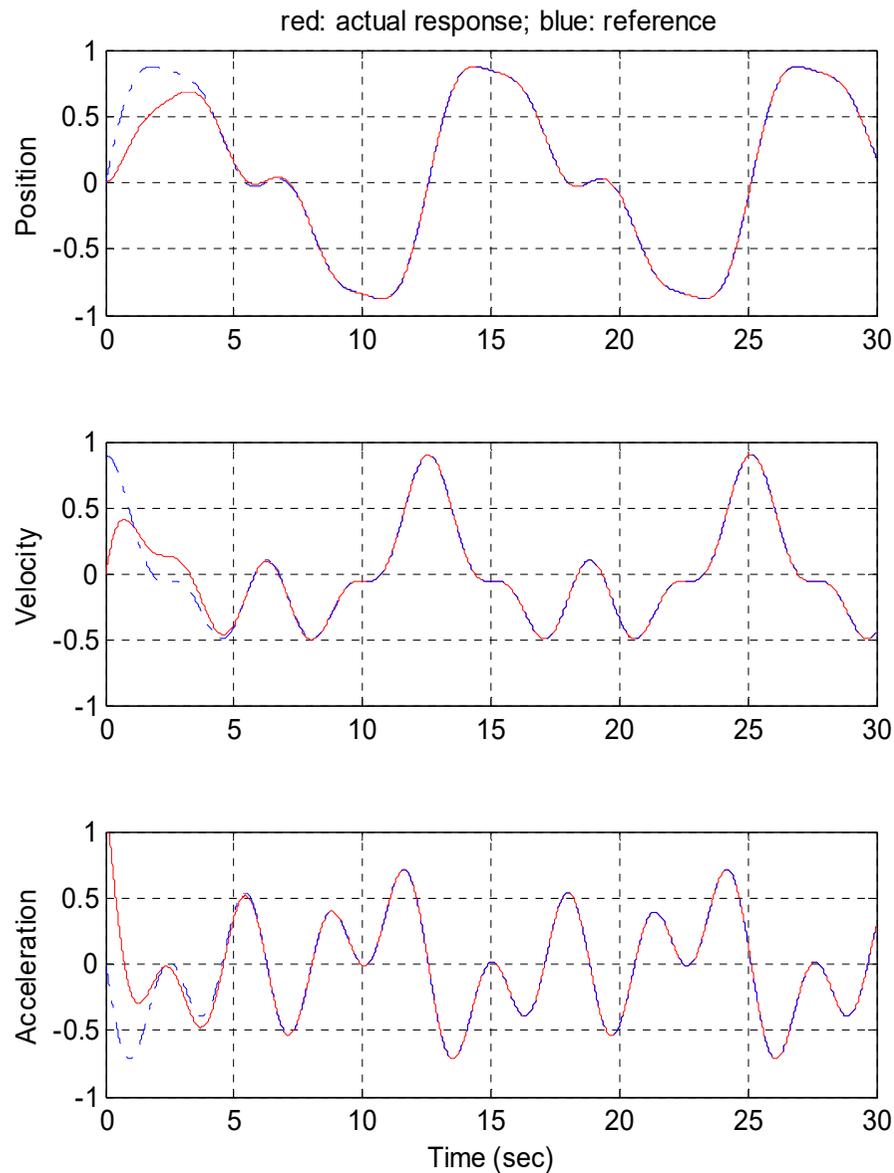
Simulation of RPT control with $\zeta = 0.7$ & $\omega_n = 1$ (cont.)
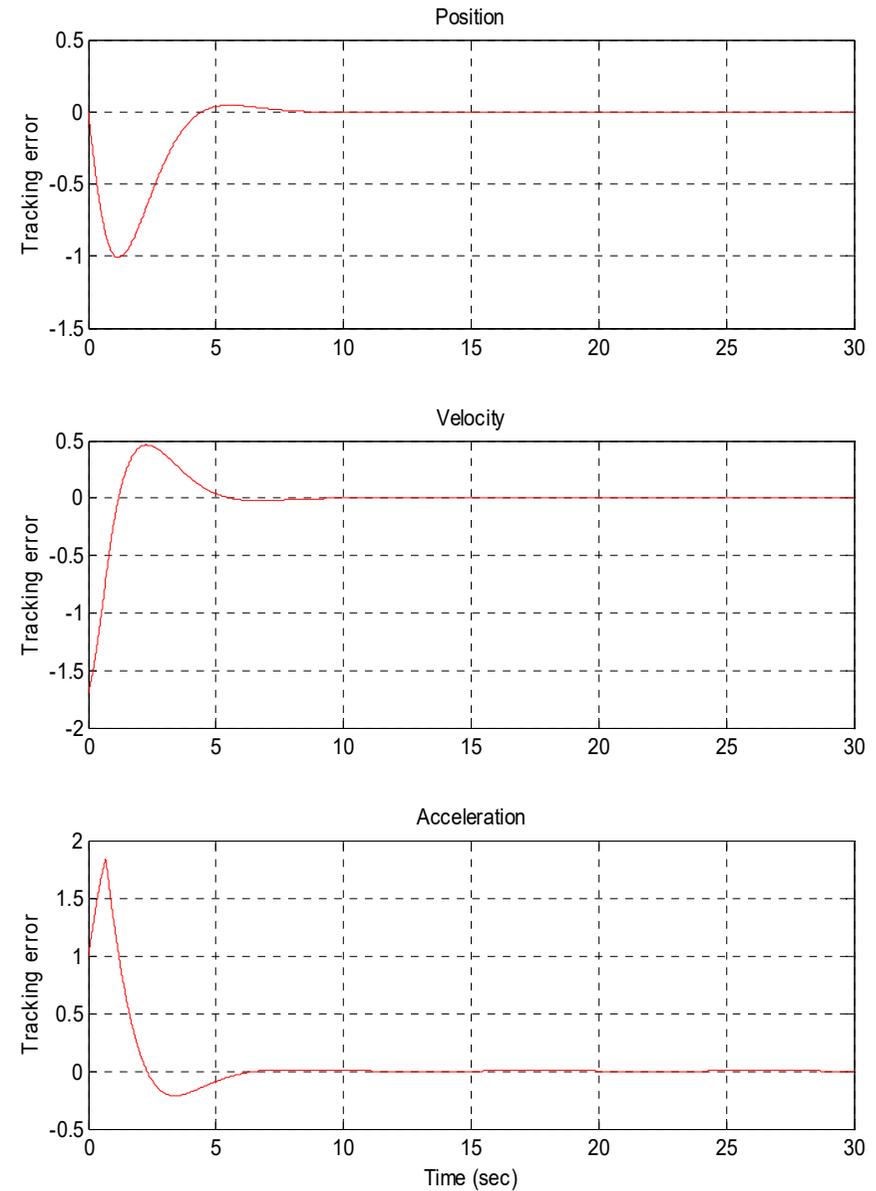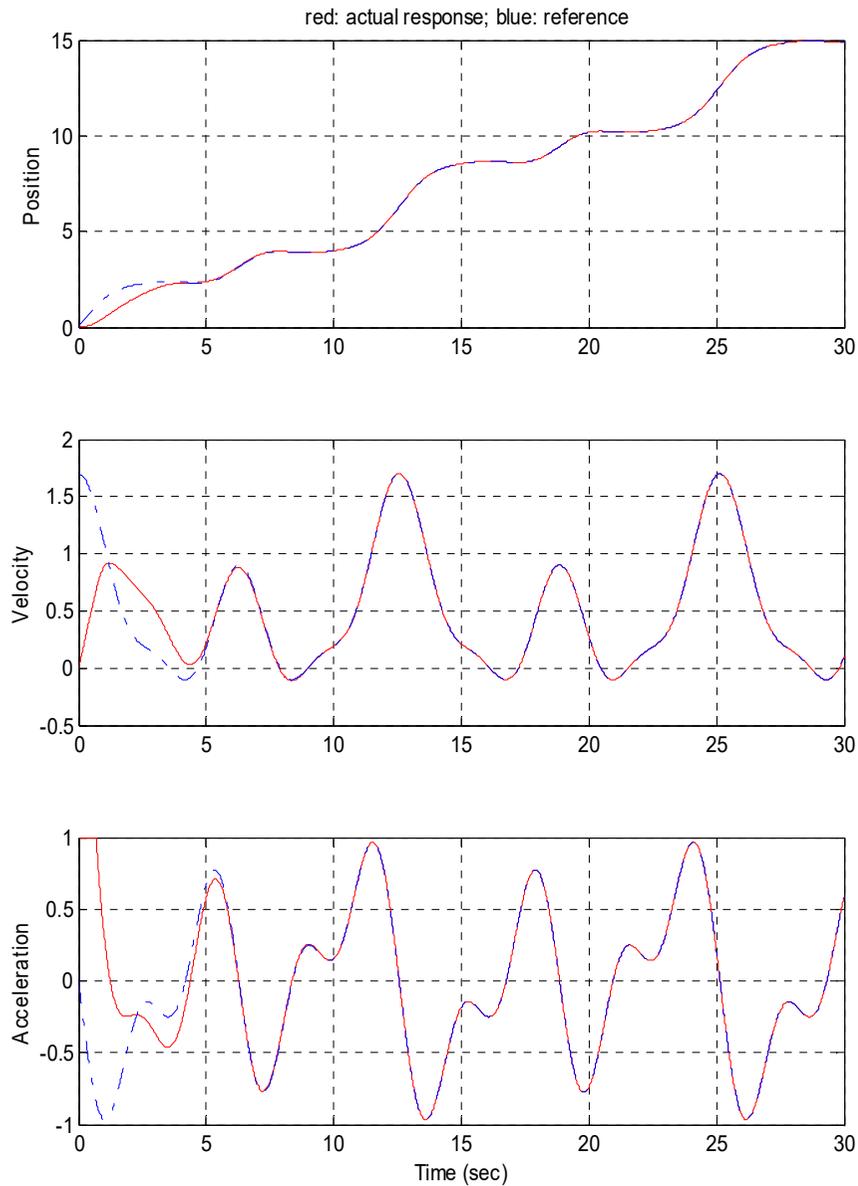
10. Loop Transfer Recovery
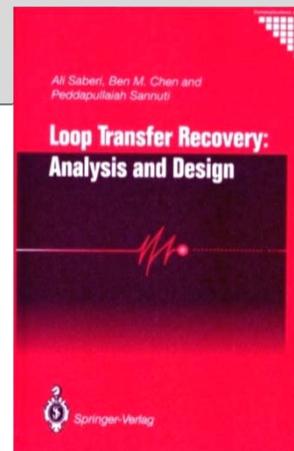
Is LQG Controller Robust?

It is now well-known that the linear quadratic regulator (LQR) has very impressive robustness properties, including guaranteed infinite gain margins and at least 60^0 phase margins in all channels. The result is only valid, however, for the full state feedback case. If observers or Kalman filters (i.e., LQG regulators) are used in implementation, no guaranteed robustness properties hold. Still worse, the closed-loop system may become unstable if you do not design the observer of Kalman filter properly. The following example given in Doyle (1978) shows the unrobustness of the LQG regulators.

Example: Consider the following system characterized by

$$\dot{x} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 1 \\ 1 \end{bmatrix} v, \quad y = \begin{bmatrix} 1 & 0 \end{bmatrix} x + w$$

where *x, u* and *y* denote the usual states, control input & measured output, and *w* and *v* are white noises with intensities 1 & $\sigma > 0$, respectively.

John Doyle
1955–

The LQG controller consists of an LQR control law + a Kalman filter.

LQR Design: Suppose we wish to minimize the performance index

$$J = \int_0^\infty (x^\mathrm{T} Q x + u^\mathrm{T} R u)\,dt, \qquad R = 1, \; Q = q \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix}, q > 0$$

It is known that the state feedback law $u = -Fx$ which minimize the performance index J is given by

$$F = R^{-1} B^\mathrm{T} P, \qquad PA + A^\mathrm{T} P - PBR^{-1} B^\mathrm{T} P + Q = 0, \quad P > 0.$$

For this particular example, we can obtain a closed-form solution,

$$F = \left(2 + \sqrt{4 + q} \right) \begin{bmatrix} 1 & 1 \end{bmatrix} = f \begin{bmatrix} 1 & 1 \end{bmatrix}.$$

It can be verified that the open loop of LQ regulator with any $q > 0$ has an infinite gain margin and a phase margin over 105 degrees. Thus, it is very robust.

It can also be shown that the Kalman filter gain for this problem can be expressed as

$$K = \left(2 + \sqrt{4 + \sigma} \right) \begin{bmatrix} 1 \\ 1 \end{bmatrix} = k \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

which together with the LQR law result an LQG controller,

$$\begin{cases} \dot{\hat{x}} = (A - BF - KC)\hat{x} + K\,y \\ u = -F\,\hat{x} \end{cases} \quad \text{or} \quad u = -F(sI - A + BF + KC)^{-1}Ky$$

Suppose that the resulting closed-loop controller has a scalar gain $1 + \varepsilon$ (nominally unity) associated with the input matrix, i.e.,

$$\text{the actual input matrix} = (1 + \varepsilon)B = \begin{bmatrix} 0 \\ 1 + \varepsilon \end{bmatrix}$$

Tedious manipulations show that the characteristic function of the closed-loop system comprising the given system an the LQG controller is given by

$$K(s) = \Pi \cdot s^4 + \Theta \cdot s^3 + \Omega \cdot s^2 + \left(2\varepsilon k f + k + f - 4 \right)s + \left(1 - \varepsilon k f \right)$$

A necessary condition for stability is that

$$2\varepsilon\,k\,f + k + f - 4 > 0 \quad \text{and} \quad 1 - \varepsilon\,k\,f > 0$$

It is easy to see that for sufficient large q and σ, the closed-loop could be unstable for a small perturbation in B in either direction. For instance, let us choose $q = \sigma = 60$. Then it is simple to verify the closed-loop system remains stable only when $-0.08 < \varepsilon < 0.01$.

The above example shows that the LQG controller is not robust at all!

What is wrong?

The answer is that the open-loop transfer function of the LQR design and the open-loop transfer function of the LQG design are totally different and thus, all the nice properties associated with the LQR design vanish in the LQG controller. It can be seen more clearly from the precise mathematical expressions of these two open-loop transfer functions, and this leads to the birth of the so-called Loop Transfer Recovery technique.

Open-Loop Transfer Function of LQR
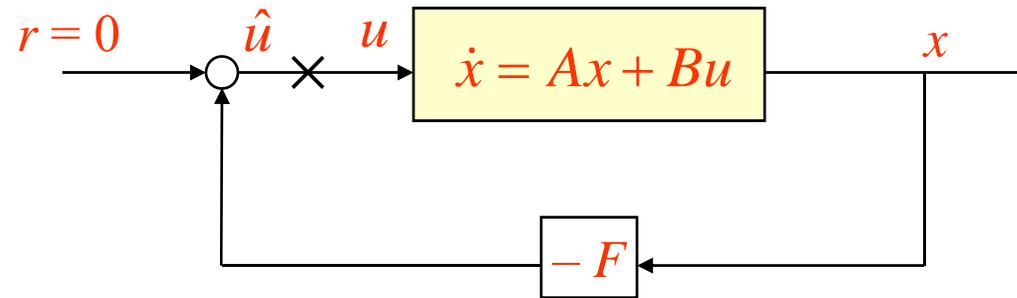
Open-loop transfer function: When the loop is broken at the input point of the plant, i.e., the point marked ×, we have

$$\hat{u} = -F(sI - A)^{-1}Bu$$

Thus, the loop transfer matrix from u to $-\hat{u}$ is given by

$$L_t(s) = F(sI - A)^{-1}B$$

We have learnt from our previous lectures that the open loop transfer $L_t(s)$ have very impressive properties if the gain matrix F comes from LQR design.

Open-Loop Transfer Function of LQG
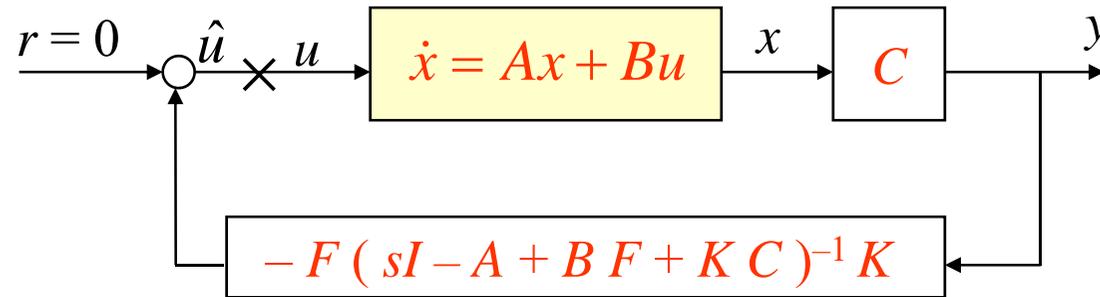
Open-loop transfer function: When the loop is broken at the input point of the plant, i.e., the point marked ×, we have

$$\hat{u} = -F(sI - A + BF + KC)^{-1}KC(sI - A)^{-1}Bu$$

Thus, the loop transfer matrix from u to $-\hat{u}$ is given by

$$L_o(s) = F(sI - A + BF + KC)^{-1}KC(sI - A)^{-1}B$$

Clearly, $L_t(s)$ and $L_o(s)$ are very different and that is why LQG in general does not have nice properties as LQR does.

Loop Transfer Recovery

The above problem can be fixed by choosing an appropriate Kalman filter gain matrix K such that $L_t(s)$ and $L_o(s)$ are exactly identical or almost matched over a certain range of frequencies. Such a technique is called **Loop Transfer Recovery**.

The idea was first pointed out by Doyle and Stein in 1979. They had given a sufficient condition under which $L_o(s) = L_t(s)$. They had also developed a procedure to design the Kalman filter gain matrix K in terms of a tuning parameter q such that the resulting $L_o(s) \to L_t(s)$ as $q \to \infty$, for invertible and minimum phase systems .

Doyle-Stein Conditions: It can be shown that $L_o(s)$ and $L_t(s)$ are identical if the observer gain K satisfies

$$K(I + C\Phi K)^{-1} = B(C\Phi B)^{-1}, \quad \Phi = (sI - A)^{-1}$$

which is equivalent to $B = 0$ (prove it!). Thus, it is impractical.

Gunter Stein

Classical LTR Design

The following procedure was proposed by Doyle and Stein in 1979 for left invertible and minimum phase systems: Define

$$Q_q = Q_0 + q^2 BVB^{\mathrm{T}}, \quad R = R_0$$

where Q_0 and R_0 are noise intensities appropriate for the nominal plant (in fact, Q_0 can be chosen as a zero matrix and $R_0 = I$), and V is any positive definite symmetric matrix (V can be chosen as an identity matrix). Then the observer (or Kalman filter) gain is given by

$$K = PC^{\mathrm{T}} R^{-1}$$

where P is the positive definite solution of

$$AP + PA^{\mathrm{T}} + Q_q - PC^{\mathrm{T}} R^{-1} CP = 0$$

It can be shown that the resulting open-loop transfer function $L_o(s)$ from the above observer or Kalman filter has

$$L_o(s) \to L_t(s), \quad as \ \ q \to \infty.$$

Example: Consider a given plant characterized by

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 35 \\ -61 \end{bmatrix} v, \qquad y = \begin{bmatrix} 2 & 1 \end{bmatrix} x + w$$

with $E[v(t)] = E[w(t)] = 0$ and $E[v(t)v(\tau)] = E[w(t)w(\tau)] = \delta(t - \tau)$.

This system is of minimum phase with one invariant zero at $s = -2$. The LQR control

law is given by

$$u = -F\, x = -\begin{bmatrix} 50 & 10 \end{bmatrix} x$$

The resulting open-loop transfer function $L_t(s)$ has an infinity gain margin and a phase

margin over 85°. We apply the Doyle-Stein LTR procedure to design an observer based

controller, i.e.,

$$u = -F[\Phi^{-1} + BF + KC]^{-1} K\, y$$

where K is computed as on the previous page with

$$Q_q = \begin{bmatrix} 35 \\ -61 \end{bmatrix} \begin{bmatrix} 35 & -61 \end{bmatrix} + q^2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix} = \begin{bmatrix} 1225 & -2135 \\ -2135 & 3721 + q^2 \end{bmatrix}.$$

Magnitude (dB) vs Frequency (rad/sec)

$L_o(s)$ with $q^2 = 10000$

$L_o(s)$ with $q^2 = 500$

$L_o(s)$ with $q^2 = 100000$

target
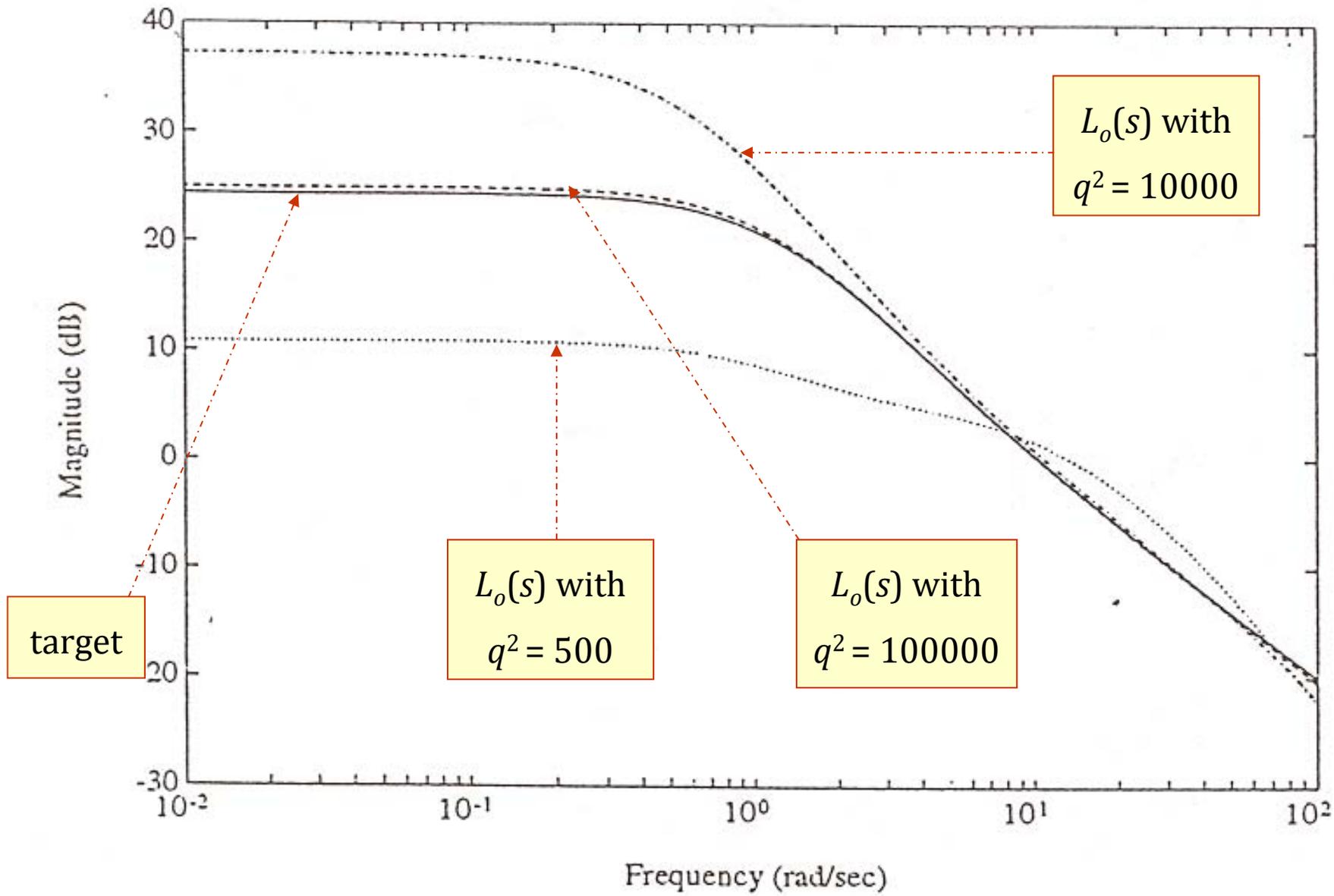
New Formulation for Loop Transfer Recovery

Consider a general stabilizable and detectable plant,

$$\begin{cases} \dot{x} = A\,x + B\,u \\ y = C\,x + D\,u \end{cases}$$

The transfer function is given by $P(s) = C\Phi B + D, \ \Phi = (sI - A)^{-1}$. Also, let F be a state feedback gain matrix such that under the state feedback control law $u = -F\,x$ has the following properties:

- the resulting closed-loop system is asymptotically stable; and

- the resulting target loop $L_t(s) = F\Phi B$ meets design specifications (GM, PM).

Such a state feedback can be obtained using LQR design or any other design methods so long as it meets your design specifications. Usually, a desired target loop would have the shape as given in the following figure.
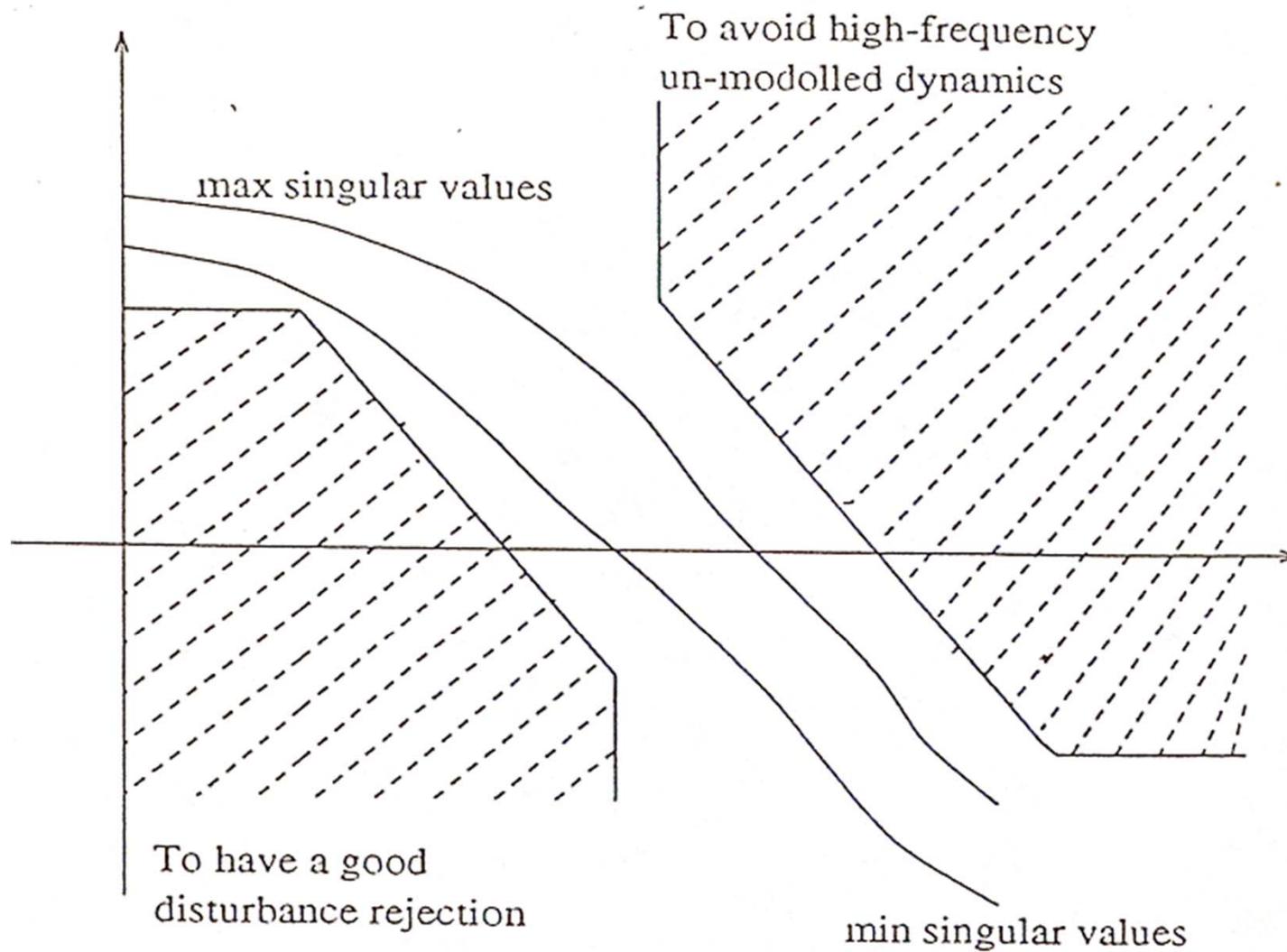
Typical desired open-loop characteristics...

The problem of loop transfer recovery (LTR) is to find a stabilizing controller $u = -C(s)y$
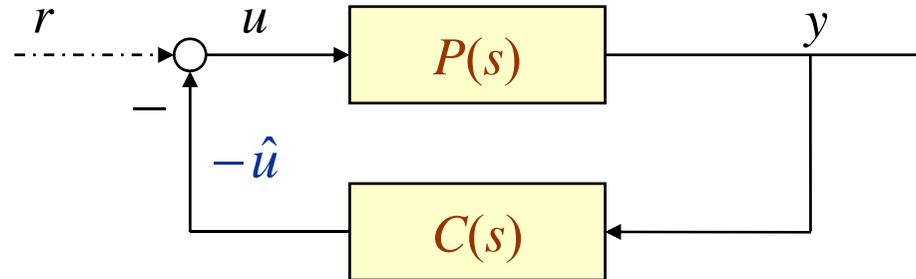
such that the resulting open-loop transfer function from u to $-\hat{u}$, i.e.,

$$L_o(s) = C(s)P(s)$$

is either exactly or approximately equal to the target loop $L_t(s)$. Let us define the recovery error as the difference between the target loop and the achieved loop, i.e.,

$$E(s) = L_t(s) - L_o(s) = F\Phi B - C(s)P(s)$$

Then, we say exact LTR is achievable if $E(s)$ can be made identically zero, or almost LTR is achievable if $E(s)$ can be made arbitrarily small.

Observer Based Structure for $C(s)$
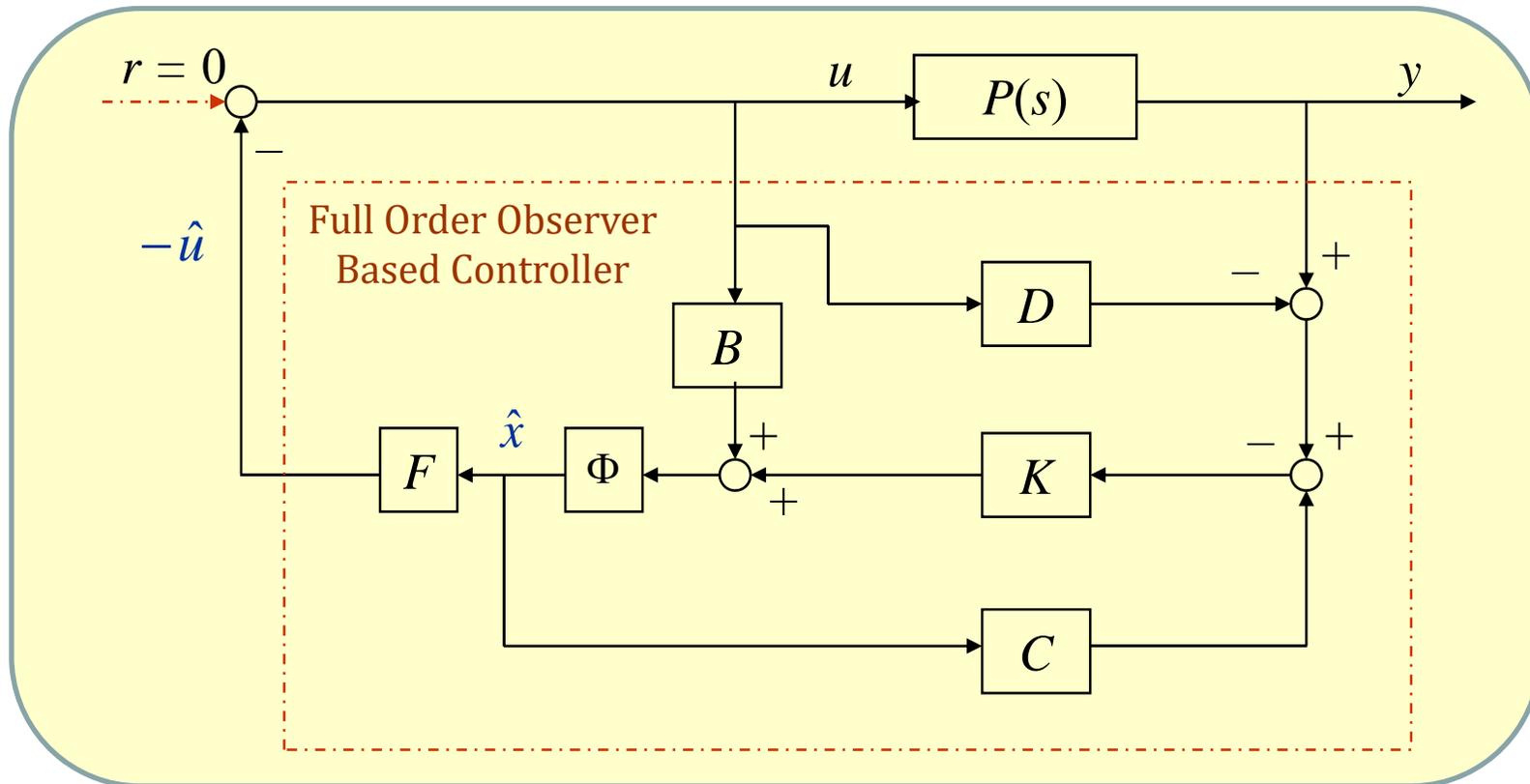
Dynamic equations of $C(s)$: $\dot{\hat{x}} = A\,\hat{x} + B\,u + K(y - C\hat{x} - D\,u), \quad \hat{u} = u = -F\,\hat{x}$

Transfer function of $C(s) = C_o(s) = F(\Phi^{-1} + BF + KC - KDF)^{-1}K$

Achieved open-loop: $L_o(s) = C_o(s)P(s) = F(\Phi^{-1} + BF + KC - KDF)^{-1}K(C\Phi B + D)$

Lemma: Recovery error, $E_o(s)$, i.e., the mismatch between the target loop and the resulting open-loop of the observer based controller is given by

$$E_o(s) = M(s)\left[I + M(s)\right]^{-1}(I + F\Phi B), \quad M(s) = F(\Phi^{-1} + KC)^{-1}(B - KD)$$

Proof.

$$L_o(s) = C_o(s)P(s) = F(\Phi^{-1} + BF + KC - KDF)^{-1}K(C\Phi B + D)$$

$$= F[I + (\Phi^{-1} + KC)^{-1}(B - KD)F]^{-1}(\Phi^{-1} + KC)^{-1}K(C\Phi B + D)$$

$$(I + AB)^{-1}A = A(I + BA)^{-1}$$

$$= [I + F(\Phi^{-1} + KC)^{-1}(B - KD)]^{-1}F(\Phi^{-1} + KC)^{-1}K(C\Phi B + D)$$

$$= [I + M(s)]^{-1}[F(\Phi^{-1} + KC)^{-1}KC\Phi B + F(\Phi^{-1} + KC)^{-1}KD]$$

$$= [I + M(s)]^{-1}\{F[I - (\Phi^{-1} + KC)^{-1}\Phi^{-1}]\Phi B + F(\Phi^{-1} + KC)^{-1}KD\}$$

Credit to G C Goodman in a master thesis conducted at MIT in 1984

$$= [I + M(s)]^{-1}[F\Phi B - F(\Phi^{-1} + KC)^{-1}B + F(\Phi^{-1} + KC)^{-1}KD]$$

$$= [I + M(s)]^{-1}[F\Phi B - F(\Phi^{-1} + KC)^{-1}(B - KD)]$$

$$= [I + M(s)]^{-1}[F\Phi B - M(s)]$$

Note that we have used $(\Phi^{-1} + KC)^{-1}KC = I - (\Phi^{-1} + KC)^{-1}\Phi^{-1}$. Thus,

Michael Athans
1937–

$$E_o = L_t - L_o = F\Phi B - [I + M]^{-1}[F\Phi B - M] = [I + M]^{-1}M(I + F\Phi B).$$

Loop Transfer Recovery Design

It is simple to observe from the above lemma that the loop transfer recovery is achievable if and only if we can design a gain matrix K such that $M(s)$ can be made either identically zero or arbitrarily small, where $M(s) = F(\Phi^{-1} + KC)^{-1}(B - KD).$

Let us define an auxiliary system

$$\Sigma_{\text{aux}} : \begin{cases} \dot{x} = A^{\text{T}}x + C^{\text{T}}u + F^{\text{T}}w \\ y = x \\ z = B^{\text{T}}x + D^{\text{T}}u \end{cases} \quad + \quad u = -K^{\text{T}}x$$

➡ Closed-loop transfer function from w to z is $(B^{\text{T}} - D^{\text{T}}K^{\text{T}})(sI - A^{\text{T}} + C^{\text{T}}K^{\text{T}})^{-1}F^{\text{T}} = M^{\text{T}}(s).$

Thus, LTR design is equivalent to design a state feedback law for the above auxiliary system such that certain norm of the resulting closed-loop transfer function is made either identically zero or arbitrarily small. As such, the H_2 and H_∞ optimization techniques can be used to solve the LTR problem. There is no need to repeat all over again once this is formulated.

What really got me interested in control was my first unintentional discovery. I was asked to simulate some examples on loop transfer recovery (LTR) in the book *Control System Design*, by Bernard Friedland. It was mentioned in the text that under the Doyle-Stein condition for LTR, the link feeding the control input signal to an observer-based control law might be omitted. When I simulated examples without satisfying the Doyle-Stein condition (which can never be met in any physical system, by the way) by removing the link to the observer, to my surprise, the recovery performance turned out to be unbelievably good. When I showed the result to my advisor, I got kicked out of his office, as apparently I had violated the common belief in control systems design—the separation principle. Nevertheless, the discovery eventually led to a new controller structure for the LTR design

A story behind a new controller structure for LTR...

Full Order Observer Based Controller

Bernard Friedland

LTR Design via CSS Architecture Based Controller
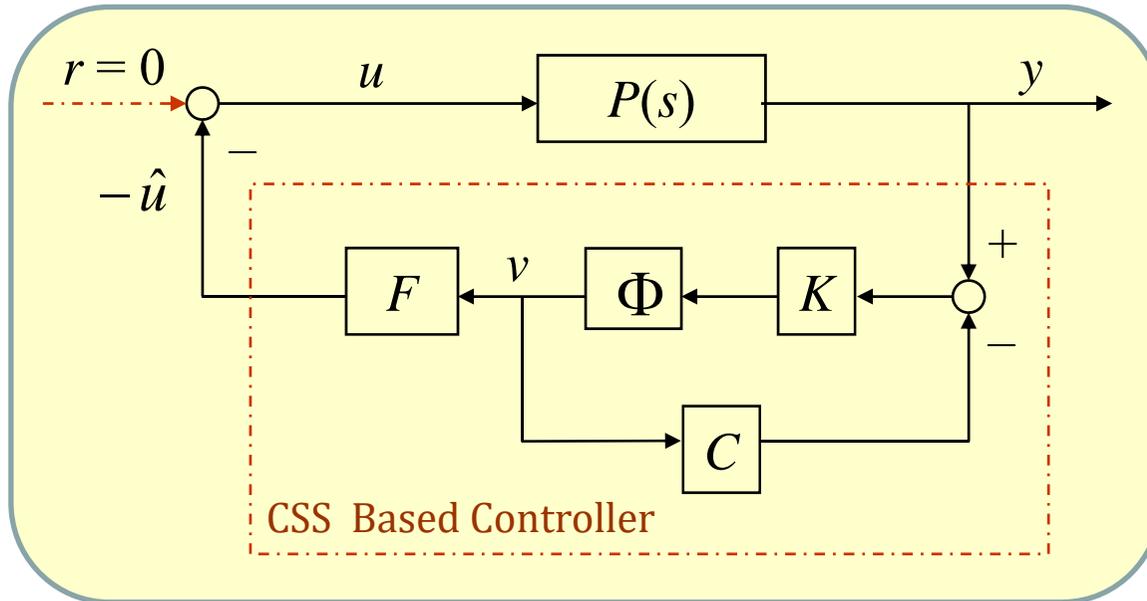
CSS Based Controller

Proposed by Chen, Saberi and Sannuti in 1991, the CSS based controller has the following characteristics:

Dynamic equations of $C(s)$: $\dot{v} = (A - KC)\,v + Ky, \quad \hat{u} = u = -F\,v$

Transfer function of $C(s) = C_c(s) = F(\Phi^{-1} + KC)^{-1}K$

Achieved open-loop: $L_c(s) = C_c(s)P(s) = F(\Phi^{-1} + KC)^{-1}K(C\Phi B + D)$

Ali Saberi
1949–

Pedda Sannuti
1941–

Lemma: Recovery error, $E_c(s)$, i.e., the mismatch between the target loop and the resulting open-loop of the CSS architecture based controller is given by

$$E_c(s) = M(s) = F(\Phi^{-1} + KC)^{-1}(B - KD)$$

Proof.
$$E_c(s) = L_t(s) - L_c(s)$$
$$= F\Phi B - F(\Phi^{-1} + KC)^{-1} K(C\Phi B + D)$$
$$= F(\Phi^{-1} + KC)^{-1}[(\Phi^{-1} + KC)\Phi B - KC\Phi B - KD]$$
$$= F(\Phi^{-1} + KC)^{-1}(B + KC\Phi B - KC\Phi B - KD)$$
$$= M(s)$$
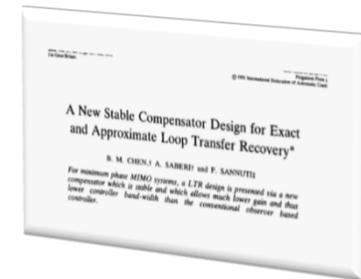
It is clear that LTR via the CSS architecture based controller is achievable if and only if one can design a gain matrix K such that the resulting $M(s)$ can be made either identically zero or arbitrarily small. This is identical to the LTR design via the observer based controller. Thus, one can again using the H_2 and H_∞ techniques to carry out the design of such a gain matrix.

- *B. M. Chen, A. Saberi and P. Sannuti, A new stable compensator design for exact and approximate loop transfer recovery, Automatica, Vol. 27, No. 2, pp. 257–280, March 1991.*

  *Collected in **Bibliography on Robust Control** by P. Dorato, R. Tempo, G. Muscato in Automatica, Vol. 29, No. 1, January 1993.*

What is the Advantage of CSS Structure?

Answer: Theorem. Consider a stabilizable and detectable system Σ characterized by (A, B, C, D) and target loop transfer function $L_t(s) = F\Phi B$. Assume that Σ is left invertible and of minimum phase, which implies that the target loop $L_t(s)$ is recoverable by both observer based and CSS architecture based controllers. Also, assume that the same gain K is used for both observer based controller and CSS architecture based controller and is such that for all $\omega \in \Omega$, where Ω is some frequency region of interest,

$$0 < \sigma_{\max}[M(j\omega)] << 1, \qquad \sigma_{\min}[L_t(j\omega)] = \sigma_{\min}[F(j\omega I - A)^{-1}B] >> 1$$

Then, for all $\omega \in \Omega$,

$$\sigma_{\max}[E_c(j\omega)] << \sigma_{\max}[E_o(j\omega)].$$

Proof. See Chen, Saberi and Sannuti, *Automatica*, vol. 27, 1991.

Remark: In order to have good command following and desired disturbance rejection properties, the target loop transfer function $L_t(j\omega)$ has to be large and consequently, the minimum singular value $\sigma_{\min}[L_t(j\omega)]$ should be relatively large in the appropriate frequency region. Thus, the assumption in the above theorem is very practical.

Example: Consider a given plant characterized by

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \quad y = \begin{bmatrix} 2 & 1 \end{bmatrix} x + 0 \cdot u$$

Let the target loop $L_t(s) = F\,\Phi B$ be characterized by a state feedback gain $F = \begin{bmatrix} 50 & 10 \end{bmatrix}$.

Using MATLAB, we know that the above system has an invariant zero at $s = -2$. Hence it is of minimum phase. Also, it is invertible. Thus, the target loop $L_t(s)$ is recoverable by both the observer based and CSS architecture based controllers.

Using the H_2 optimization method, we obtain matrix $K = \begin{bmatrix} 6.9 \\ 84.6 \end{bmatrix}$.
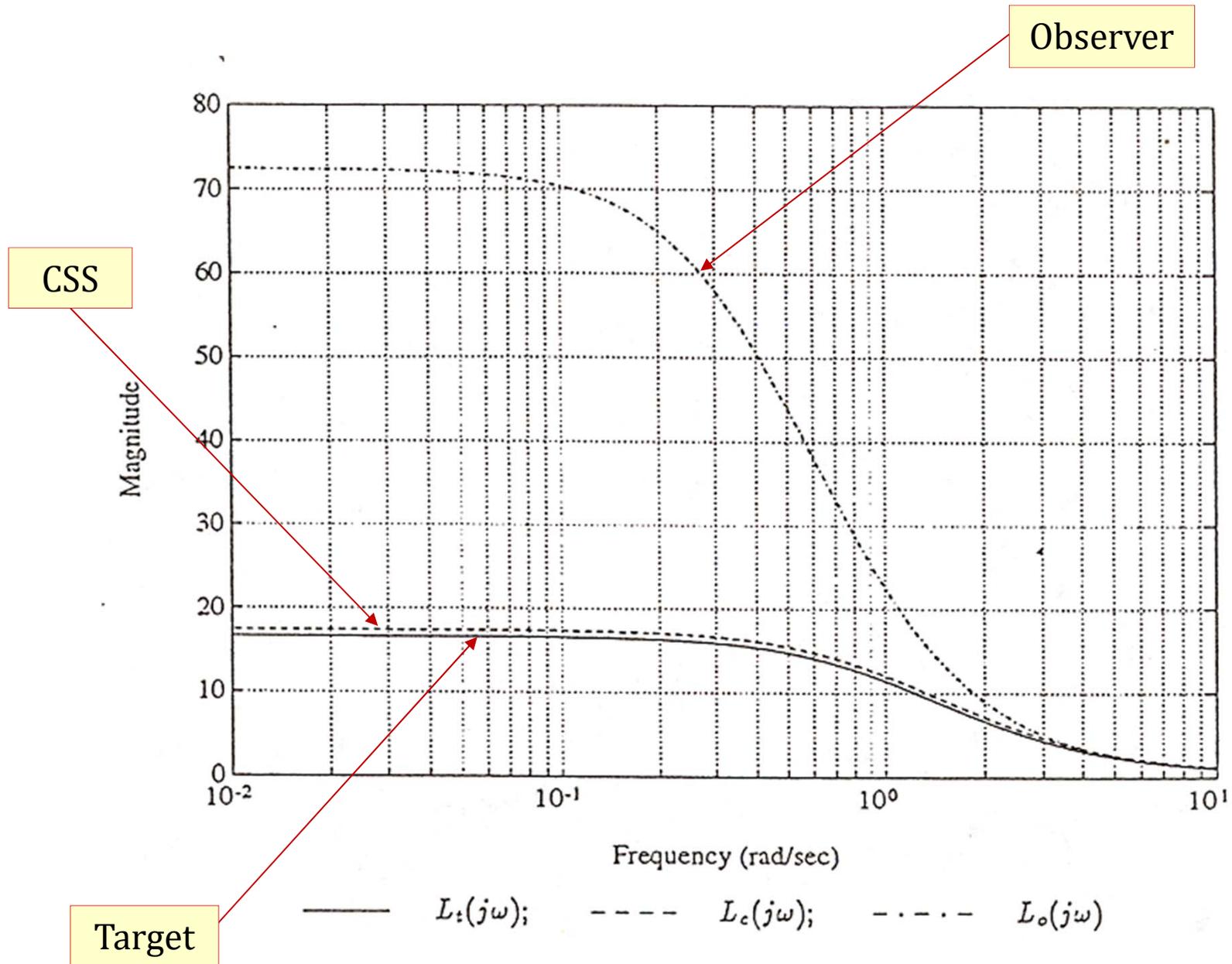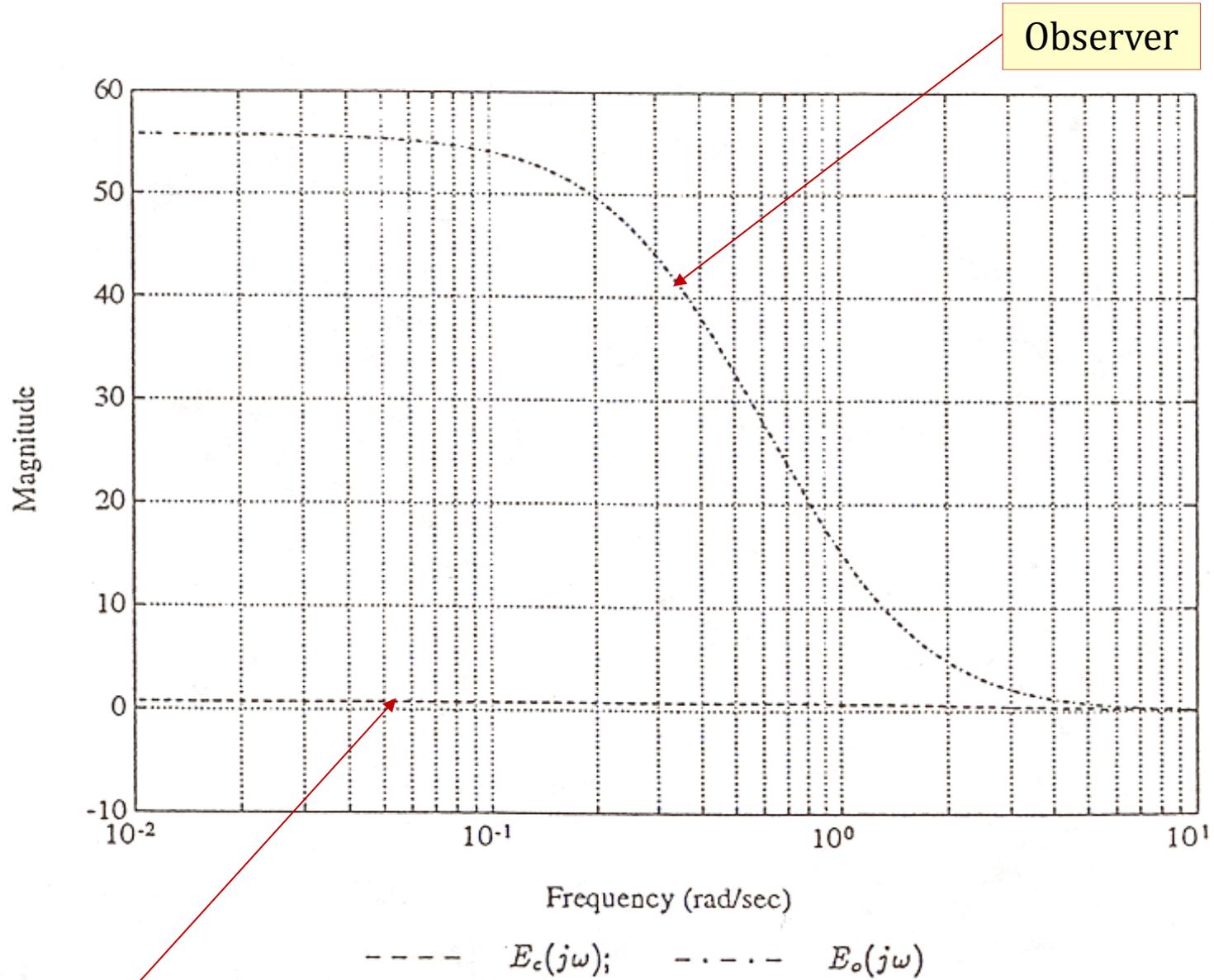
Example: Consider a given plant characterized by

$$
\mathbf{A} = \begin{bmatrix} 2 & 0 & 1 & 0 & 0 & 1 & 1 \\ -1 & 2 & 0 & 0 & 0 & 0 & 0 \\ 3 & 2 & -1 & 0 & 0 & -2 & 1 \\ 2 & -2 & 0 & -4 & 2 & 0 & -1 \\ 0 & 2 & 3 & 0 & -2 & 1 & -1 \\ 1 & 0 & 2 & -3 & 2 & 2 & 0 \\ -1 & -1 & 1 & 0 & 0 & -1 & 1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} I_3 & 0_{3\times 4} \end{bmatrix}.
$$

The target loop $L_t(s) = F\,\Phi B$ is characterized by an LQR state feedback gain with $Q = I_7$

and $0.001 \times I_3$. The observer and CSS controller gain matrices are chosen thru the

classical LTR design technique with tuning parameter summarized in the table below.

	Tuning parameter	Supremum $\sigma_{\max}\{E_0(j\omega)\}$	Supremum $\sigma_{\max}\{E_c(j\omega)\}$
Case 1	$\sigma = 10$	203.0387	19.9622
Case 2	$\sigma = 100$	136.1517	2.1660
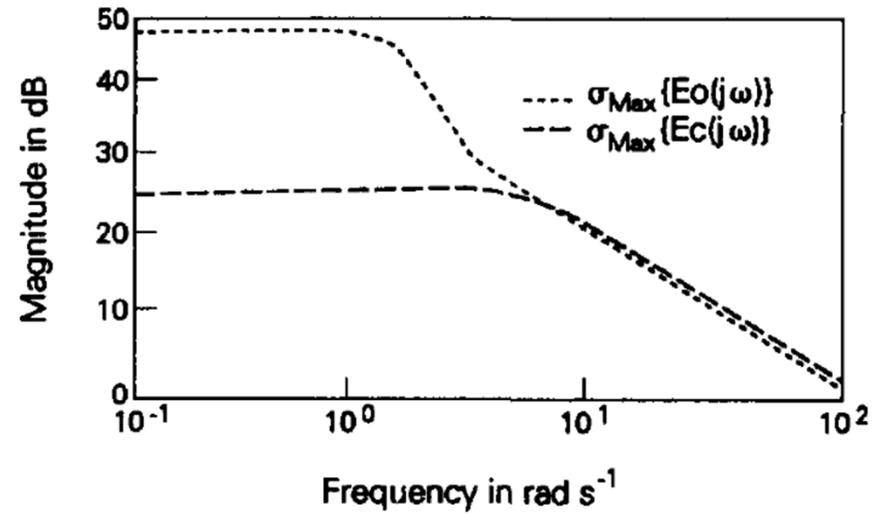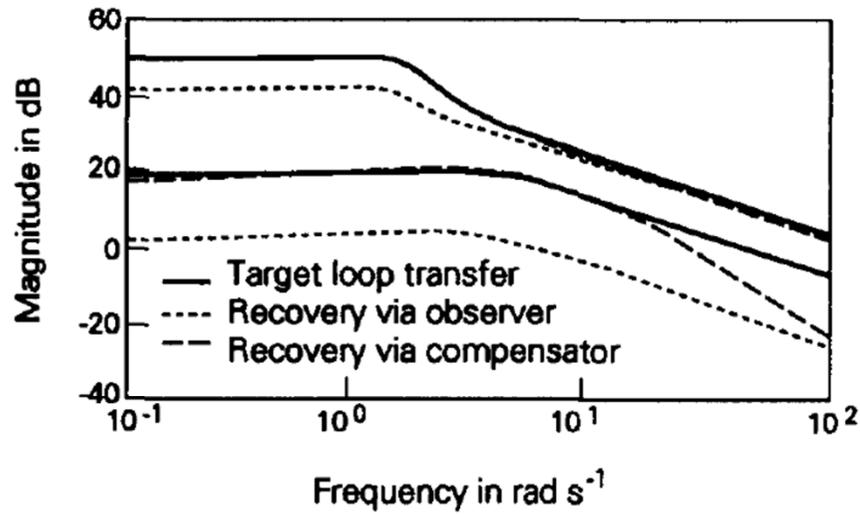
Case 1
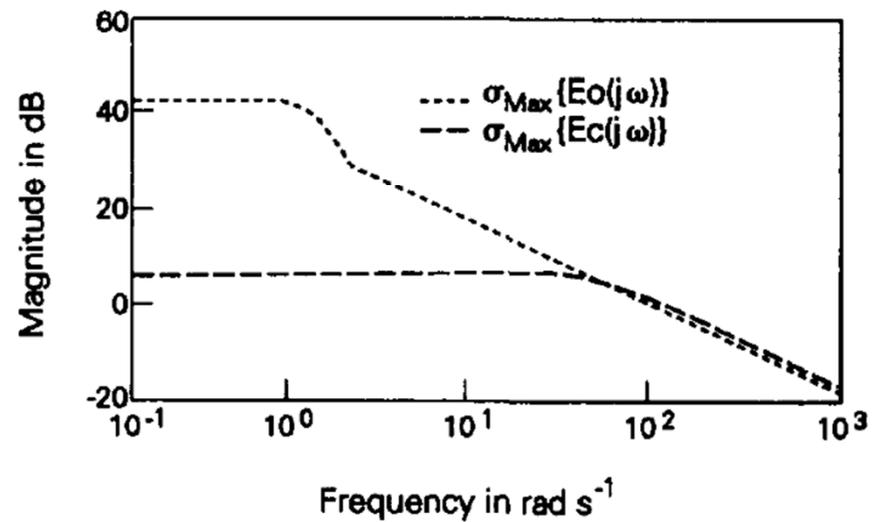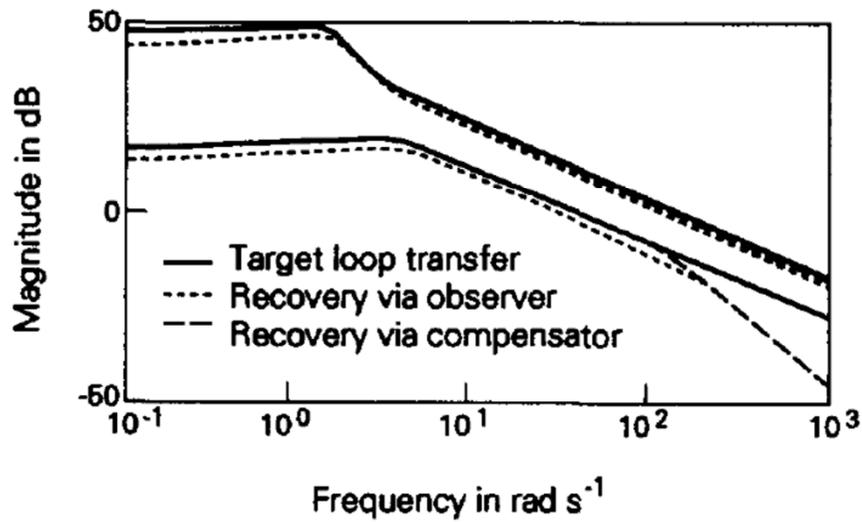
Case 2

Homework Assignment 3:

Using the loop transfer recovery control technique to design appropriate measurement feedback control laws that meet all the design specification specified in the (HDD or helicopter) problem.

Show all the detailed calculation and simulate your design using MATLAB and Simulink. Give all the necessary plots that show the evidence of your design.

11. Concluding Remarks

Some Personal Viewpoints on Control Systems Design

在上一期"有问必答"栏目里，我们刊登了华东交通大学杨辉教授团队提出的一个常常问起的控制问题：如今各类型的先进控制算法这么多，为什么实际应用最常见的还是PID控制？

答（B.M.C.）：在实际应用当中，因为PID控制简单易调，自然就成为工程设计人员的首选控制器。如果被控系统无法被PID所控，设计人员一般首先考虑是重新设计被控的系统（如重新设计机械架构或重新调配驱动器和传感器），而不是直接采用先进的控制方法，如此这般，PID就成了实际应用中常见的控制器。不过PID也非万能，由于PID控制器结构上的缺陷，在PID控制的系统当中，我们一般无法优化系统的整体性能；在诸多多变量系统中，PID往往也是束手无策的。在设计多变量控制系统时，同样因为简易，LQR则是最常见的控制方法。顺便提一句，在实际应用中，许多所谓的先进控制方法其实都是大同小异的（此处可以有石头、鸡蛋飞来）。

中国〖TCCT通讯〗2014年 第4期

Question: *Why is PID control still dominating in practical applications even though there are so many advanced control techniques out there?*

Answer (B.M.C.): PID control is always the first choice of practicing engineers because it is structurally simple and it is easy to tune. If a system cannot be controlled by a PID controller, the first thing that engineers would do is to redesign the system (such as to restructure the system mechanical part or to reselect and/or replace the system sensors and actuators), instead of trying an advanced control technique. As such, PID is dominating in practical applications. However, this does not mean that PID is superior. Because of its structural limitation, it is generally difficult to push for an optimal performance of the PID controlled system. Furthermore, many multivariable systems cannot even be stabilized by PID control laws. For MIMO systems, LQR control on the other hand is the most popular choice among all the control techniques.

By the way, many advanced control methods do not make much difference in controlling practical systems.

TCCT Newsletter, April 2014

03.2013

09.2013

03.2014

2016

2017

A novel hybrid UAV that may change the way people operate drones

I personally believe that a good control system design should not start from differential equations but should be down to earth and start from the hardware level, including the selection and placement of sensors and actuators. BMC

A more advanced course in linear systems and control...

Linear Systems and Control

Ben M. Chen

Professor of Electrical & Computer Engineering
National University of Singapore

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Email: bmchen@nus.edu.sg   http://www.bmchen.net

... this is not a regular course at NUS. it is designed just for my own graduate students ...

Course material is available online at www.bmchen.net

The course is aimed to answer the following questions:

- ➢ Why is the commonly used PID a bad controller?

- ➢ What control performance can one expect from a given system?

- ➢ Why are system nonminimum phase zeros bad for control?

- ➢ What else are bad to be controlled?

- ➢ When an airplane passes through turbulences, why can it maintain its position while its body is shaking badly?

- ➢ When and how can disturbances, uncertainties and nonlinearities be attenuated through proper control system design?

- ➢ What is the best way to design a control system?
  - ○ to design a good control law? or
  - ○ to design a good system?

- ➢ How to design a good system through sensor and actuator selection?

- ➢ Why is PID not bad at all after all?

- ➢ How to improve control performance?
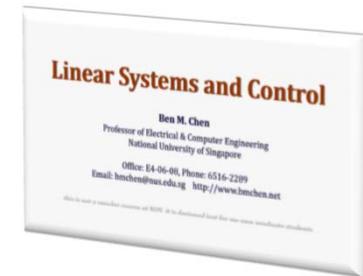
That's all, folks!

Thank You!

Bugs Bunny
1940–