EE3331C: Feedback Control Systems

Part 2: Frequency Domain Methods

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Introduction & Background Materials

Text and References

- G. F. Franklin, J. D. Powell and A. Emami-Naeini, *Feedback Control of Dynamic Systems*, Pearson Prentice Hall
- R. C. Dorf and R. H. Bishop, *Modern Control Systems*, Pearson Prentice Hall

Consultation

- No fixed time for consultation
- Appointment through email is recommended

Motivation

The design of feedback control systems in industry is probably accomplished using frequency-response methods more often than any other. This approach provides good designs in the face of uncertainty in the plant model. The so-called frequency response of a system lies at the core of these methods.

An introduction to frequency response is covered in this unit.

Learning Objectives

- Ability to find the gain and the phase of a system when the sinusoidal waveforms at its input and output are given
- Ability to compute the gain and the phase of a transfer function at a given frequency
- Ability to find the steady-state output of a transfer function for a given sinusoidal input
- Ability to compute the crossover frequencies

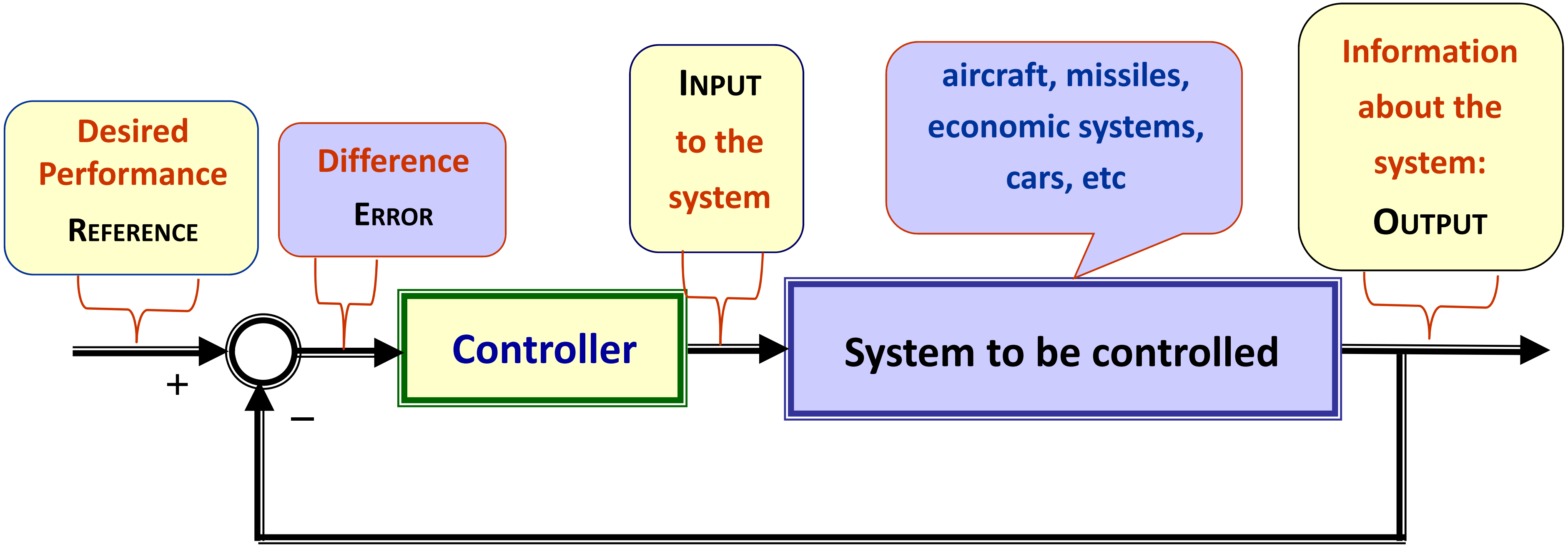
Final Grades for Part 2

Final Grade = 80% $\times$ Final exam marks for Part 2 (max = 50) + …

20% $\times$ Part 2 Test marks (max = 50)

There will be a in class (10–11:30am in LT4) test for Part 2 on Tuesday, 24 October 2017.

What is a control system?

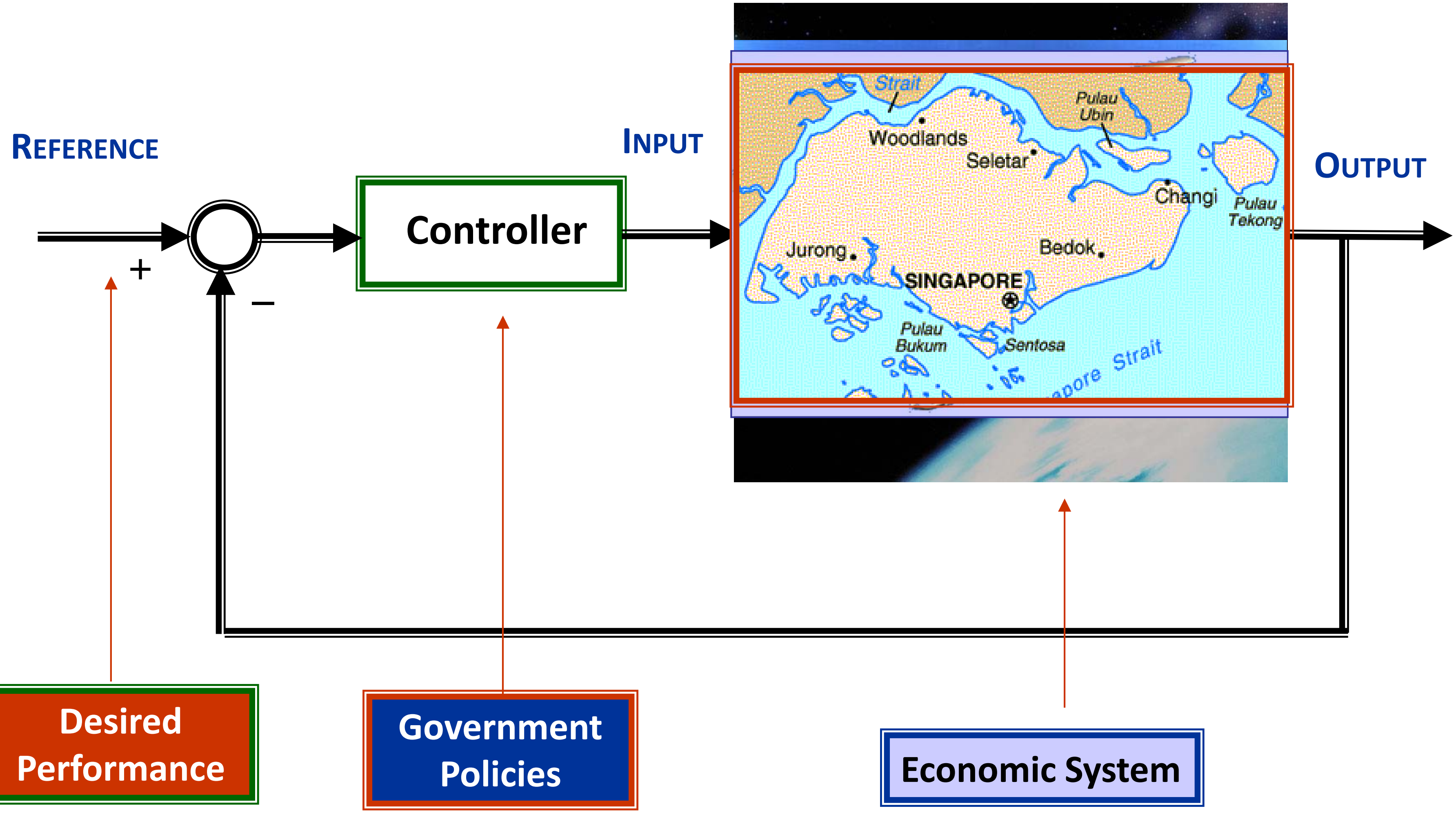


Objective: To make the system **OUTPUT** and the desired **REFERENCE** as close as possible, i.e., to make the **ERROR** as small as possible.

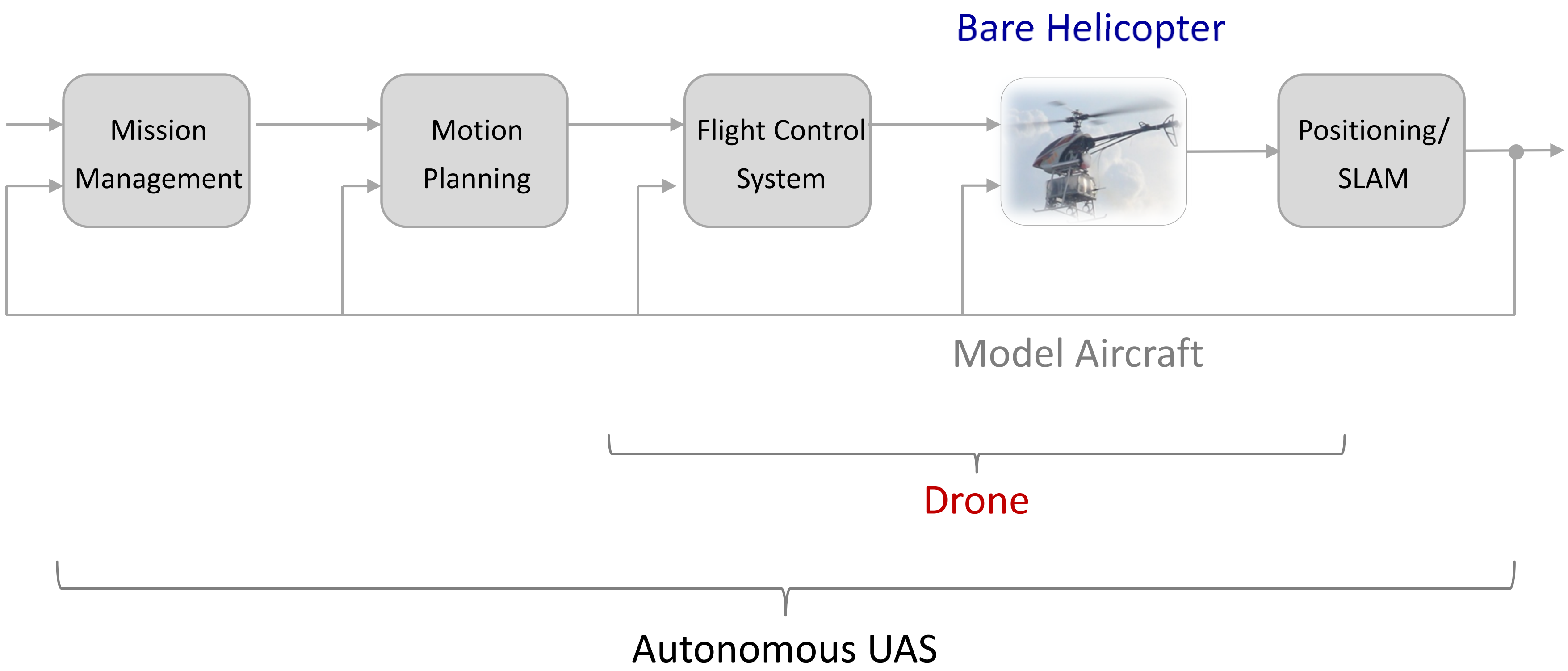
Key Issues: (1) How to describe the system to be controlled? **(Modeling)**

(2) How to design the controller? **(Control)**

Some Control Systems Examples



A Control System Design Example – Unmanned Helicopter



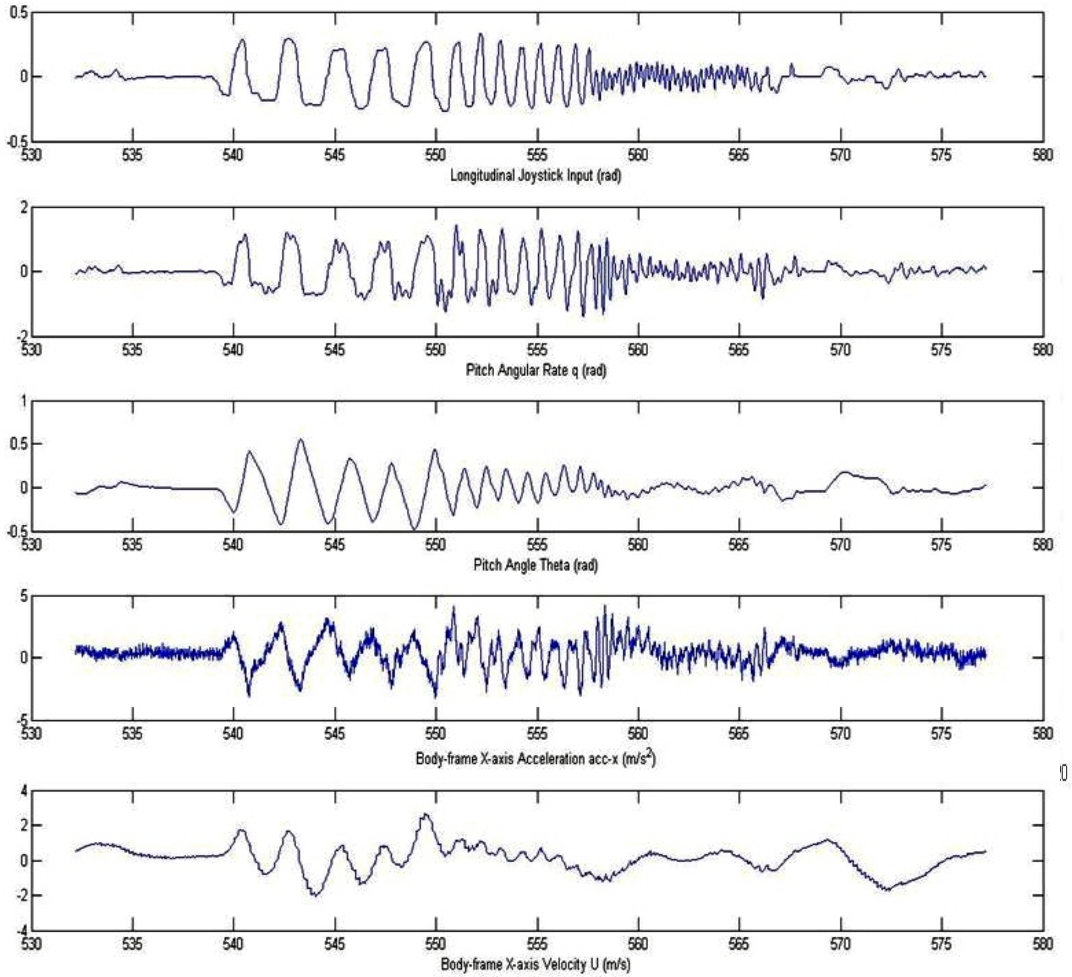
Modeling – Data Collection

Data Collection Procedure:

1). Chirp-like signal issued in single channel;

2). Chirp-like signal issued in multi-channels;

3). Step-like and random signals issued for validation.



Chirp-like signal and corresponding responses

Modeling – Test Flights

Flight testing for modeling purpose

Frequency Domain Responses

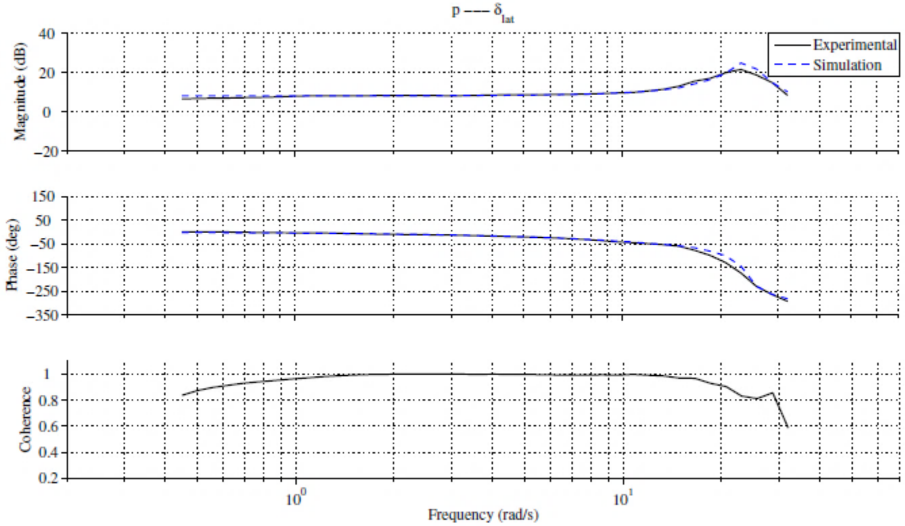


Fig. 6.15 Response comparison using frequency-sweep input (δ_{lat} — p)

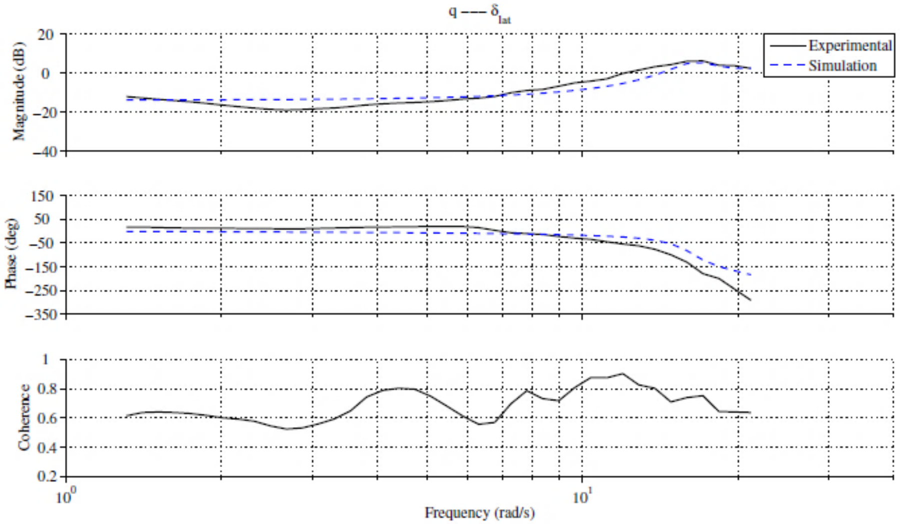


Fig. 6.16 Response comparison using frequency-sweep input (δ_{lat} — q)

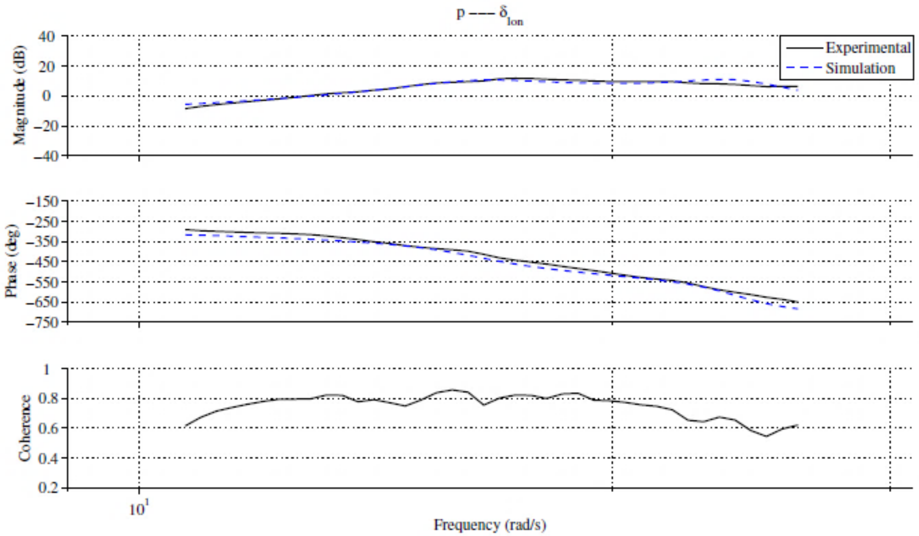


Fig. 6.17 Response comparison using frequency-sweep input (δ_{lon} — p)

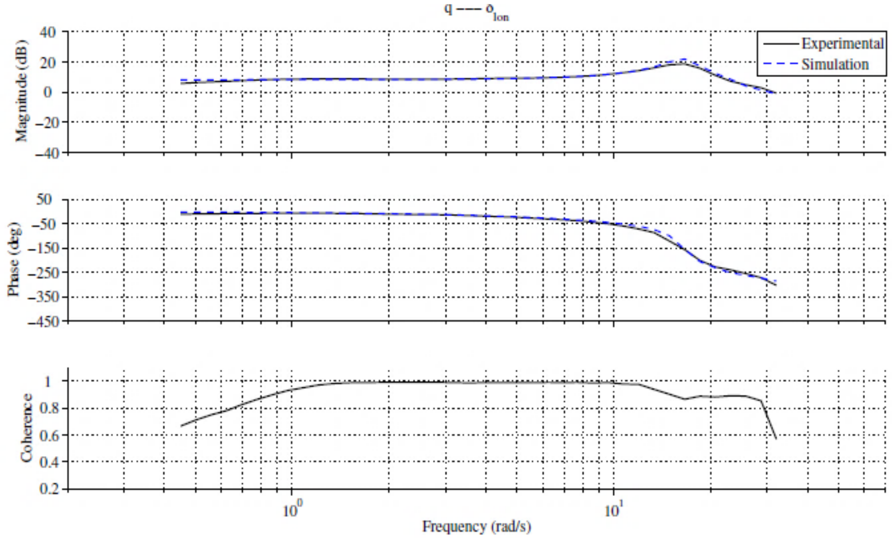


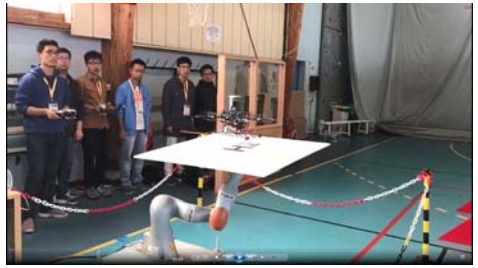
Fig. 6.18 Response comparison using frequency-sweep input (δ_{lon} — q)

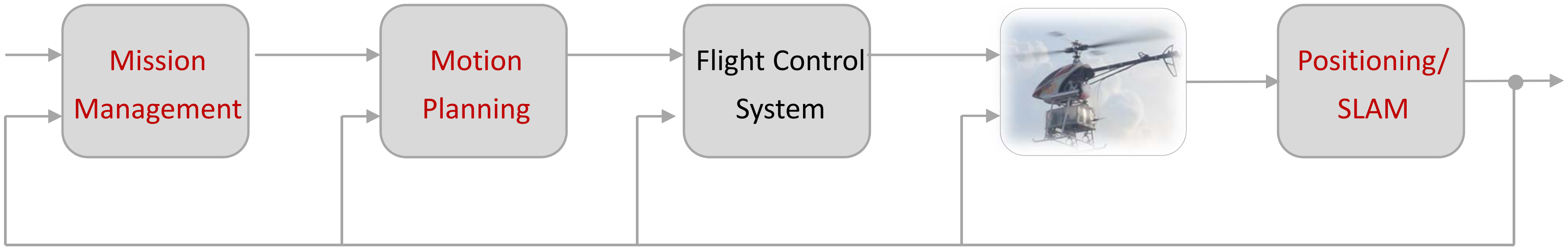
Unmanned Helicopter Control System

A Hybrid UAV Control System

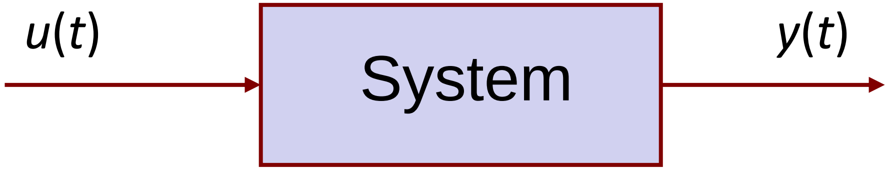
Auto-landing on moving platform…

NUS Team Instinct Cougar





Linear Systems



Let $y_1(t)$ be the output produced by an input signal $u_1(t)$ and $y_2(t)$ be the output produced by another input signal $u_2(t)$. Then, the system is said to be linear if

a) the input is $\alpha\, u_1(t)$, the output is $\alpha\, y_1(t)$, where α is a scalar; and

b) the input is $u_1(t) + u_2(t)$, the output is $y_1(t) + y_2(t)$.

Or equivalently, the input is $\alpha\, u_1(t) + \beta\, u_2(t)$, the output is $\alpha\, y_1(t) + \beta\, y_2(t)$. Such a property is called ***superposition***. For the circuit example on the previous page,

$$y(t) = \frac{R_2}{R_1 + R_2} \cdot \left[\alpha\, u_1(t) + \beta\, u_2(t)\right] = \alpha \frac{R_2}{R_1 + R_2} u_1(t) + \beta \frac{R_2}{R_1 + R_2} u_2(t) = \alpha\, y_1(t) + \beta\, y_2(t)$$

It is a linear system! We will mainly focus on linear systems in this course.

Summary of Laplace transform properties

Property	$f(t)$	$F(s)$
Linearity	$a_1 f_1(t) + a_2 f_2(t)$	$a_1 F_1(s) + a_2 F_2(s)$
Scaling	$f(at)$	$\frac{1}{a} F(\frac{s}{a})$
Time shift	$f(t-a)u(t-a)$	$e^{-as} F(s)$
Frequency shift	$e^{-at} f(t)$	$F(s+a)$
Time derivative	$\frac{d^n f(t)}{dt^n}$	$s^n F(s) - s^{n-1} f(0^-) - s^{n-2} f'(0^-) - \ldots - s^0 f^{(n-1)}(0^-)$
Time integration	$\int_0^t f(\zeta) d\zeta$	$\frac{1}{s} F(s)$
Time periodicity	$f(t) = f(t+nT)$	$\frac{F_1(s)}{1-e^{-sT}}$
Initial value	$f(0^-)$	$\lim_{s\to\infty} [sF(s)]$
Final value	$f(\infty)$	$\lim_{s\to 0} [sF(s)]$
Convolution	$f_1(t) \otimes f_2(t)$	$F_1(s) F_2(s)$

Some commonly used Laplace transform pairs

$f(t)$	$\Leftrightarrow$	$F(s)$
$\delta(t)$	$\Leftrightarrow$	1
$1(t)$	$\Leftrightarrow$	$\dfrac{1}{s}$
t	$\Leftrightarrow$	$\dfrac{1}{s^2}$
t^n	$\Leftrightarrow$	$\dfrac{n!}{s^{n+1}}$
e^{-at}	$\Leftrightarrow$	$\dfrac{1}{s+a}$
te^{-at}	$\Leftrightarrow$	$\dfrac{1}{(s+a)^2}$

$f(t)$	$\Leftrightarrow$	$F(s)$
$\sin \omega t$	$\Leftrightarrow$	$\dfrac{\omega}{s^2+\omega^2}$
$\cos \omega t$	$\Leftrightarrow$	$\dfrac{s}{s^2+\omega^2}$
$\sin(\omega t+\theta)$	$\Leftrightarrow$	$\dfrac{s\sin\theta+\omega\cos\theta}{s^2+\omega^2}$
$\cos(\omega t+\theta)$	$\Leftrightarrow$	$\dfrac{s\cos\theta-\omega\sin\theta}{s^2+\omega^2}$
$e^{-at}\sin \omega t$	$\Leftrightarrow$	$\dfrac{\omega}{(s+a)^2+\omega^2}$
$e^{-at}\cos \omega t$	$\Leftrightarrow$	$\dfrac{s+a}{(s+a)^2+\omega^2}$

Operations of complex numbers

Coordinates: Cartesian Coordinate and Polar Coordinate

$$12 + j5 = 13\, e^{j0.39} = \sqrt{12^2 + 5^2}\; e^{j \tan^{-1}\left(\frac{5}{12}\right)}$$

real part imaginary part magnitude argument

Euler's Formula: $\boxed{e^{j\theta} = \cos\theta + j\sin\theta}$

Additions: It is easy to do additions (subtractions) in Cartesian coordinate.

$$\boxed{(a + jb) + (v + jw) = (a + v) + j(b + w)}$$

Multiplication's: It is easy to do multiplication's (divisions) in Polar coordinate.

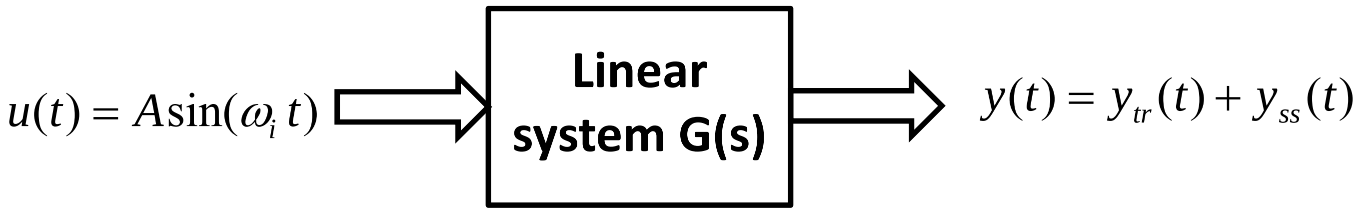
$$\boxed{re^{j\theta} \cdot ue^{j\omega} = (ru)e^{j(\theta+\omega)}} \qquad \boxed{\frac{re^{j\theta}}{ue^{j\omega}} = \frac{r}{u}\, e^{j(\theta-\omega)}}$$

Frequency Responses

Frequency Response

The **frequency response** of a system is defined as the steady-state response of the system to a sinusoidal input signal.

Ref: Modern Control Systems, Richard C. Dorf & Robert H. Bishop



When a stable linear system is subject to sinusoidal input of frequency ω_i rad/s

the output and all intermediate signals are sinusoidal of frequency ω_i rad/s in the steady state

the output and the intermediate signals differ from the input in amplitude and phase.

$$y_{ss}(t) = B\sin(\omega_i t + \phi)$$

Frequency Response Example

Example: Response of a 1st-order system to sinusoidal input

$$G(s)=\frac{1}{s+1}, \quad u(t)=\sin(10t) \quad \Rightarrow \quad U(s)=\frac{10}{s^2+100}$$

Solution:
$$Y(s)=\frac{1}{(s+1)}\frac{10}{(s^2+100)} = \frac{A}{(s+1)}+\frac{Bs+C}{(s^2+100)}$$

$$10 = A(s^2+100)+Bs(s+1)+C(s+1) = (A+B)s^2+(B+C)s+(100A+C)$$

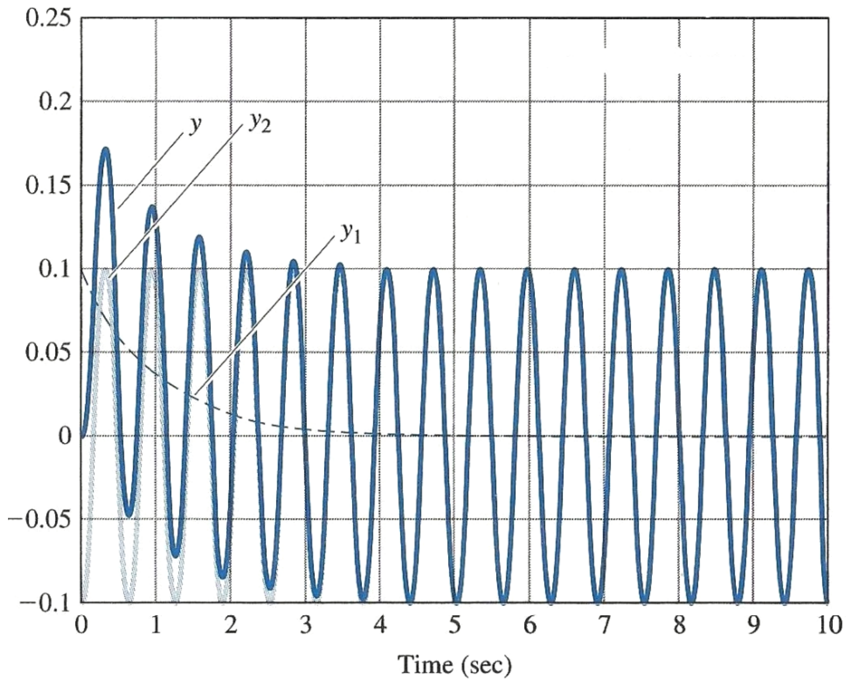
$$A+B=0, \quad B+C=0, \quad 100A+C=10$$

$$A=\frac{10}{101}, \quad B=-\frac{10}{101}, \quad C=\frac{10}{101}$$

$$Y(s)=\frac{10/101}{(s+1)}-\frac{10}{101}\frac{s}{(s^2+10^2)}+\frac{1}{101}\frac{10}{(s^2+10^2)}$$

$$y(t)=\frac{10}{101}e^{-t}-\frac{10}{101}\cos(10t)+\frac{1}{101}\sin(10t)$$

$$y(t)=\frac{10}{101}e^{-t}+\frac{1}{\sqrt{101}}\sin(10t-84°)$$

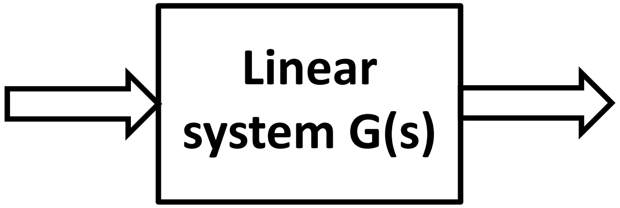


$$y_1=\frac{10}{101}e^{-t}$$

$$y_2=\frac{1}{\sqrt{101}}\sin(10t-84°)$$

Frequency Response

$u(t) = A\sin(\omega_i t)$



$y(t) = y_{tr}(t) + y_{ss}(t)$

$y_{ss}(t) = B\sin(\omega_i t + \phi)$

Gain of the system: $M = \dfrac{B}{A}$ [amplitude scaling]

Phase of the system: ϕ [shift in time]

Gain & phase vary with the frequency of the input sinusoid

$M(\omega)$ and $\phi(\omega)$ define the **frequency response** of the linear system

Frequency Response

$u(t) = A\sin(\omega_i t)$ ⟹ **Linear system G(s)** ⟹ $y(t) = y_{tr}(t) + y_{ss}(t)$

$$y_{ss}(t) = B\sin(\omega_i t + \phi)$$

Measurement of frequency response:

a) Apply a sinusoidal input to the system under test
b) Allow sufficient time for transient to decay
c) Measure amplitudes of the input and the output; ratio of amplitudes is the gain
d) Measure time-shift of the output with reference to the input and determine the phase
e) Repeat steps (a) to (d) for different frequencies $\omega_{\text{min}} < \omega_i < \omega_{\text{max}}$

ω	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	ω_7	ω_8	…..
Gain	M_1	M_2	M_3	M_4	M_5	M_6	M_7	M_8	….
Phase	ϕ_1	ϕ_2	ϕ_3	ϕ_4	ϕ_5	ϕ_6	ϕ_7	ϕ_8	….

In modern equipment, the measurement steps are automated

Frequency Response (cont.)

$G(s)$: Transfer function of a linear system

$M(\omega_i)$: Gain of the system at frequency ω_i rad/s

$\phi(\omega_i)$: Phase of the system at frequency ω_i rad/s

Then,

$$M(\omega_i) = |G(j\omega_i)|, \quad \phi(\omega_i) = \angle G(j\omega_i)$$

Example: Find the gain and the phase at ω = 4 rad/s for $G(s) = \dfrac{10}{s^2 + 3s + 9}$

$$G(j4) = \frac{10}{(j4)^2 + 3(j4) + 9} = \frac{10}{-7 + j12} = \frac{10}{\sqrt{7^2 + 12^2}\angle \tan^{-1}\left(\dfrac{12}{-7}\right)} = 0.72\angle -120.25^\circ$$

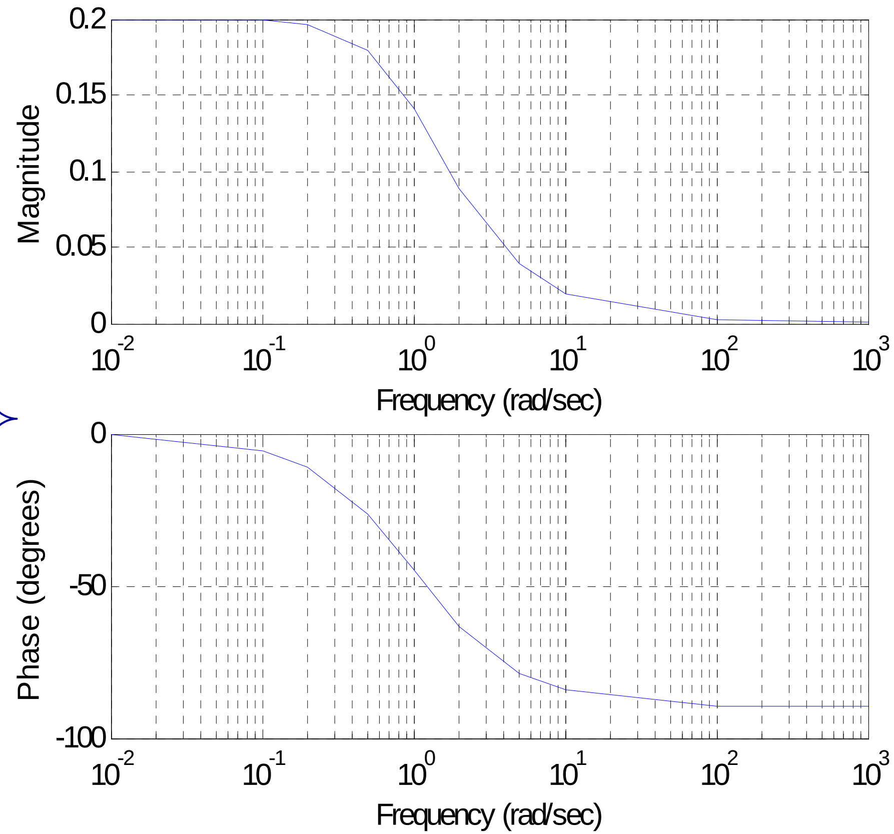
Example:

Let us a series RL circuit with the following transfer function:

$$G(s)=\frac{0.2}{s+1} \quad \Rightarrow \quad G(j\omega)=\frac{0.2}{j\omega+1}=|G(j\omega)|\cdot\angle G(j\omega)=\frac{0.2}{\sqrt{1+\omega^2}}\angle-\tan^{-1}\omega$$

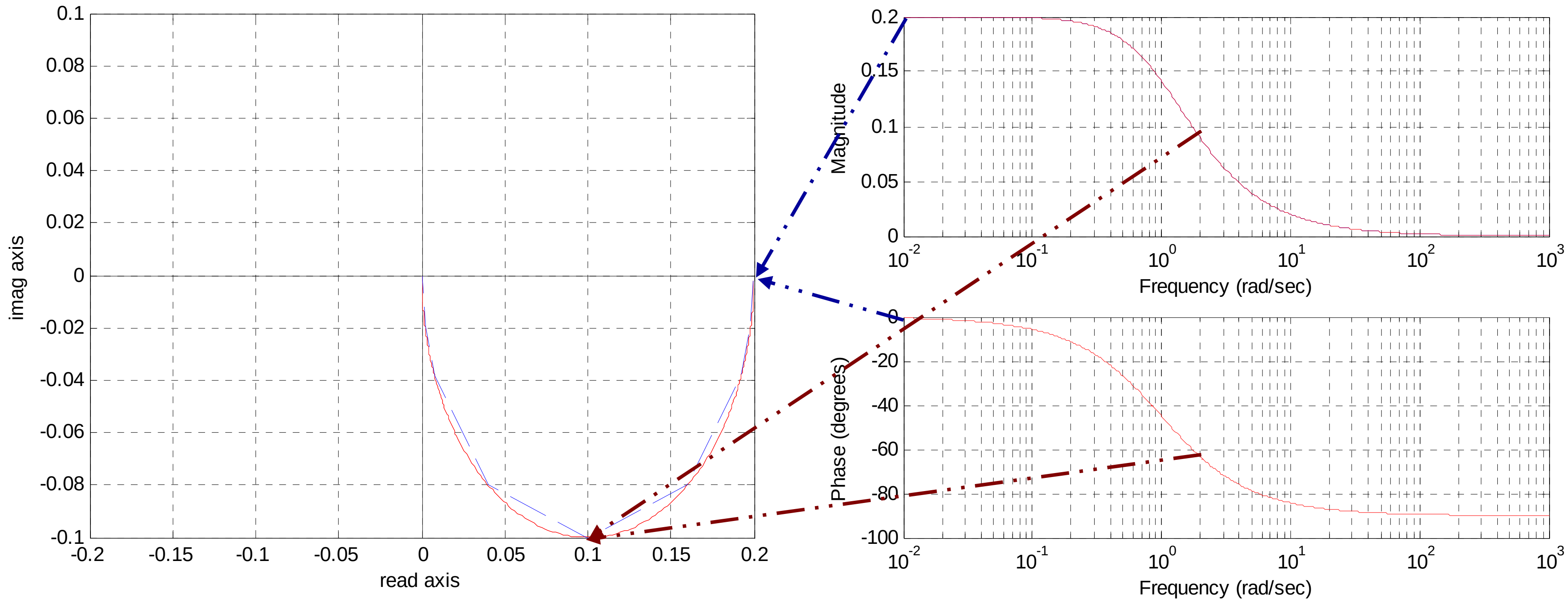
Thus, it is straightforward, but very tedious, to compute its amplitude and phase.

$$\omega=0.01 \quad \Rightarrow \quad |G(j\omega)|=0.2,\ \angle G(j\omega)=-0.57^\circ$$
$$\omega=0.1 \quad \Rightarrow \quad |G(j\omega)|=0.199,\ \angle G(j\omega)=-5.71^\circ$$
$$\omega=0.2 \quad \Rightarrow \quad |G(j\omega)|=0.196,\ \angle G(j\omega)=-11.31^\circ$$
$$\omega=0.5 \quad \Rightarrow \quad |G(j\omega)|=0.179,\ \angle G(j\omega)=-26.57^\circ$$
$$\omega=1 \quad \Rightarrow \quad |G(j\omega)|=0.141,\ \angle G(j\omega)=-45^\circ$$
$$\omega=2 \quad \Rightarrow \quad |G(j\omega)|=0.089,\ \angle G(j\omega)=-63.43^\circ$$
$$\omega=5 \quad \Rightarrow \quad |G(j\omega)|=0.039,\ \angle G(j\omega)=-78.69^\circ$$
$$\omega=10 \quad \Rightarrow \quad |G(j\omega)|=0.020,\ \angle G(j\omega)=-84.29^\circ$$
$$\omega=100 \quad \Rightarrow \quad |G(j\omega)|=0.002,\ \angle G(j\omega)=-89.43^\circ$$
$$\omega=1000 \quad \Rightarrow \quad |G(j\omega)|=0.0002,\ \angle G(j\omega)=-89.94^\circ$$



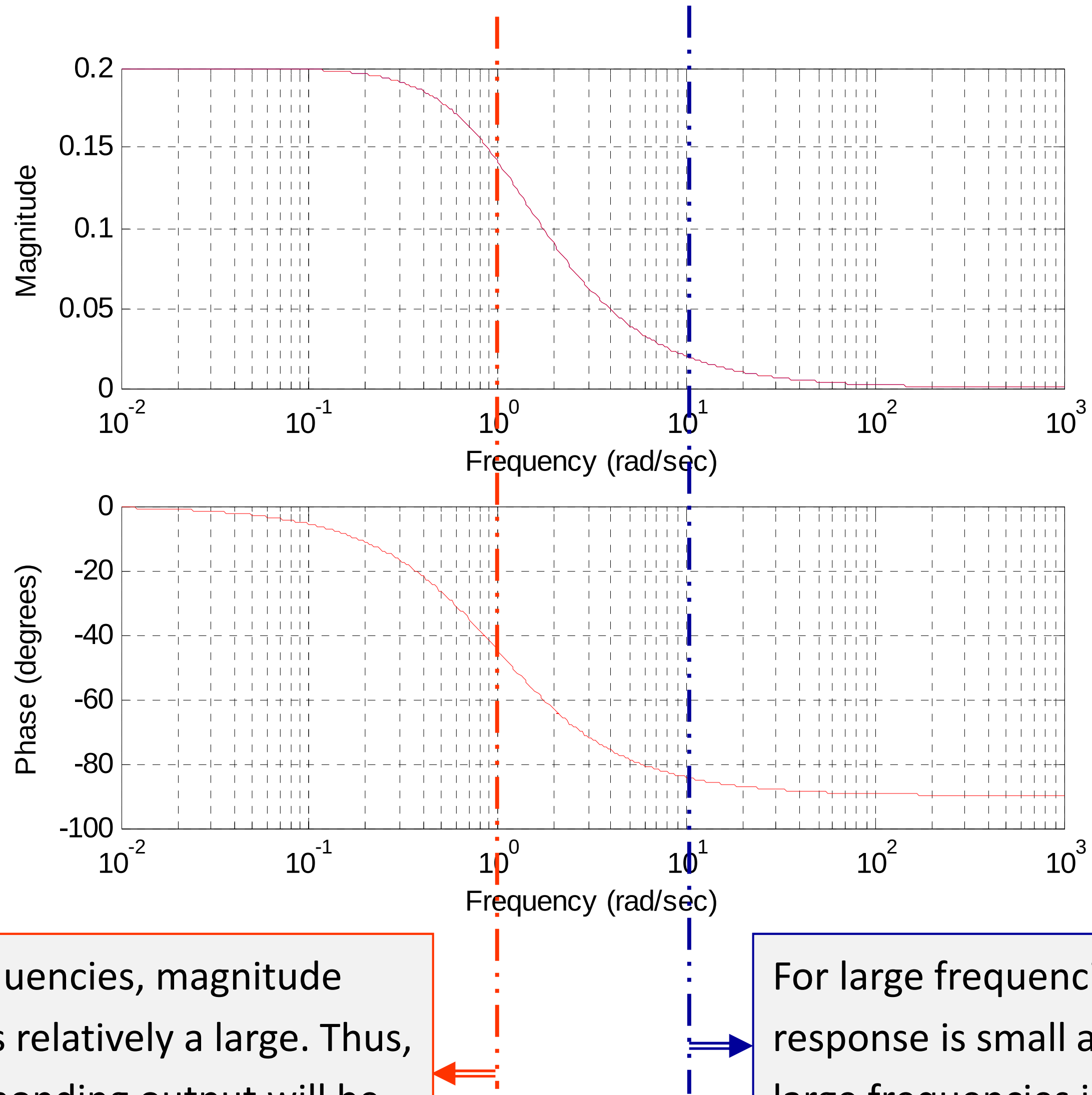
Note that in the plots of the magnitude and phase responses, we use a log scale for the frequency axis. If we draw in a normal scale, the responses will look awful.

There is another way to draw the frequency response. i.e., directly draw both magnitude and phase on a complex plane, which is called the polar plot. For the example considered, its polar plot is given as follows:



The red-line curves are more accurate plots using MATLAB.

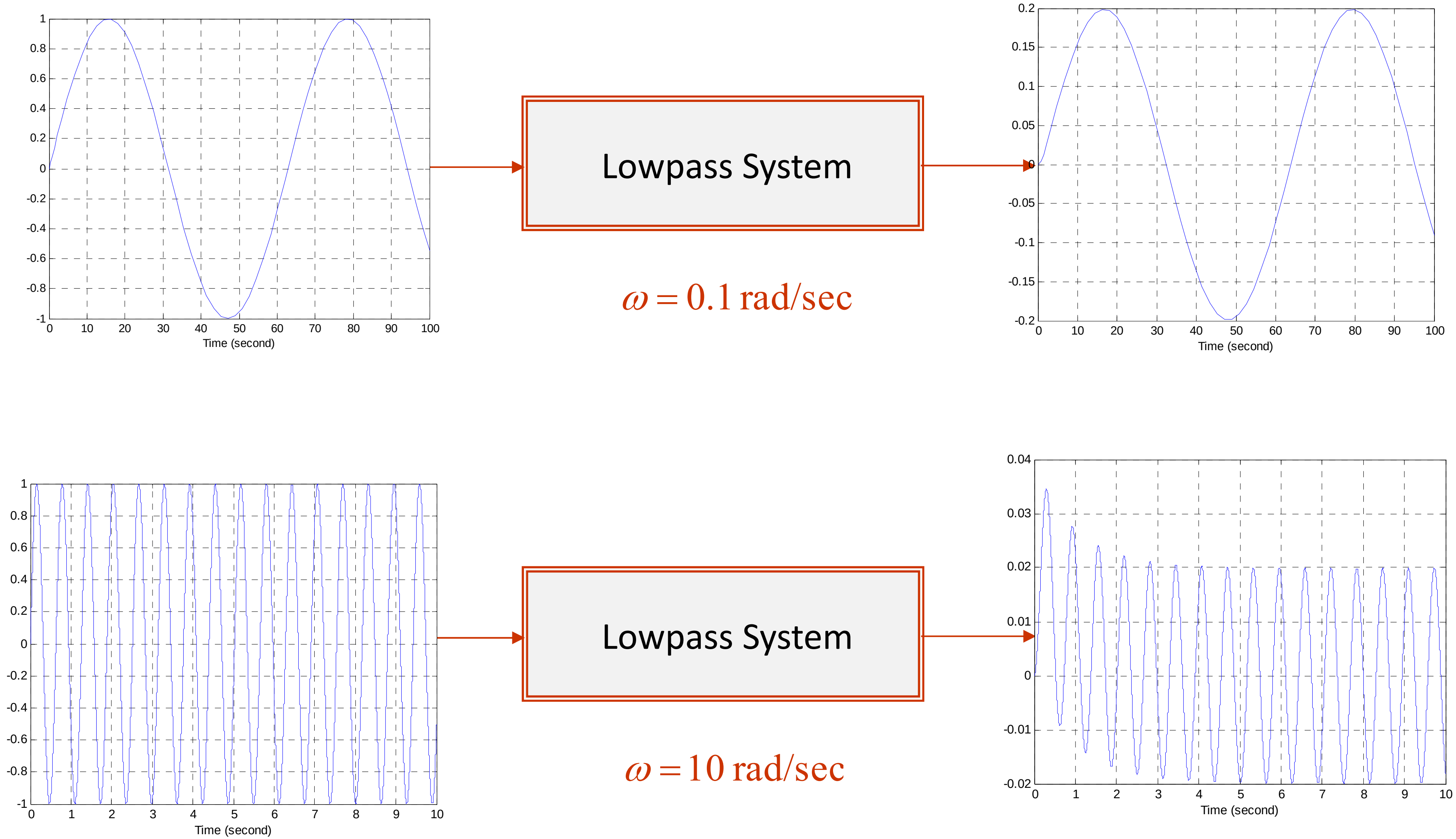
What can we observe from the frequency response?



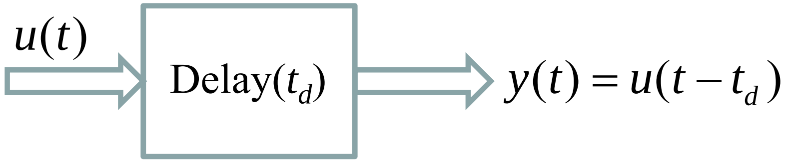
At low frequencies, magnitude response is relatively a large. Thus, the corresponding output will be large.

For large frequencies, the magnitude response is small and thus signals with large frequencies is attenuated or blocked.

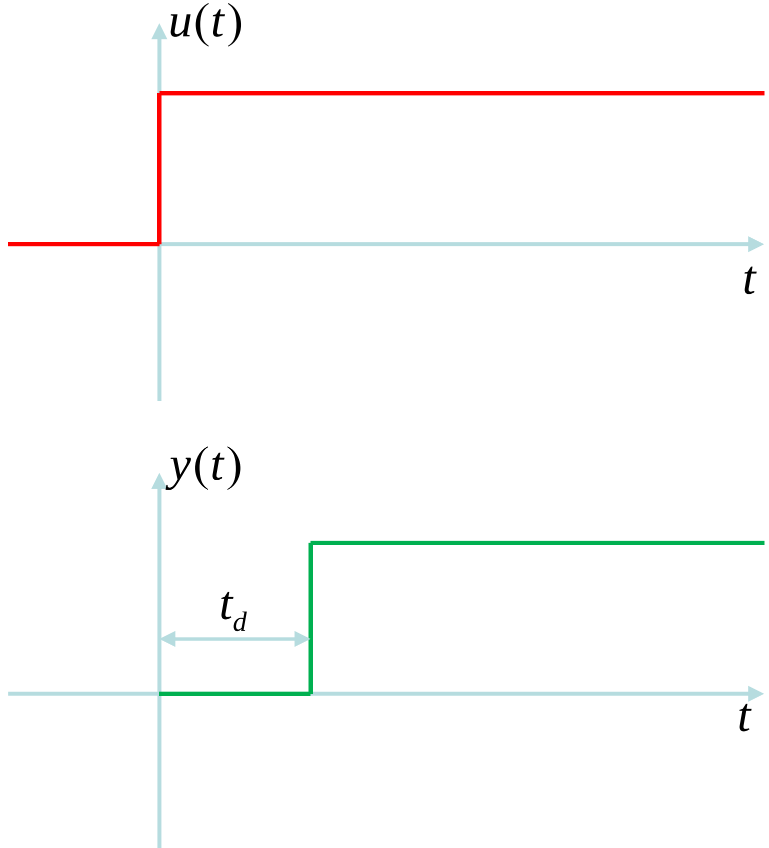
Lowpass Systems



Time Delay



Element causing delay of t_d seconds



$$u(t) \underset{L.T.}{\Rightarrow} U(s)$$

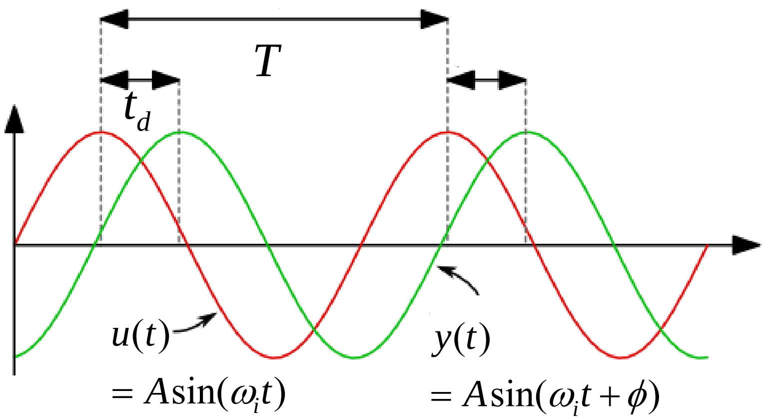
$$u(t - t_d) \underset{L.T.}{\Rightarrow} e^{-st_d} U(s)$$

$$G_{\text{delay}}(s) = \frac{Y(s)}{U(s)} = \frac{e^{-st_d} U(s)}{U(s)} = e^{-st_d}$$

$$G_{\text{delay}}(j\omega) = e^{-j\omega t_d}$$

$$\Rightarrow \begin{cases} M = \left| e^{-j\omega t_d} \right| = 1 \\ \phi = -\omega t_d \end{cases}$$

Time Delay



No difference between amplitudes of two signals

$$M = 1$$

The phase of the delay unit is ϕ rad

$$y(t) = A\sin(\omega_i t + \phi) = A\sin(\omega_i\left(t + \frac{\phi}{\omega_i}\right))$$

$$\Longrightarrow \quad t_d = -\frac{\phi}{\omega_i} \quad \Rightarrow \quad \phi = -\omega_i t_d = -2\pi f t_d = -\frac{2\pi t_d}{T}$$

Phase of delay unit is $-\omega_i t_d \text{ rad} = -\omega_i t_d \left(\frac{180^\circ}{\pi}\right) \text{degree}$

Exercise 1-1

A sine wave of unity amplitude and frequency 5 Hz is applied to a linear system described by the transfer function

$$G(s)=\frac{(s+10)}{(s+1)(s+50)}$$

What is the steady-state output?

Solution:

$$G(s)=\frac{s+10}{(s+1)(s+50)},\quad u(t)=\sin(2\pi\, f\, t)=\sin(10\pi t)\quad\Rightarrow\quad U(s)=\frac{10\pi}{s^2+100\pi^2}$$

$$Y(s)=G(s)U(s)=\frac{s+10}{(s+1)(s+50)}\cdot\frac{10\pi}{s^2+100\pi^2}=\frac{A}{s+1}+\frac{B}{s+50}+\frac{Cs+D}{s^2+100\pi^2}$$

$$A=\left.\frac{s+10}{s+50}\cdot\frac{10\pi}{s^2+100\pi^2}\right|_{s=-1}=\frac{9}{49}\cdot\frac{10\pi}{1+100\pi^2}=0.00584059$$

$$B=\left.\frac{s+10}{s+1}\cdot\frac{10\pi}{s^2+100\pi^2}\right|_{s=-50}=\frac{-40}{-49}\cdot\frac{10\pi}{2500+100\pi^2}=0.00735473$$

$$Y(s)=G(s)U(s)=\frac{s+10}{(s+1)(s+50)}\cdot\frac{10\pi}{s^2+100\pi^2}=\frac{0.00584059}{s+1}+\frac{0.00735473}{s+50}+\frac{Cs+D}{s^2+100\pi^2}$$

$$=\frac{0.00584059(s+50)(s^2+100\pi^2)+0.00735473\,(s+1)(s^2+100\pi^2)+(s+1)(s+50)(Cs+D)}{(s+1)(s+50)(s^2+100\pi^2)}$$

$$=\frac{(0.00584059+0.00735473+C)s^3+\cdots+(0.00584059\times 5000\pi^2+0.00735473\times 100\pi^2+50D)}{(s+1)(s+50)(s^2+100\pi^2)}$$

$$\Rightarrow \quad 0.00584059+0.00735473+C=0 \quad \Rightarrow \quad C=-0.01319532$$

$$\Rightarrow \quad 0.00584059\times 5000\pi^2+0.00735473\times 100\pi^2+50D=100\pi \quad \Rightarrow \quad D=0.37357712$$

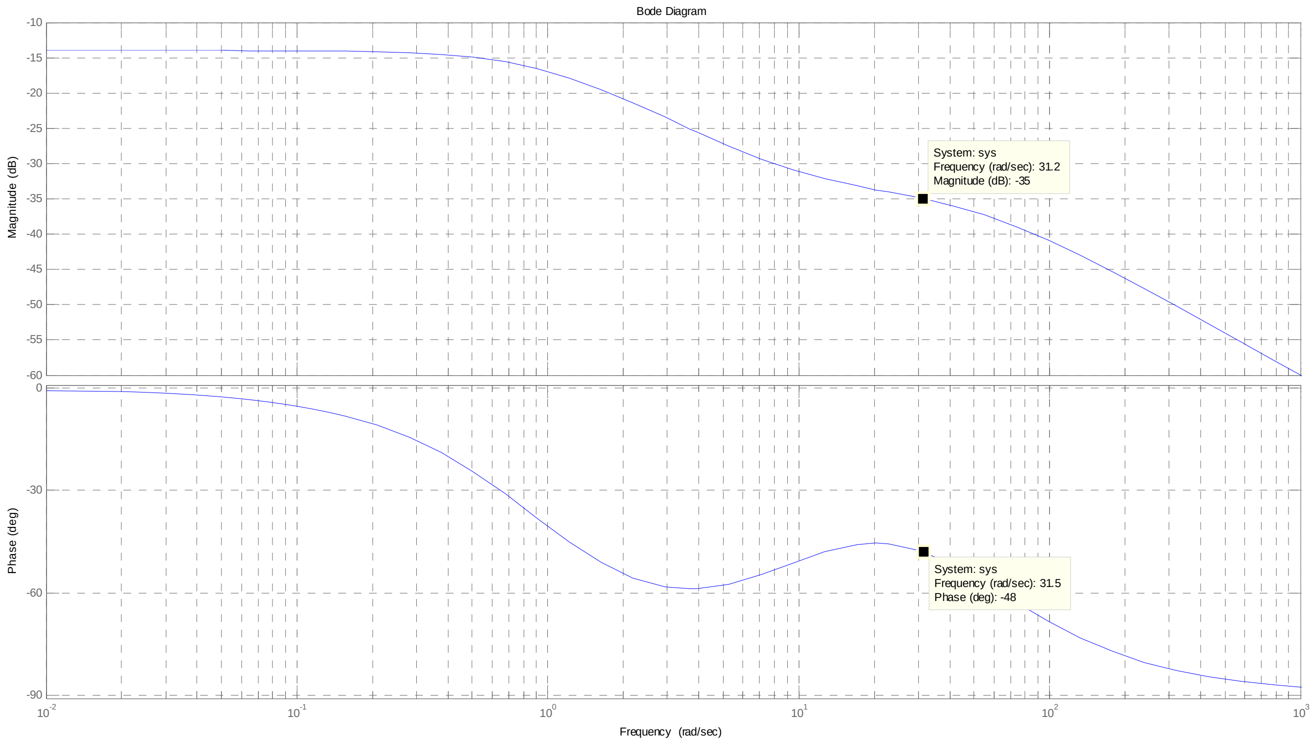
$$Y(s)=\frac{0.00584059}{s+1}+\frac{0.00735473}{s+50}-\frac{0.01319532s}{s^2+100\pi^2}+\frac{0.37357712}{s^2+100\pi^2}$$

$$y(t)=0.00584059e^{-t}+0.00735473e^{-50t}-0.01319532\cos(10\pi t)+0.01189133\sin(10\pi t)$$
$$=0.00584059e^{-t}+0.00735473e^{-50t}+\underline{0.01776289}\sin(10\pi t\underline{-48^\circ})$$

Alternatively, we calculate

$$G(j10\pi) = \frac{j10\pi + 10}{50 - (10\pi)^2 + j51 \times 10\pi} = \frac{32.9691\angle 72.34°}{1856.065\angle 120.32°} = 0.01776289\angle -48°$$

$$\Rightarrow \quad y_{ss}(t) = 0.01776289 \sin(10\pi t - 48°)$$



It is even easier to use MATLAB…

Gain-crossover frequency

For any system, there may exist one or more frequencies at which the gain of the system is unity (1). Such a frequency is called the **gain-crossover frequency**. We'll use the symbol ω_{cg} to represent it, i.e.,

$$\left|G(j\omega_{cg})\right| = 1$$

Example: Find the gain-crossover frequency ω_{cg} of $G(s) = \dfrac{10}{s^2 + 3s + 9}$

Solution:

$$\left|G(j\omega_{cg})\right| = \left|\frac{10}{(j\omega_{cg})^2 + 3j\omega_{cg} + 9}\right| = \left|\frac{10}{9 - \omega_{cg}^2 + j3\omega_{cg}}\right| = \frac{10}{\sqrt{\left(9 - \omega_{cg}^2\right)^2 + \left(3\omega_{cg}\right)^2}} = 1$$

$$\Rightarrow \quad \sqrt{\left(9 - \omega_{cg}^2\right)^2 + \left(3\omega_{cg}\right)^2} = 10 \quad \Rightarrow \quad \left(9 - \omega_{cg}^2\right)^2 + \left(3\omega_{cg}\right)^2 = 100$$

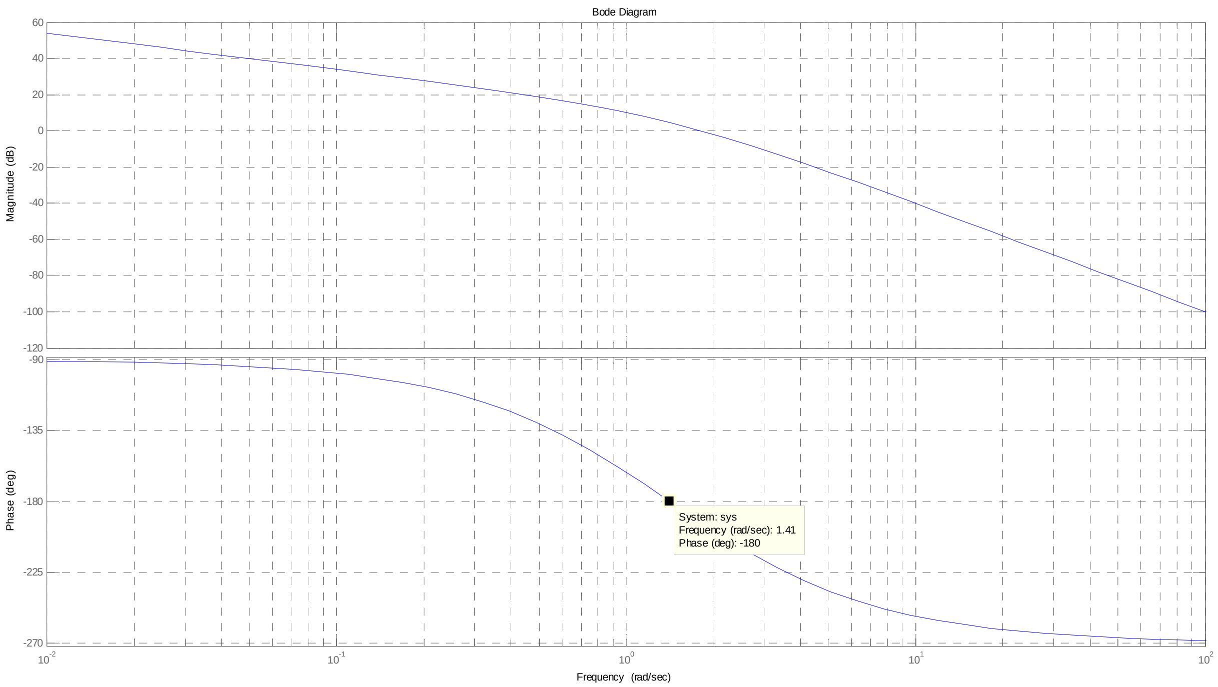
$$\Rightarrow \quad \left(\omega_{cg}^2\right)^2 - 9\omega_{cg}^2 - 19 = 0 \quad \Rightarrow \quad \omega_{cg}^2 = 10.765 \quad \Rightarrow \quad \omega_{cg} = 3.281 \text{ rad}$$

Phase-crossover frequency

For any system, there may exist one or more frequencies at which the phase of the system is ±180°. Such a frequency is called the **phase-crossover frequency**. We'll use the symbol $\boldsymbol{\omega_{cp}}$ to represent it, i.e.,

$$\angle G(j\omega_{cp}) = \pm 180^\circ$$

Exercise 1-3: Find the phase-crossover frequency ω_{cp} of $G(s) = \dfrac{10}{s(s+1)(s+2)}$



Frankly, it is really troublesome to calculate the crossover frequencies (gain and frequency) using pen and paper…

Exercise 1-2

Find the gain-crossover frequency of the following transfer function

$$G(s) = 100\frac{(s+10)}{(s+1)(s+50)}$$

Use MATLAB to verify your result.

Solution:

$$|G(j\omega)|^2 = 100^2 \frac{|j\omega+10|^2}{|j\omega+1|^2 \cdot |j\omega+50|^2}$$

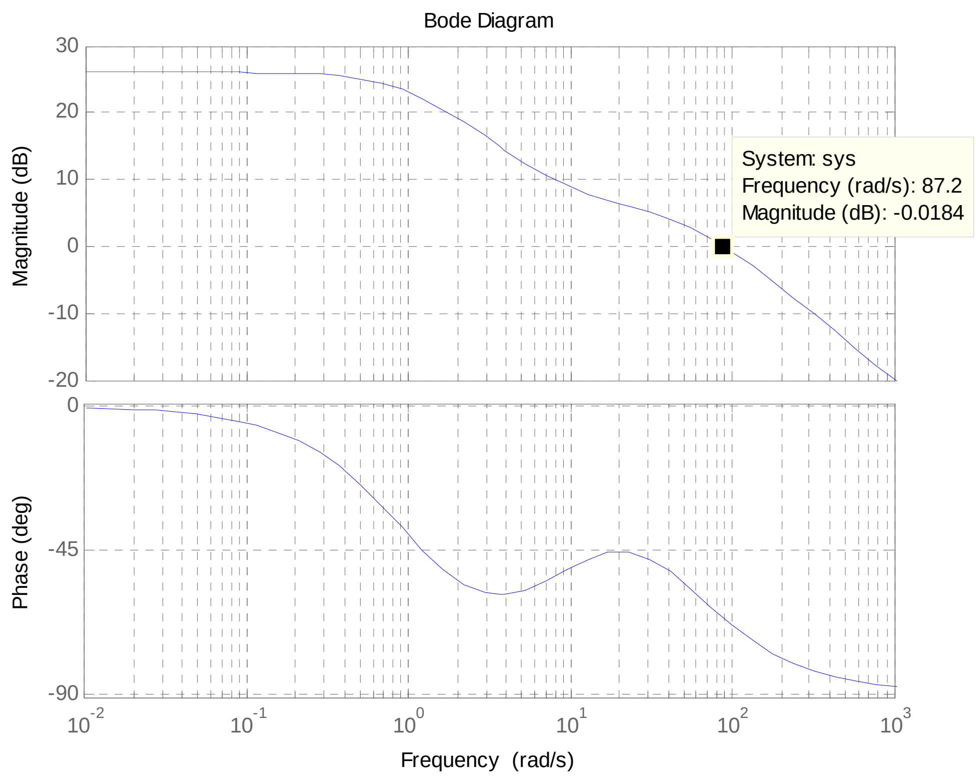
$$= 100^2 \frac{\omega^2+10^2}{(\omega^2+1)\cdot(\omega^2+50^2)}$$

$$= 1$$

$$\Rightarrow \quad (\omega^2)^2 - 7499\omega^2 - 997500 = 0$$

$$\Rightarrow \quad \omega^2 = 7629.7384$$

$$\Rightarrow \quad \omega = 87.35 \ \text{ rad/s}$$



Key Concepts Learnt

- Frequency response is defined for linear systems only
- If input to a stable linear system is sinusoidal of frequency ω_i, then its output in the steady-state is also sinusoidal of frequency ω_i
  - Input an output may differ in amplitude and phase
- Gain of the linear system is the ratio between the output (steady-state) amplitude and the input amplitude
  - Gain varies with frequency
- Phase defines the delay between the input sinusoid and the output sinusoid
  - Phase varies with frequency
- If a system's transfer function is $G(s)$, then its gain and phase at ω_i are

$$M(\omega_i) = |G(j\omega_i)|, \quad \phi(\omega_i) = \angle G(j\omega_i)$$

- For a transfer function $G(s)$, the crossover frequencies are defined as

$$|G(j\omega_{cg})| = 1, \quad \angle G(j\omega_{cp}) = \pm 180°$$

Bode plot

The plots of the magnitude and phase responses of a transfer function are called the Bode plot. The easiest way to draw Bode plot is to use MATLAB (i.e., bode function). However, there are some tricks that can help us to sketch Bode plots (approximation) without computing detailed values. To do this, we need to introduce a scale called dB (decibel). Given a positive scalar a, its decibel is defined as $20 \cdot \log_{10}(a)$. For example,

$$a = 1 \quad \Rightarrow \quad 20 \cdot \log_{10}(a) = 0 \quad \Rightarrow \quad a = 0 \text{ dB}$$

$$a = 10 \quad \Rightarrow \quad 20 \cdot \log_{10}(a) = 20 \quad \Rightarrow \quad a = 20 \text{ dB}$$

$$a = 100 \quad \Rightarrow \quad 20 \cdot \log_{10}(a) = 40 \quad \Rightarrow \quad a = 40 \text{ dB}$$

$$a = \alpha \cdot \beta \quad \Rightarrow \quad 20 \cdot \log_{10}(\alpha \cdot \beta) = 20 \cdot \log_{10}(\alpha) + 20 \cdot \log_{10}(\beta) \quad \Rightarrow \quad a = \alpha \text{ in dB} + \beta \text{ in dB}$$

$$a = {}^{\alpha}\!/_{\beta} \quad \Rightarrow \quad 20 \cdot \log_{10}({}^{\alpha}\!/_{\beta}) = 20 \cdot \log_{10}(\alpha) - 20 \cdot \log_{10}(\beta) \quad \Rightarrow \quad a = \alpha \text{ in dB} - \beta \text{ in dB}$$

In the dB scale, the product of two scalars becomes an addition and the division of two scalars becomes a subtraction.

Bode plot – an integrator

We start with finding the Bode plot asymptotes for a simple system characterized by

$$G(s)=\frac{1}{s} \quad \Rightarrow \quad G(j\omega)=\frac{1}{j\omega}=|G(j\omega)|\angle G(j\omega)=\frac{1}{|\omega|}\angle -90^\circ$$

Examining the amplitude in dB scale, i.e.,

$$20\cdot\log_{10}|G(j\omega)|=20\cdot\log_{10}\frac{1}{|\omega|}=-20\cdot\log_{10}|\omega| \text{ dB}$$

it is simple to see that

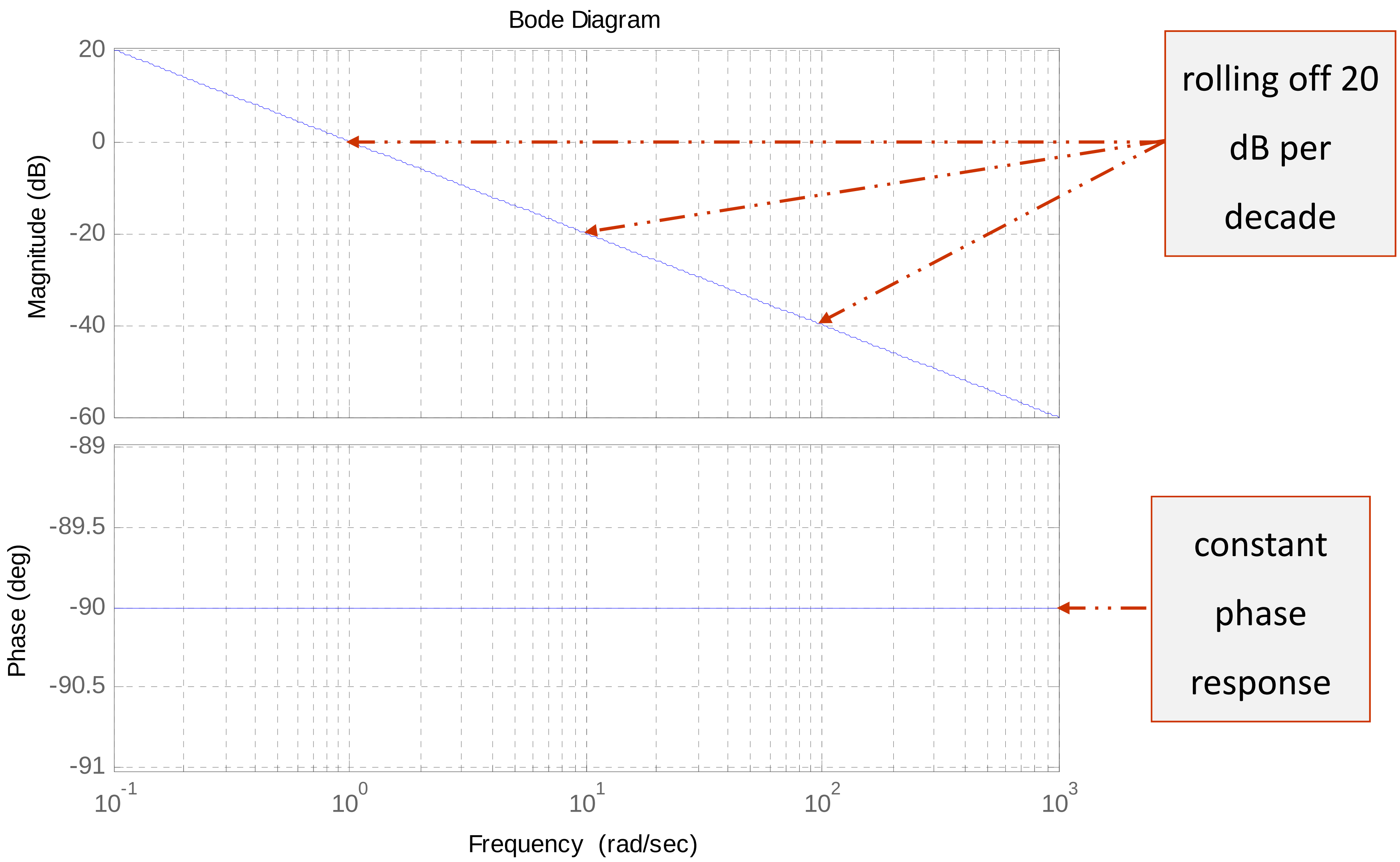
$$\omega=1 \quad \Rightarrow \quad 20\cdot\log_{10}|G(j1)|=-20\cdot\log_{10}1=0 \text{ dB}$$

$$\omega=10 \quad \Rightarrow \quad 20\cdot\log_{10}|G(j10)|=-20\cdot\log_{10}10=-20 \text{ dB}$$

$$\begin{aligned}\omega=\omega_2=10\omega_1 \quad \Rightarrow \quad 20\cdot\log_{10}|G(j\omega_2)| &= \underline{-20\cdot\log_{10}\omega_2} = -20\cdot\log_{10}10\omega_1 \\ &= -20\cdot\log_{10}10-20\cdot\log_{10}\omega_1 = \underline{-20-20\cdot\log_{10}\omega_1 \text{ dB}}\end{aligned}$$

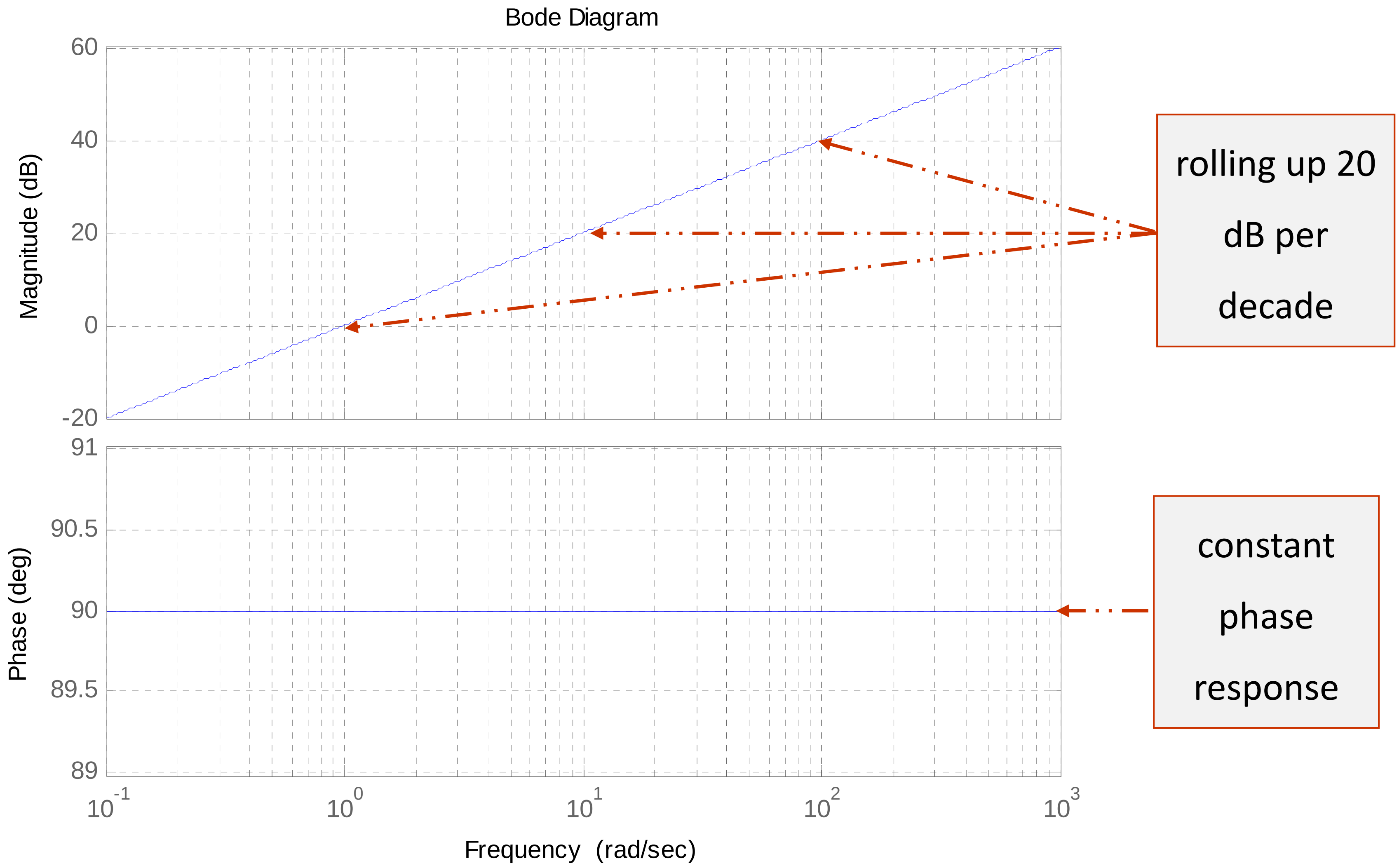
Thus, the above expressions clearly indicate that the magnitude is reduced by – 20 dB when the frequency is increased by 10 times. It is equivalent to say that the magnitude is rolling off 20 dB per decade.

The phase response of an integrator is – 90 degrees, a constant. The Bode plot of an integrator is given by



Bode plot – an differentiator

The Bode plot of $G(s) = s$ can be done similarly…



Bode plot – a general first order system

The Bode plot of a first order system characterized by a simple pole, i.e.,

$$G(s) = \frac{\omega_1}{s+\omega_1} = \frac{1}{1+ {}^{s}\!/_{\omega_1}} \quad \Rightarrow \quad G(j\omega) = \frac{1}{1+j\left({}^{\omega}\!/_{\omega_1}\right)} = \frac{1}{\sqrt{1+\left({}^{\omega}\!/_{\omega_1}\right)^2}} \angle -\tan^{-1}\left({}^{\omega}\!/_{\omega_1}\right)$$

Let us examine the following situations.

$$\omega = \omega_1 \quad \Rightarrow \quad G(j\omega_1) = \frac{1}{\sqrt{1+\left({}^{\omega_1}\!/_{\omega_1}\right)^2}} \angle -\tan^{-1}\left({}^{\omega_1}\!/_{\omega_1}\right) = 0.707\angle -45^\circ \quad (0.707 = -3\text{ dB})$$

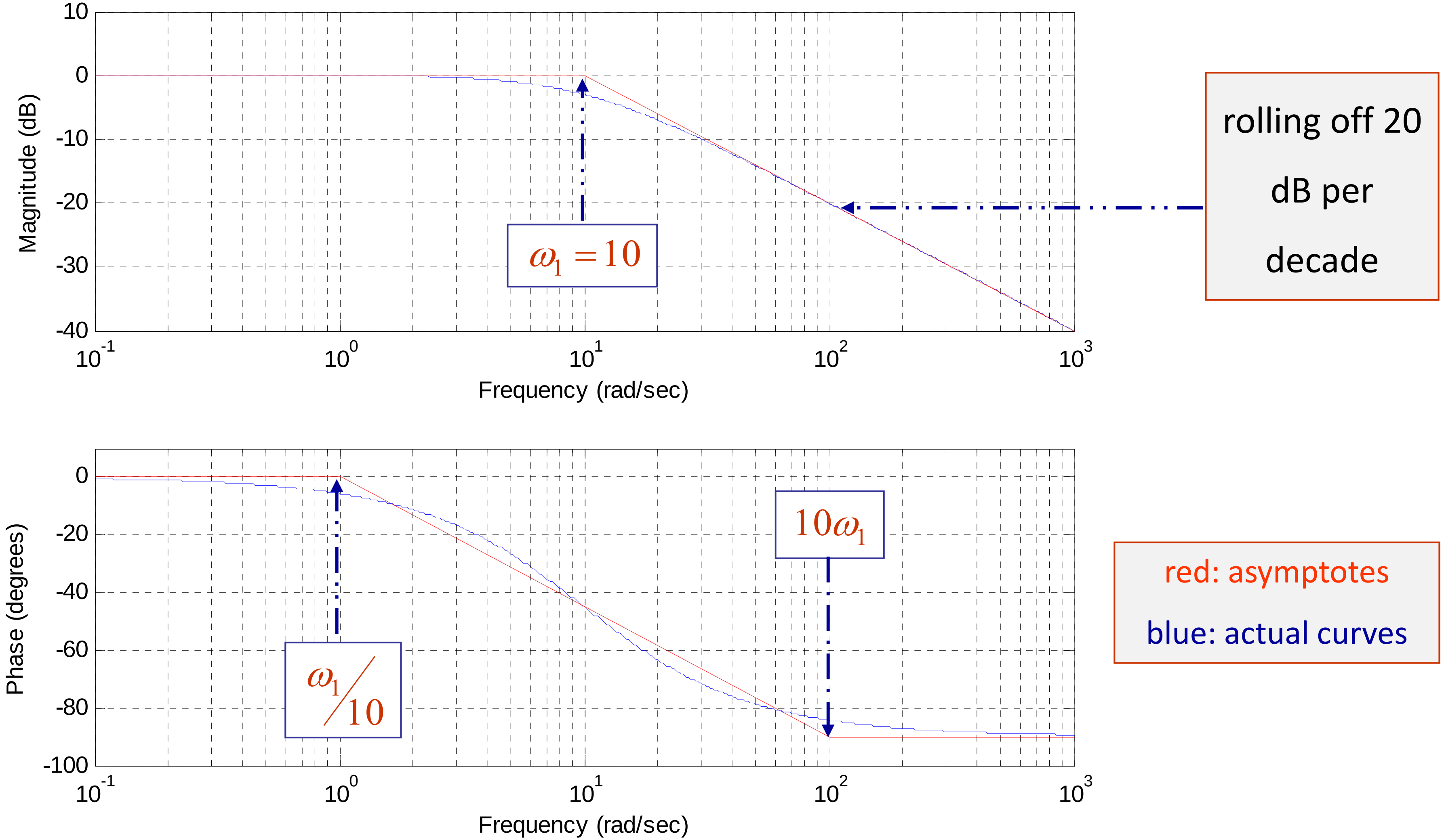
$$\omega << \omega_1 \quad \Rightarrow \quad G(j\omega) = \frac{1}{\sqrt{1+\left({}^{\omega}\!/_{\omega_1}\right)^2}} \angle -\tan^{-1}\left({}^{\omega}\!/_{\omega_1}\right) \approx 1\angle 0^\circ$$

$$\omega >> \omega_1 \quad \Rightarrow \quad G(j\omega) = \frac{1}{\sqrt{1+\left({}^{\omega}\!/_{\omega_1}\right)^2}} \angle -\tan^{-1}\left({}^{\omega}\!/_{\omega_1}\right) \approx \frac{\omega_1}{\omega} \angle -90^\circ$$

These give us the approximation (asymptotes) of the Bode curves…

Example

Consider a 1st order system $G(s) = \dfrac{1}{1 + s/10}$ where $\omega_1 = 10$ rad/sec is called the corner frequency.



Bode plot – a general first order unstable system

The Bode plot of a first order system characterized by a simple pole, i.e.,

$$G(s) = \frac{\omega_1}{s-\omega_1} = \frac{-1}{1-{s}/{\omega_1}} \quad \Rightarrow \quad G(j\omega) = \frac{-1}{1-j\left({\omega}/{\omega_1}\right)} = \frac{1}{\sqrt{1+\left({\omega}/{\omega_1}\right)^2}} \angle -180^\circ + \tan^{-1}\left({\omega}/{\omega_1}\right)$$

Let us examine the following situations.

$$\omega = \omega_1 \quad \Rightarrow \quad G(j\omega_1) = \frac{1}{\sqrt{1+\left({\omega_1}/{\omega_1}\right)^2}} \angle \tan^{-1}\left({\omega_1}/{\omega_1}\right) = 0.707 \angle -135^\circ \quad (0.707 = -3\text{ dB})$$

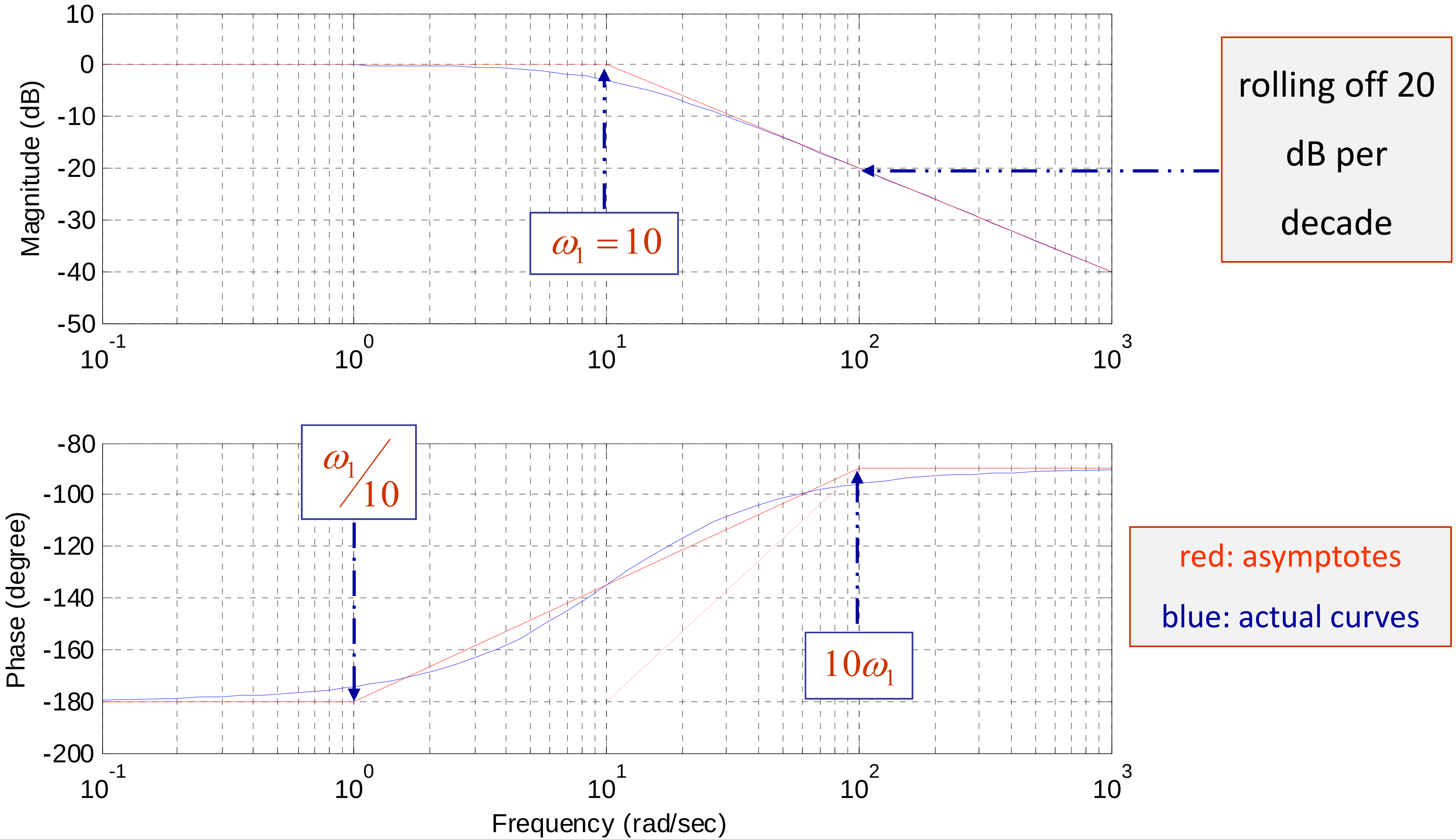
$$\omega << \omega_1 \quad \Rightarrow \quad G(j\omega) = \frac{1}{\sqrt{1+\left({\omega}/{\omega_1}\right)^2}} \angle -180^\circ + \tan^{-1}\left({\omega}/{\omega_1}\right) \approx 1 \angle -180^\circ$$

$$\omega >> \omega_1 \quad \Rightarrow \quad G(j\omega) = \frac{1}{\sqrt{1+\left({\omega}/{\omega_1}\right)^2}} \angle -180^\circ + \tan^{-1}\left({\omega}/{\omega_1}\right) \approx \frac{\omega_1}{\omega} \angle -90^\circ$$

These give us the approximation (asymptotes) of the Bode curves…

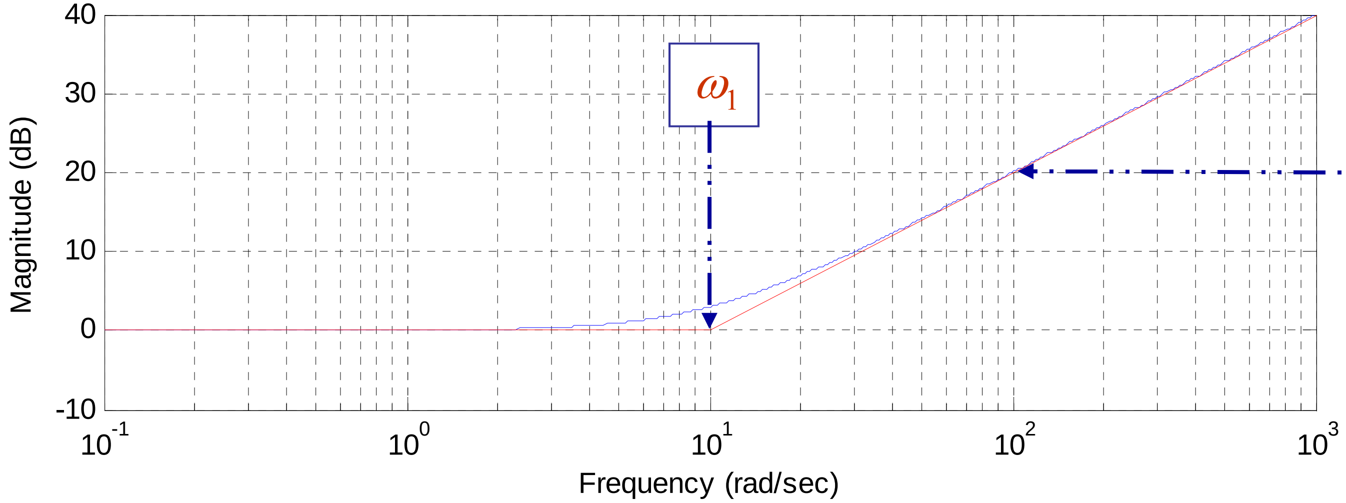
Example

Consider a 1st order system $G(s) = \dfrac{-1}{1 - s/10}$ where $\omega_1 = 10$ rad/sec is called the corner frequency.

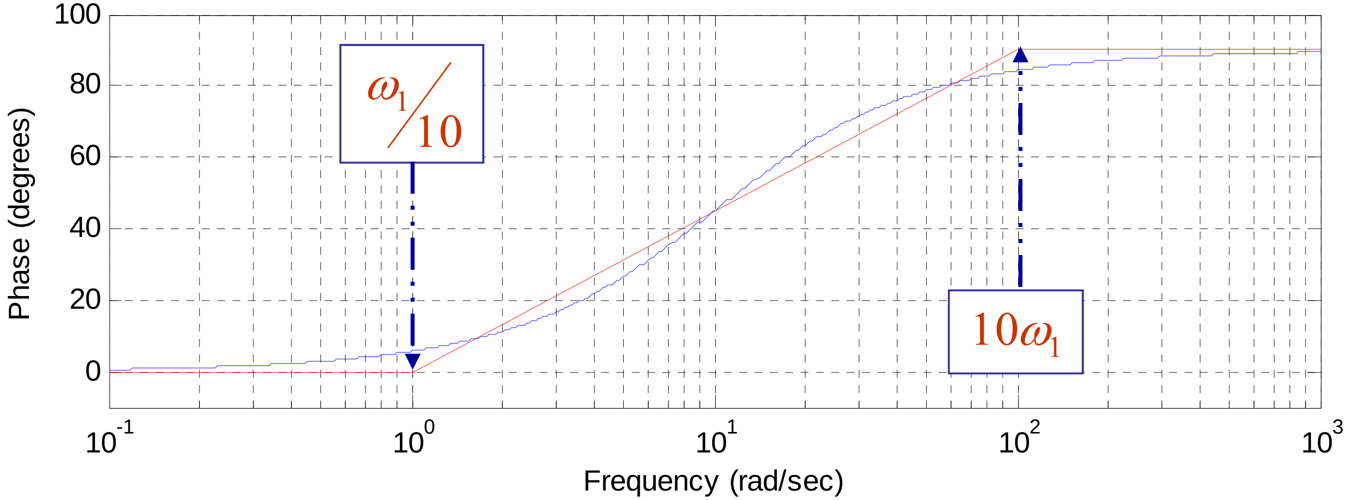


Bode plot – a simple zero factor

The Bode plot of $G(s) = 1 + {}^{s}\!/_{\omega_1}$ can be done similarly…



rolling up 20 dB per decade

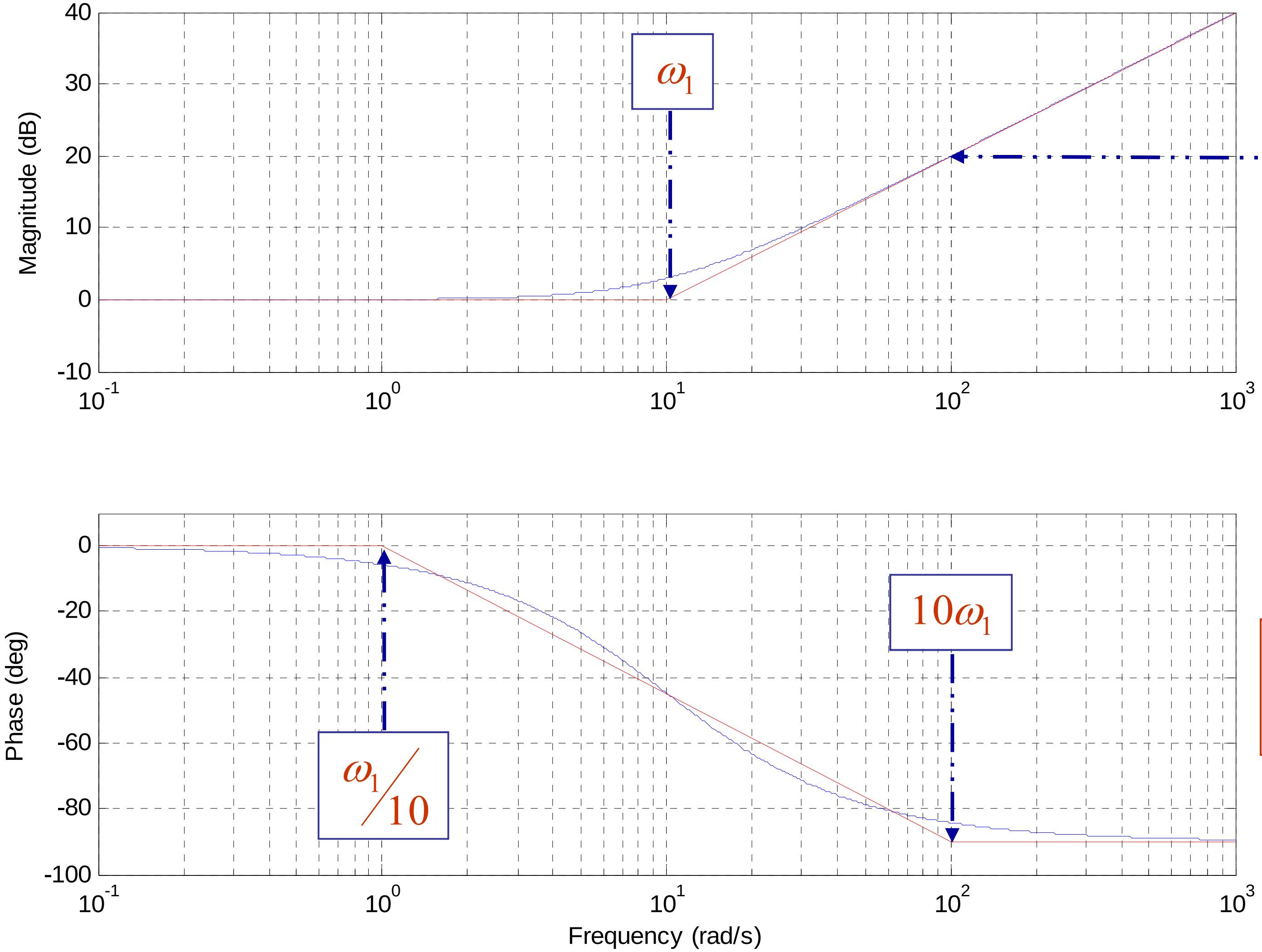


red: asymptotes

blue: actual curves

Bode plot – a simple unstable zero factor

The Bode plot of $G(s) = 1 - {s}/{\omega_1}$ can be done similarly…



rolling up 20 dB per decade

red: asymptotes

blue: actual curves

Bode plot – putting all together

Assume a given system has only simple poles and zeros, i.e.,

$$G(s)=\frac{b_m s^m+b_{m-1}s^{m-1}+\cdots+b_1 s+b_0}{s^n+a_{n-1}s^{n-1}+\cdots+a_1 s+a_0}=\frac{b_m(s+z_1)\cdots(s+z_m)}{(s+p_1)\cdots(s+p_m)}$$

or

$$\boxed{G(s)=\frac{b_m(s+z_1)\cdots(s+z_m)}{(s+p_1)\cdots(s+p_m)}=\frac{k(1+{}^{s}\!/_{z_1})\cdots(1+{}^{s}\!/_{p_{m_1}})}{s^q(1+{}^{s}\!/_{p_1})\cdots(1+{}^{s}\!/_{p_{n_1}})}}$$

In the dB scale, we have

$$\boxed{|G(j\omega)|\,\mathrm{dB}=|k|\text{ in dB}+\left|1+{}^{j\omega}\!/_{z_1}\right|\text{ in dB}+\cdots+\left|1+{}^{j\omega}\!/_{z_{m_1}}\right|\text{ in dB}-|\omega|\text{ in dB}\times q-\left|1+{}^{j\omega}\!/_{p_1}\right|\text{ in dB}-\cdots-\left|1+{}^{j\omega}\!/_{p_{n_1}}\right|\text{ in dB}}$$

Similarly,

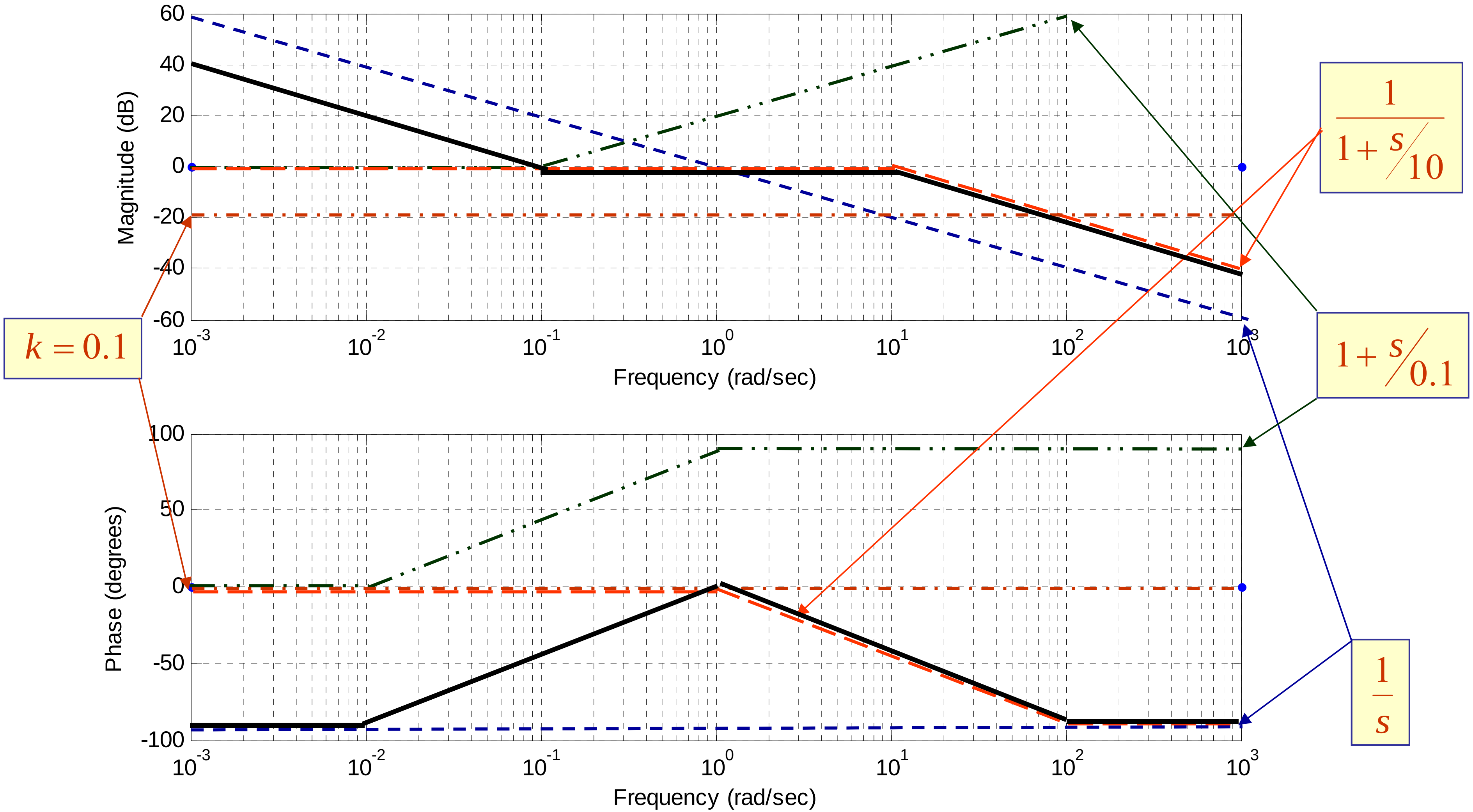
$$\boxed{\angle G(j\omega)=\angle k+\angle\left(1+{}^{j\omega}\!/_{z_1}\right)+\cdots+\angle\left(1+{}^{j\omega}\!/_{z_{m_1}}\right)-90^\circ\times q-\angle\left(1+{}^{j\omega}\!/_{p_1}\right)-\cdots-\angle\left(1+{}^{j\omega}\!/_{p_{n_1}}\right)}$$

Thus, the Bode plot of a complex system can be broken down to the additions and subtractions of some simple systems…

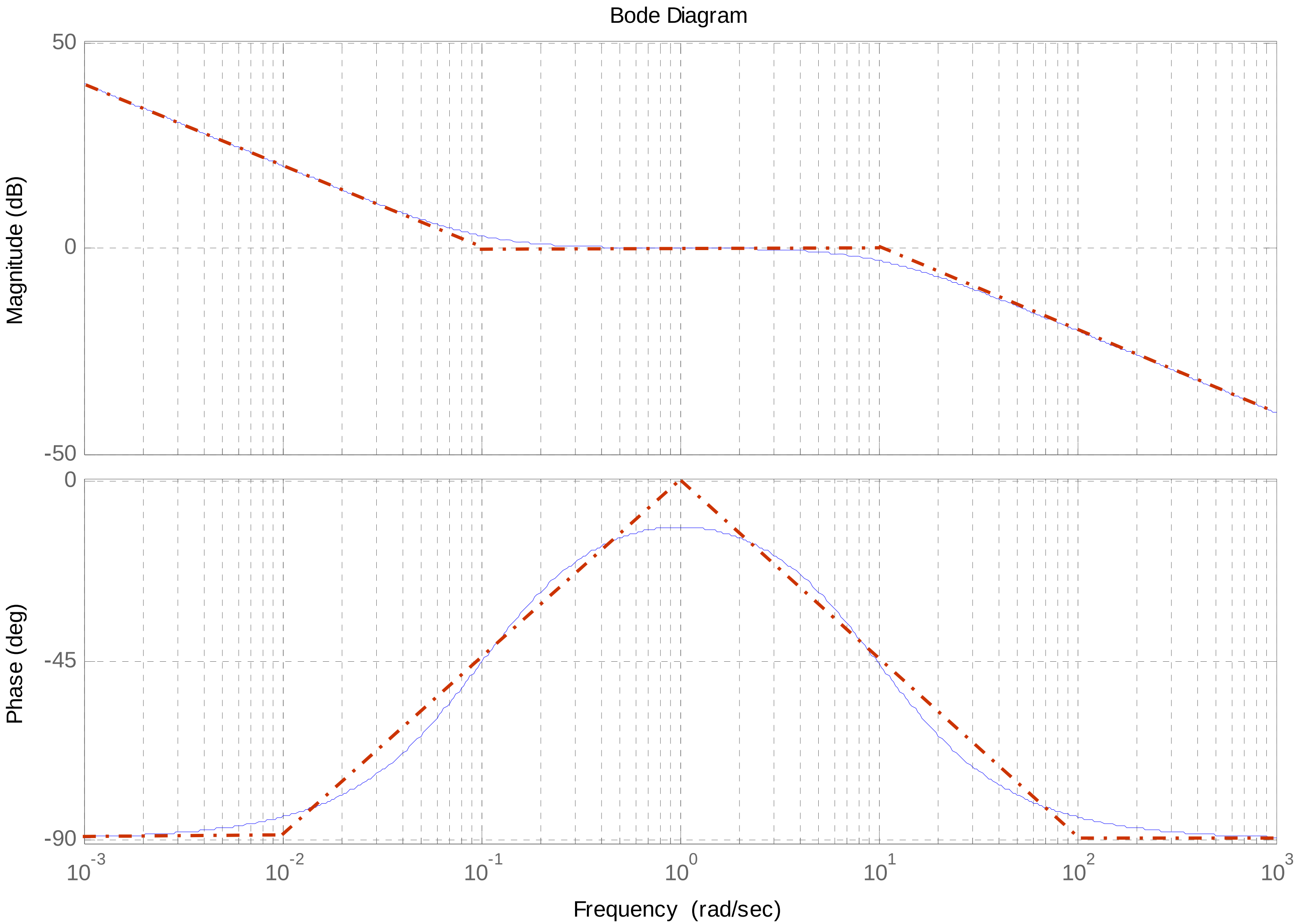
Example

We consider a system characterized by

$$G(s)=\frac{10(s+0.1)}{s(s+10)}=\frac{0.1(1+s/0.1)}{s(1+s/10)}$$



The actual Bode plot

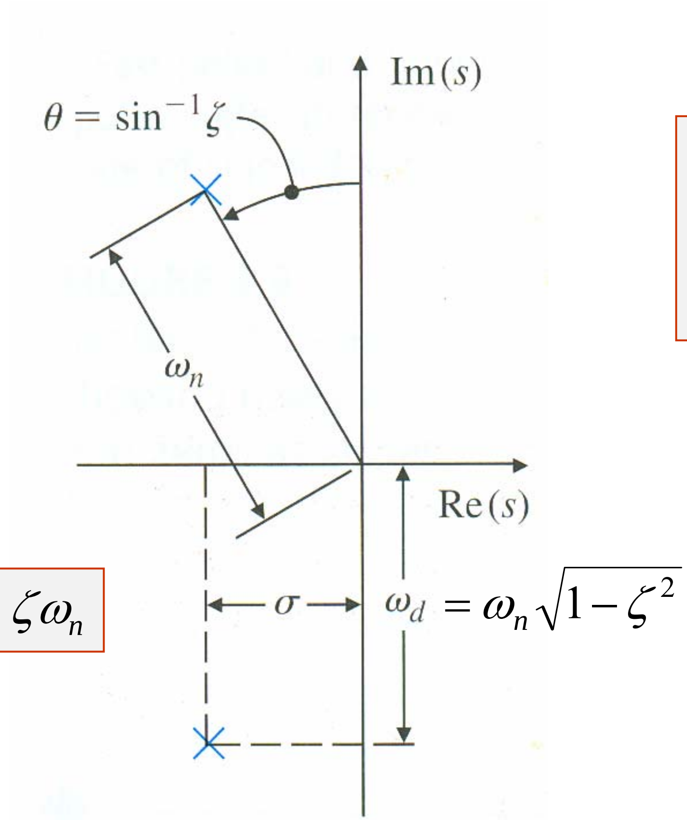


Bode plot of a typical 2nd order system with complex poles

So far, we haven't touched the case when the system has complex poles. Consider

$$G(s)=\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}=\frac{\omega_n^2}{(s+\zeta\omega_n)^2+\omega_n^2(1-\zeta^2)}=\frac{1}{1+2\zeta\left(s/\omega_n\right)+\left(s/\omega_n\right)^2}$$

When $\zeta < 1$, it has two complex conjugated poles at



$$\sigma=\zeta\omega_n$$

ζ is called the damping ratio of the system

ω_n is called the natural frequency

This 2nd order prototype is the most important system for classical control.

Examining

$$G(j\omega)=\frac{1}{1+2\zeta\left(j\omega/\omega_n\right)+\left(j\omega/\omega_n\right)^2}=\frac{1}{\left[1-\left(\omega/\omega_n\right)^2\right]+j2\zeta\left(\omega/\omega_n\right)}$$

we have

$$\omega \ll \omega_n \quad\Rightarrow\quad G(j\omega)=\frac{1}{\left[1-\left(\omega/\omega_n\right)^2\right]+j2\zeta\left(\omega/\omega_n\right)}\to 1=1\angle 0^\circ$$

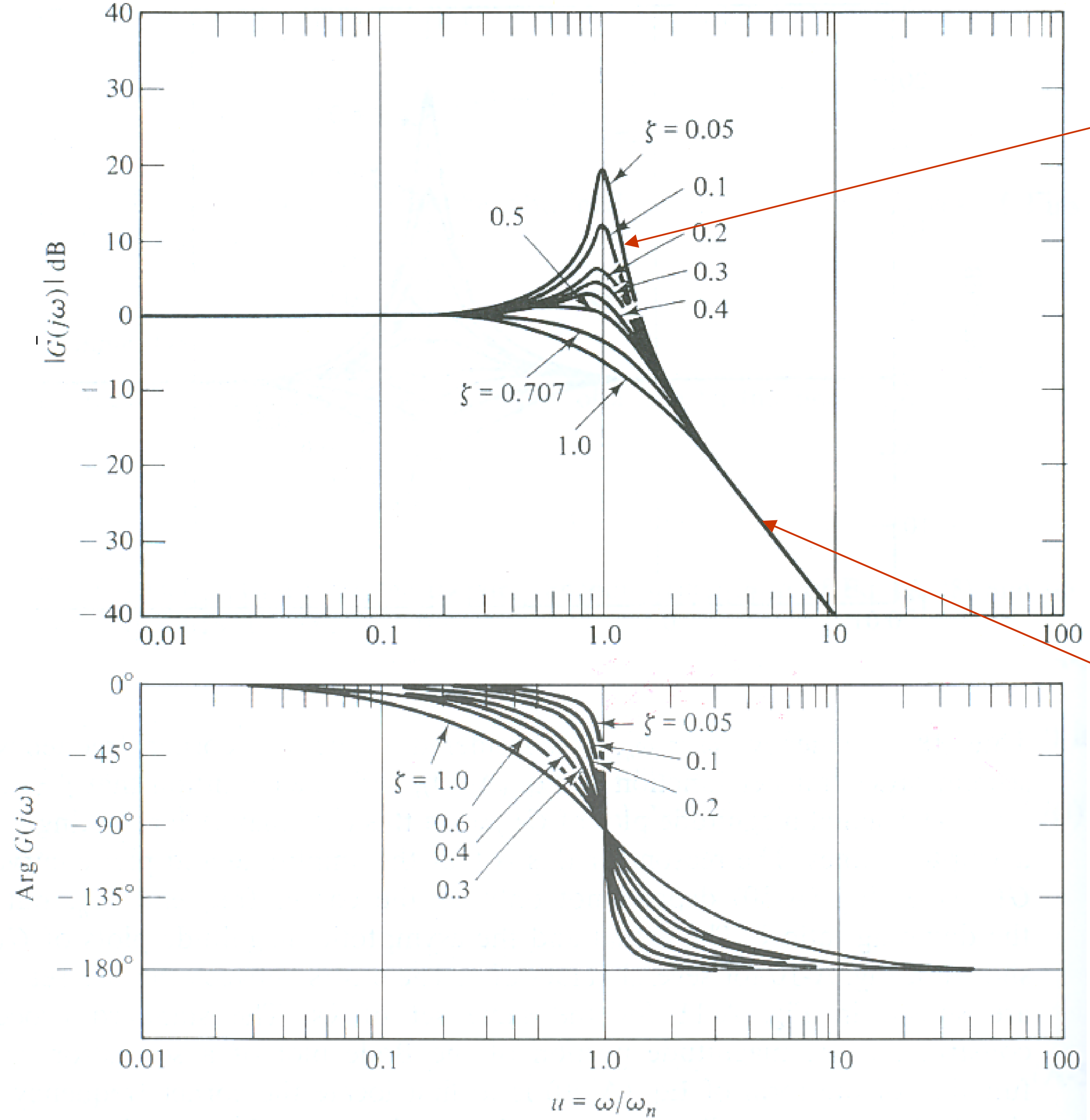
0 dB in magnitude and 0 degree phase at low frequencies

$$\omega \gg \omega_n \quad\Rightarrow\quad G(j\omega)=\frac{1}{\left[1-\left(\omega/\omega_n\right)^2\right]+j2\zeta\left(\omega/\omega_n\right)}\to -\left(\omega_n/\omega\right)^2=\left(\omega_n/\omega\right)^2\angle -180^\circ$$

roll off 40 dB per decade at high frequencies

$$\omega=\omega_n \quad\Rightarrow\quad \begin{cases} G(j\omega)=\dfrac{1}{\left[1-\left(\omega/\omega_n\right)^2\right]+j2\zeta\left(\omega/\omega_n\right)}=\dfrac{1}{j2\zeta}=\dfrac{1}{2\zeta}\angle -90^\circ \\[2ex] |G(j\omega)| \text{ in dB}=20\log_{10}\left(\dfrac{1}{2\zeta}\right)=20\log_{10}\left(\dfrac{1}{2}\right)+20\log_{10}\left(\dfrac{1}{\zeta}\right)=-6\text{ dB}-20\log_{10}\zeta \end{cases}$$

Bode plot of the 2nd order prototype



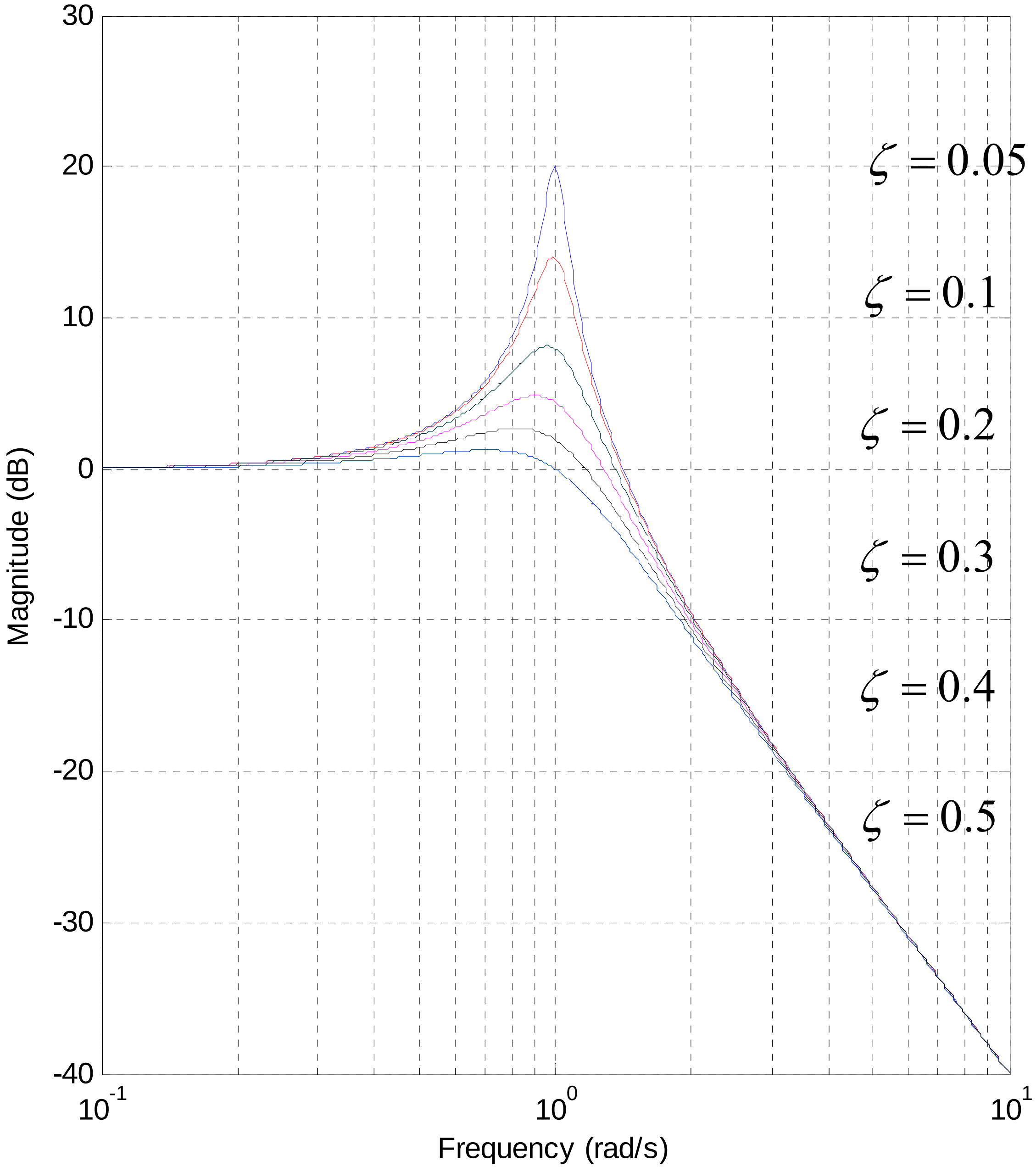
If $\zeta < 0.707$, then the peak (resonant) frequency is given by

$$\omega_r = \omega_n \sqrt{1 - 2\zeta^2}$$

The peak magnitude is

$$M_r = \frac{1}{2\zeta\sqrt{1-\zeta^2}}$$

rolling off 40 dB per decade



	w_r	M_r	M_n
$\zeta = 0.05$	1 rad/s	20 dB	20 dB
$\zeta = 0.1$	0.99 rad/s	14.1 dB	14 dB
$\zeta = 0.2$	0.96 rad/s	8.3 dB	8 dB
$\zeta = 0.3$	0.91 rad/s	5.3 dB	4.5 dB
$\zeta = 0.4$	0.82 rad/s	3.6 dB	2.0 dB
$\zeta = 0.5$	0.71 rad/s	3 dB	0.02 dB

Resonant frequency ω_r is considerably lower than natural frequency ω_n when $\zeta > 0.2$.

Modeling and System Identification

Modeling from the Bode Plot

Motivation

Frequency response data can be generated by performing experiments. It is a common practice to find a transfer function model of the dynamic system from the experimentally obtained frequency response. In this unit, we learn how to estimate a transfer function model from the Bode plot.

Learning Objectives

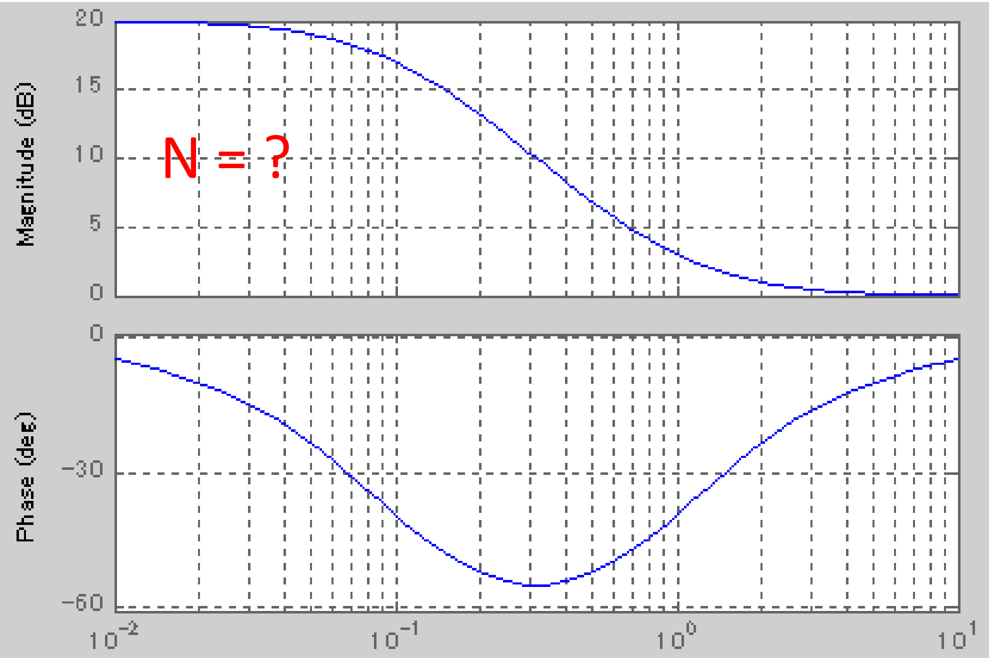
Ability to estimate the transfer function model of a system from its Bode plot

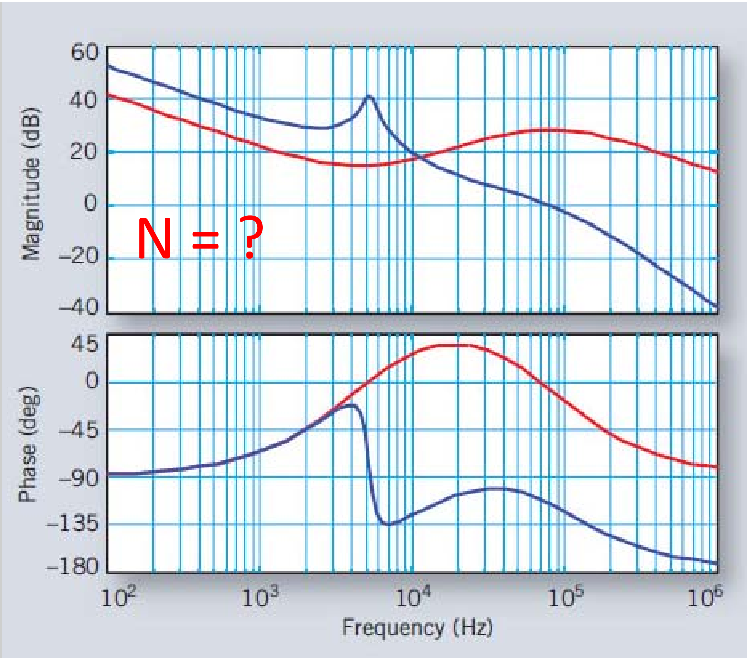
Features of Bode (Magnitude) Plot

a) Low frequency gradient: $(\text{gradient})_{\text{low\_freq}} = N \times (-20)$ dB/decade

N: Number of integrators in G(s)
negative value of N implies differentiator

- By measuring the low-frequency gradient of Bode (magnitude) plot, we can determine the value of N





b) High frequency gradient: $(\text{gradient})_{\text{high\_freq}} = \text{n} \times (-20) + \text{m} \times (+20) \quad \text{dB/decade}$

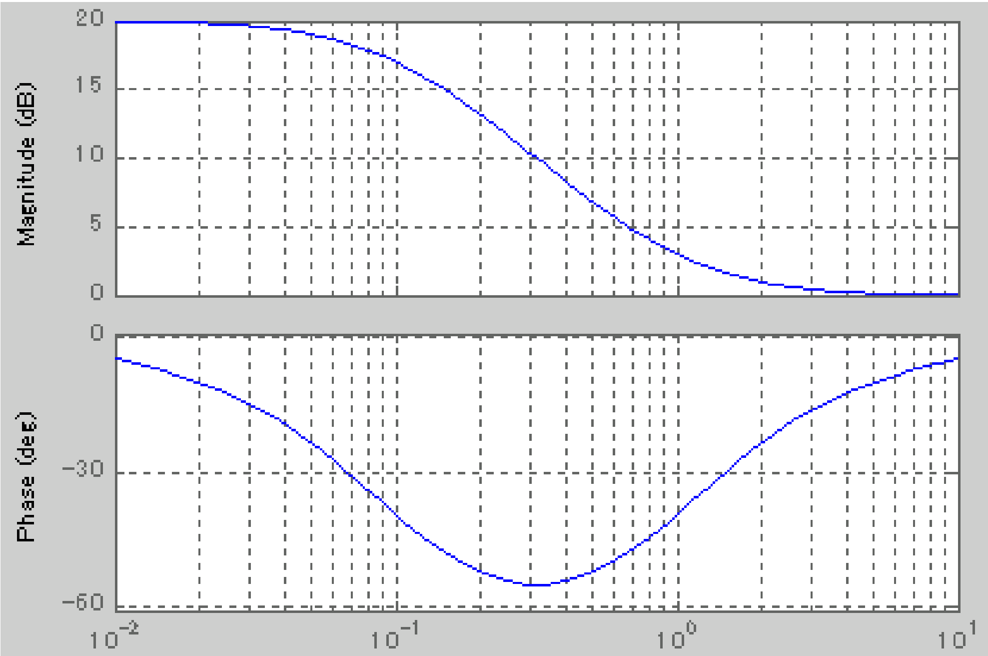
$$= (\text{n} - \text{m}) \times (-20) \text{ dB/decade}$$

n: Number of poles in G(s)

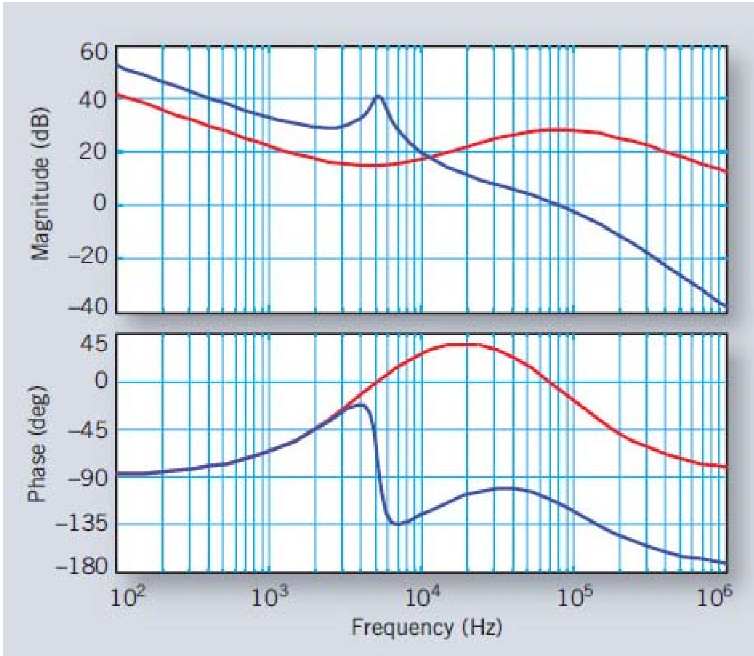
m: Number of zeros in G(s)

By measuring the high frequency gradient of Bode (magnitude) plot, we can determine the value of (n-m) $\Rightarrow$ **Pole Excess**

(n-m) = ?



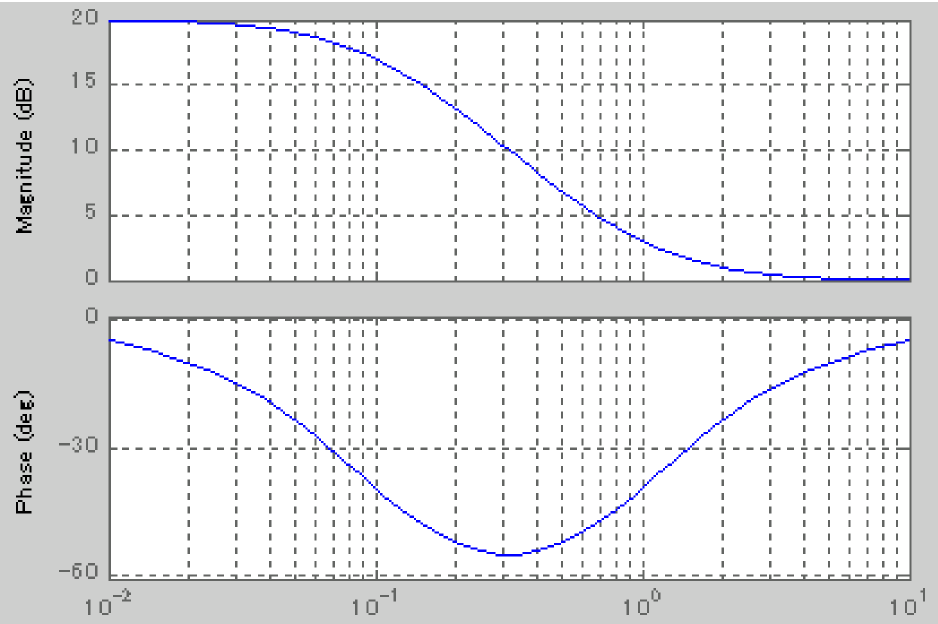
(n-m) = ?

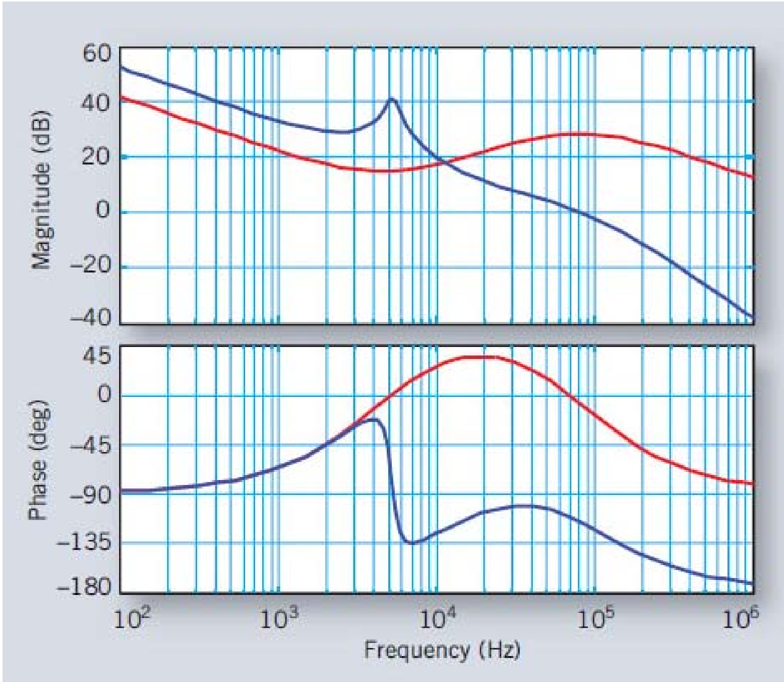


c) At the corner frequency of a zero, the slope is changed by +20 dB/decade

d) At the corner frequency of a pole, the slope is changed by -20 dB/decade

- o Change in the gradient is an indicator of the presence of corner frequency or natural frequency
- o Bending upward indicates presence of zero
- o Bending downward means pole





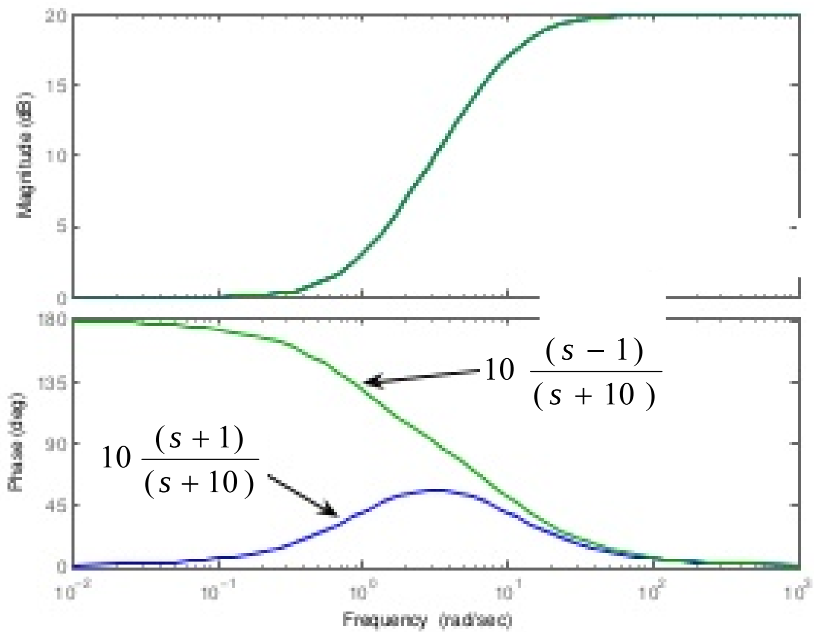
Phase of System with RHS Pole/ Zero

Two 1st-order factors shown below have identical magnitude plots but different phase plots

$$(s+a) \xrightarrow{s=j\omega} (j\omega+a) \Rightarrow \sqrt{a^2+\omega^2}\angle \tan^{-1}\frac{\omega}{a}$$

$$(s-a) \xrightarrow{s=j\omega} (j\omega-a) \Rightarrow \sqrt{a^2+\omega^2}\angle (180^\circ - \tan^{-1}\frac{\omega}{a})$$

A transfer function with RHS zeros is called **non-minimum phase** system



- Two transfer functions can have identical magnitude plots though their phase plots are different

Phase of System with Delay

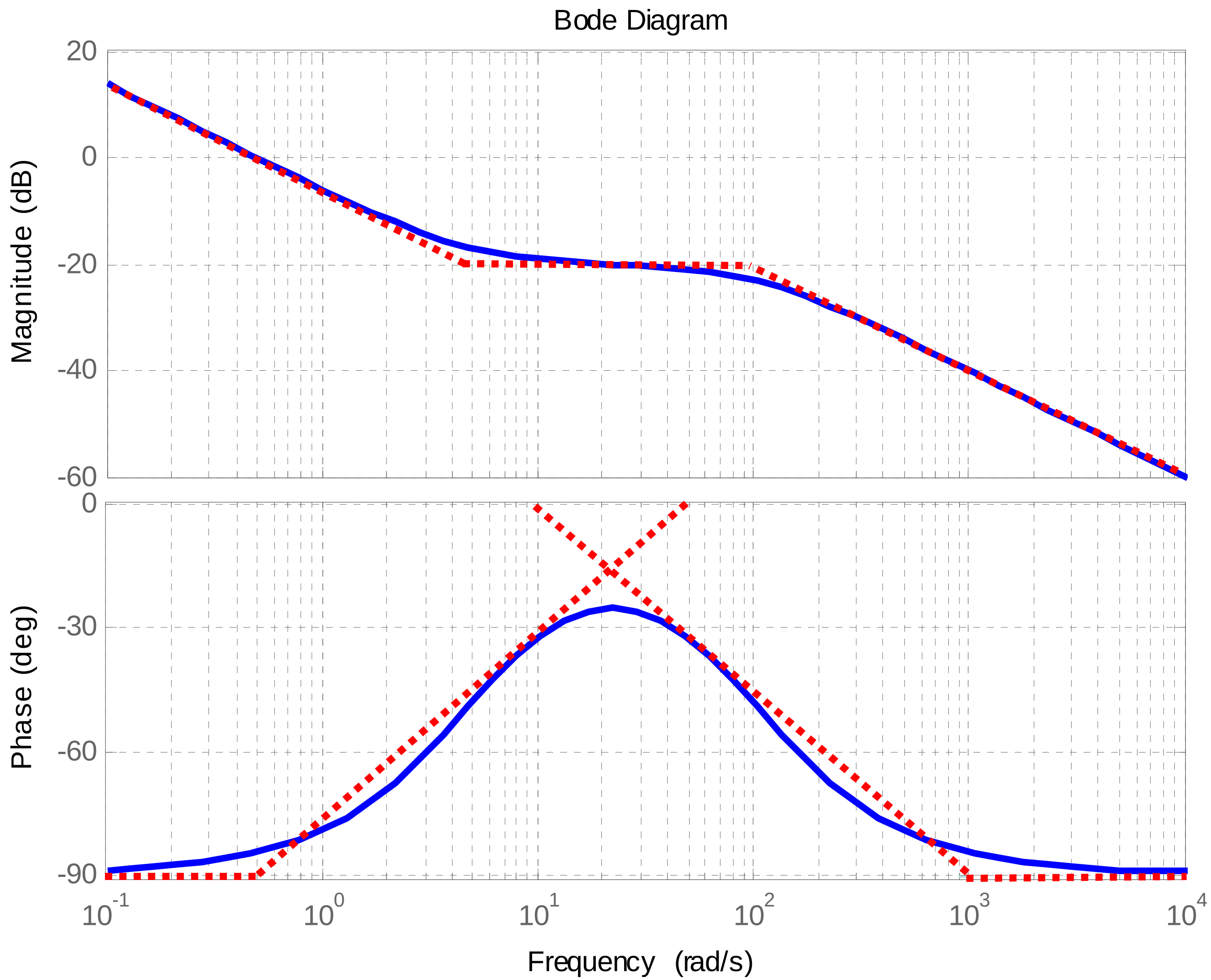
Two functions may have identical magnitude plots but different phase plots if transmission delays are different

$$(s+a) \overset{s=j\omega}{\Longrightarrow} (j\omega+a) \Rightarrow \sqrt{a^2+\omega^2}\angle \tan^{-1}\frac{\omega}{a}$$

$$(s+a)e^{-st_d} \overset{s=j\omega}{\Longrightarrow} (j\omega+a)e^{-j\omega t_d} \Rightarrow \sqrt{a^2+\omega^2}\angle\left(\tan^{-1}\frac{\omega}{a}-\omega t_d\times\frac{180^\circ}{\pi}\right)$$

Magnitude plot alone is not enough for identification of model

Exercise 2-1: Find the transfer function model from the Bode plot given

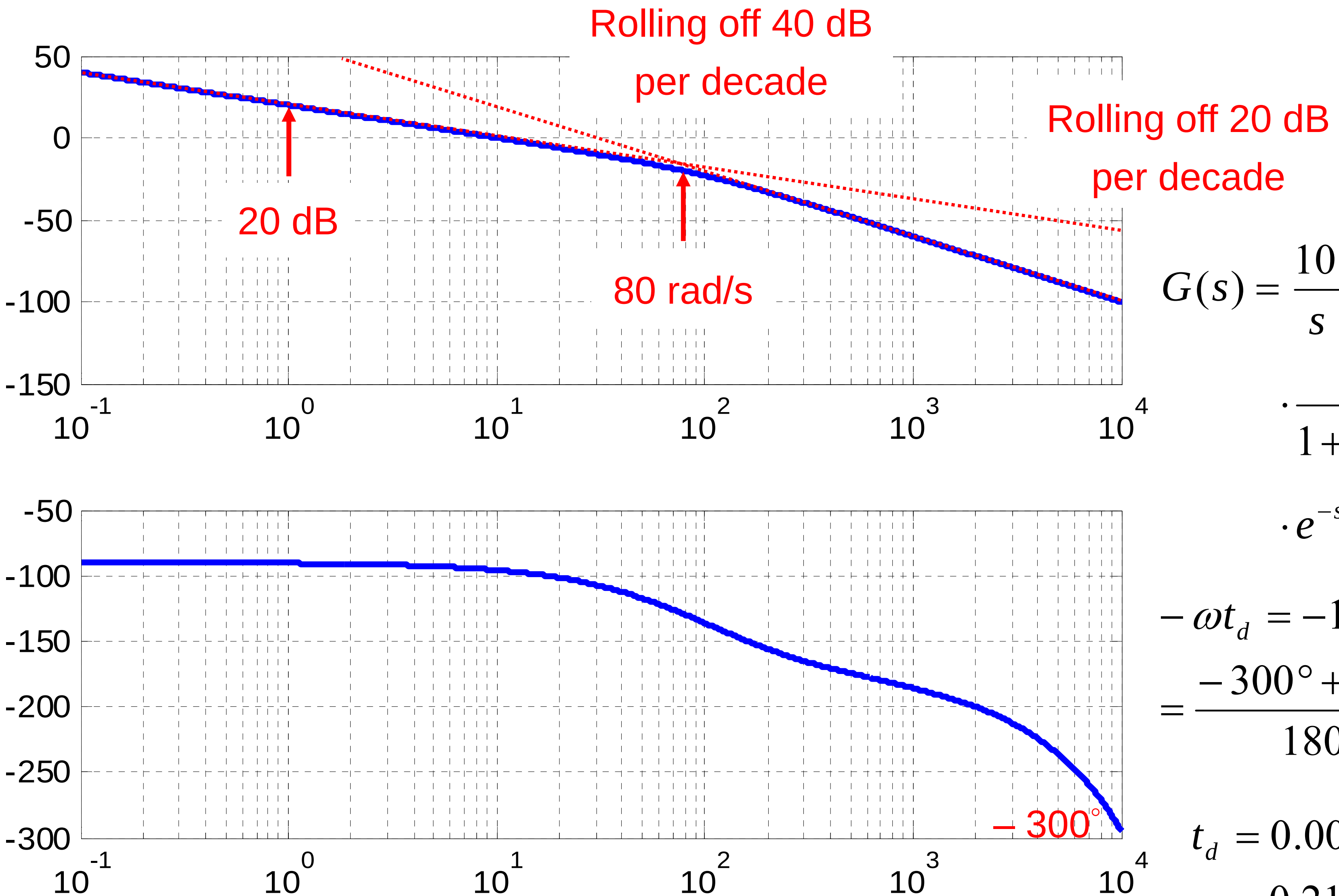


$$G(s) = \frac{k}{s} \cdot \left(1 + \frac{s}{4}\right) \cdot \frac{1}{1 + \frac{s}{100}}$$

$$k = -8 \text{ dB} = 10^{-8/20} = 0.4$$

$$G(s) = \frac{10(s+4)}{s(s+100)}$$

Exercise 2-2: Find the transfer function model from the Bode plot given



$$G(s) = \frac{10}{s} \cdot \frac{1}{1 + s/80} \cdot e^{-st_d}$$

$$-\omega t_d = -10000\, t_d = \frac{-300^\circ + 180^\circ}{180}\pi$$

$$t_d = 0.00021 \text{ s} = 0.21 \text{ ms}$$

Steady State Responses

Understanding Design Specifications (Steady-State Error)

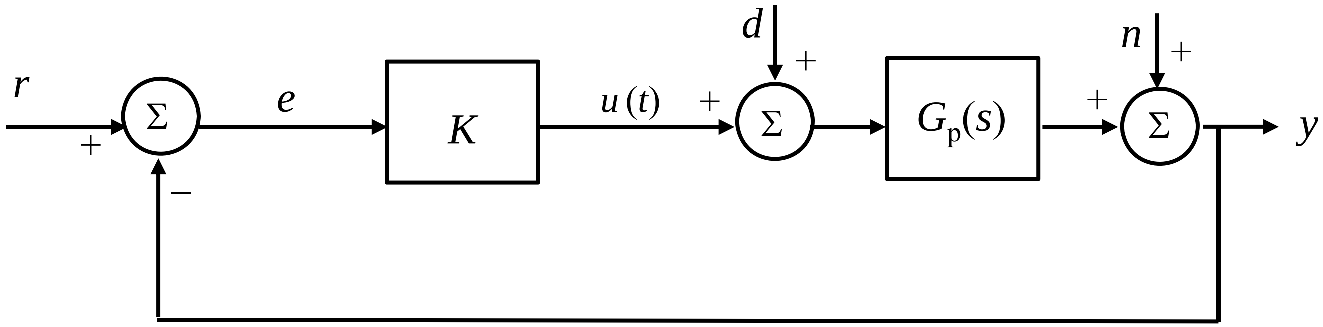
Motivation

Putting a controller in cascade with the plant effectively modifies the properties of the loop transfer function and hence its frequency response. What modification is desirable and what is not? It is important to know the properties of the loop transfer function that would give the desired performance.

Learning Objectives

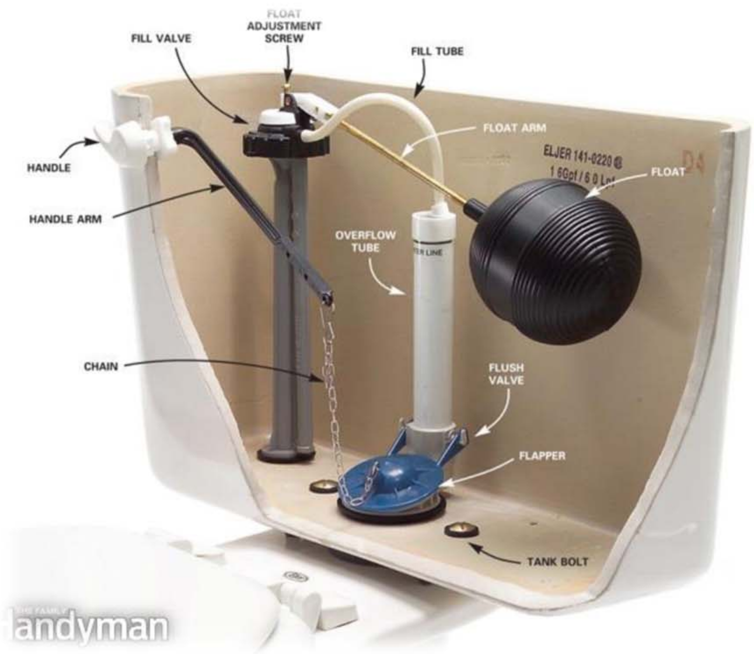
1) Ability to determine the target properties of the loop transfer function, given the desired specifications on steady-state error
2) Ability to design the feedback loop so that it meets the steady-state error specifications (only)

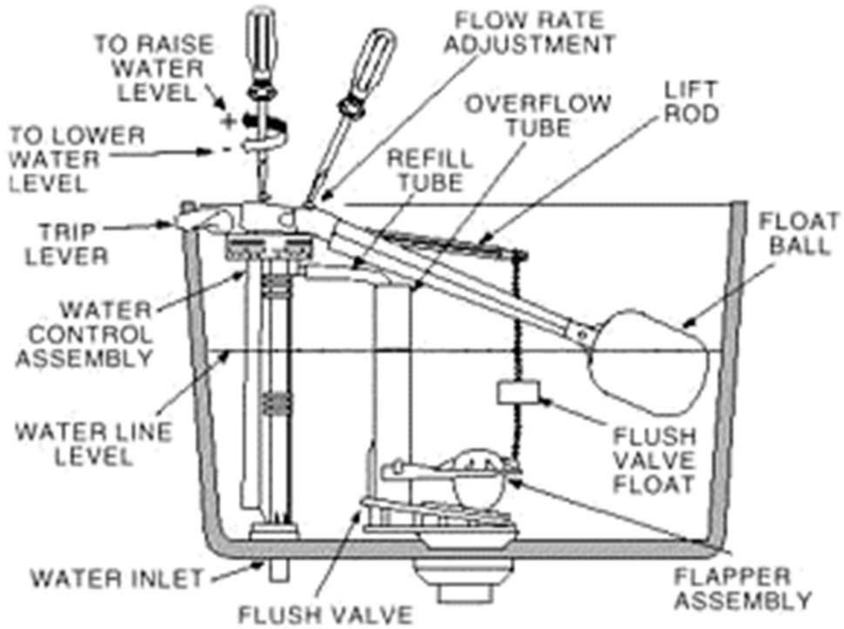
Design of feedback: **start with the simplest feedback loop possible**



- Only one parameter (K) to choose
- Needs
  - Minimum number of components (in analog implementation) or
  - Minimum computational delay (in digital implementation)

A simple control system: Toilet water tank...





Can we meet the design requirements with the simplest solution?

How do we specify the design requirements?

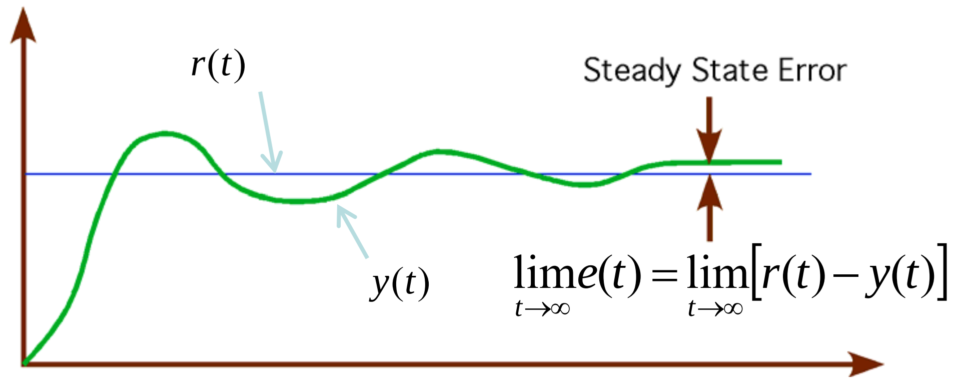
1. Tracking Performance: How closely does the output follow the reference?

   If the desired speed of a motor is 100 RPM, does it spin at 100 RPM or 99.5 RPM?

During transient, the output is still changing towards the final steady-state value. Tracking performance is defined by the difference between output and reference in the steady-state.

$$e_{ss} = r - y_{ss}$$

Reference is a step function

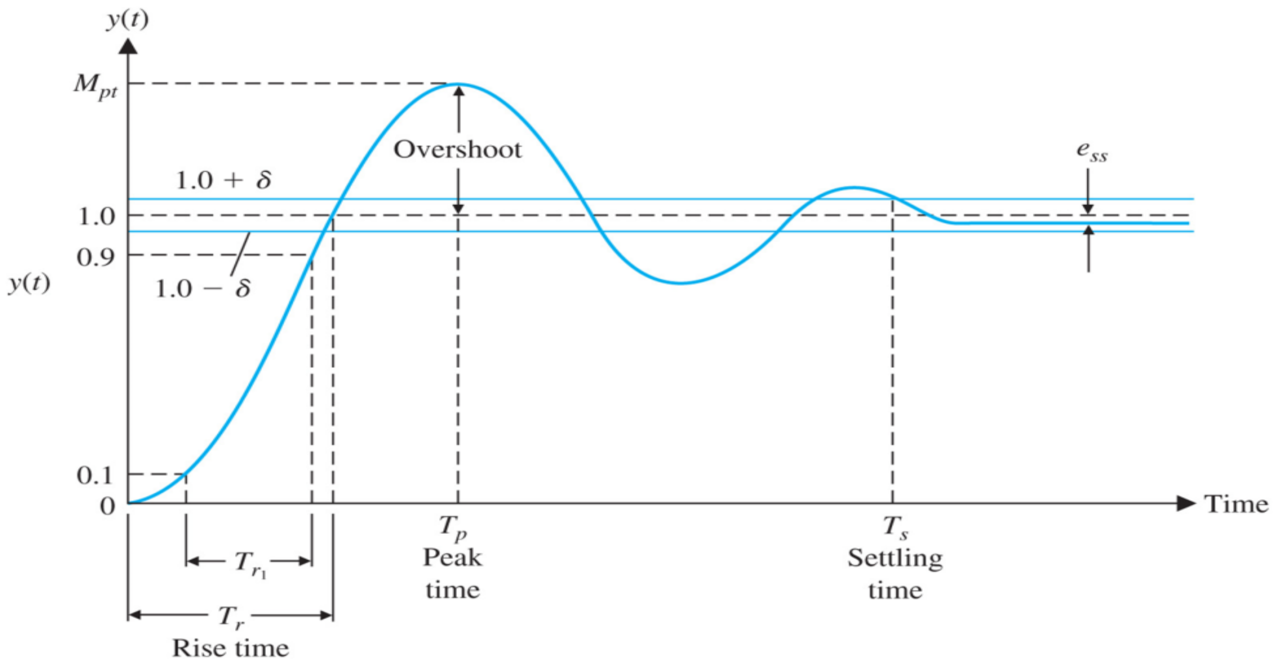


How do we specify the design requirements?

2. Transient Performance:

   How **fast** does the output reaches the steady-state?

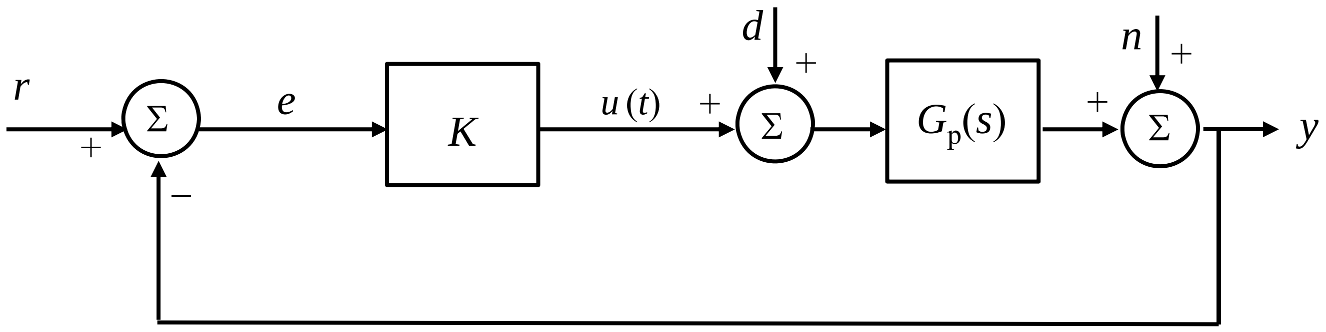
   What is the nature of the transient response? Does the response reach steady-state exponentially? Does it oscillate?



In this unit, we consider only the tracking performance. We'll also consider stability issue. Transient specifications are discussed later.

The output signal may or may not follow the reference in a feedback control system. If the output is not following the reference, there is an error.

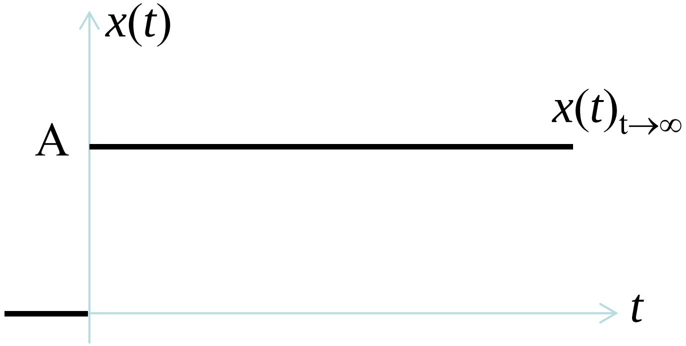
How do we find the steady-state error?



$$e(t) = r(t) - y(t)$$

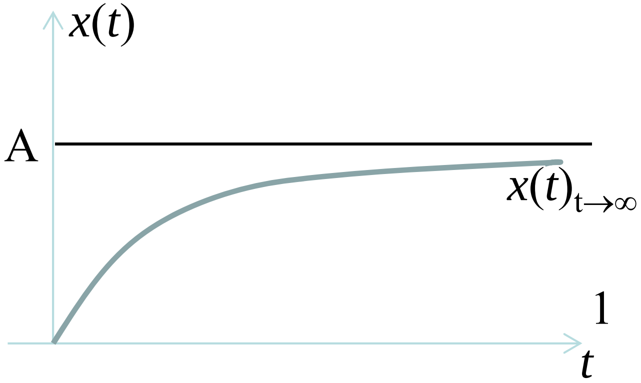
According to the Final Value Theorem, for any signal $x(t)$, $x(t)_{t\to\infty} = \lim_{s\to 0} sX(s)$

Final Value Theorem (Illustrations)



$$X(s)=\frac{A}{s}$$

$$\lim_{s\to 0} sX(s)=\lim_{s\to 0} s\frac{A}{s}$$
$$=A$$
$$=x(t)_{t\to\infty}$$



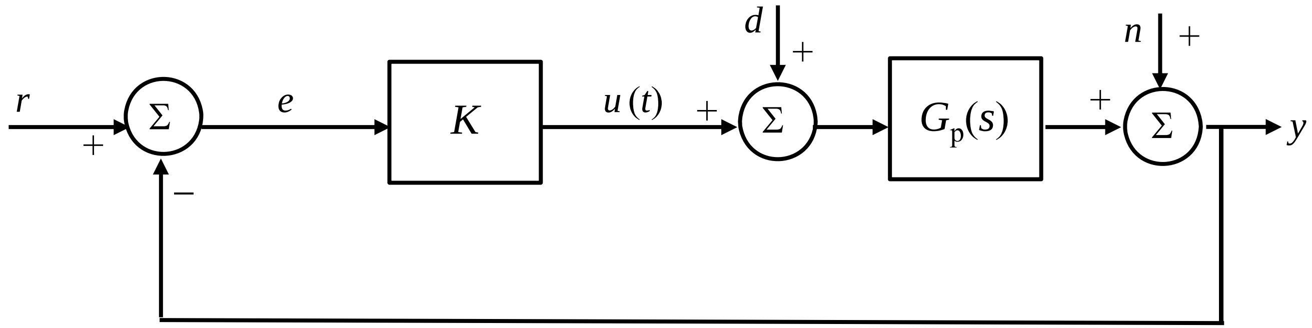
$$x(t)=A\left(1-e^{-at}\right)$$

$$X(s)=\frac{A}{s}-\frac{A}{s+a}$$

$$x(t)_{t\to\infty}=A\left(1-e^{-a\infty}\right)$$
$$=A$$

$$\lim_{s\to 0} sX(s)=\lim_{s\to 0} s\frac{A}{s}-\lim_{s\to 0} s\frac{A}{s+a}$$
$$=A$$

How do we find the steady-state error?



$L(s) = KG_p(s)$ is called open loop transfer function

$$U = KE$$
$$Y = G_p(D+U)+N$$

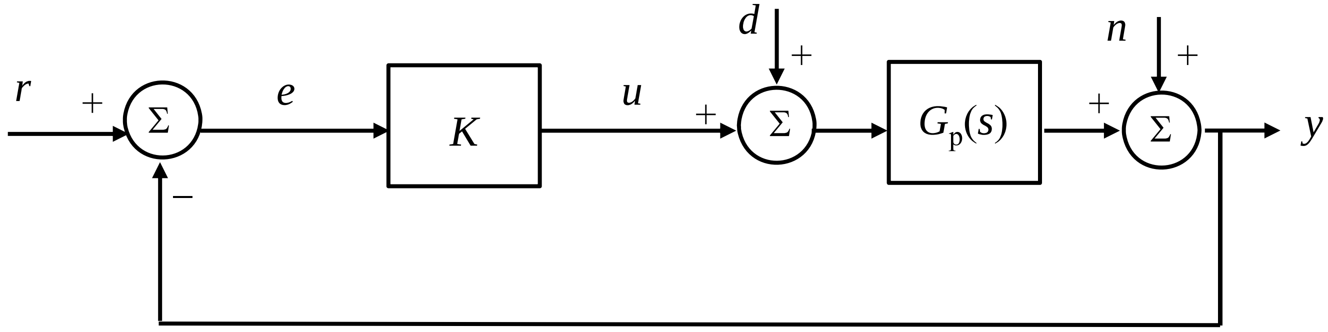
$$\Rightarrow \quad E = R - Y$$
$$= R - G_p D - G_p U - N$$
$$= R - G_p D - G_p KE - N$$

$$\Rightarrow \quad (1+H_p K)E = R - G_p D - N$$

$$\Rightarrow \quad E(s) = \frac{1}{1+KG_p(s)}R(s) - \frac{G_p(s)}{1+KG_p(s)}D(s) - \frac{1}{1+KG_p(s)}N(s)$$
$$= \frac{1}{1+L(s)}R(s) - \frac{G_p(s)}{1+L(s)}D(s) - \frac{1}{1+L(s)}N(s)$$

$$e(t)_{t\to\infty} = \lim_{s\to 0} sE(s)$$

How do we find the steady-state error?



It is easily concluded from the previous slide that, the steady-state error depends on

a) the Open Loop Transfer Function *L*(*s*) and b) the type of reference signal

$$e(t)_{t\to\infty} = \lim_{s\to 0} s \cdot \frac{1}{1+L(s)} \cdot R(s)$$

If $r(t)$ is a **step function,** $r(t) = At^0, \quad \Rightarrow \quad R(s) = \frac{A}{s}$

$$e(t)_{t\to\infty} = \lim_{s\to 0} \frac{A}{1+L(s)} = \frac{A}{1+\lim\limits_{s\to 0} L(s)}$$

$$\lim_{s\to 0} L(s) = k_p$$

(Position Error Constant)

$$e(t)_{t\to\infty} = \lim_{s\to 0} s \cdot \frac{1}{1+L(s)} \cdot R(s)$$

If $r(t)$ is a **ramp function**, $r(t) = At^1, \quad \Rightarrow \quad R(s) = \frac{A}{s^2}$

$$e(t)_{t\to\infty} = \lim_{s\to 0} s \frac{1}{1+L(s)} \frac{A}{s^2}$$

$$= \lim_{s\to 0} \frac{A}{s+sL(s)}$$

$$= \frac{A}{\lim_{s\to 0} sL(s)}$$

$$\lim_{s\to 0} sL(s) = k_v$$

(Velocity Error Constant)

If $r(t)$ is a **parabolic function**, $r(t) = At^2, \quad \Rightarrow \quad R(s) = \frac{2A}{s^3}$

$$e(t)_{t\to\infty} = \lim_{s\to 0} s \frac{1}{1+L(s)} \frac{2A}{s^3}$$

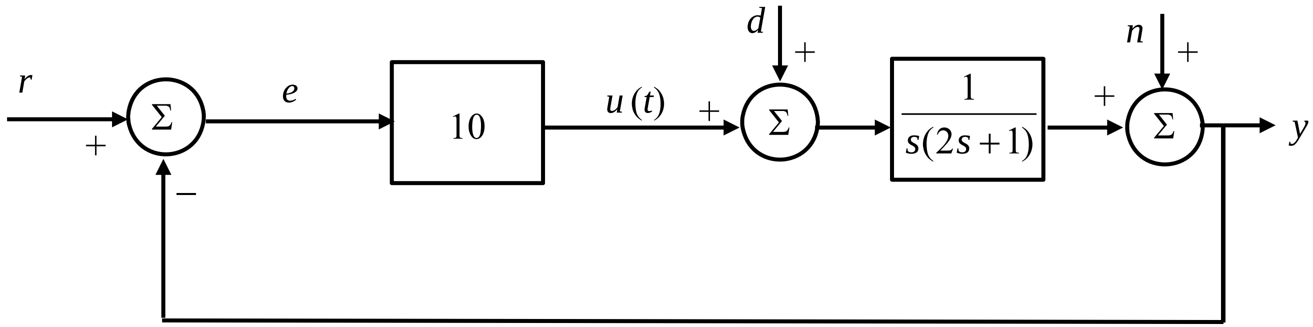
$$= \lim_{s\to 0} \frac{2A}{s^2+s^2L(s)}$$

$$= \frac{2A}{\lim_{s\to 0} s^2L(s)}$$

$$\lim_{s\to 0} s^2L(s) = k_a$$

(Acceleration Error Constant)

Exercise 3-1: Find the steady-state error for ramp-input in the following system

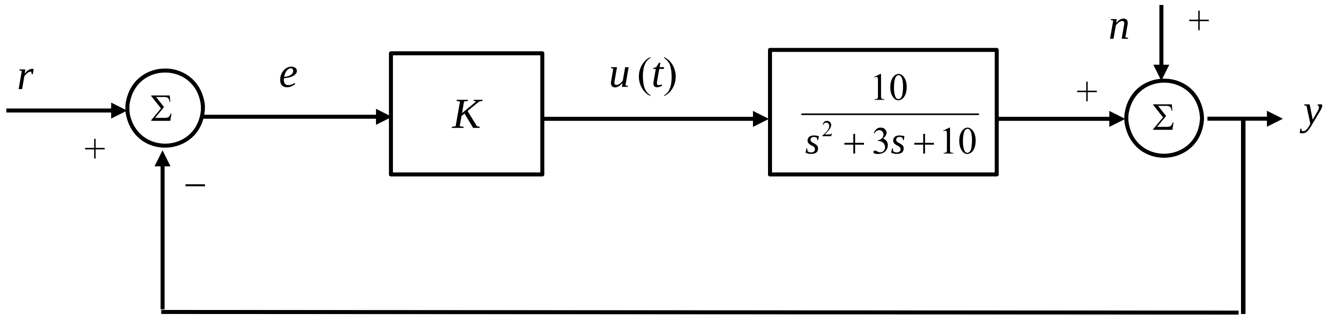


$$L(s)=\frac{10}{s(2s+1)}$$

$$\lim_{s\to 0} sL(s)=\lim_{s\to 0} s\frac{10}{s(2s+1)}=10$$

$$e(t)_{t\to\infty}=\frac{1}{k_v}=\frac{1}{10}=0.1$$

Exercise 3-2: Find the value of *K* so that steady-state error for step-input is < 10%



$$L(s) = \frac{10K}{s^2 + 3s + 10}$$

$$k_p = \lim_{s \to 0} L(s) = \lim_{s \to 0} \frac{10K}{s^2 + 3s + 10} = K$$

$$e(t)_{t \to \infty} = \frac{1}{1 + k_p} < 0.1 \quad \Rightarrow \quad K = k_p > \frac{1 - 0.1}{0.1} = 9$$

Exercise 3-3: Find the closed loop poles of the simple proportional control feedback system from the previous slide.

- What is the damping coefficient?
- Sketch the step response of this feedback system?
- Is this a good design?

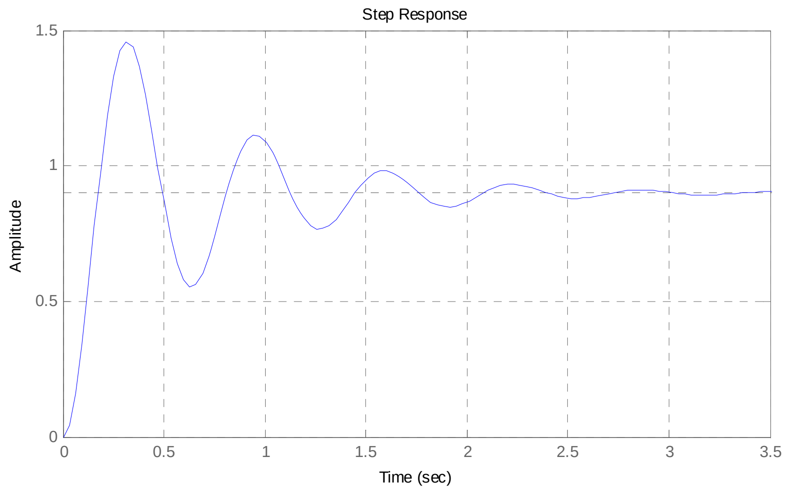
$$E(s)=\frac{1}{1+KG_p(s)}R(s)=R(s)-Y(s) \quad \Rightarrow \quad Y(s)=\left[1-\frac{1}{1+KG_p(s)}\right]R(s)=\frac{KG_p(s)}{1+KG_p(s)}R(s)$$

$$\Rightarrow \quad \frac{Y(s)}{R(s)}=\frac{KG_p(s)}{1+KG_p(s)}=\frac{\frac{90}{s^2+3s+10}}{1+\frac{90}{s^2+3s+10}}=\frac{90}{s^2+3s+100}=\frac{90}{s^2+2\times 0.15\times 10s+10^2}$$

The closed loop poles of the control feedback system are

$$\varsigma=0.15, \quad \omega_n=10$$

$$s_{1,2}=-1.5\pm j9.9$$



Steady-state error for different input signals:

	Step	Ramp	Parabolic
Steady-state error	$\dfrac{A}{1+\lim\limits_{s\to 0} L(s)} = \dfrac{A}{1+k_p}$	$\dfrac{A}{\lim\limits_{s\to 0} sL(s)} = \dfrac{A}{k_v}$	$\dfrac{2A}{\lim\limits_{s\to 0} s^2 L(s)} = \dfrac{2A}{k_a}$

If zero-error is desired for step-input, k_p must be infinity which requires the loop transfer function $L(s)$ to have at least one integrator

$$L(s) = \frac{1}{s^N} P(s), \quad N \geq 1$$

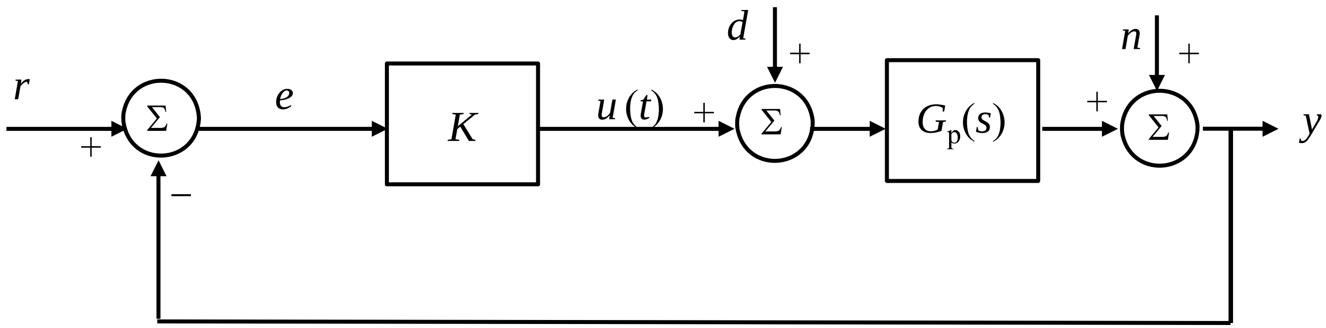
P(*s*) has no integrator in it

Number of integrators present in the loop transfer function is called **System Type**. A Type 0 has no integrator, Type 1 has one integrator, and so on

To achieve zero steady-state error,

- **Step input**: the loop transfer function $L(s)$ must be **Type 1 or higher**
- **Ramp input**: $L(s)$ must be **Type 2 or higher**
- **Parabolic input**: $L(s)$ must be **Type 3 or higher**

Going back to the slide where we said, "**start with the simplest feedback loop possible**", such design may not satisfy many other design specifications. We may even end up with an unstable system.

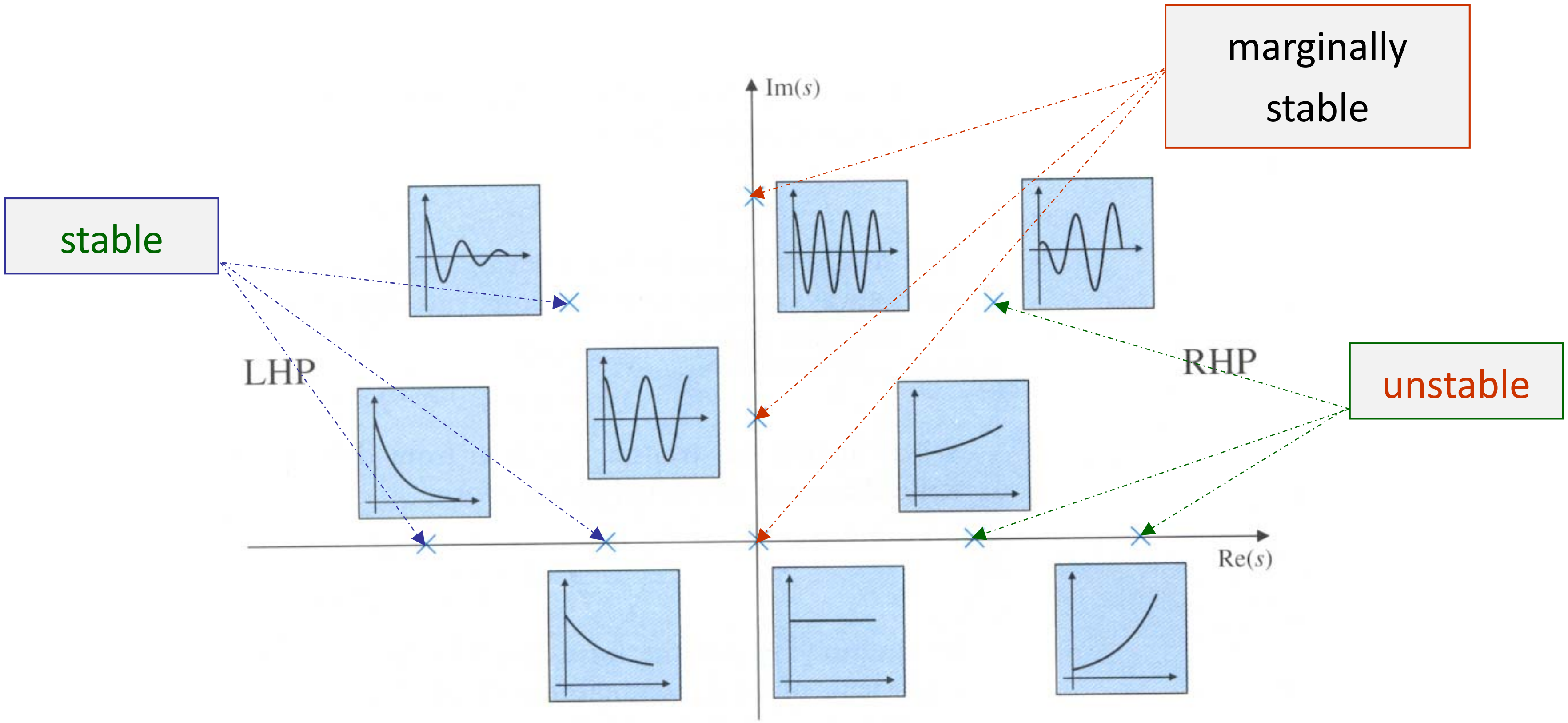


So, we need to answer two more questions:

- How to test stability of the closed loop system?
- How to predict the transient response of the closed loop?

Stability Margins

A given system is *stable* if the system does not have poles on the right-half plane (RHP). It is *unstable* if it has at least one pole on the RHP. In particular,



The above diagram also shows the relationship the locations of poles and natural responses.

Motivation

- While designing a feedback system, we start with a known model of plant and then add a controller such that target specifications (for example, steady-state error specification) are met.
- We didn't consider the stability issue in earlier.
- How to test stability?
  - We can derive the closed loop transfer function and find its poles to test stability.
  - But it is more convenient if we can assess closed loop stability from the open loop transfer function.
- We also need to determine the range of stable operation as model parameters are not accurate for any practical system.

Learning Objectives

- Ability to assess the closed loop stability from the knowledge of the open loop transfer function $L(s)$
- Ability to find the gain margin and phase margin of a given $L(s)$
- Ability to explain the significance of the stability margins

Recap:

- o Gain-crossover frequency (ω_{cg}) – frequency at which gain is 1 or unity (0 dB)
- o Phase-crossover frequency (ω_{cp}) – frequency at which phase is $\pm 180°$
- o If ω_{cg} and ω_{cp} are the same frequency for a loop transfer function $L(s)$ then the resulting closed loop has poles on the imaginary axis
  - Closed loop will be marginally stable if no closed loop pole lies in the right hand side of the complex s-plane

Recall the transfer function of the closed-loop system is given by

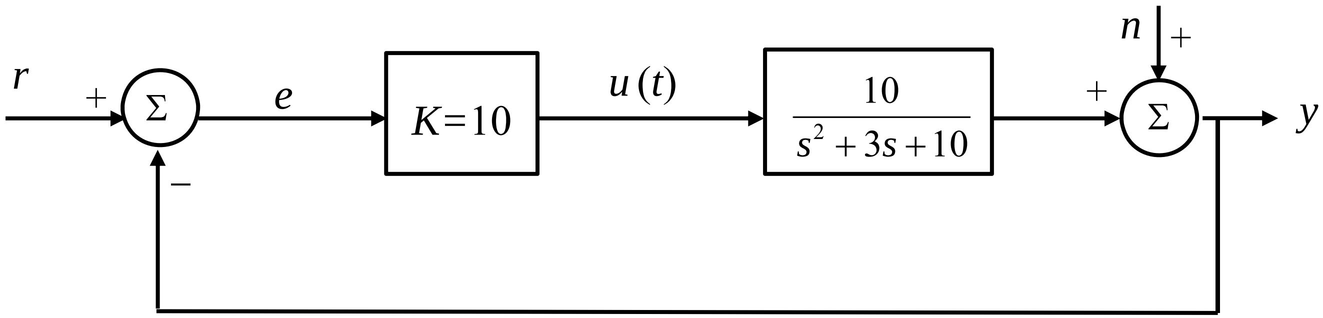
$$\frac{Y(s)}{R(s)} = \frac{KG_p(s)}{1+KG_p(s)} = \frac{L(s)}{1+L(s)}$$

The closed-loop poles can be found by solving

$$1+L(s)=0 \quad \Rightarrow \quad L(s)=-1=1\angle \pm 180°$$

If we have an ω such that such that $L(j\omega) = 1 \angle \pm 180°$, which implies $j\omega$ is a pole.

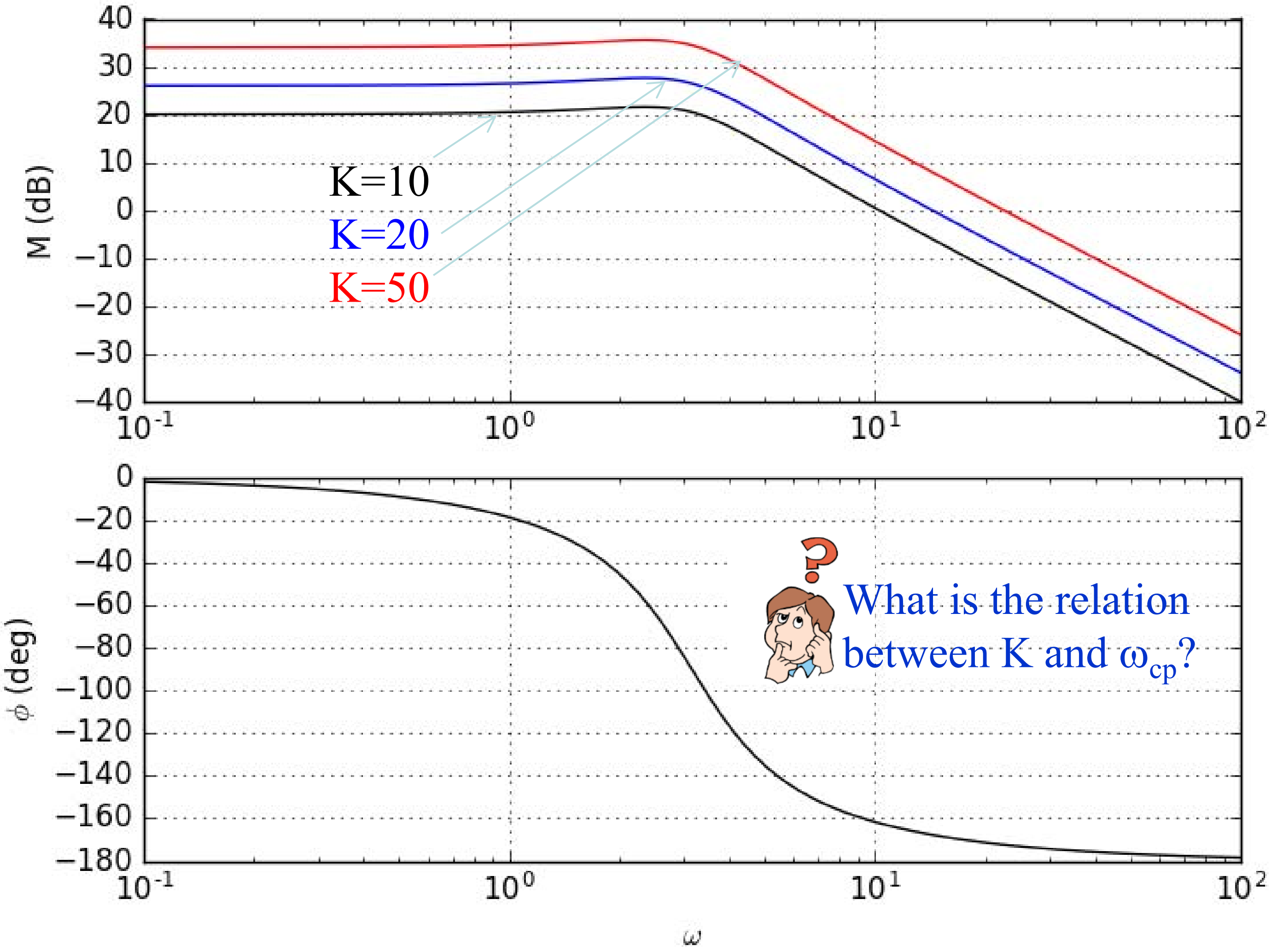
Exercise 4-1: Find the crossover frequencies of the system of Exercise 3-2



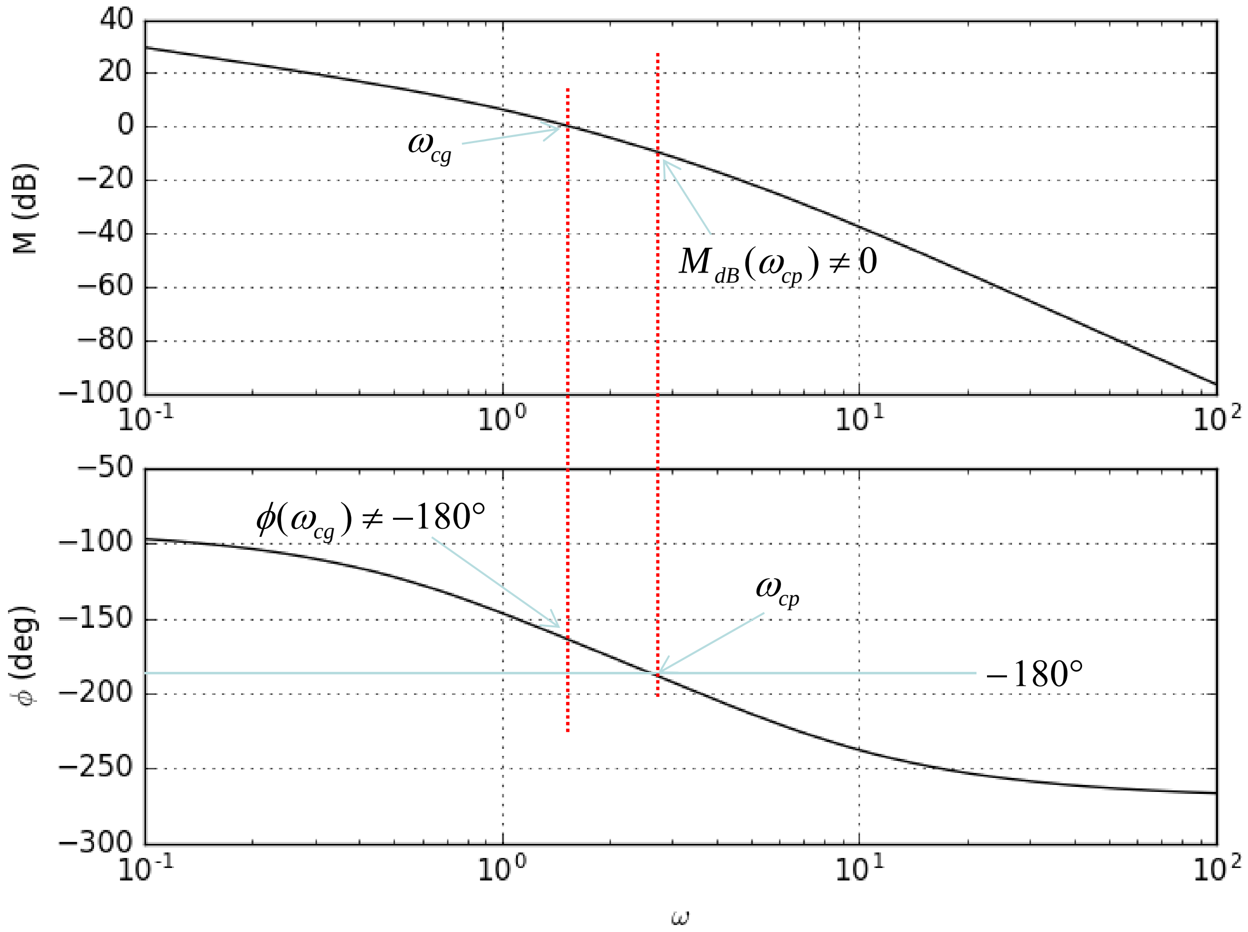
Bode plot of *L*(*s*) for different values of K

$$L(s)=\frac{10K}{s^2+3s+10}$$

What is the effect of varying K on ω_{cg}?



What is the relation between K and ω_{cp}?



$$L(s) = \frac{15}{s(s^2 + 6s + 5)}$$

For this case:

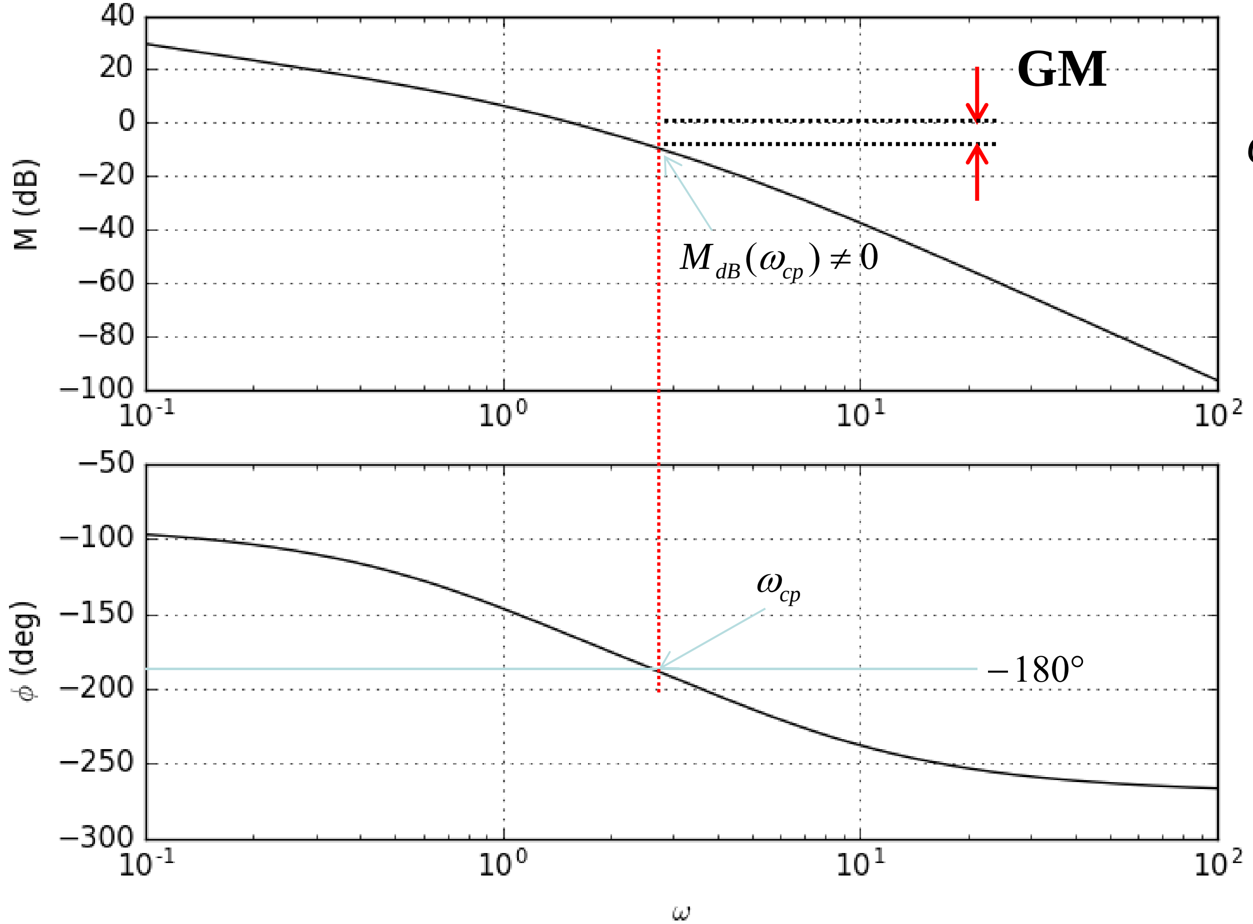
$$\omega_{cg} \neq \omega_{cp}$$

The resulting closed loop will not have poles on the $j\omega$-axis.

In this example, gain can be increased before ω_{cg} and ω_{cp} become equal

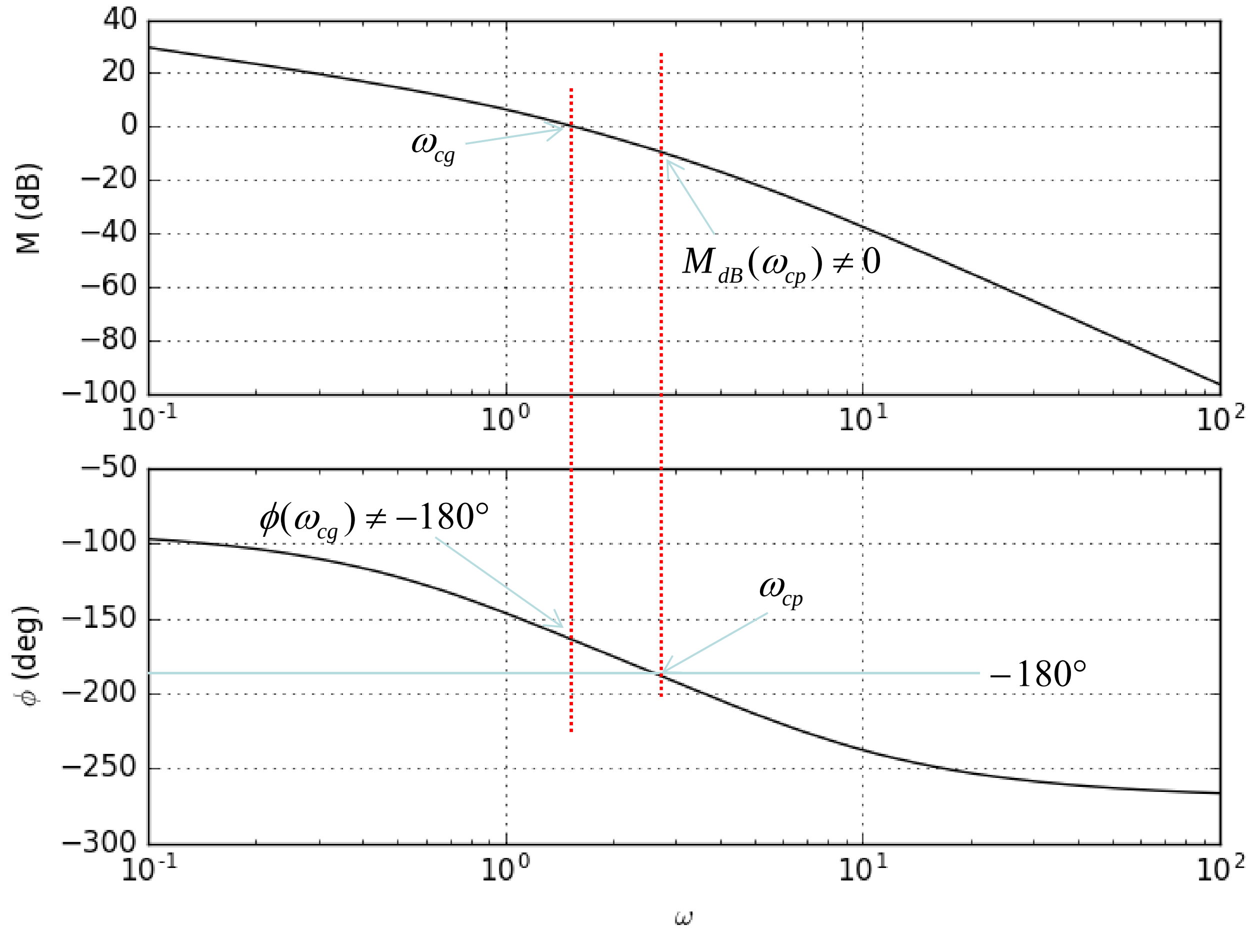
Gain Margin (GM): Variation in gain that can be allowed before the condition for marginal stability is met

$$L(s)=\frac{15}{s(s^2+6s+5)}$$



$$GM = 0 - M_{dB}(\omega_{cp})$$
$$= -20\log\left|L(j\omega_{cp})\right|$$

Next slide shows numerical example of computing gain margin

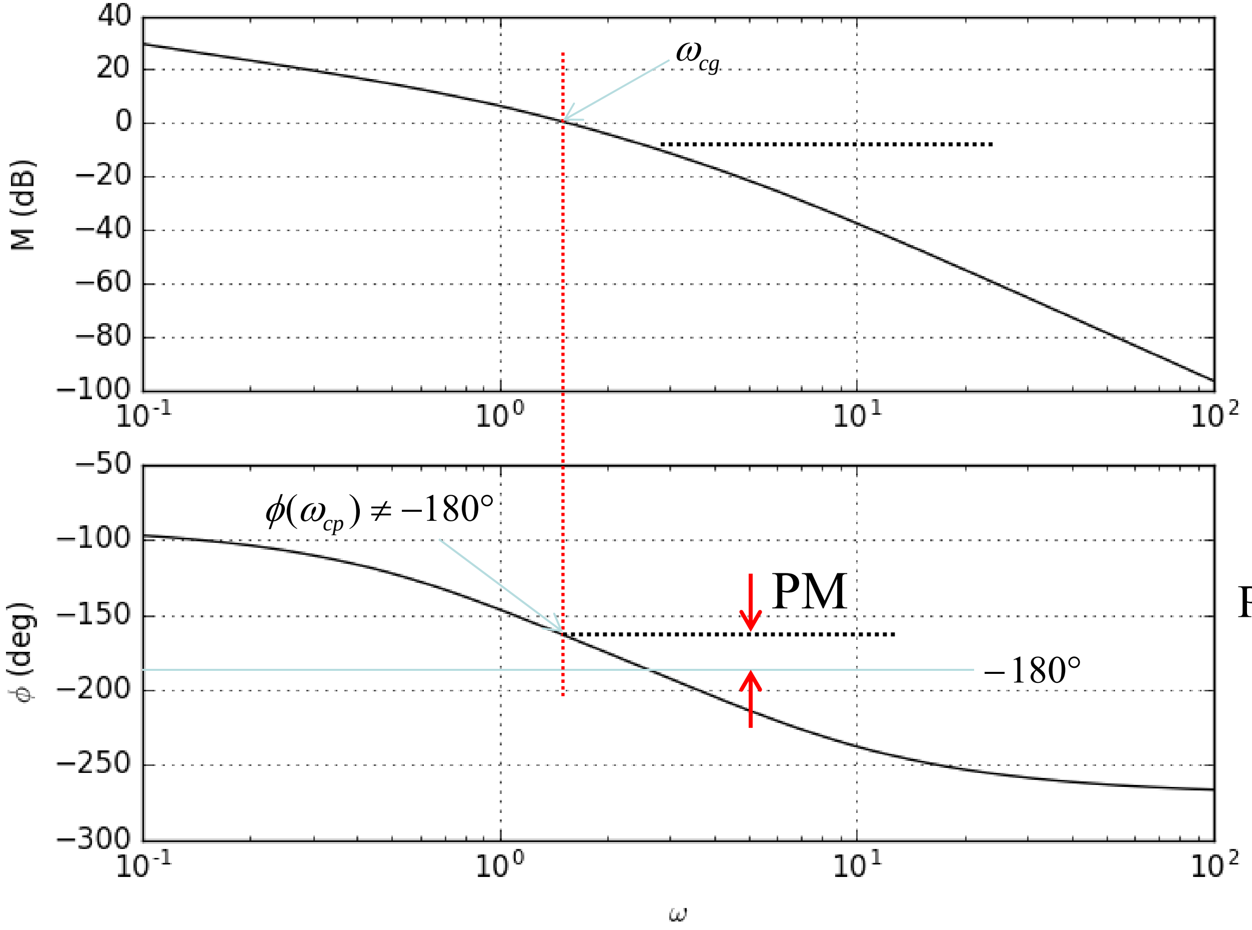


$$L(s) = \frac{15}{s(s^2 + 6s + 5)}$$

In this example, ω_{cg} & ω_{cp} can become equal due to delay (negative phase).

Phase Margin (PM): Variation in phase that can be allowed before the condition for marginal stability is met

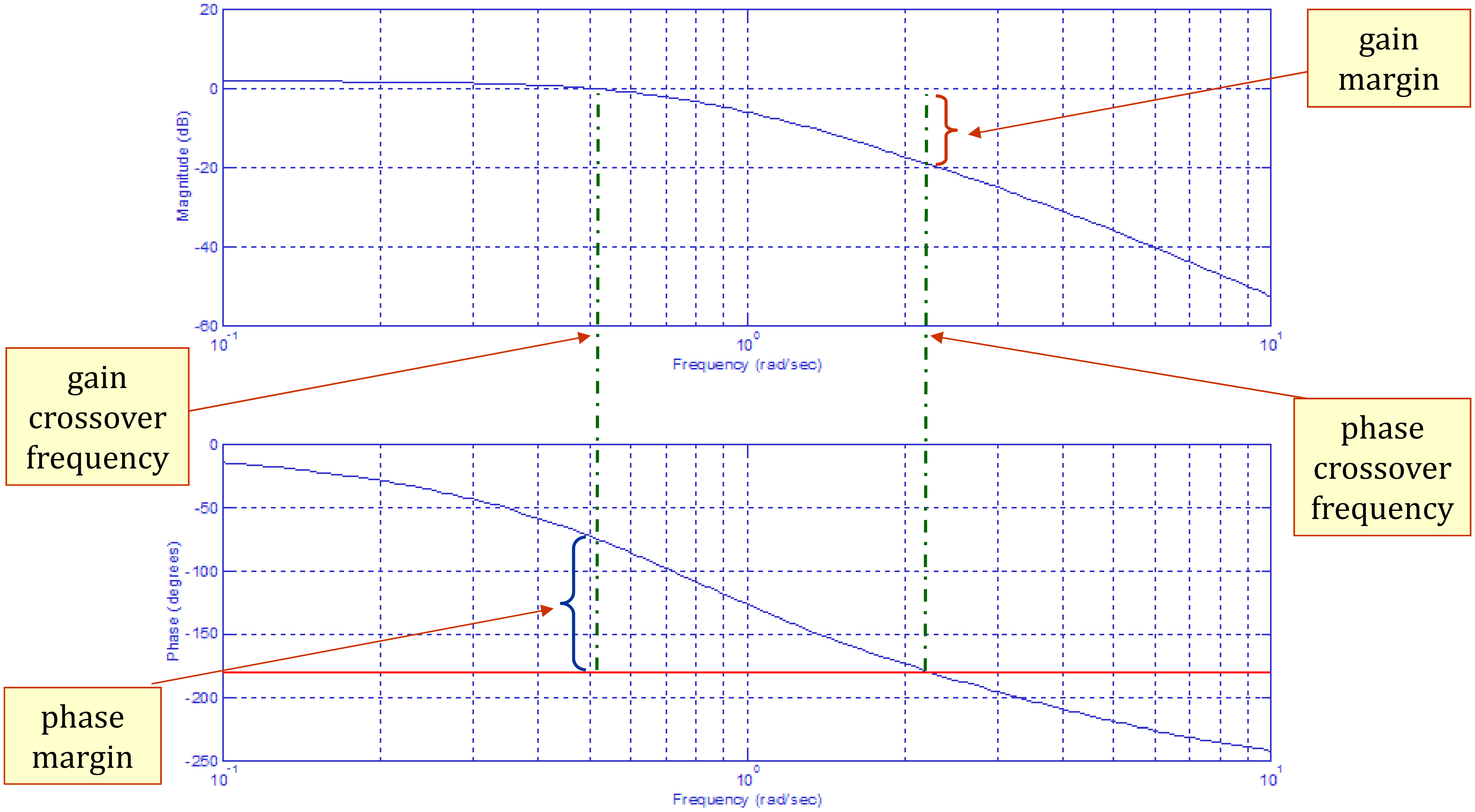
$$L(s) = \frac{15}{s(s^2+6s+5)}$$



GM expressed in dB

$$\text{PM} = \angle L(j\omega_{cg}) - (-180°)$$

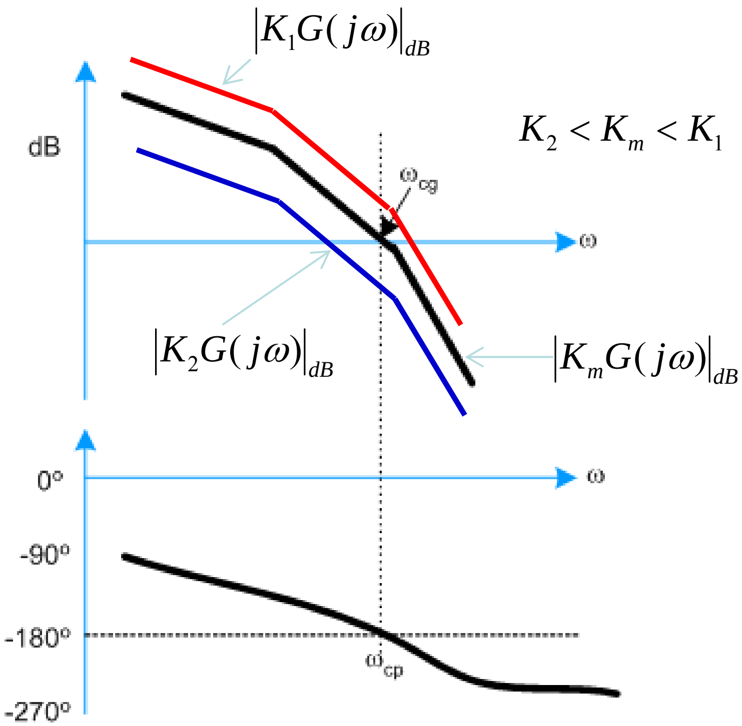
Gain and phase margins (recap)



The stability margins (GM and PM) can be used as indicators of closed loop stability

For most practical systems (second order or higher), increasing loop gain makes the system more oscillatory and eventually may lead to instability.

We use this assumption to find a relation between GM/PM and closed loop stability.



For $L(j\omega) = K_m G(j\omega)$

$$\omega_{cg} = \omega_{cp}$$

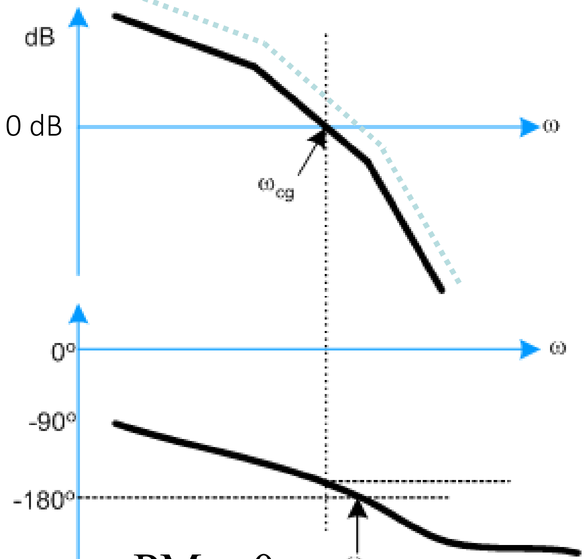
Closed loop will have poles on the jω-axis

Based on the assumption made above,

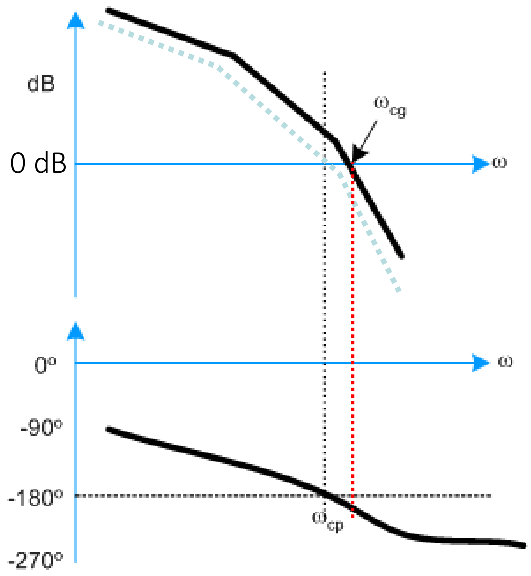
$K_1 G(j\omega)$ Unstable CL

$K_2 G(j\omega)$ Stable CL

$$\text{PM} = \angle L(j\omega_{cg}) - (-180^\circ)$$

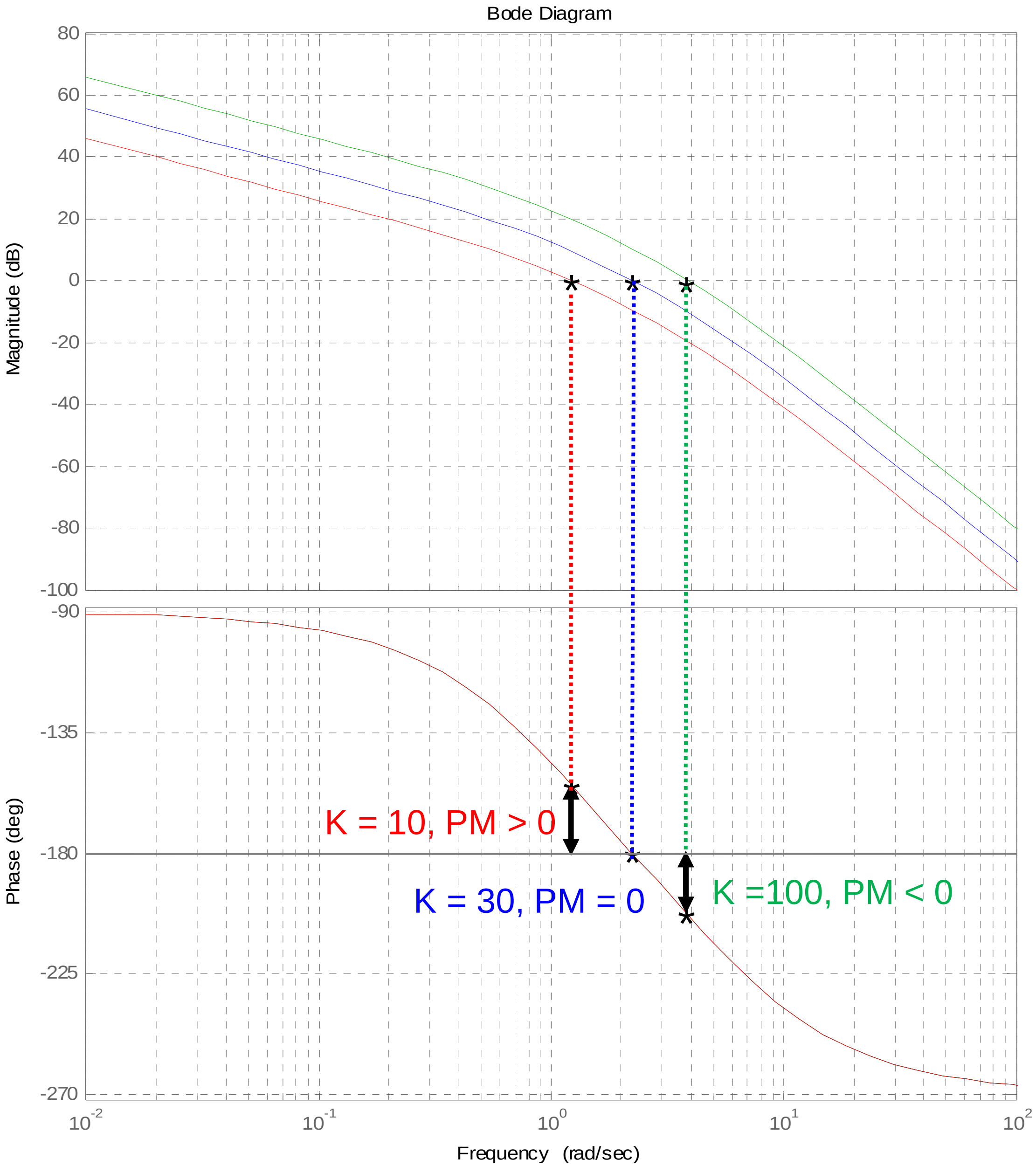


PM > 0



PM < 0

Stable	Unstable
PM > 0	**PM < 0**



Example

$$L(s) = \frac{K}{s(s^2 + 6s + 5)}$$

Closed-loop Poles

K = 10

$-5.42; -0.29 \pm j\,1.33$

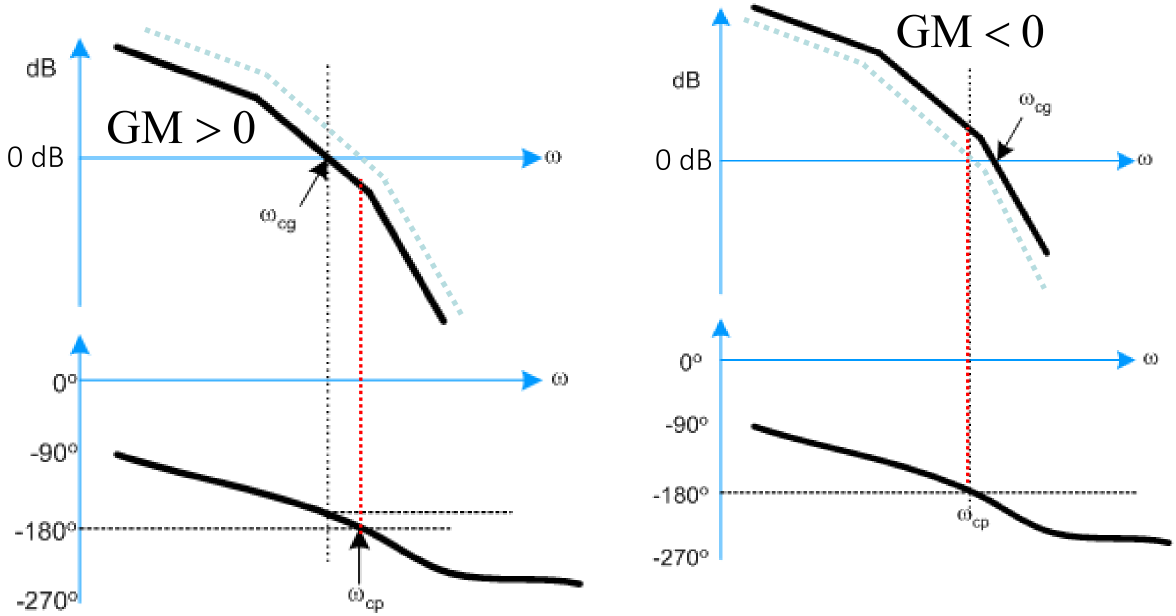
K = 30

$-6;\ 0 \pm j\,2.24$

K = 100

$$-7.22;\ \underbrace{0.61 \pm j\,3.67}_{\text{Unstable!}}$$

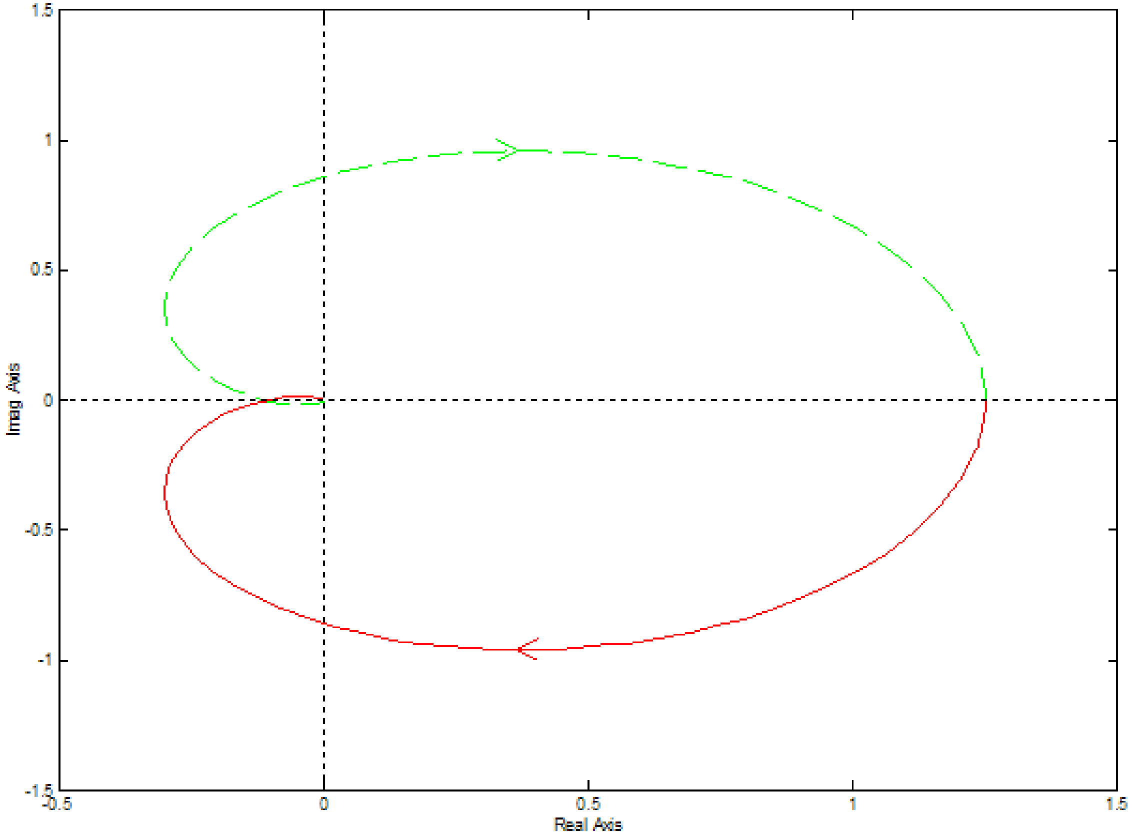
$$\text{GM} = 0 - \left| L(j\omega_{cp}) \right|_{dB} = -20\log\left| L(j\omega_{cp}) \right|$$



Stable	Unstable
GM > 0	**GM < 0**

Nyquist Plot

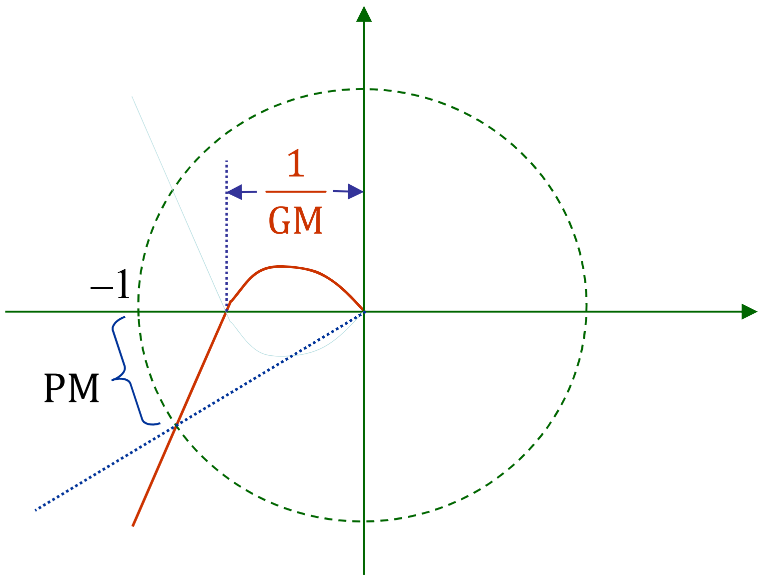
Instead of separating into magnitude and phase diagrams as in Bode plots, Nyquist plot maps the open-loop transfer function $L(s)$ directly onto a complex plane, e.g.,



Harry Nyquist
1889–1976

Gain and phase margins

The gain margin and phase margin can also be found from the Nyquist plot by zooming in the region in the neighbourhood of the origin.

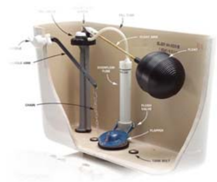


Remark: Gain margin is the maximum additional gain you can apply to the closed-loop system such that it will still remain stable. Similarly, phase margin is the maximum phase you can tolerate to the closed-loop system such that it will still remain stable.

Summary

Stable	CL poles on the jω-axis	Unstable
$K < K_m$	$K = K_m$	$K > K_m$
$\omega_{cg} < \omega_{cp}$	$\omega_{cg} = \omega_{cp}$	$\omega_{cg} > \omega_{cp}$
$GM > 0$	$GM = 0$	$GM < 0$
$\lvert L(j\omega_{cp})\rvert < 1$	$\lvert L(j\omega_{cp})\rvert = 1$	$\lvert L(j\omega_{cp})\rvert > 1$
$PM > 0$	$PM = 0$	$PM < 0$
$\angle L(j\omega_{cg})\rvert > -180°$	$\angle L(j\omega_{cg})\rvert = -180°$	$\angle L(j\omega_{cg})\rvert < -180°$

- These conditions are based on the assumption that increasing gain makes a marginally stable system unstable and decreasing gain makes it stable
- This assumption holds for most practical systems
- However, there are cases when it is violated



Transient Responses

Motivation

If we know how the frequency domain properties of the open loop transfer function $L(s)$ are related to the transient response of the closed loop, we can modify the loop by adding controller in a way to meet the transient specifications.

Learning Objectives

Ability to determine the target properties of the loop transfer function, given the desired specifications on transient response

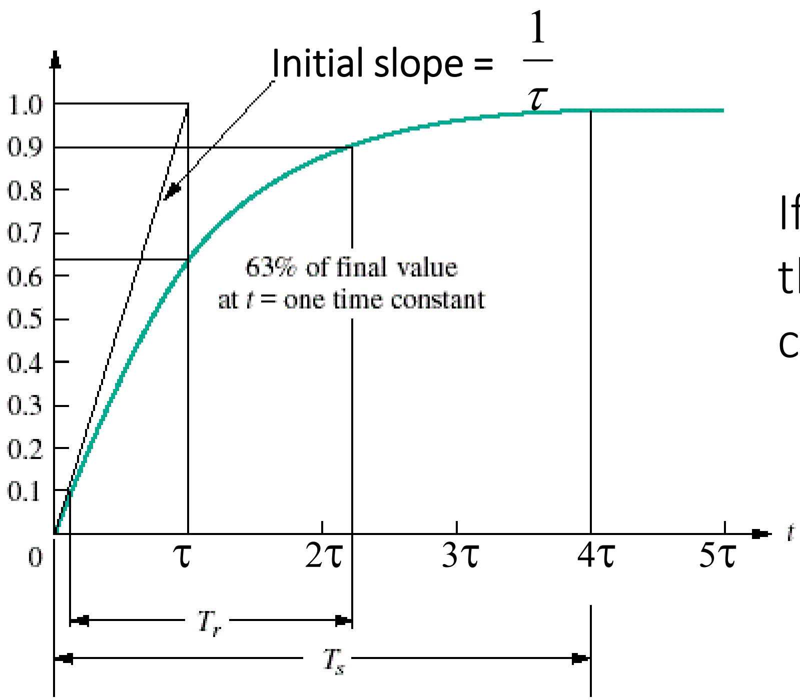
We have learnt how to modify the loop transfer function to meet specifications on steady-state error with polynomial input signal.

Taking the process output from one level to another is not the only need.

How it reaches the final level is just as important.

Transient profile is used to define the trajectory of the output in transition.

Transient specifications are typically described in terms of step response.

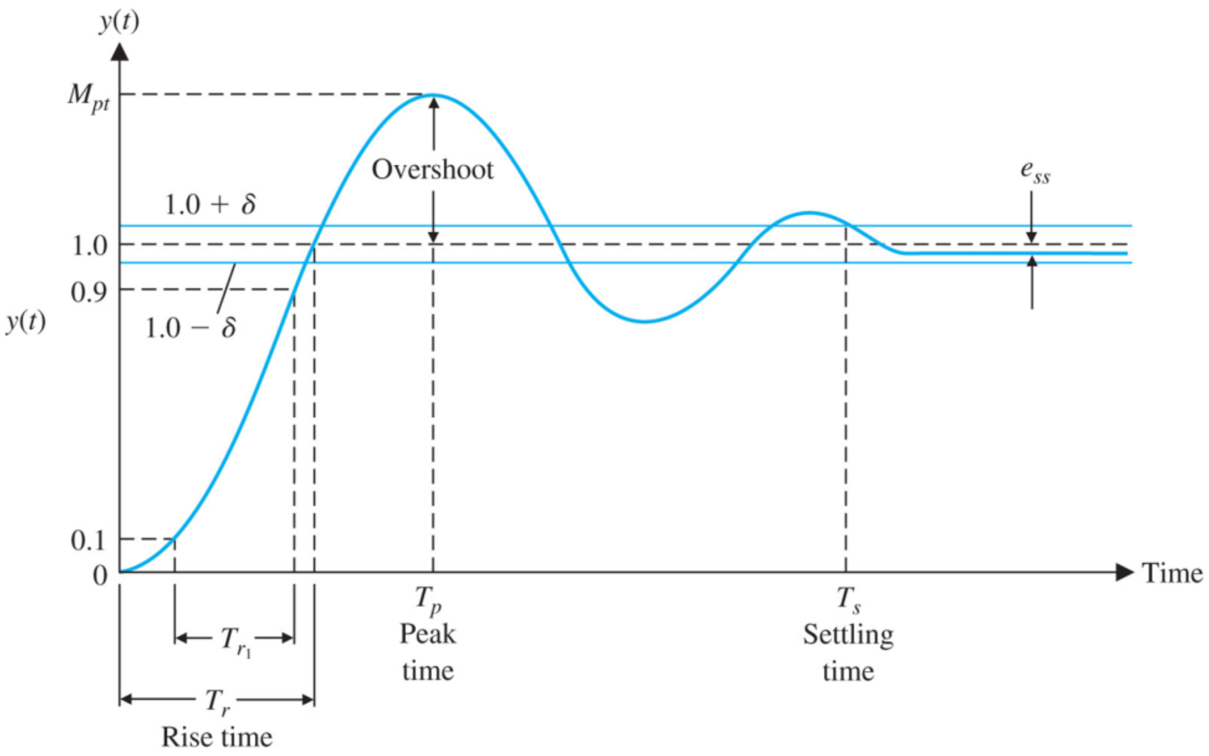


If the closed loop is dominant 1st-order, then its transient response is characterized by the time constant.

If the closed loop transfer function is dominant 2nd-order, then its transient is characterized by the rise time, the settling time and the overshoot.

In this note, closed loop represented by dominant 2nd-order transfer function (two complex poles) is considered for quantifying the frequency-domain specifications.

$$G_{CL}(s) = \frac{K\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$



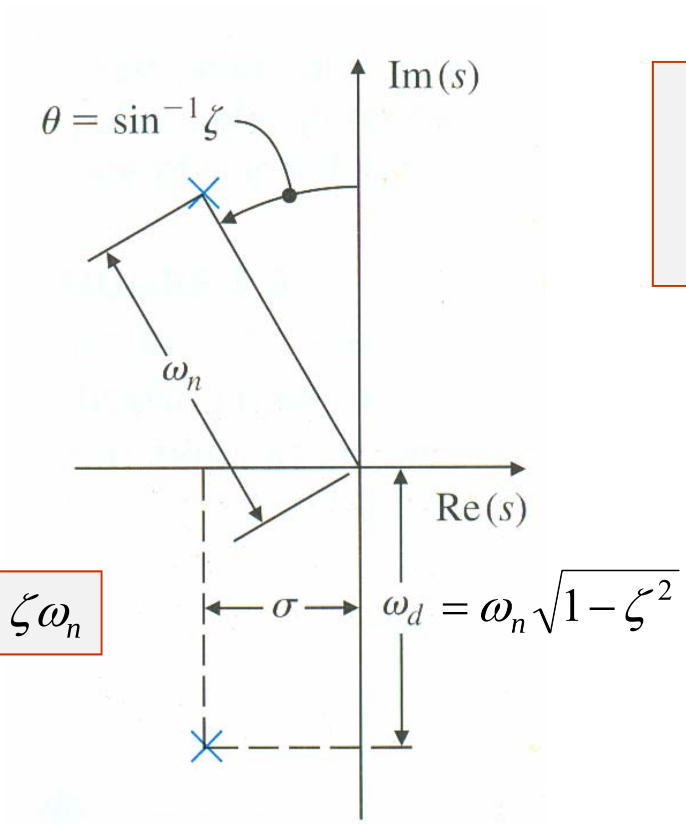
Step response of a typical 2nd order system

Unit step response for the 2nd order prototype

This is very important for the 2nd part of this course in designing a meaningful control system.

We consider the 2nd order prototype

$$H(s)=\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}=\frac{\omega_n^2}{(s+\zeta\omega_n)^2+\omega_n^2(1-\zeta^2)}$$



$\sigma=\zeta\omega_n$

ζ is the damping ratio of the system

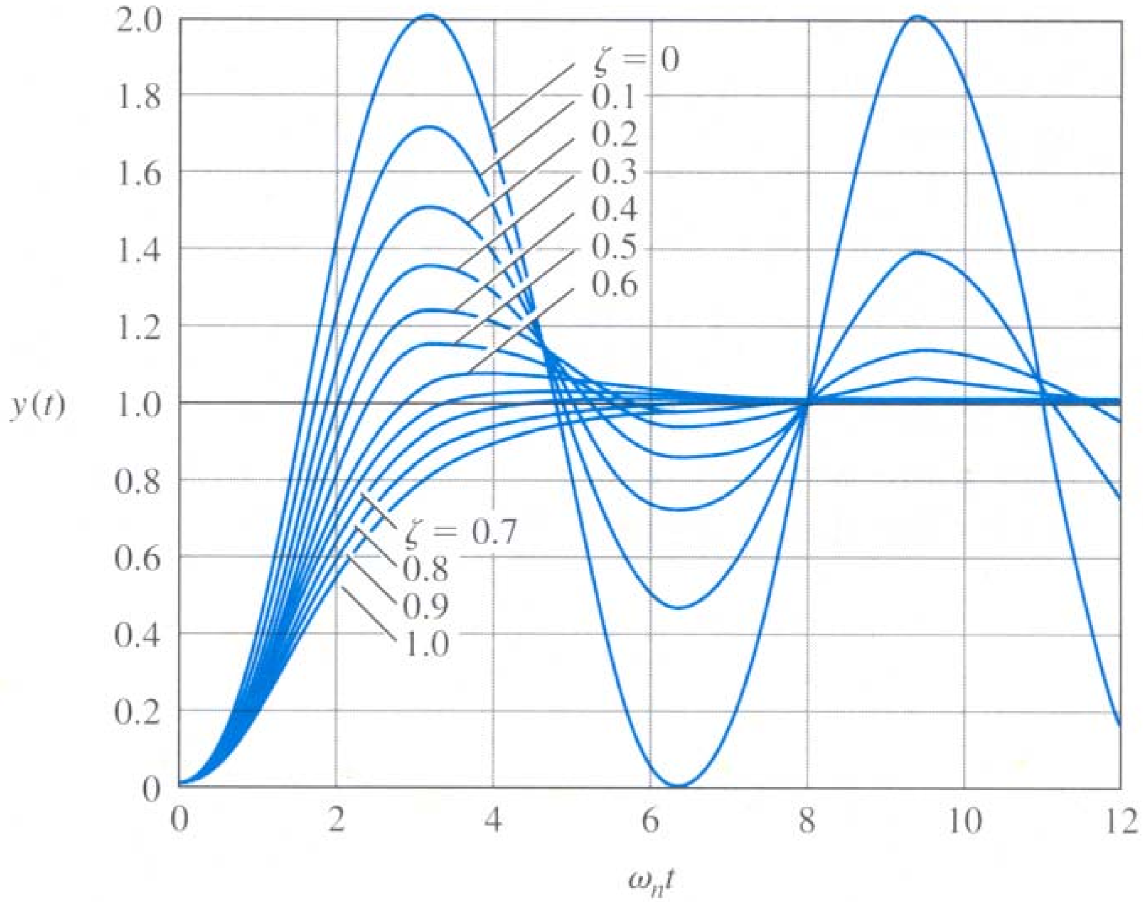
ω_n is the natural frequency

It can be shown that its unit step response is given as

$$y(t)=1-e^{-\sigma t}\left(\cos\omega_d t+\frac{\sigma}{\omega_d}\sin\omega_d t\right)$$

Unit step response for the 2nd order prototype (cont.)

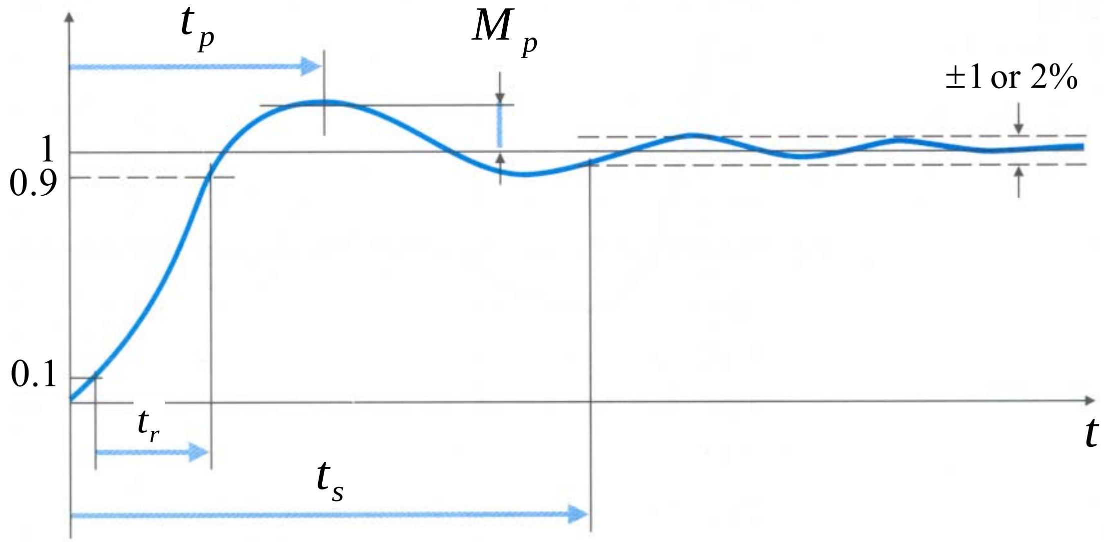
Graphically,

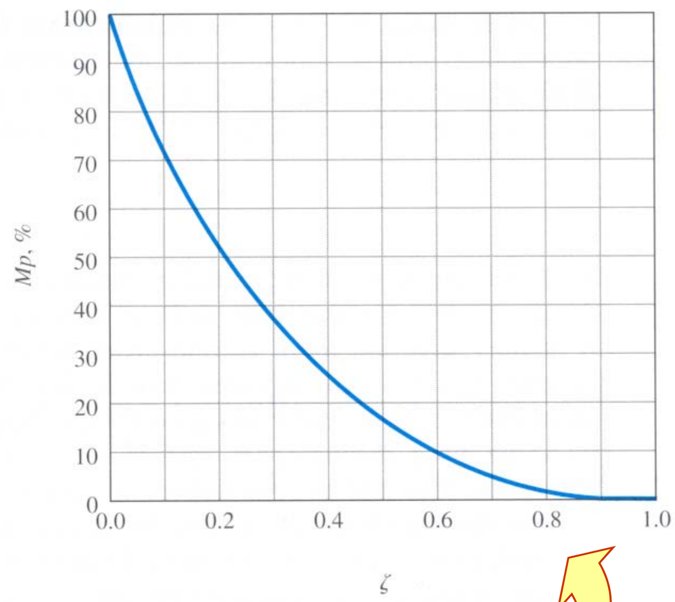


It can be observed that the smaller damping ratio yields larger overshoot.

Unit step response for the 2nd order prototype (cont.)

The typical step response can be depicted as





$$y(t) = 1 - e^{-\sigma t}\left(\cos\omega_d t + \frac{\sigma}{\omega_d}\sin\omega_d t\right)$$

overshoot $M_p = e^{-\pi\zeta/\sqrt{1-\zeta^2}}$

rise time $t_r \cong \dfrac{1.8}{\omega_n}$

1% settling time $t_s \cong \dfrac{4.6}{\zeta\omega_n}$

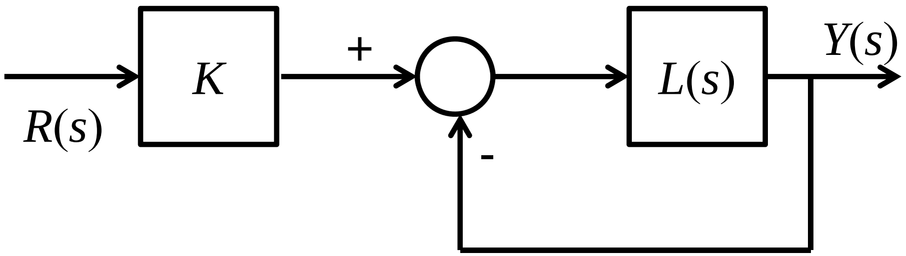
2% settling time $t_s \cong \dfrac{4}{\zeta\omega_n}$

peak time $t_p \cong \dfrac{\pi}{\omega_n\sqrt{1-\zeta^2}}$

all in sec

A loop transfer function with two poles gives a closed loop transfer function of this type

$$G_{CL}(s) = K\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$



$$G_{CL}(s) = K\frac{L(s)}{1+L(s)}$$

$$L(s) = \frac{\omega_n^2}{s(s+2\zeta\omega_n)}$$

$$G_{CL}(s) = K\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$L(s) = \frac{1}{(s+a)(s+b)}$$

$$G_{CL}(s) = K\frac{1}{s^2 + (a+b)s + ab + 1}$$

Examples

Let’s find the gain-crossover frequency and the phase margin of the loop transfer function

$$L(s)=\frac{\omega_n^2}{s(s+2\zeta\omega_n)}$$

$$L(j\omega)=\frac{\omega_n^2}{(j\omega)(j\omega+2\zeta\omega_n)} \qquad L(j\omega)=\frac{1}{\dfrac{j\omega}{\omega_n}\left(\dfrac{j\omega+2\zeta\omega_n}{\omega_n}\right)}=\frac{1}{j\dfrac{\omega}{\omega_n}\left(j\dfrac{\omega}{\omega_n}+2\zeta\right)}$$

$$\upsilon=\frac{\omega}{\omega_n} \quad \Rightarrow \quad L(j\upsilon)=\frac{1}{j\upsilon\left(j\upsilon+2\zeta\right)}$$

$$\left|L(j\upsilon)\right|=\frac{1}{\upsilon\sqrt{\upsilon^2+4\zeta^2}}$$

Gain-crossover frequency can be obtained by solving

$$\left|L(j\upsilon)\right|=\frac{1}{\upsilon\sqrt{\upsilon^2+4\zeta^2}}=1$$

$$\frac{1}{\upsilon_{cg}\sqrt{\upsilon_{cg}^2+4\zeta^2}}=1 \quad \Rightarrow \quad \upsilon_{cg}\sqrt{\upsilon_{cg}^2+4\zeta^2}=1$$

$$\Rightarrow \quad \upsilon_{cg}^4+4\zeta^2\upsilon_{cg}^2-1=0 \quad \Rightarrow \quad \upsilon_{cg}^2=-2\zeta^2+\sqrt{4\zeta^4+1}$$

$$\Rightarrow \quad \upsilon_{cg}=\sqrt{-2\zeta^2+\sqrt{4\zeta^4+1}}$$

$$\upsilon_{cg}=\frac{\omega_{cg}}{\omega_n} \quad \Rightarrow \quad \omega_{cg}=\omega_n\sqrt{-2\zeta^2+\sqrt{4\zeta^4+1}}$$

Conclusion:

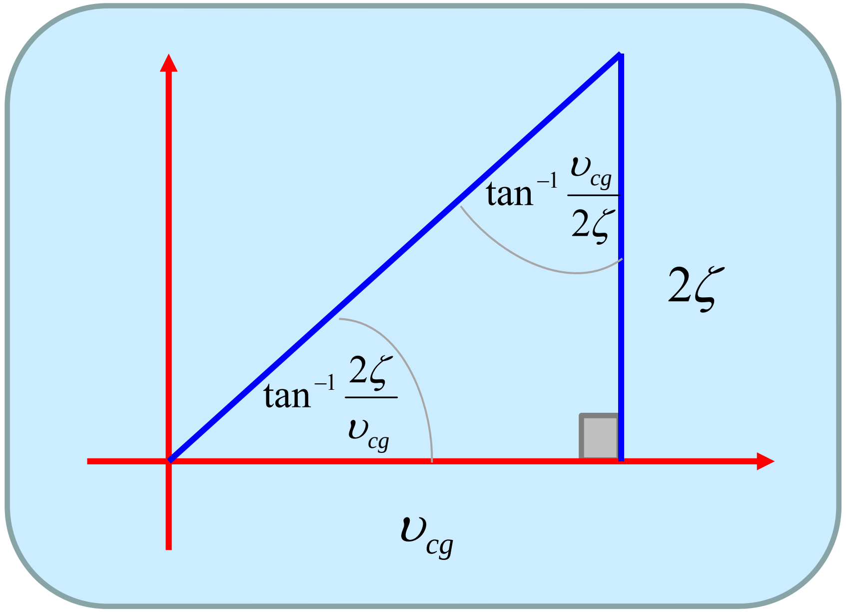
Higher the gain-crossover frequency of the loop transfer function, higher is the natural frequency (ω_n) of the dominant 2nd-order CL poles $\Rightarrow$ shorter rise time

This can be used as a design guideline

$$L(j\upsilon)=\frac{1}{j\upsilon(j\upsilon+2\zeta)},\quad \upsilon=\frac{\omega}{\omega_n}$$

$$\angle L(j\upsilon)=-90^\circ-\tan^{-1}\frac{\upsilon}{2\zeta}$$

$$\angle L(j\upsilon_{cg})=-90^\circ-\tan^{-1}\frac{\upsilon_{cg}}{2\zeta}$$



$$\mathrm{PM}=180^\circ+\angle L(j\upsilon_{cg})$$

$$=90^\circ-\tan^{-1}\frac{\upsilon_{cg}}{2\zeta}$$

$$=\tan^{-1}\frac{2\zeta}{\upsilon_{cg}}$$

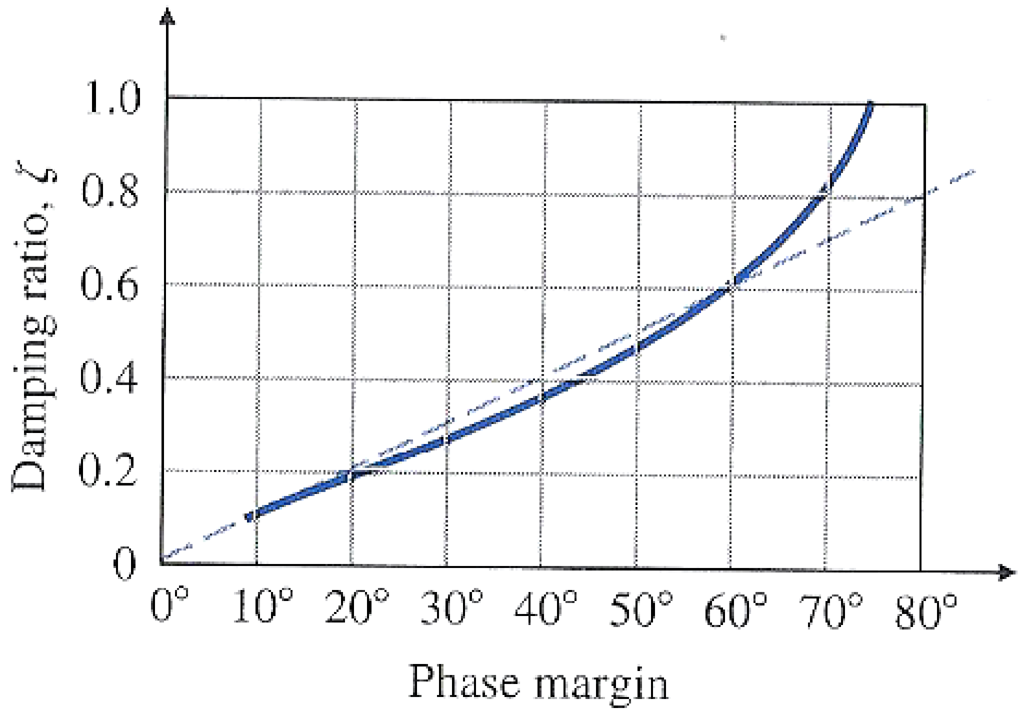
$$\mathrm{PM}=\tan^{-1}\frac{2\zeta}{\sqrt{-2\zeta^2+\sqrt{4\zeta^4+1}}}$$

Conclusion:

PM is related to the damping factor of the dominant closed loop pole

$$L(s) = \frac{\omega_n^2}{s(s + 2\zeta\omega_n)}$$

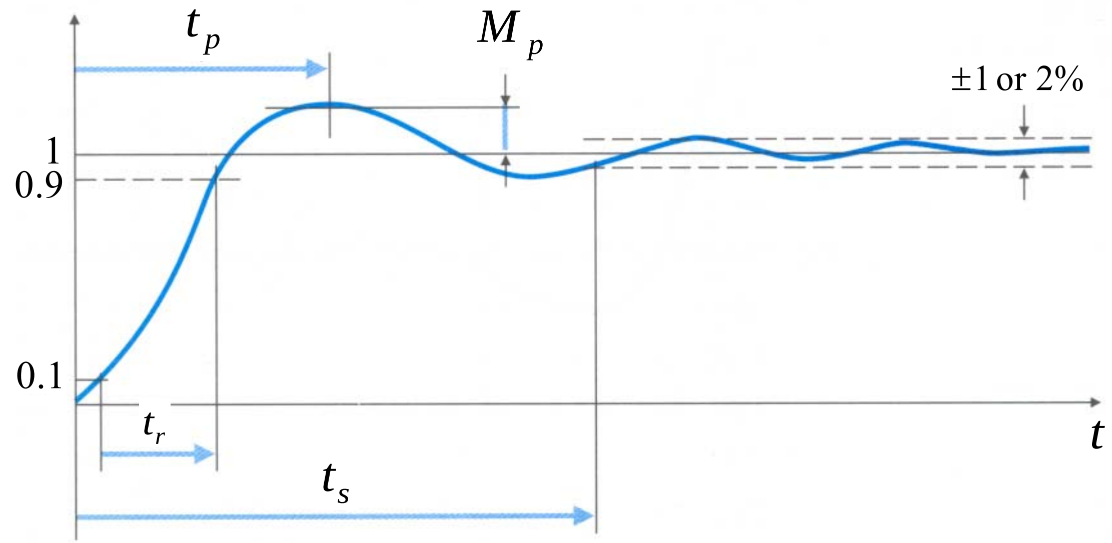
$$\text{PM} = \tan^{-1} \frac{2\zeta}{\sqrt{-2\zeta^2 + \sqrt{4\zeta^4 + 1}}}$$



The relation between PM and the damping factor of the closed loop poles can be approximated by the following for PM < 60°

$$\zeta \cong \frac{\text{PM}}{100}$$ (This is commonly used as the design formula)

This example shows that if we try to meet the time-domain specifications, the frequency domain parameters (ω_{cg} and PM) are also fixed…



overshoot $M_p = e^{-\pi\zeta/\sqrt{1-\zeta^2}}$

rise time $t_r \cong \dfrac{1.8}{\omega_n}$

2% settling time $t_s \cong \dfrac{4}{\zeta\omega_n}$

peak time $t_p \cong \dfrac{\pi}{\omega_n\sqrt{1-\zeta^2}}$

$\boxed{\text{Time-domain specs}}$

Frequency-domain parameters

$$\omega_{cg} = \omega_n\sqrt{-2\zeta^2 + \sqrt{4\zeta^4 + 1}}$$

$$\mathrm{PM} = \tan^{-1}\frac{2\zeta}{\sqrt{-2\zeta^2 + \sqrt{4\zeta^4 + 1}}}$$

Lead and Lag Compensators

Motivation

- Feedback design is usually carried out in stages starting with the attempt to meet the steady-state error which can be done by adjusting DC-gain (gain-adjustment phase). However, it doesn’t take into consideration the transient response or stability.
- A compensator is a circuit/system that compensates for the discrepancy remaining after the gain-adjustment stage.

Learning Objectives

1) Ability to explain the role of compensator in a feedback system
2) Ability to explain the effects of 1^{st}-order compensator on the frequency response of the loop transfer function
3) Ability to recognize the difference between two types of 1^{st}-order compensator

Recap:

The steady state error with polynomial signals depends on different error constants

For step input	For ramp input	For parabolic input
$e_{ss} = \dfrac{1}{1+k_p},$	$e_{ss} = \dfrac{1}{k_v},$	$e_{ss} = \dfrac{1}{k_a},$
$k_p = \lim\limits_{s\to 0} L(s)$	$k_v = \lim\limits_{s\to 0} sL(s)$	$k_a = \lim\limits_{s\to 0} s^2 L(s)$

- These error constants are limiting values of certain transfer function as $s \to 0$

  In frequency domain, they represent the DC gain of the corresponding transfer function

$$k_p = \lim_{s\to 0} L(s) \quad \overset{s=j\omega}{\Rightarrow} \quad k_p = \lim_{\omega\to 0} L(j\omega)$$

Recap:

Parameters that characterize the transient response (for example, ζ and ω_n) are related to the gain-crossover frequency (ω_{cg}) and phase-margin (PM) of the loop transfer function.

$$\omega_{cg} = \omega_n \sqrt{-2\zeta^2 + \sqrt{4\zeta^4 + 1}}$$

$$\mathrm{PM} = \tan^{-1} \frac{2\zeta}{\sqrt{-2\zeta^2 + \sqrt{4\zeta^4 + 1}}} \quad \Rightarrow \quad \zeta \cong \frac{\mathrm{PM}}{100}, \quad \mathrm{PM} < 60^\circ$$

Phase margin is an indicator of stability and stability margin.

Design in stages:

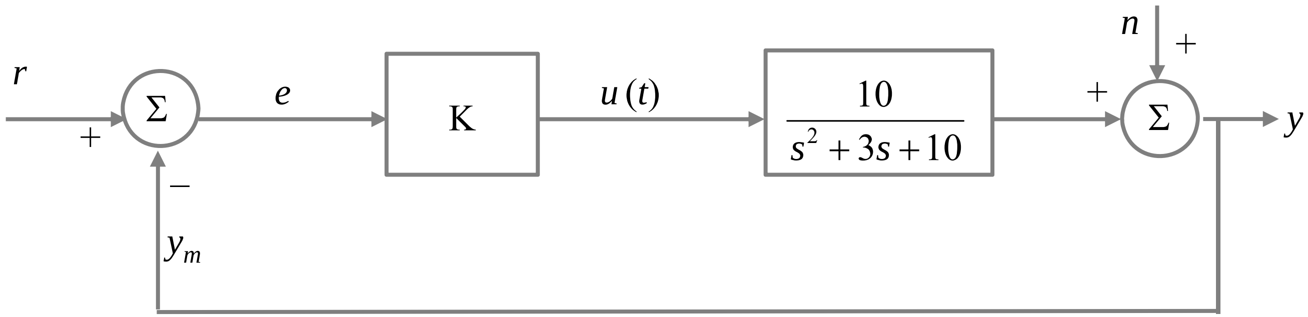
- Gain adjustment to meet the steady-state error specifications

  Effect in frequencies around ω_{cg} are not considered at this stage

- Design compensator to modify the loop around crossover frequency

  ω_{cg} and PM can also be changed by adjusting the gain but such adjustment affects the DC-gain as well and result of gain-adjustment stage is therefore undone.

Exercise 3-2 revisited:



Desired steady-state error is less the 10% for step input:

$$e_{ss,step} < 0.1$$

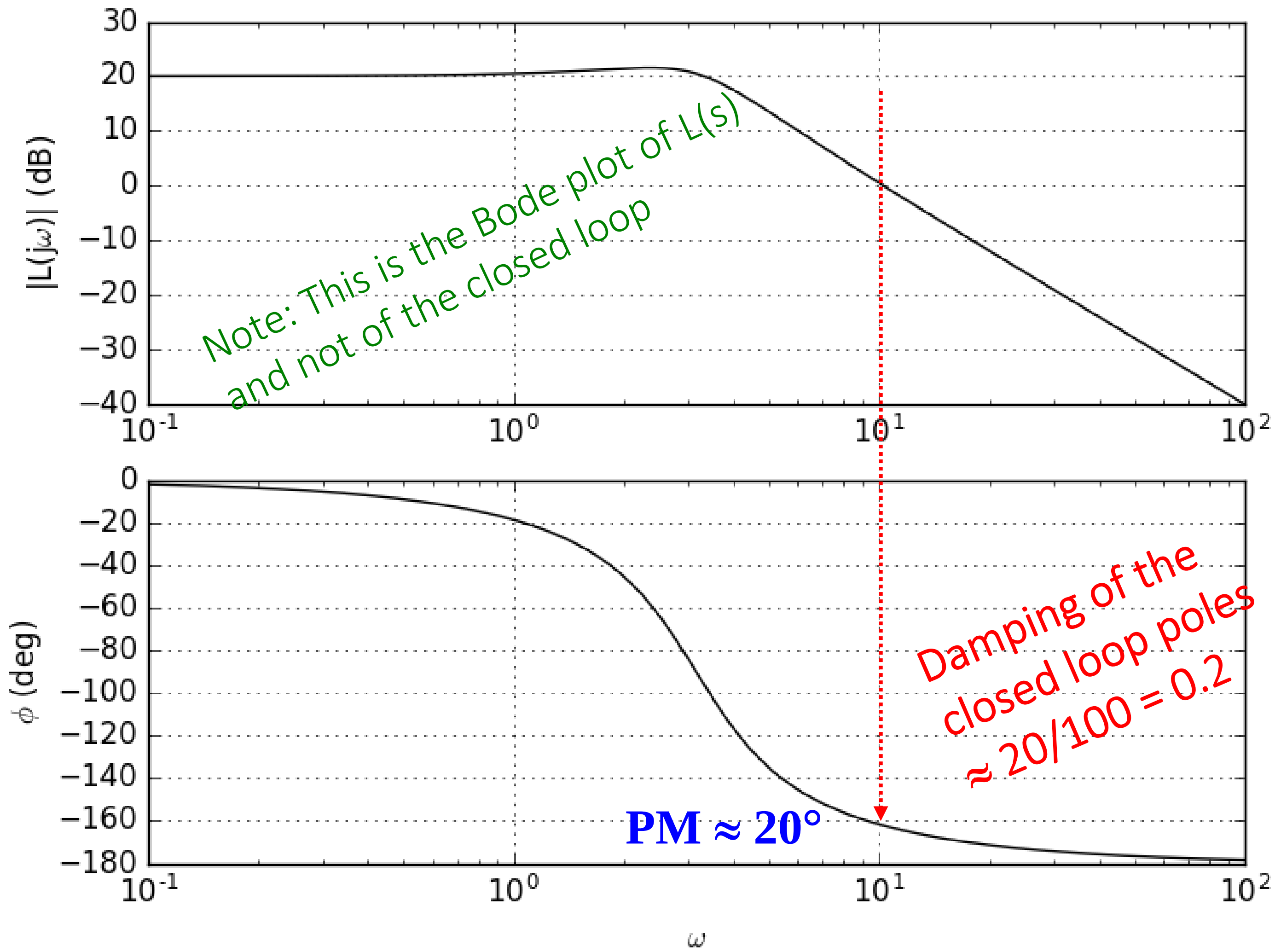
$$\frac{1}{1+k_p} < 0.1, \quad \Rightarrow \quad k_p > 9$$

Position error constant: $$k_p = \lim_{s \to 0} \frac{10K}{s^2 + 3s + 10} = K$$

$$k_p > 9 \qquad \Rightarrow \qquad K > 9, \quad \text{let} \quad K = 10$$

$$L(s) = \frac{100}{s^2 + 3s + 10}$$

Bode Plot of $L(s)$ of this example:

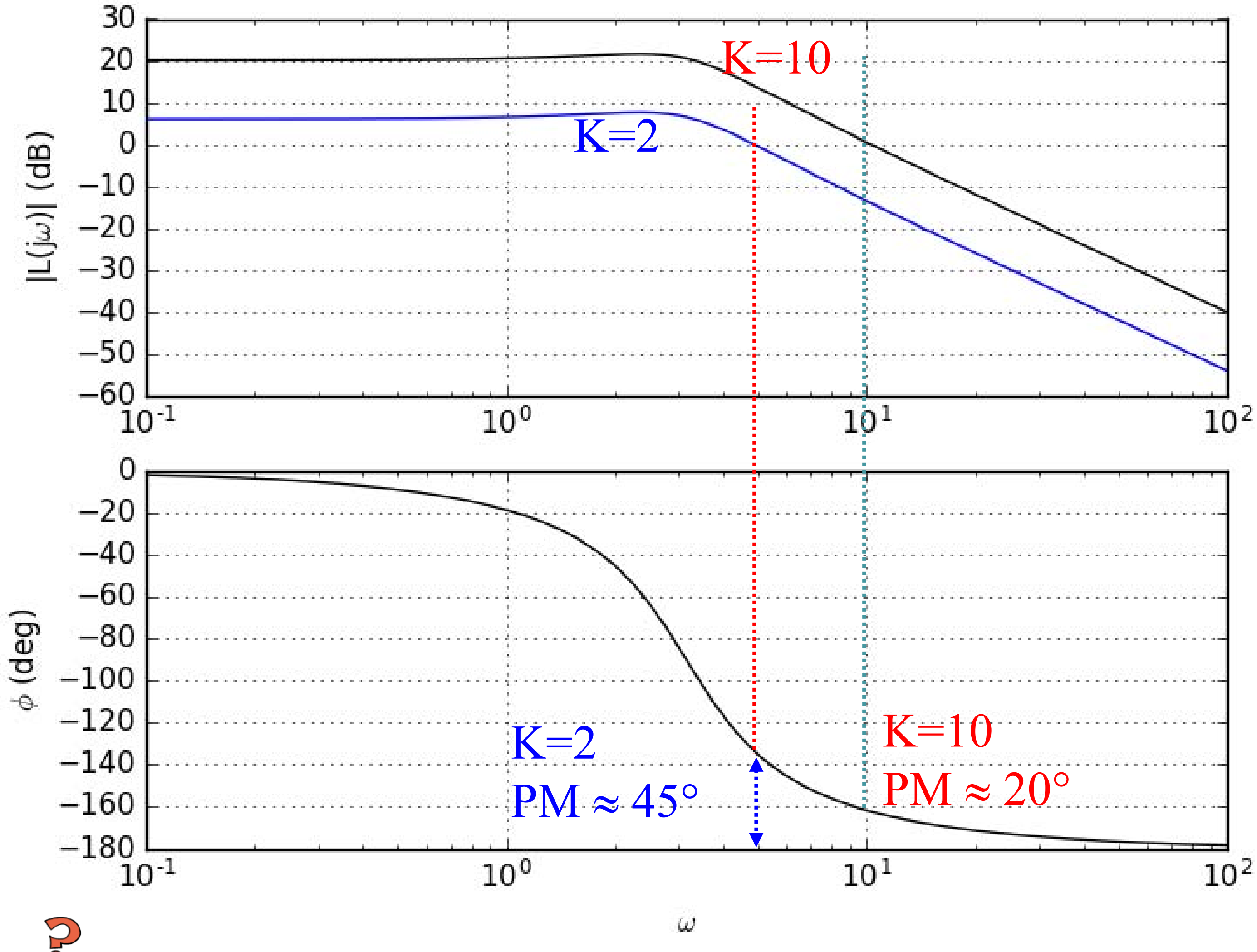


Can you find PM numerically?

What kind of transient response will the closed loop give?

What is the GM for this case?

What can be done to achieve higher PM?

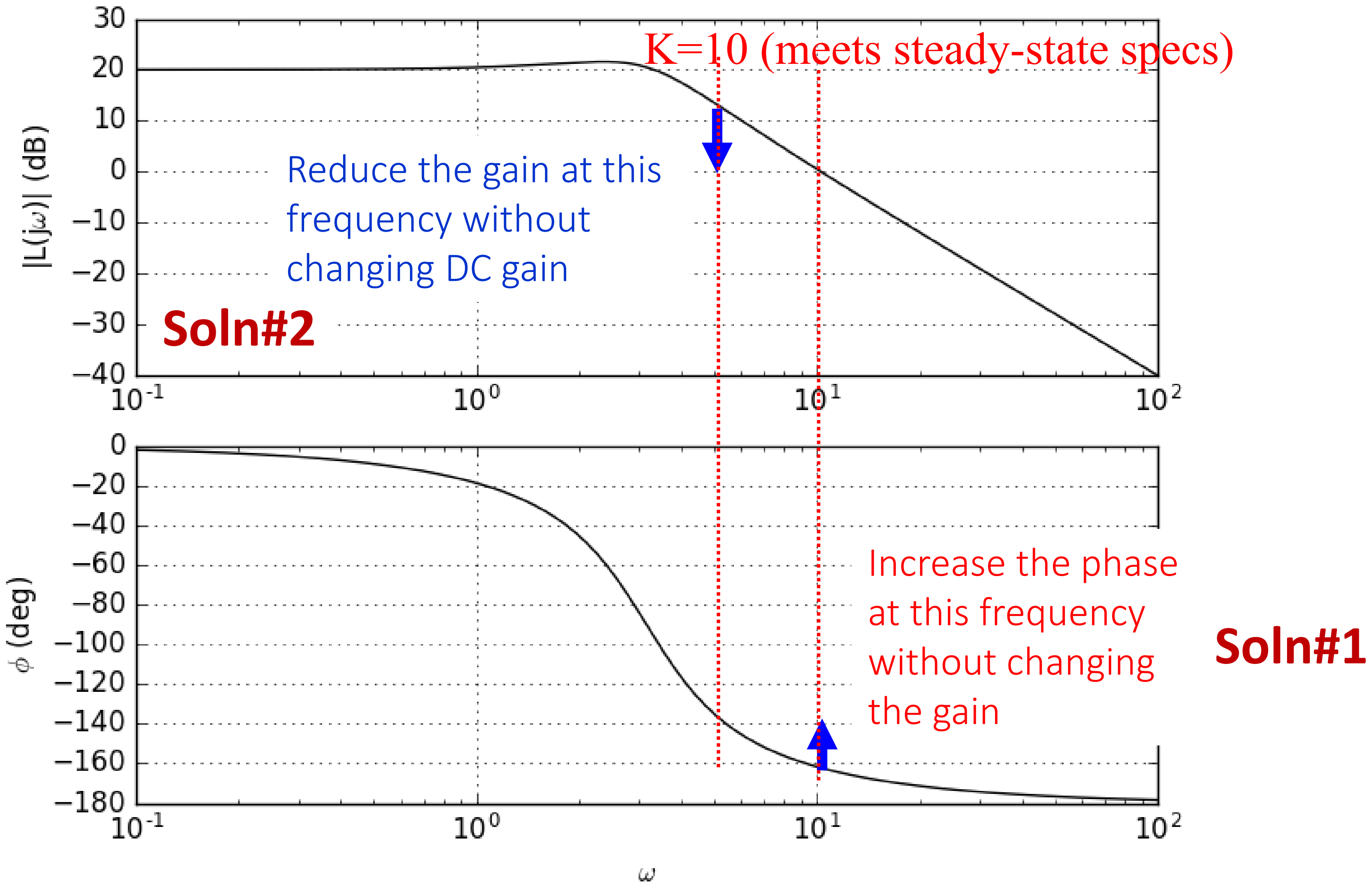


Use lower value of K so that ω_{cg} is shifted to a lower frequency

Change in K has no effect on phase

What will happen to steady-state error performance if lower value of K is used?

Ideally what we want?

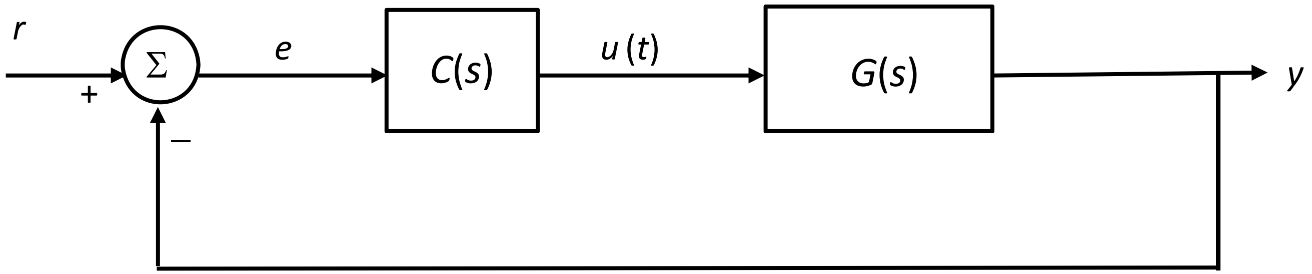


These solutions are not feasible by varying K.

Compensator gives a realizable approximation of these solutions by modifying gain/phase in a selected range of frequency.

Solution?

Add a compensator…



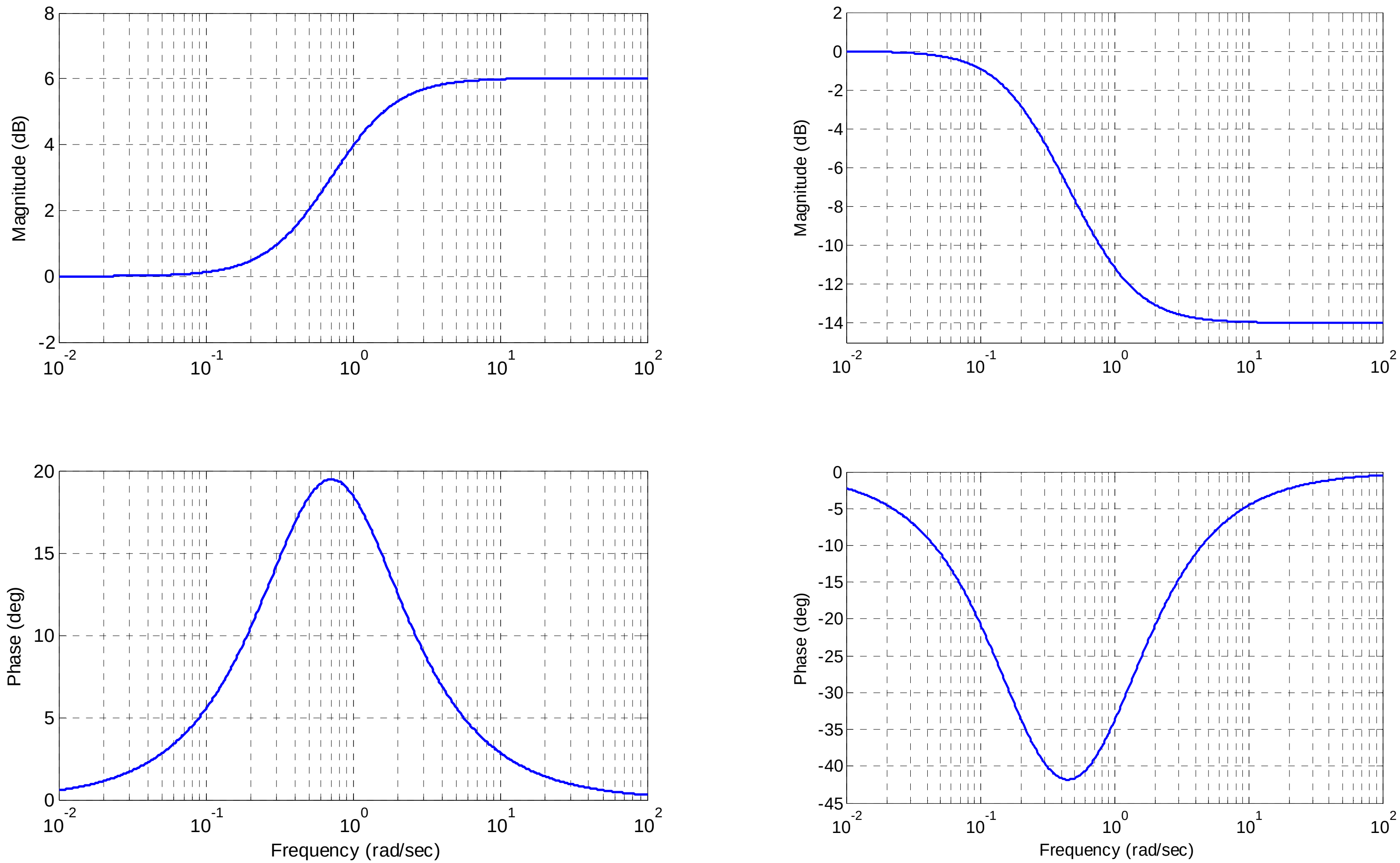
$$C(s) = \frac{Ts+1}{\alpha Ts+1}, \quad T > 0$$

- If $0 < \alpha < 1$, $C(s)$ is called a lead compensator as it will add a positive phase to the resulting open loop transfer function $L(s)$.

- If $\alpha > 1$, $C(s)$ is called a lag compensator as it will add a negative phase to the resulting open loop transfer function $L(s)$.

Compensators: Sketch the Bode plot of the following two transfer functions:

$$C_1(s) = \frac{2s+1}{s+1} = \frac{s/0.5 + 1}{s+1} \qquad\qquad C_2(s) = \frac{s+1}{5s+1} = \frac{s+1}{s/0.2 + 1}$$



Transfer functions shown on the previous slide can be generalized in this form

$$C(s) = \frac{Ts+1}{\alpha Ts+1}, \quad T>0, \quad 0<\alpha<1$$

Such a transfer function is a 1st-order compensator.

- The value of T determines the corner frequency of the zero (let's name it ω_z)

$$\omega_z = \frac{1}{T}$$

- The values of α and T determine the corner frequency of the pole (let's name it ω_p)

$$\omega_p = \frac{1}{\alpha T}$$

- The value of α determines whether $\omega_z > \omega_p$ or $\omega_p > \omega_z$ and also the separation between these two frequencies

$$\omega_z = \alpha \omega_p$$

Two examples:

- $\alpha < 1$

$$C_1(s) = \frac{2s+1}{s+1} = \frac{2s+1}{0.5\times 2s+1}$$

$$T = 2, \quad \omega_z = \frac{1}{2} = 0.5 \quad \text{rad/s}$$

$$\omega_p > \omega_z$$

$$\alpha = 0.5, \quad \omega_p = \frac{1}{0.5\times 2} = 1.0 \text{ rad/s}$$

- $\alpha > 1$

$$C_2(s) = \frac{s+1}{5s+1} = \frac{s+1}{5\times s+1}$$

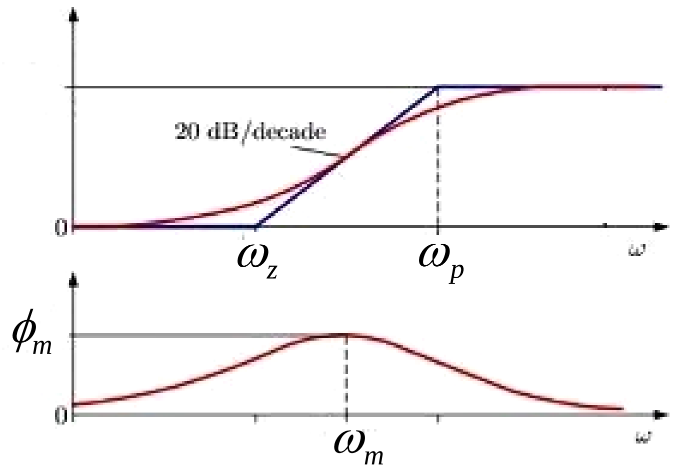
$$T = 1, \quad \omega_z = \frac{1}{1} = 1.0 \quad \text{rad/}s$$

$$\omega_z > \omega_p$$

$$\alpha = 5, \quad \omega_p = \frac{1}{5\times 1} = 0.2 \text{ rad/s}$$

Compensator with 0 < α < 1

$$C(s) = \frac{Ts+1}{\alpha Ts+1}, \quad 0 < \alpha < 1$$

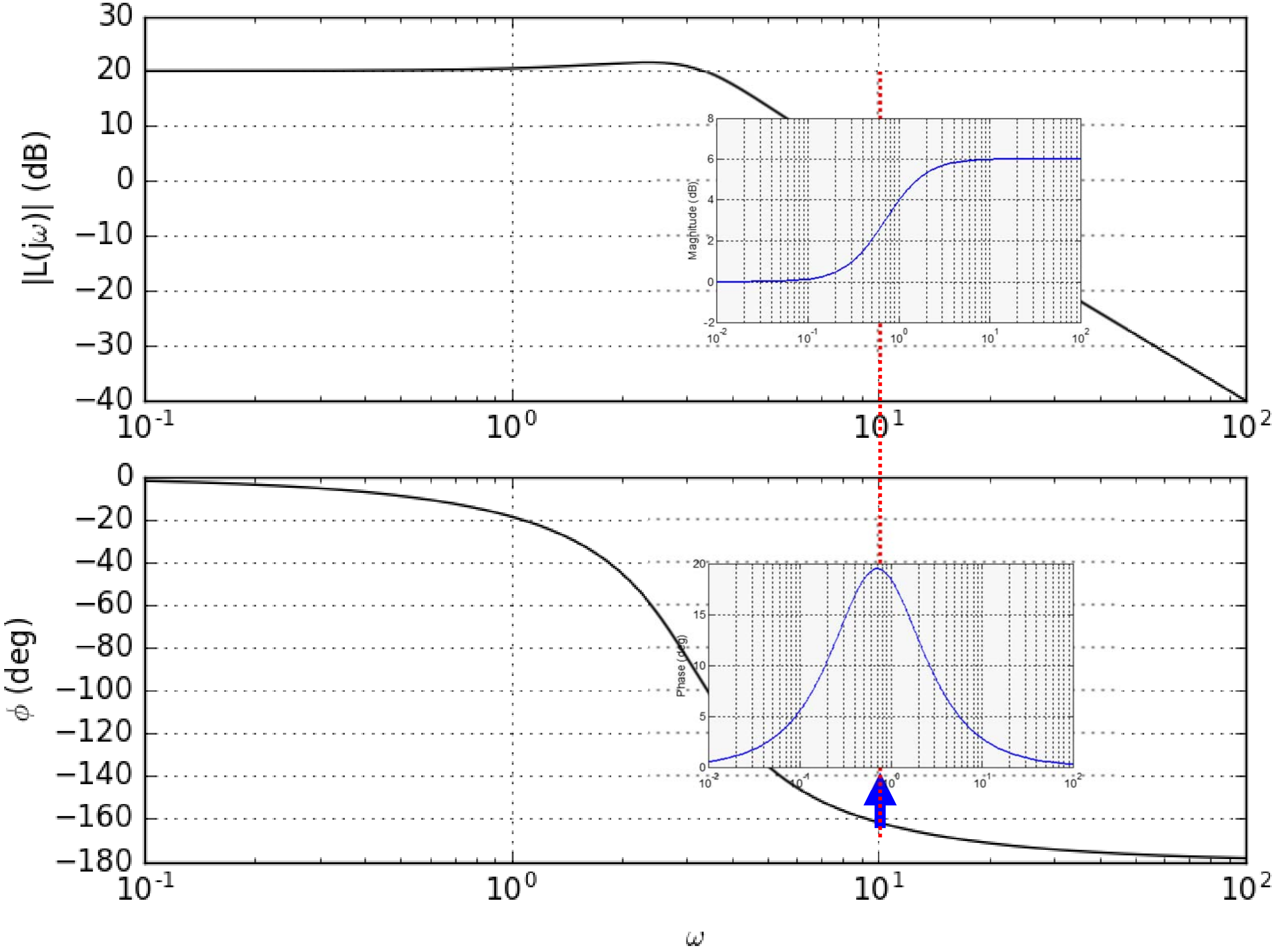


It adds positive phase (phase lead). So it is called Lead Compensator.

It can be used to improve phase margin by choosing ω_m of the compensator at a frequency near gain-crossover frequency of the loop before compensation. Positive phase of the lead compensator is the feature that we want to use.

Soln#1 can be approximated by this!

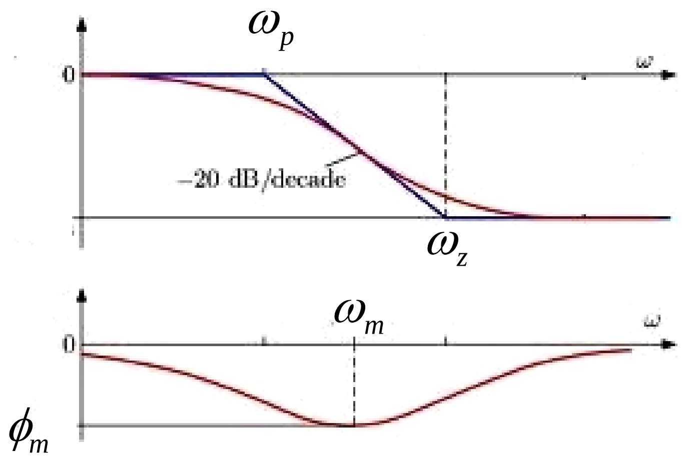
Soln#1: By a lead compensation…



Increase the phase at this frequency without changing much in the gain

Compensator with $\alpha > 1$

$$C(s) = \frac{Ts+1}{\alpha Ts+1}, \quad \alpha > 1$$



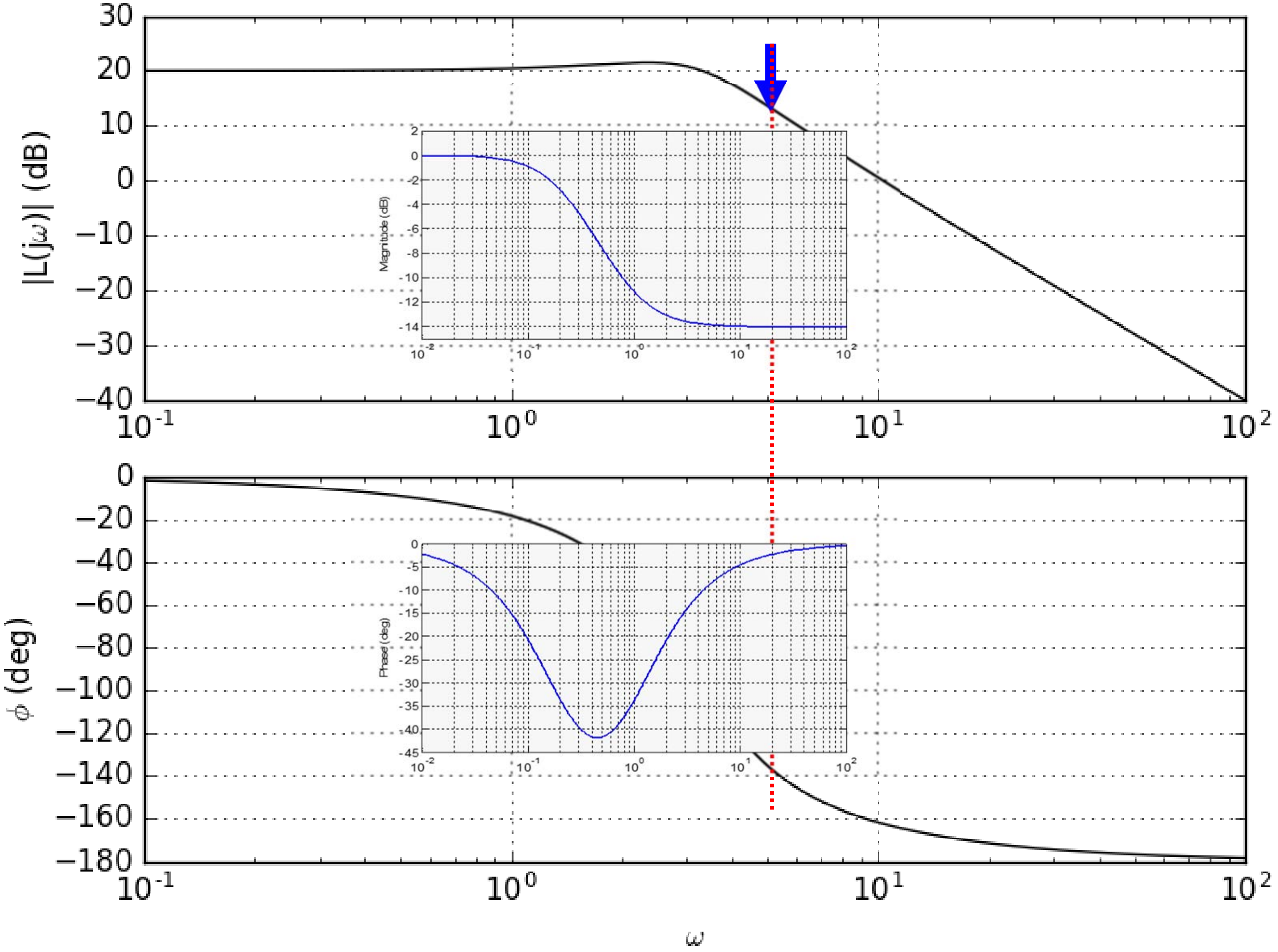
It adds negative phase (phase lag). So it is called Lag Compensator.

- Gain is negative dB and fairly constant for $\omega \gg \omega_z$.
- Phase is negligibly small for $\omega \gg \omega_z$.
- DC gain is 0 dB.

If we need to reduce the gain at some frequency ω_x without changing the phase significantly and without affecting DC gain, we can use a lag compensator such that $\omega_z \ll \omega_x$.

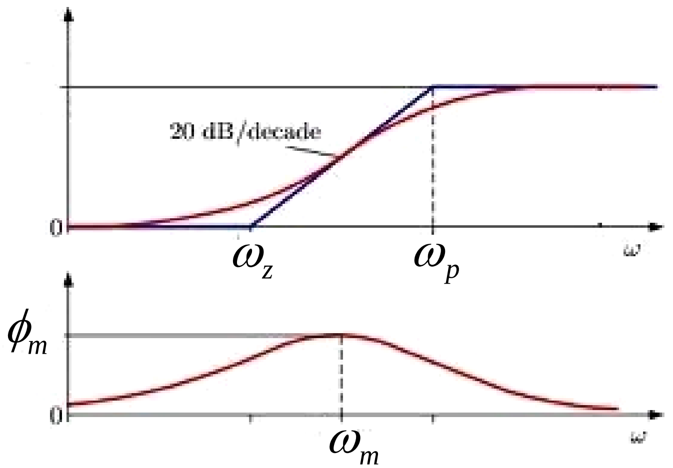
Soln#2 can be approximated by this!

Soln#2: By a lag compensation…

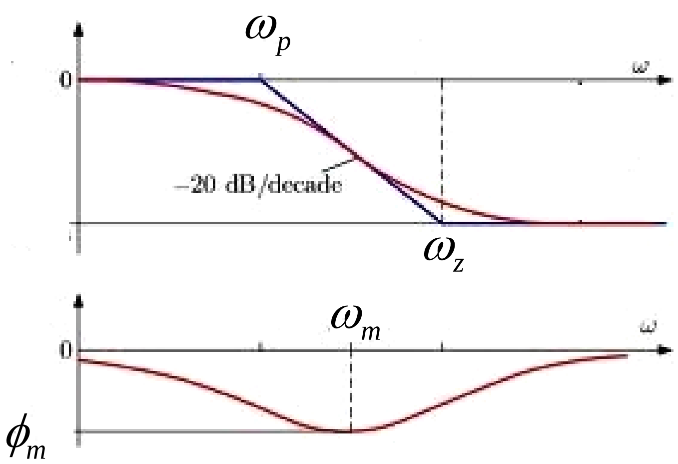


Reduce the gain at this frequency without changing DC gain

Undesirable (but Unavoidable) Consequence:



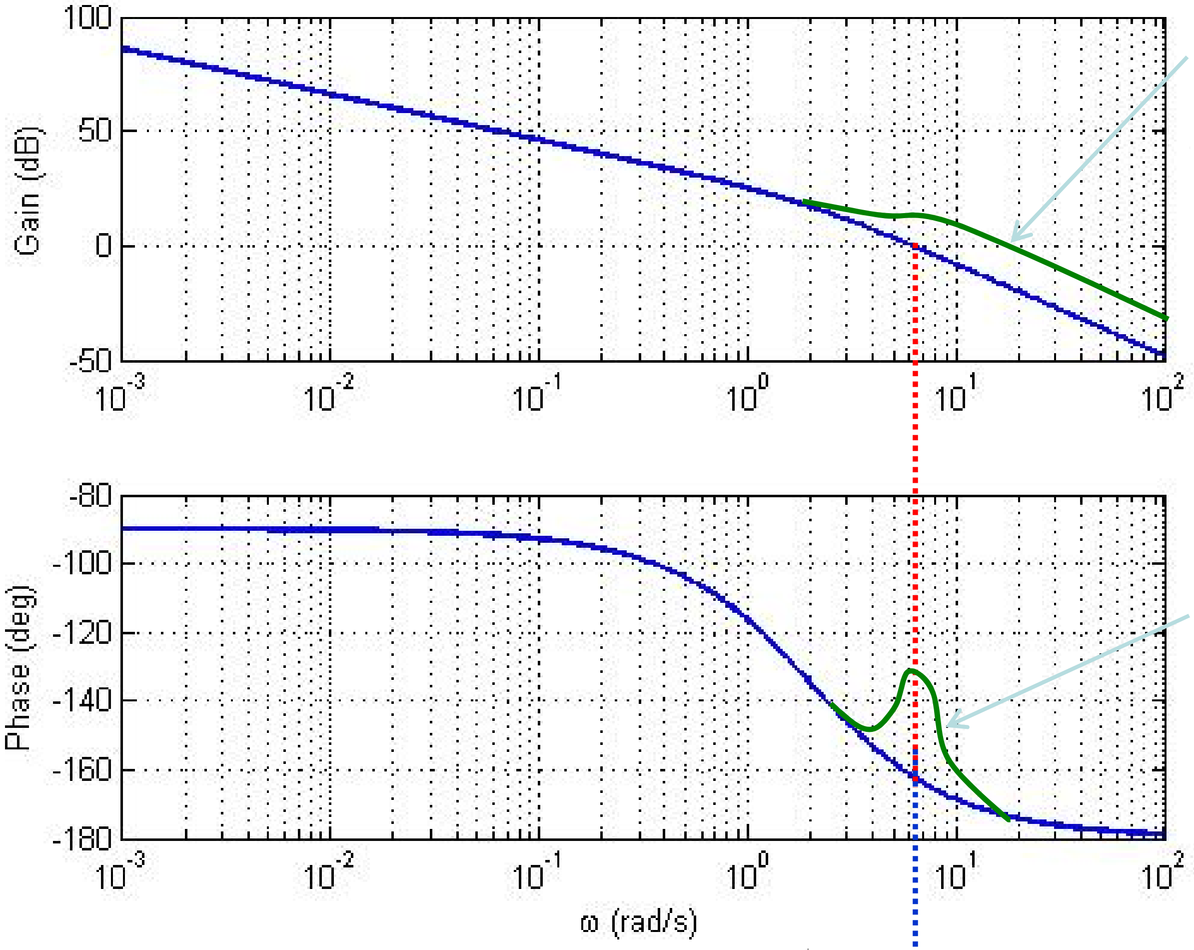
Phase lead is the desirable effect. But the gain is also altered in frequencies greater than ω_z. Attention must be paid to this when we design lead compensator.



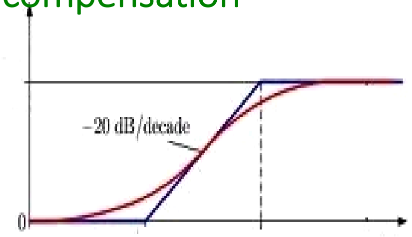
To achieve gain reduction without significant change in phase, ω_z must be chosen much lower than the frequency at which gain reduction is desired.

This affect low-frequency gain of the loop though the DC-gain is unaffected.

Issues to Consider while Designing a Lead Compensator



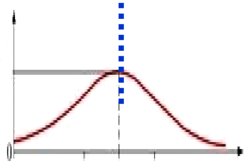
ω_{cg} gets shifted after compensation



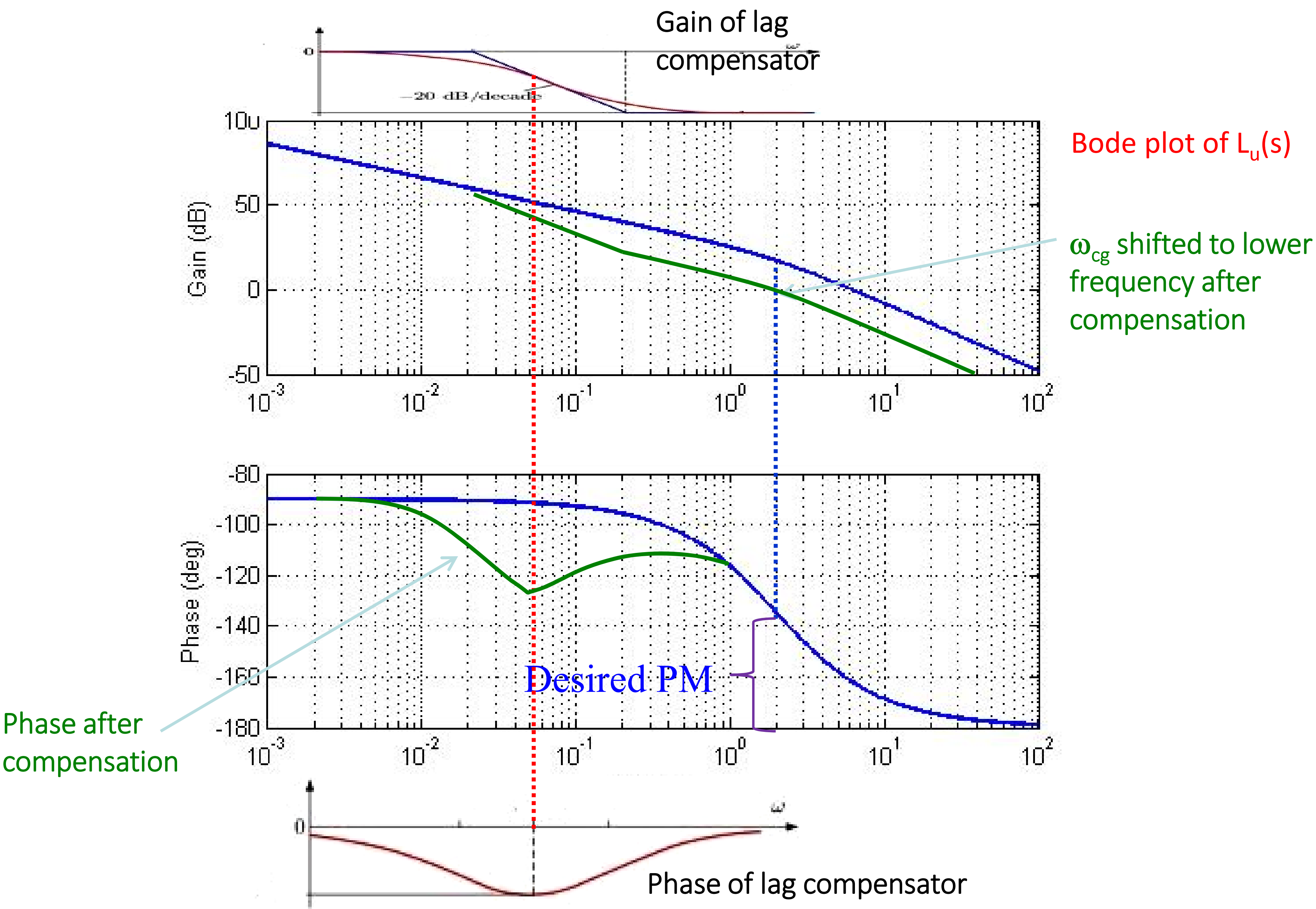
Gain of lead compensator

Phase after compensation

Phase of lead compensator



Issues to Consider while Designing a Lag Compensator



How to design a lag compensator?

Example: Consider the 1st-order process

$$G_{\mathrm{p}}(s) = \frac{1}{0.5s + 1}$$

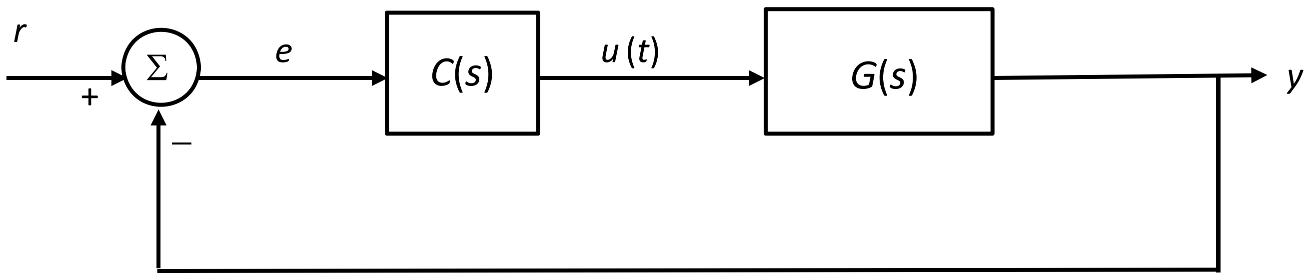
Design a controller to meet the following specifications -

a) Zero steady-state error for step input

b) 5% error for ramp input

c) Phase margin of 45°

First use gain-adjustment so that specifications (a) and (b) are met.



Loop transfer function that meets specs (a) and (b):

$$L_u(s) = \frac{K}{s} G_p(s) = \frac{20}{s(0.5s+1)}$$

The subscript u is used to imply that the loop is uncompensated.

Does it meet the 3rd specification of PM (related to transient response)?

$$L_u(j\omega) = \frac{20}{j\omega\,(j0.5\omega+1)}$$

To find PM, we need to know the gain-crossover frequency. $\omega_{cg} \cong 6.2 \text{ rad/s}$

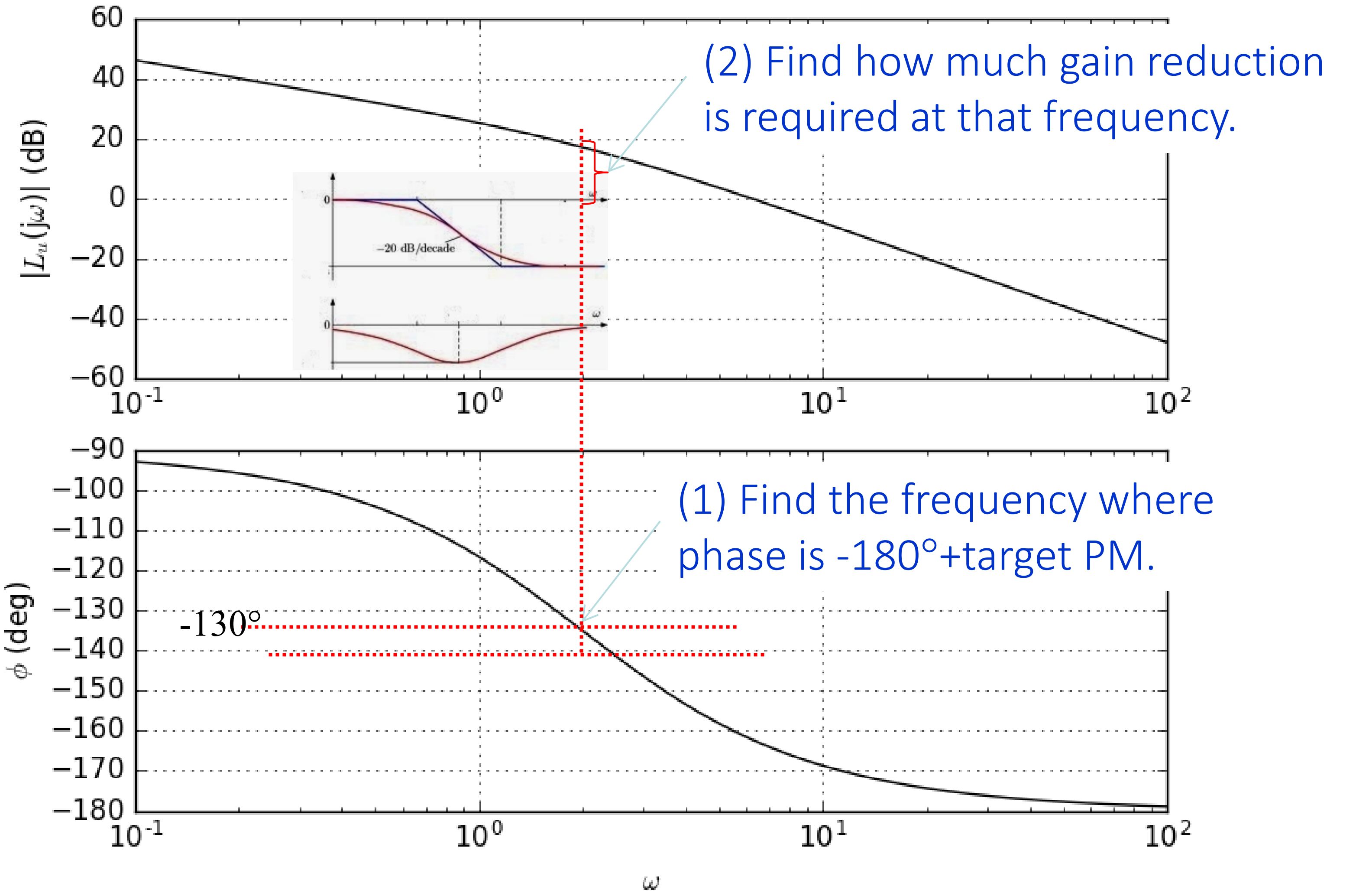
Phase at the gain-crossover frequency: $\angle L_u(j\omega_{cg}) = -162^\circ$

Phase margin: $-162^\circ - (-180^\circ) = 18^\circ$

Let's use the symbol PM_u for this.

$$\text{PM}_u = 18^\circ$$

Lag compensator can be used to reduce gain in a range of frequencies



Caution: Lag compensator introduces a small phase lag. Choose target PM slightly greater (5°) than the one given in the specifications. (50° and not 45°)

Find the frequency at which phase of L_u is – 130°

$$L_u(j\omega) = \frac{20}{j\omega(j0.5\omega+1)} \quad \Rightarrow \quad \phi = -90° - \tan^{-1}(0.5\omega)$$

$$-90° - \tan^{-1}(0.5\omega_x) = -130°$$

$$\tan^{-1}(0.5\omega_x) = 40°$$

$$\omega_x \cong 1.7 \text{ rad/s}$$

Step 1 of previous slide

Find the uncompensated gain at this frequency $\quad |L_u(j\omega_x)| = \dfrac{20}{\omega_x\sqrt{0.25\omega_x^2+1}}$

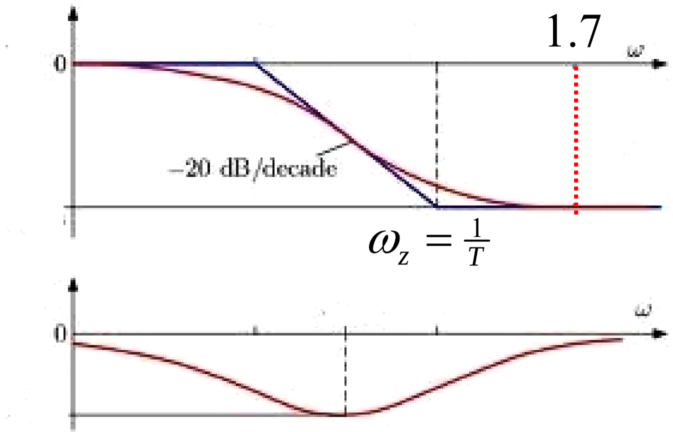
Step 2 of previous slide

$$|L_u(j1.7)| = \frac{20}{1.7\sqrt{0.25(1.7)^2+1}} \approx 9$$

We want to make the compensated gain at this frequency to be 1 (equivalent to 0 dB).

$$|C(j1.7)| \times |L_u(j1.7)| = 1 \quad \Rightarrow \quad |C(j1.7)| = \frac{1}{|L_u(j1.7)|} = \frac{1}{9}$$

Where do I go from here?



$$C(s) = \frac{Ts+1}{\alpha Ts+1} \quad \Rightarrow \quad C(j\omega) = \frac{j\omega T+1}{j\omega\alpha T+1}$$

$$C(j1.7) = \frac{j1.7T+1}{j1.7\alpha T+1}$$

If we choose ω_z of the compensator << 1.7 rad/s,

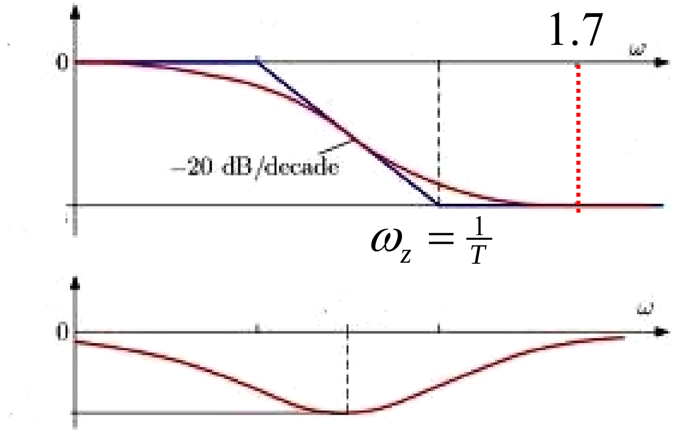
$$1.7 >> \omega_z \quad \Rightarrow \quad 1.7 >> \frac{1}{T} \quad \Rightarrow \quad 1.7T >> 1$$

Under this condition,

$$C(j1.7) = \frac{j1.7T+1}{j1.7\alpha T+1} \approx \frac{j1.7T}{j1.7\alpha T} = \frac{1}{\alpha}, \qquad \alpha > 1$$

Required reduction in gain at this frequency,

$$|C(j1.7)| = \frac{1}{|L_u(j1.7)|} = \frac{1}{9} \longrightarrow \alpha = 9$$



$$C(s) = \frac{Ts+1}{\alpha Ts+1} \quad \Rightarrow \quad C(j\omega) = \frac{j\omega T+1}{j\omega\alpha T+1}$$

$$C(j1.7) = \frac{j1.7T+1}{j1.7\alpha T+1}$$

How to find the 2nd parameter T ?

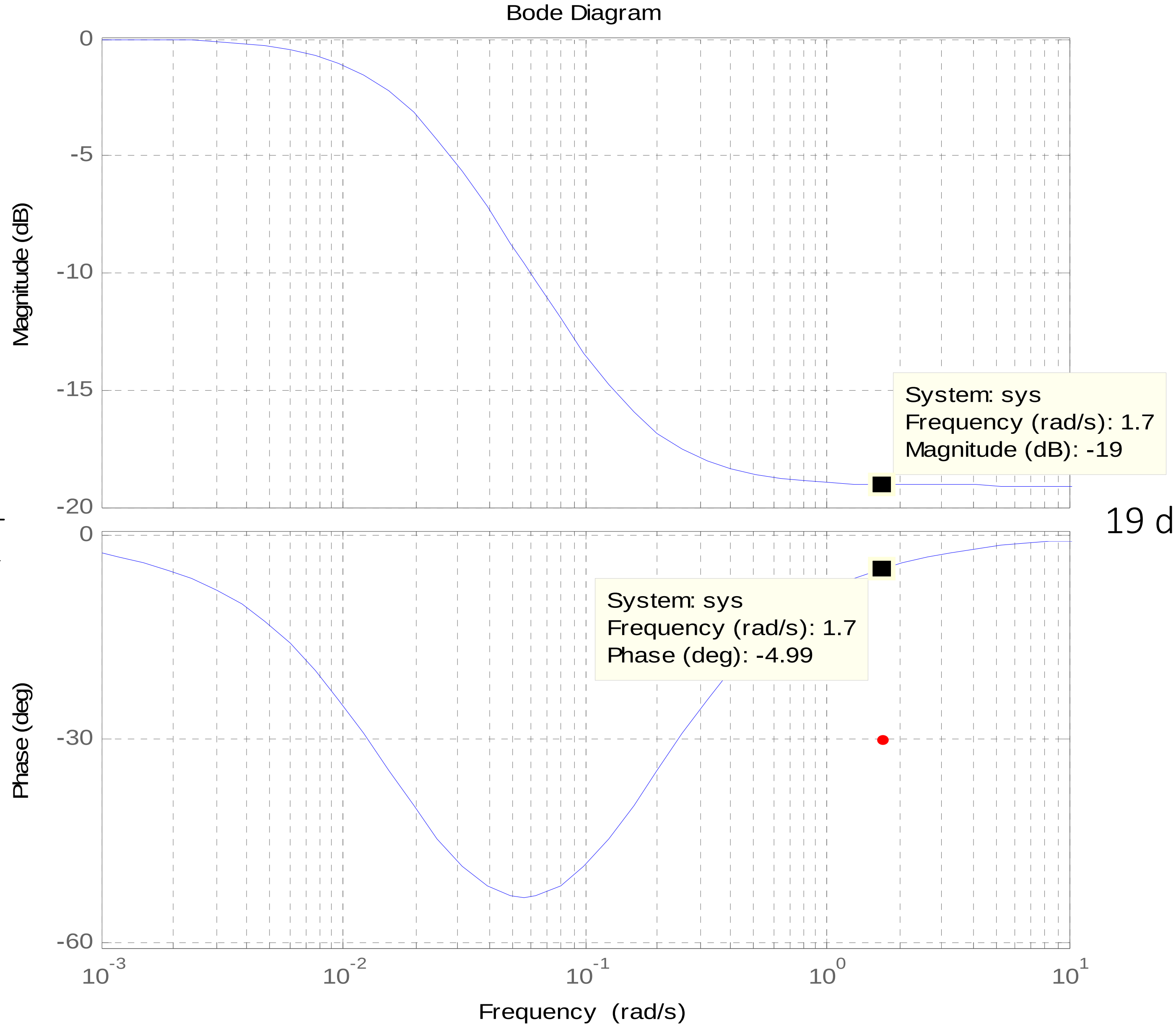
$$1.7 >> \omega_z = \frac{1}{T} \quad \Rightarrow \quad 1.7T >> 1$$

A factor of 10 is usually good enough,

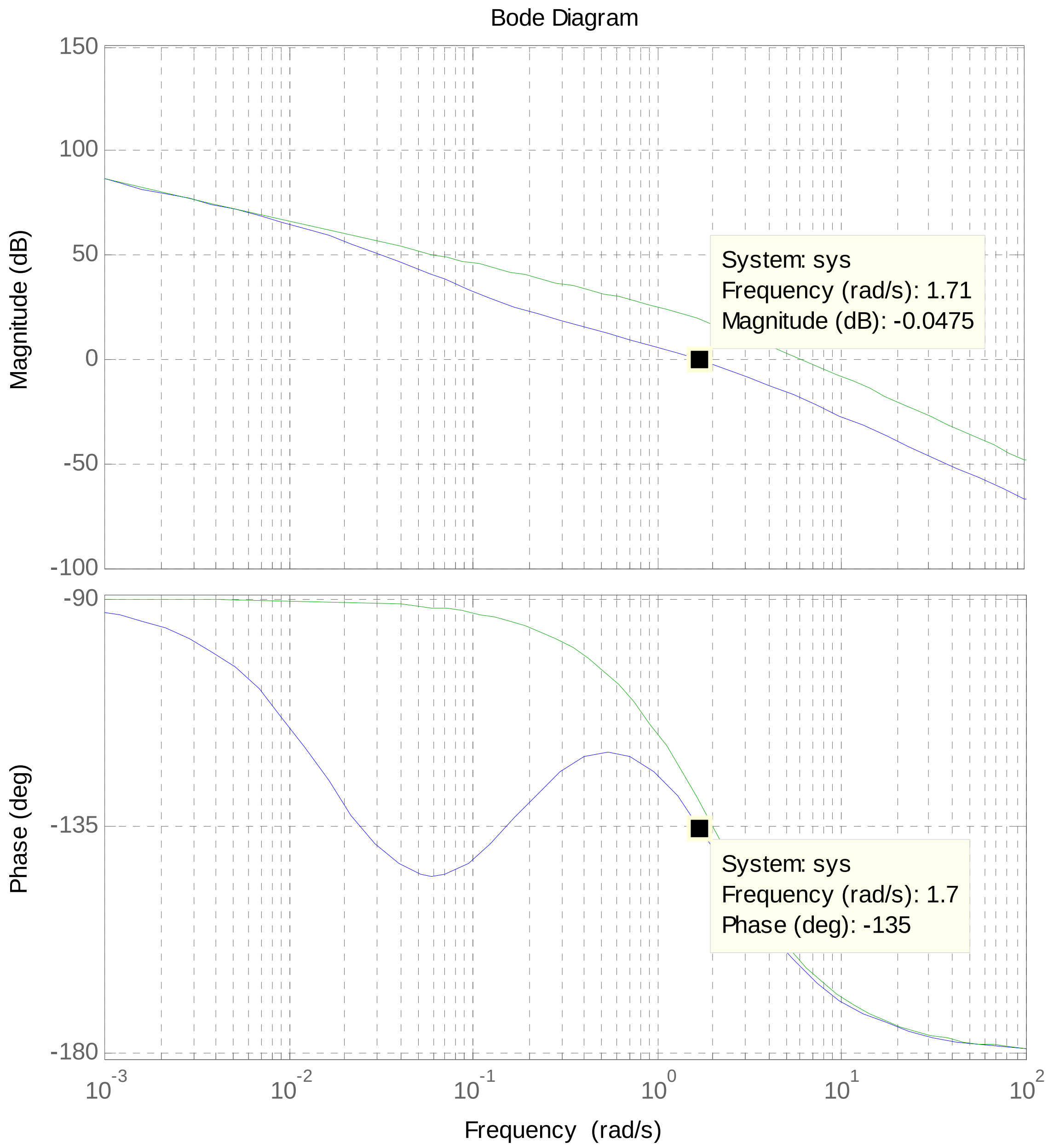
$$1.7T = 10 \quad \Rightarrow \quad T = \frac{10}{1.7} \approx 6 \quad \Rightarrow \quad C(s) = \frac{Ts+1}{\alpha Ts+1} = \frac{6s+1}{54s+1}$$

Homework: Find gain and phase of C(s) at ω = 1.7 rad/s and verify if they satisfy the requirements, i.e., gain equal to 1/9 and small negative phase.

$$C(s)=\frac{6s+1}{54s+1}$$



19 dB = 8.9



Green Curve:

Uncompensated open loop transfer function

ω_{cg} = 6.2 rad/s

PM = 18°

Blue Curve:

Compensated open loop transfer function

$\omega_{cg,new}$ = 1.7 rad/s

PM = 45°

Procedure to design a lag compensator…

i. Find the phase margin of the uncompensated system and compare it to the design specs on phase margin (say, ϕ_{req}). To be safe, we usually choose a value $\phi_m > \phi_{req}$ to start with the design.

ii. Find the frequency ω_x such that the uncompensated loop at which has a phase response of $-180° + \phi_m$.

iii. Find the corresponding gain value (say, G_x) of the uncompensated loop at ω_x.

iv. Compute $\alpha = G_x$ and $T = \dfrac{10}{\omega_x}$.

v. The required lag compensator is given as $C(s) = \dfrac{Ts+1}{\alpha Ts+1}$.

vi. Verify your result. Repeat the above steps if necessary.

How to design a lead compensator?

Example: Consider the same problem but repeat the design using lead compensator

$$G_p(s) = \frac{1}{0.5s + 1}$$

Specifications:

a) Zero steady-state error for step input

b) 5% error for ramp input

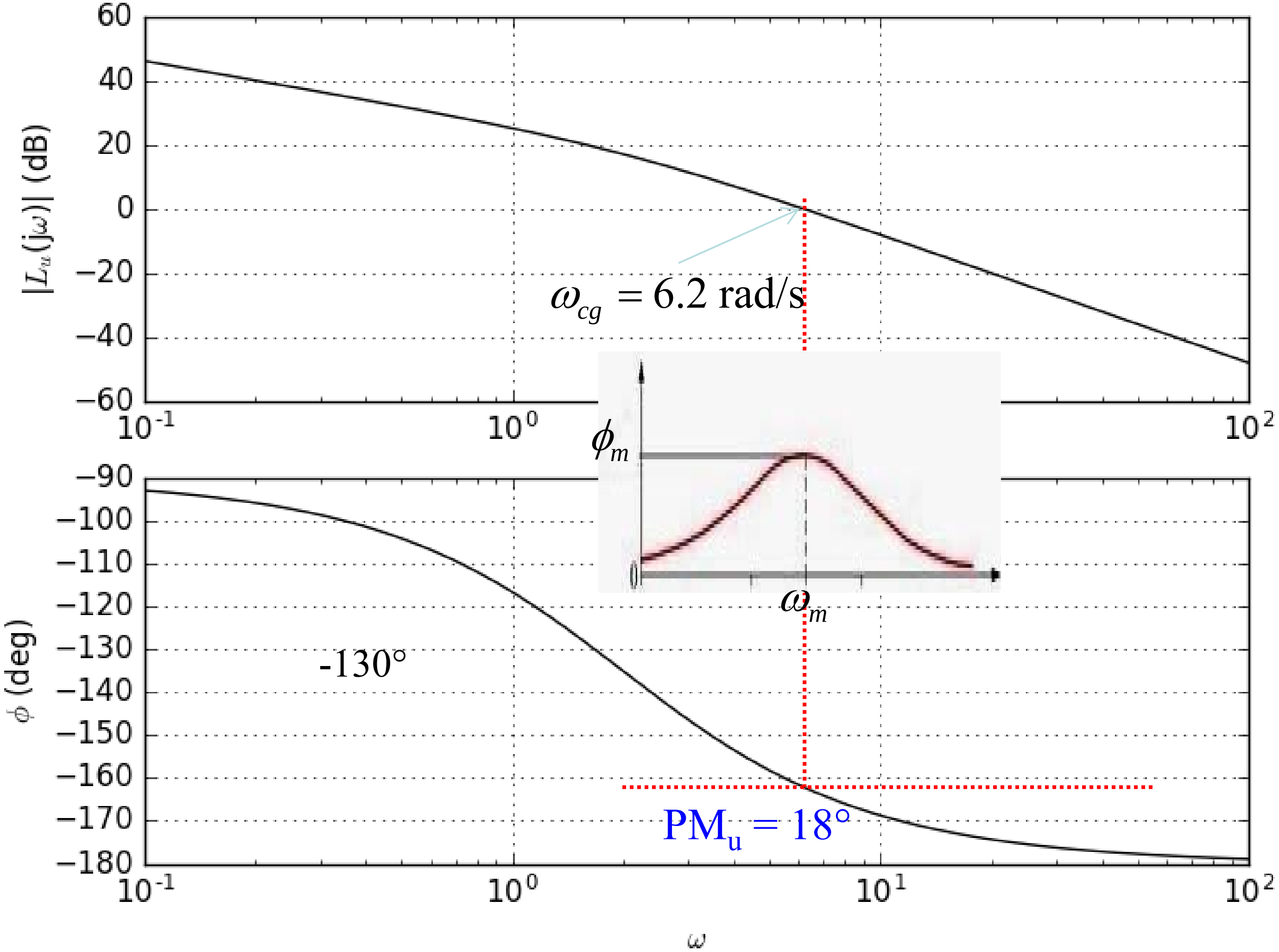
c) Phase margin of 45°

Gain-adjustment stage is same as in the previous example.

$$L_u(s) = \frac{K}{s} G_p(s) = \frac{20}{s(0.5s+1)}$$

First two specifications are met but phase margin is only 18°. We now use a lead compensator to improve phase margin to ≈ 45°.

Lead compensator can be used to increase phase lead



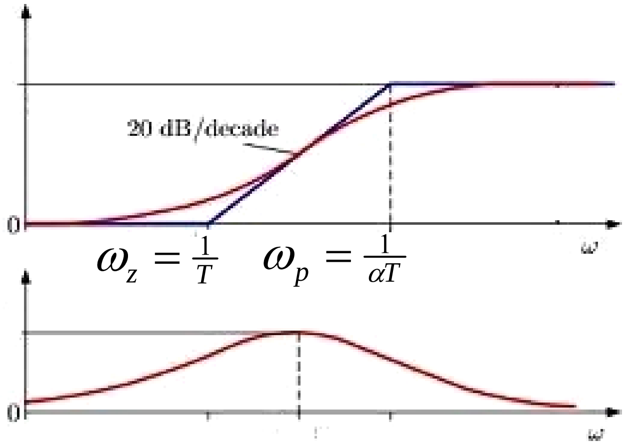
$$\phi_{\text{req,app}} = \text{PM}_{\text{spec}} - \text{PM}_{\text{u}} = 27°$$

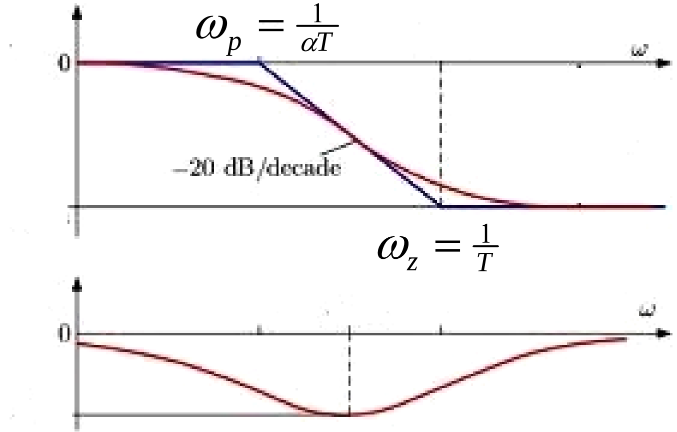
Try a compensator with $\omega_m = \omega_{cg}$ that gives $\phi_m = 27°$.

Caution: It is not going to work

How ϕ_m and ω_m of compensator (**lead** or **lag**) are related to the compensator parameters T and α?

$$C(s) = \frac{Ts+1}{\alpha Ts+1} \quad \Rightarrow \quad C(j\omega) = \frac{j\omega T+1}{j\alpha\omega T+1}$$





One way to find ω_m is to express ϕ as function of ω and solve $\frac{d\phi}{d\omega} = 0$, where

$$\phi = \tan^{-1}\omega T - \tan^{-1}\alpha\omega T$$

Alternatively,

$$\omega_z = \frac{1}{T}, \quad \omega_p = \frac{1}{\alpha T}$$

$$\log\omega_m = \frac{\log\omega_z + \log\omega_p}{2} = \frac{1}{2}\log(\omega_z\omega_p) \quad \Rightarrow \quad \omega_m = \sqrt{\omega_z\omega_p} \quad \Rightarrow \quad \boxed{\omega_m = \frac{1}{T\sqrt{\alpha}}}$$

Find the relation between ϕ_m and compensator parameters

$$C(s)=\frac{Ts+1}{\alpha Ts+1} \quad \Rightarrow \quad C(j\omega)=\frac{j\omega T+1}{j\alpha\omega T+1}$$

$$\phi=\tan^{-1}\omega T-\tan^{-1}\alpha\omega T$$

$$\phi_m=\tan^{-1}\omega_m T-\tan^{-1}\alpha\omega_m T$$

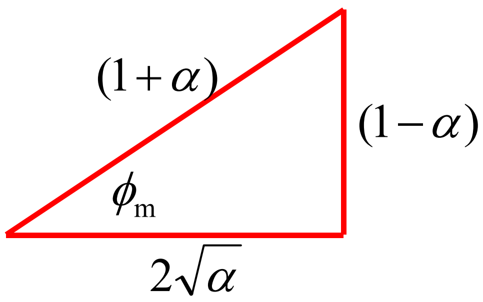
$$\boxed{\omega_m=\frac{1}{T\sqrt{\alpha}} \quad \Rightarrow \omega_m T=\frac{1}{\sqrt{\alpha}}}$$

$$\phi_m=\tan^{-1}\frac{1}{\sqrt{\alpha}}-\tan^{-1}\sqrt{\alpha}$$

$$\tan\phi_m=\frac{\dfrac{1}{\sqrt{\alpha}}-\sqrt{\alpha}}{1+1}$$

$$\boxed{\begin{aligned}\tan(A+B)&=\frac{\tan A+\tan B}{1-\tan A\tan B}\\ \tan(A-B)&=\frac{\tan A-\tan B}{1+\tan A\tan B}\end{aligned}}$$

$$\tan\phi_m=\frac{1-\alpha}{2\sqrt{\alpha}}$$



$$\boxed{\sin\phi_m=\frac{1-\alpha}{1+\alpha}}$$

Let's choose a compensator that gives ϕ_m = 27° at ω_m = 6.2 rad/s

$$\sin\phi_m = \frac{1-\alpha}{1+\alpha}$$

$$\sin 27^\circ = \frac{1-\alpha}{1+\alpha}$$

$$0.45 = \frac{1-\alpha}{1+\alpha} \quad \Rightarrow \quad \alpha = 0.4$$

$$\omega_m = \frac{1}{T\sqrt{\alpha}}$$

$$6.2 = \frac{1}{T\sqrt{0.4}}$$

$$T \approx 0.25$$

$$C(s) = \frac{0.25s+1}{0.1s+1}$$

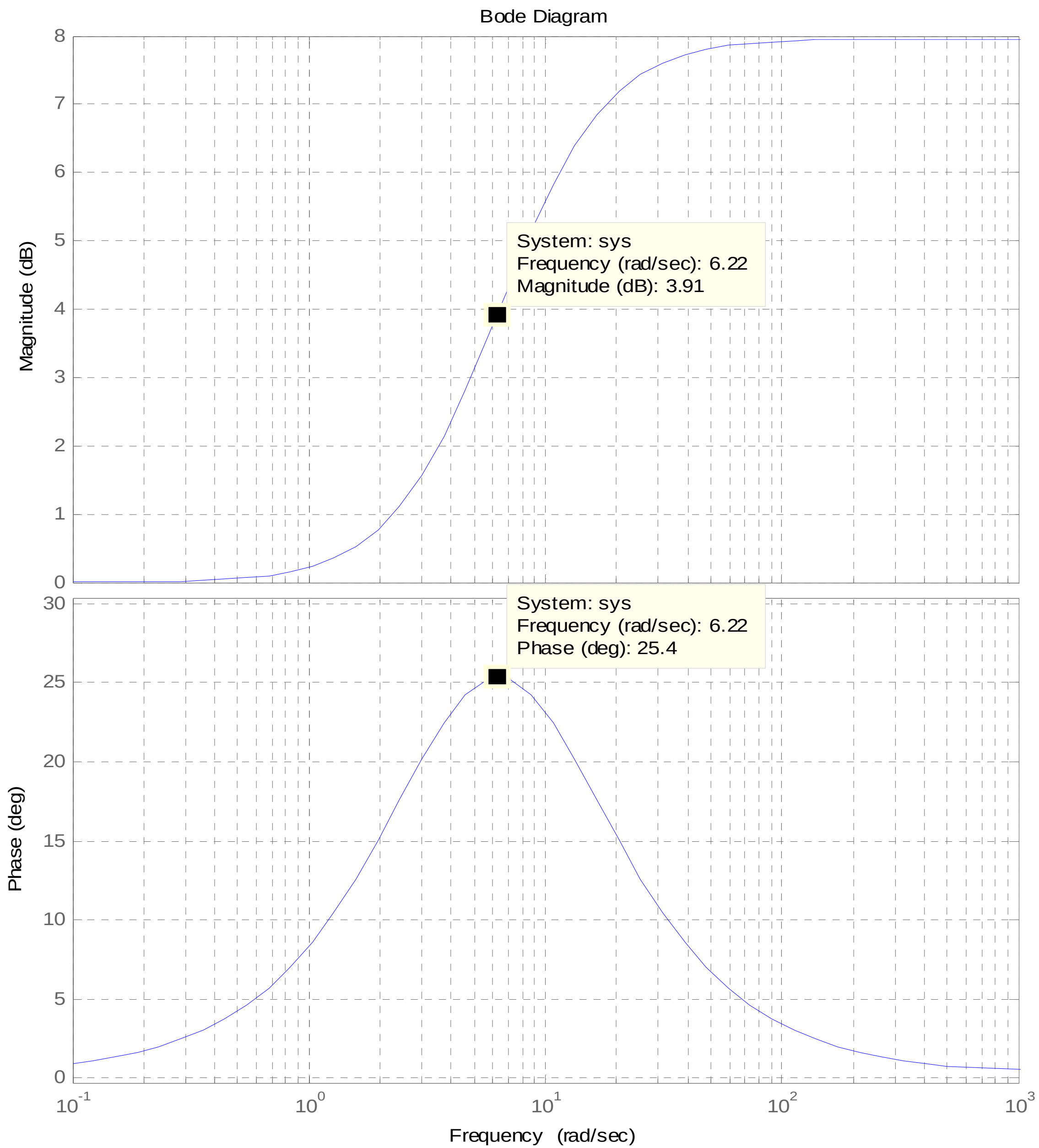
Lead Compensator

$$L_u(s) = \frac{20}{s(0.5s+1)}$$

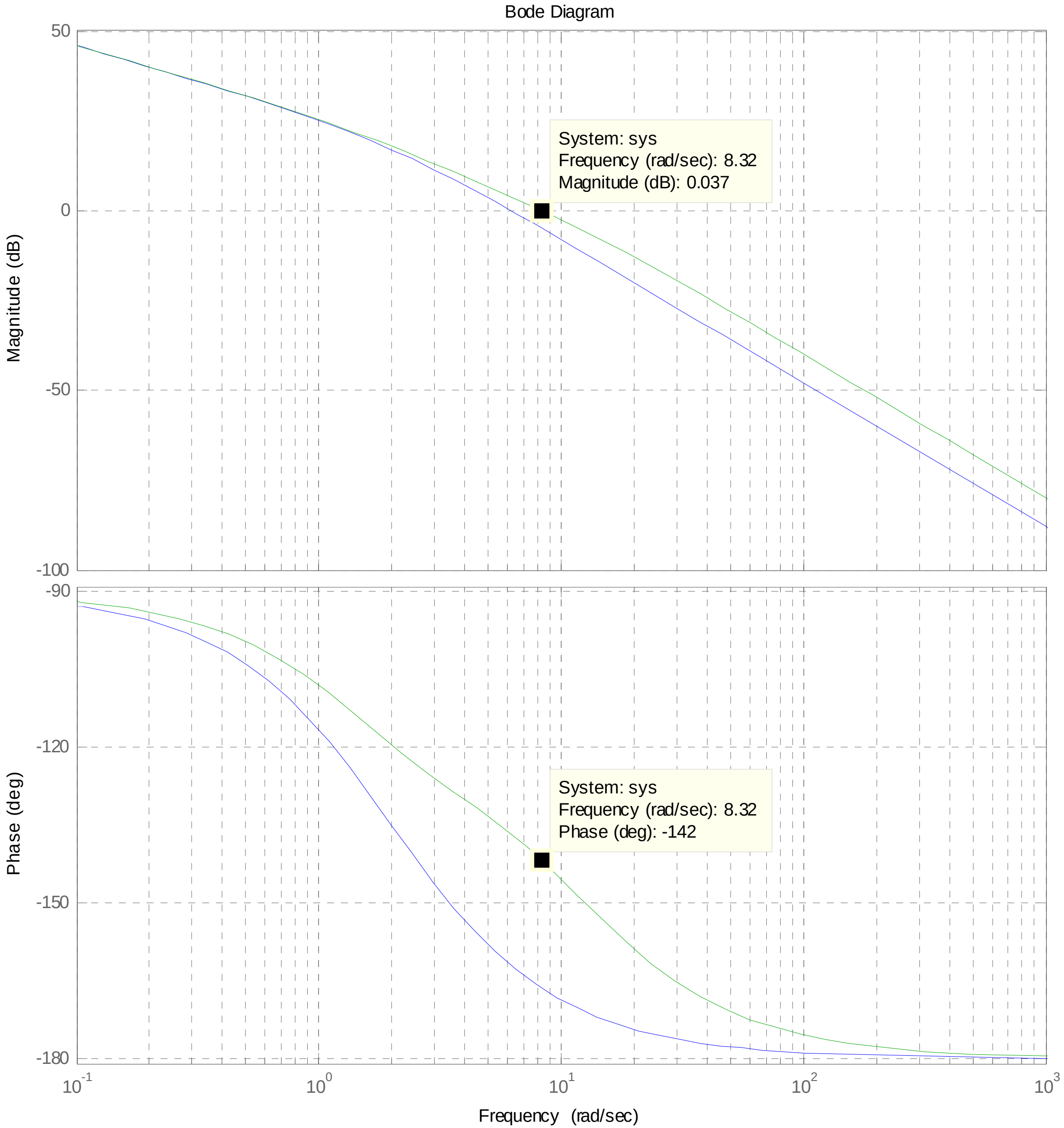
Uncompensated open loop

$$L(s) = \frac{20(0.25s+1)}{s(0.5s+1)(0.1s+1)}$$

Compensated open loop



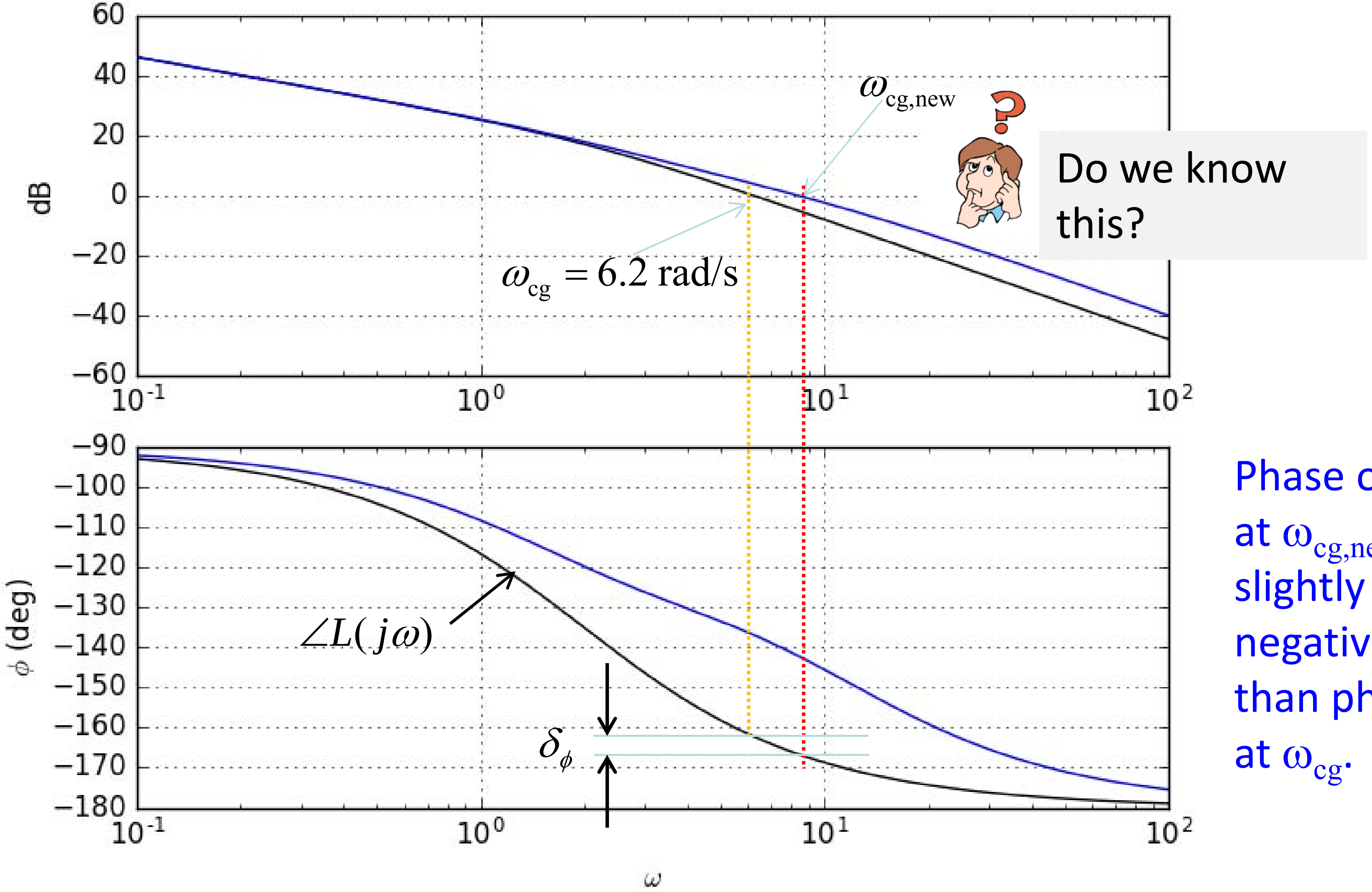
$$C(s) = \frac{0.25s + 1}{0.1s + 1}$$



The new gain cross-over frequency is about

$$\omega_{cg,new} = 8.3 \text{ rad/s}$$

$$\text{PM} = 38°$$



Phase of L_u at $\omega_{cg,new}$ is slightly more negative than phase at ω_{cg}.

We get the desired phase at the original ω_{cg}. However, the gain-crossover is shifted to higher frequency as the lead compensator adds positive dB gain.

Rule of Thumb:

$$\phi_{\text{req}} = \phi_{\text{req,app}} + 0.1 \times \phi_{\text{req,app}}$$
$$= 1.1\phi_{\text{req,app}}$$

$$\phi_{\text{req}} = 1.1(\text{PM}_{\text{spec}} - \text{PM}_{\text{u}})$$
$$= 1.1 \times (45^\circ - 18^\circ)$$
$$\cong 30^\circ$$

We need a compensator that would add maximum phase

$$\phi_{\text{m}} = 30^\circ$$

$$\sin \phi_{\text{m}} = \frac{1-\alpha}{1+\alpha} \quad \Rightarrow \quad \alpha = 0.3$$

Then we choose ω_{m} in a way so that the compensated gain becomes 1 at the frequency where compensator phase is maximum.

$$|C(j\omega_{\text{m}})| \times |L_{\text{u}}(j\omega_{\text{m}})| = 1$$

Do we know any of these two terms?

What is $|C(j\omega_{\mathrm{m}})|$ of a compensator?

$$C(j\omega)=\frac{j\omega T+1}{j\alpha\omega T+1} \quad\Rightarrow\quad C(j\omega_{\mathrm{m}})=\frac{j\omega_{\mathrm{m}}T+1}{j\alpha\omega_{\mathrm{m}}T+1}$$

$$\omega_{\mathrm{m}}=\frac{1}{T\sqrt{\alpha}} \quad\Rightarrow\quad \omega_{\mathrm{m}}T=\frac{1}{\sqrt{\alpha}} \quad\Longrightarrow\quad C(j\omega_{\mathrm{m}})=\frac{j\frac{1}{\sqrt{\alpha}}+1}{j\alpha\frac{1}{\sqrt{\alpha}}+1}=\frac{\sqrt{\alpha}+j}{\sqrt{\alpha}+j\alpha}$$

$$|C(j\omega_{\mathrm{m}})|=\frac{\sqrt{\alpha+1}}{\sqrt{\alpha+\alpha^2}}=\frac{1}{\sqrt{\alpha}}$$

As we have already chosen α, we know what $|C(j\omega_{\mathrm{m}})|$ is.

$$|C(j\omega_{\mathrm{m}})|=\frac{1}{\sqrt{\alpha}}=\frac{1}{\sqrt{0.3}}$$

$$|C(j\omega_{\mathrm{m}})|\times|L_{\mathrm{u}}(j\omega_{\mathrm{m}})|=1 \quad\Longrightarrow\quad |L_{\mathrm{u}}(j\omega_{\mathrm{m}})|=\frac{1}{|C(j\omega_{\mathrm{m}})|}=\sqrt{0.3}$$

As the uncompensated transfer function L_{u} is known, we can find the frequency at which this condition is met.

Find ω_m by solving $\left|L_u(j\omega_m)\right| = \sqrt{0.3}$

$$L_u(j\omega) = \frac{20}{j\omega(j0.5\omega+1)} \quad \Rightarrow \quad \left|L_u(j\omega)\right| = \frac{20}{\omega\sqrt{0.25\omega^2+1}}$$

$$\frac{20}{\omega_m\sqrt{0.25\omega_m^2+1}} = \sqrt{0.3} \quad \Rightarrow \quad \omega_m^2(0.25\omega_m^2+1) = \frac{20^2}{0.3} \approx 1333$$

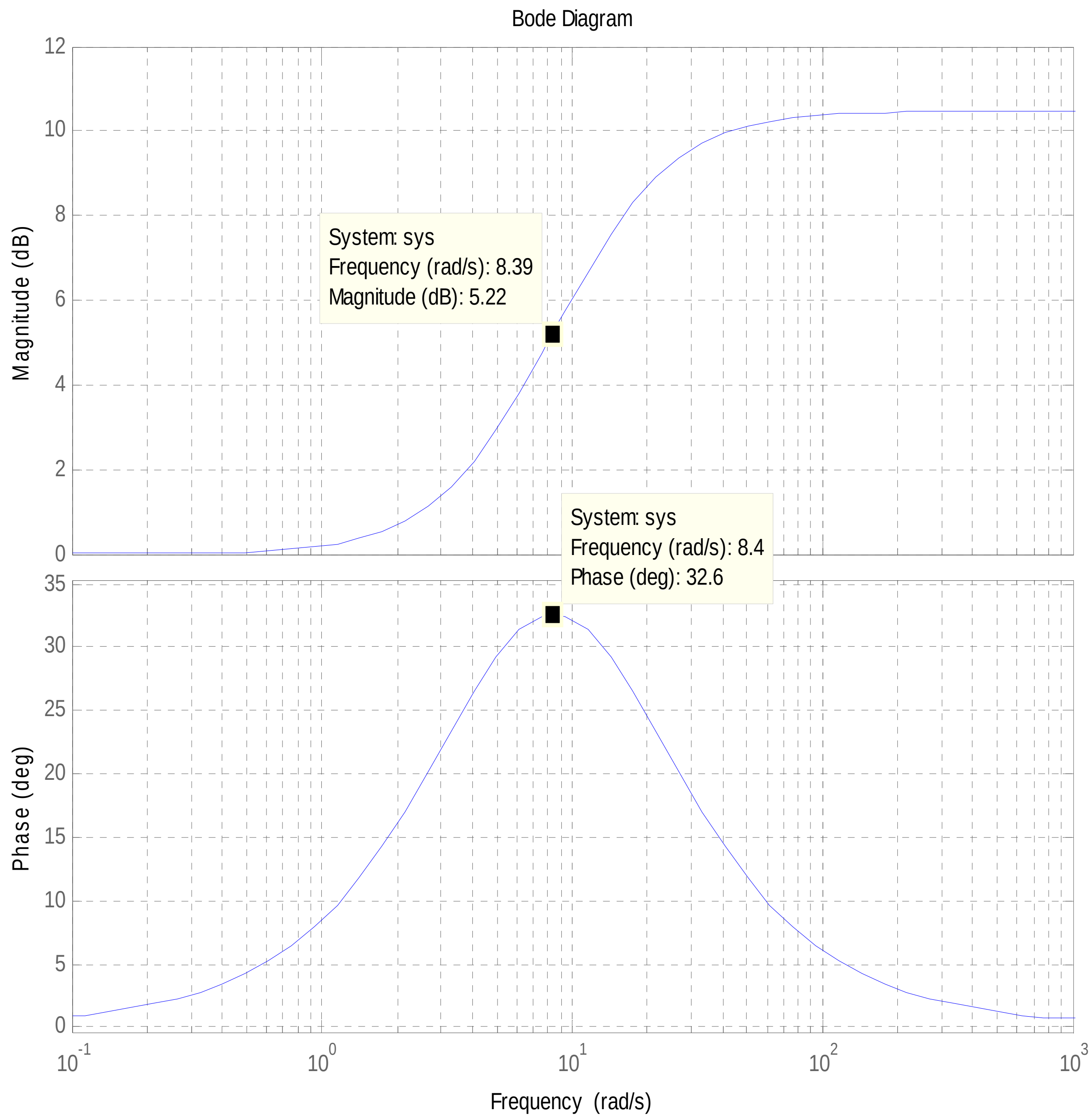
$$\omega_m^4 + 4\omega_m^2 = 5332$$

By solving this equation: $\omega_m^2 \approx 71 \quad \text{or} \quad -75$

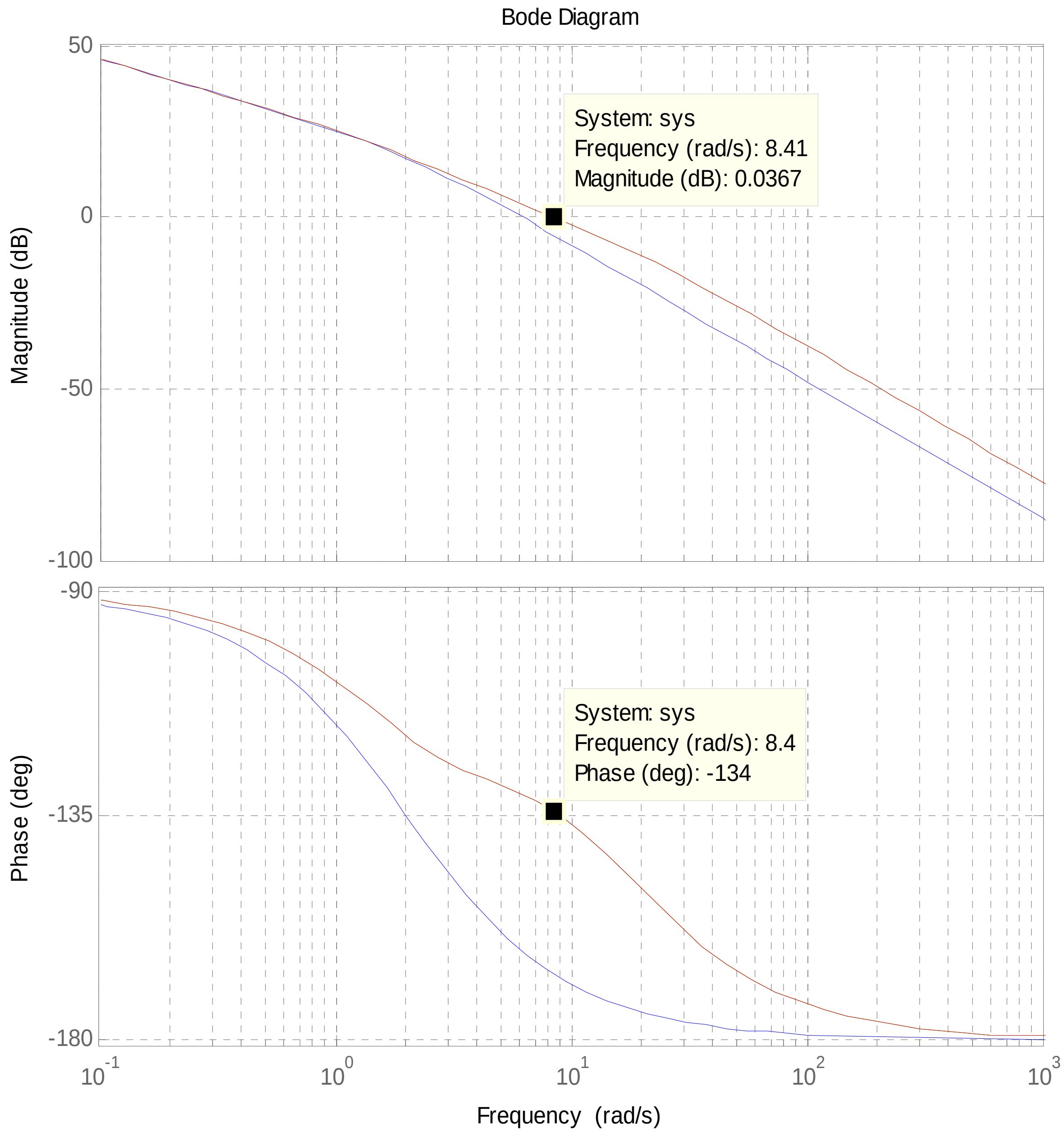
$$\omega_m = \sqrt{71} \approx 8.4 \text{ rad/s}$$

$$\omega_m = \frac{1}{T\sqrt{\alpha}} \quad \Rightarrow \quad T = \frac{1}{8.4\times\sqrt{0.3}} \approx 0.217$$

$$C(s) = \frac{0.217s+1}{0.065s+1} \qquad L_u(s) = \frac{20}{s(0.5s+1)} \qquad L(s) = \frac{20(0.217s+1)}{s(0.5s+1)(0.065s+1)}$$



$$C(s) = \frac{0.217s + 1}{0.065s + 1}$$



The new gain cross-over frequency is about

$\omega_{cg,new}$ = 8.4 rad/s

PM = 46°

✓

Procedure to design a lead compensator…

i. Find the phase margin of the uncompensated system and compare it to the design specs to obtain the necessary phase needed (say, ϕ_{req}) to be added to the loop. To be safe, we usually choose $\phi_m > \phi_{req}$ to start with the design.

ii. From $\sin\phi_{\mathrm{m}} = \dfrac{1-\alpha}{1+\alpha}$, we find α.

iii. We look for a frequency (ω_m) that has an uncompensated gain of $\sqrt{\alpha}$.

iv. Computer $T = \dfrac{1}{\omega_{\mathrm{m}}\sqrt{\alpha}}$.

v. The required lead compensator is given as $C(s) = \dfrac{Ts+1}{\alpha Ts+1}$.

vi. Verify your result. Repeat the above steps if necessary.

Step Response of the Closed Loop

$$L_u(s) = \frac{20}{s(0.5s+1)}$$

Without compensator

$$G_{CL}(s) = \frac{20}{0.5s^2 + s + 20}$$

With lag compensator

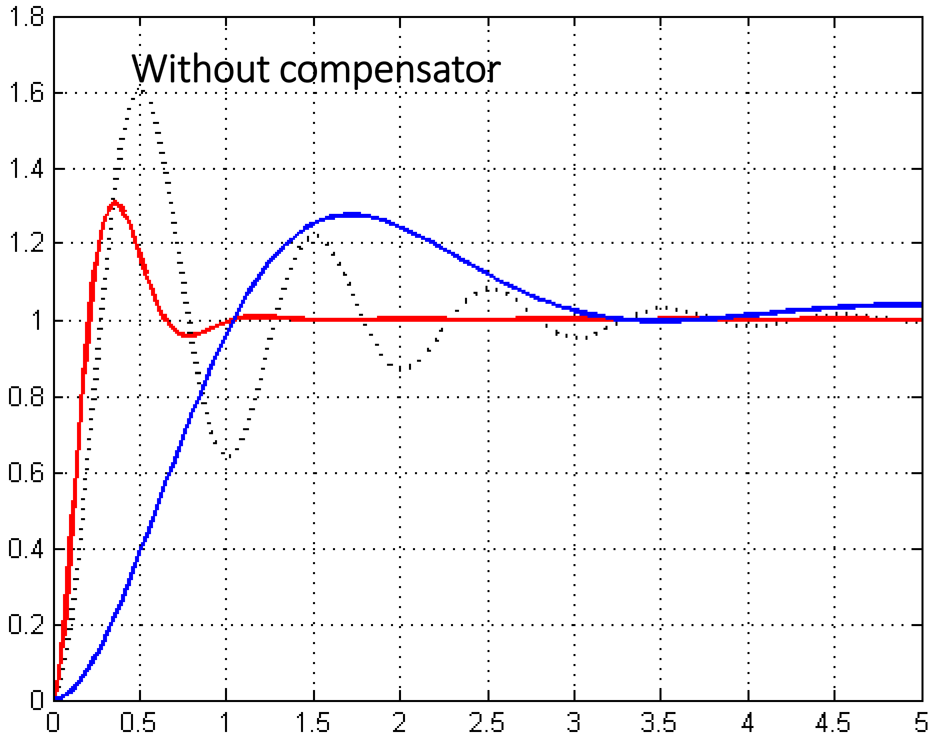
$$C_{lag}(s) = \frac{6s+1}{54s+1}$$

$$G_{CL}(s) = \frac{20(6s+1)}{27s^3 + 54.5s^2 + 121s + 20}$$

With lead compensator

$$C_{lead}(s) = \frac{0.2s+1}{0.06s+1}$$

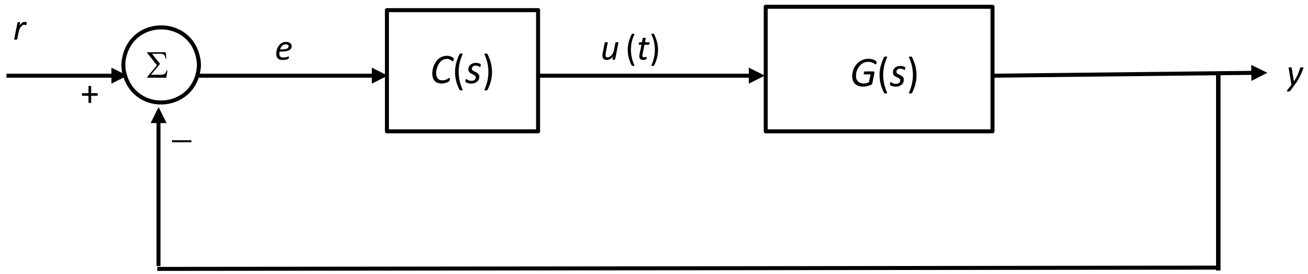
$$G_{CL}(s) = \frac{133(s+5)}{s^3 + 18.7s^2 + 166.4s + 665}$$



Which one is with lead compensator?

Which one is with lag compensator?

Example A.1: For the system given below



$$G(s) = \frac{K}{s(s+2)(s+1000)}$$

1) Design a lead compensator C(s) and find K such that

$$\text{PM} \ge 45^\circ$$
$$e_{ss} \text{ (ramp)} \le 0.02$$

2) Obtain the step response of the uncompensated and compensated systems.

Solution: (1) Since the compensator we studied in this module has a unity DC gain, i.e.,

$$\lim_{s\to 0} C(s) = \lim_{s\to 0} \frac{Ts+1}{\alpha Ts+1} = 1$$

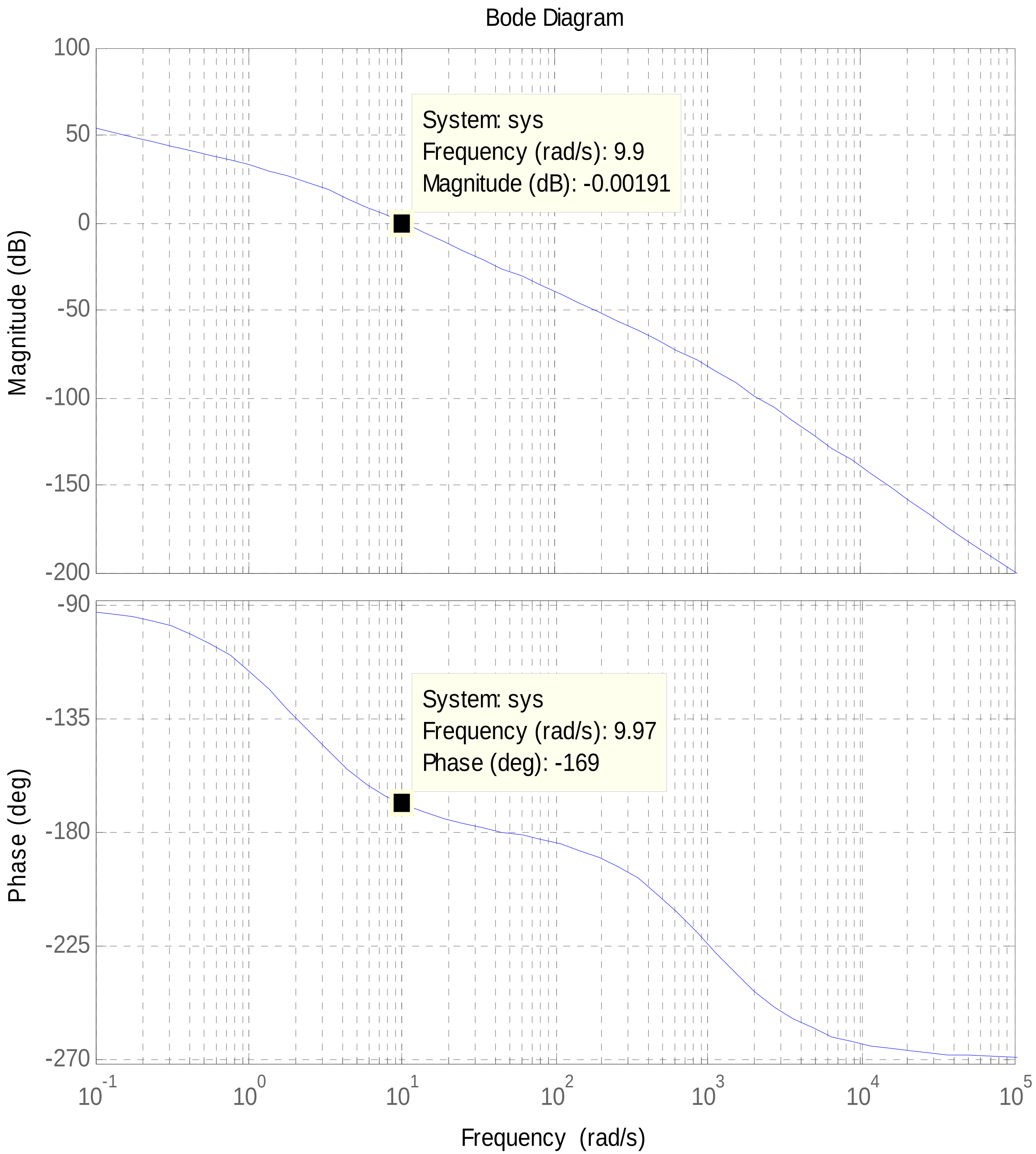
The velocity error constant is then given by

$$k_{\mathrm{v}} = \lim_{s\to 0} s\cdot C(s)\cdot G(s) = \lim_{s\to 0} C(s)\cdot \lim_{s\to 0} \frac{K}{(s+2)(s+1000)} = \frac{K}{2000} \ge \frac{1}{0.02} \quad \Rightarrow \quad K = 100{,}000$$

i. Find the phase margin of the uncompensated system and compare it to the design specs to obtain the necessary phase needed (say, ϕ_{req}) to be added to the loop. To be safe, we usually choose $\phi_m > \phi_{req}$ to start with the design.

From the Bode plot of the uncompensated open loop transfer function given on the next page, we have

PM = 180° – 169° = 11° (at frequency around 10 rad/s)



To be safe, we choose to add an additional phase of

$$\phi_{req} = (45^\circ - 11^\circ) + 5^\circ = 39^\circ$$

ii. From $\sin\phi_{\mathrm{m}} = \dfrac{1-\alpha}{1+\alpha}$, we find α.

$$\frac{1-\alpha}{1+\alpha} = \sin 39^\circ = 0.63 \quad \Rightarrow \quad \alpha = 0.227$$

iii. We look for a frequency (ω_{m}) that has an uncompensated gain of $\sqrt{\alpha}$.

$$\sqrt{\alpha} = \sqrt{0.227} = -6.44\ \mathrm{dB} \quad \Rightarrow \quad \omega_{\mathrm{m}} = 14.6\ \mathrm{rad/s}$$

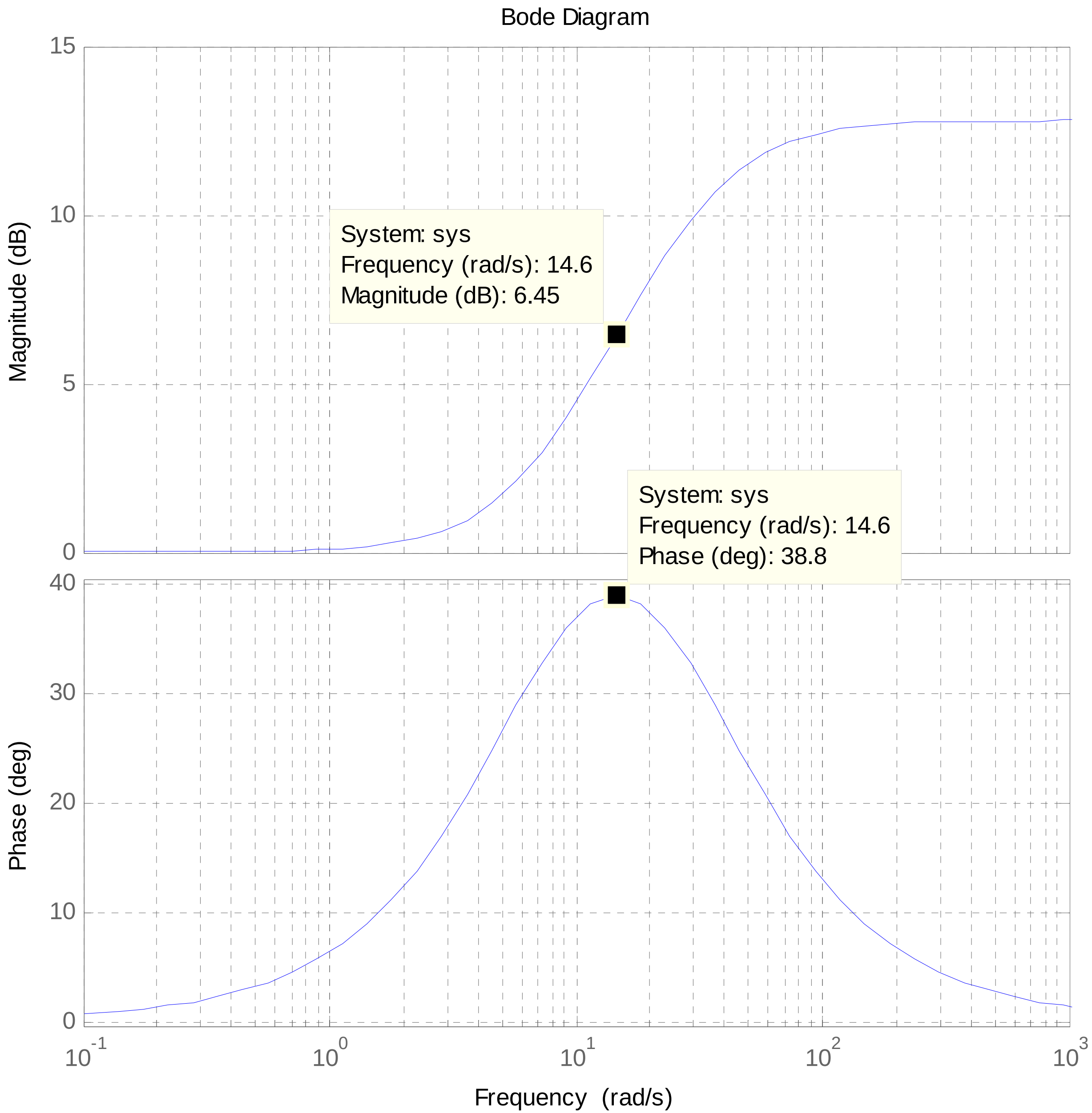
iv. Computer $T = \dfrac{1}{\omega_{\mathrm{m}}\sqrt{\alpha}}$

$$T = \frac{1}{\omega_{\mathrm{m}}\sqrt{\alpha}} = \frac{1}{14.6\sqrt{0.227}} = 0.144$$

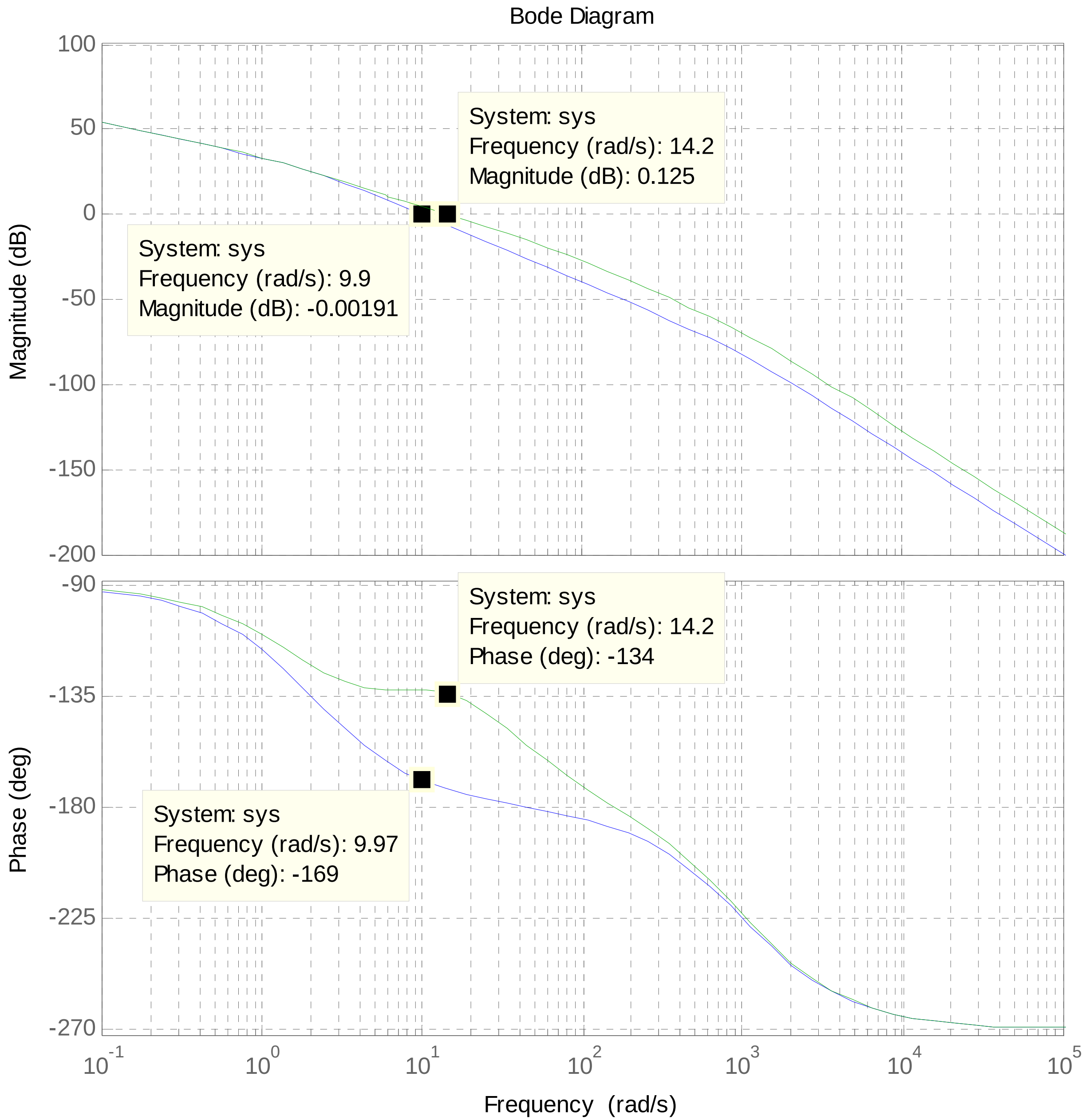
iv. The required lead compensator is given as

$$C(s) = \frac{Ts+1}{\alpha Ts+1} = \frac{0.144s+1}{0.227\times 0.144s+1} = \frac{0.144s+1}{0.033s+1}$$

iv. Verify your result. Repeat the above steps if necessary.



$$C(s) = \frac{0.144s + 1}{0.033s + 1}$$



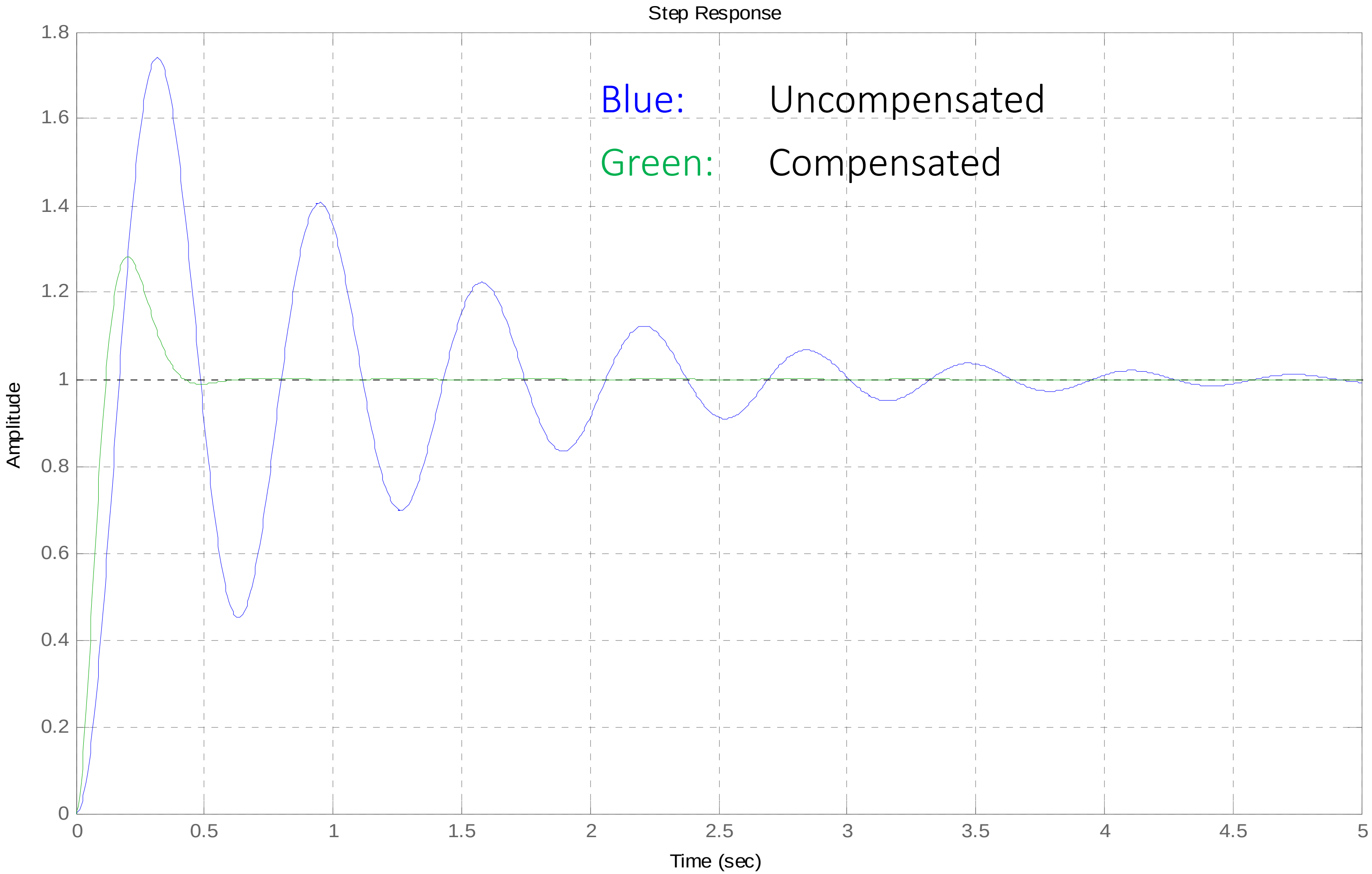
The new gain cross-over frequency is about

$\omega_{cg,new}$ = 14.3 rad/s

PM = 46°

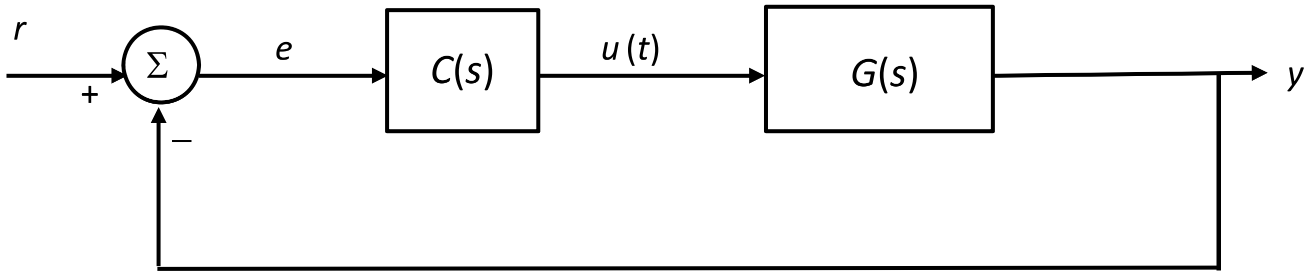
✓

(2) Obtain the step response of the uncompensated and compensated systems.



Exercise A.1: Solve the problem in Example A.1 using a lag compensator.

Example A.2: For the system given below



$$G(s) = \frac{K}{s(s+4)(s+5)}$$

1) Design a lag compensator C(s) and find K such that

$$\text{PM} \ge 45^\circ$$
$$K_\text{v} \le 5$$

2) Obtain the step response of the uncompensated and compensated systems.

Solution: (1) The velocity error constant is then given by

$$k_v = \lim_{s\to 0} s\cdot C(s)\cdot G(s) = \lim_{s\to 0} C(s)\cdot \lim_{s\to 0} \frac{K}{(s+4)(s+5)} = \frac{K}{20} = 5 \quad \Rightarrow \quad K = 100$$

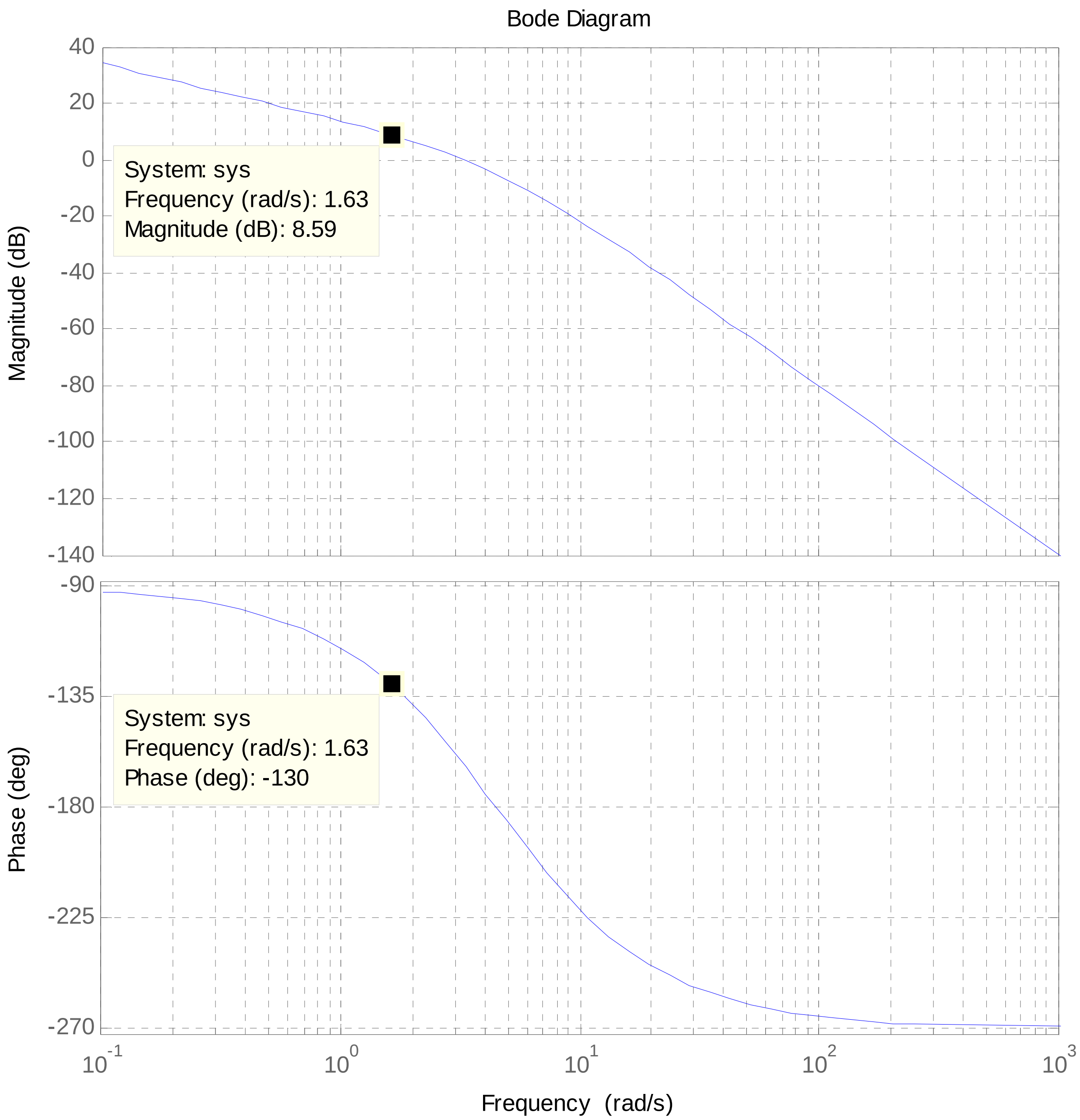
i. Find the phase margin of the uncompensated system and compare it to the design specs on phase margin (say, ϕ_{req}). To be safe, we usually choose a value $\phi_m > \phi_{req}$ to start with the design.

To be safe, we choose

$$\phi_m = 45° + 5° = 50°$$

ii. Find the frequency ω_x such that the uncompensated loop at which has a phase response of $-180° + \phi_m$.

$$\omega_x = 1.63 \text{ rad/s}$$



Looking for a frequency that gives us the required ϕ_m.

iii. Find the corresponding gain value (say, G_x) of the uncompensated loop at ω_x.

$G_x = 8.59$ dB $\Rightarrow$ $G_x = 2.7$

iv. Compute $\alpha = G_x$ and $T = \dfrac{10}{\omega_x}$.

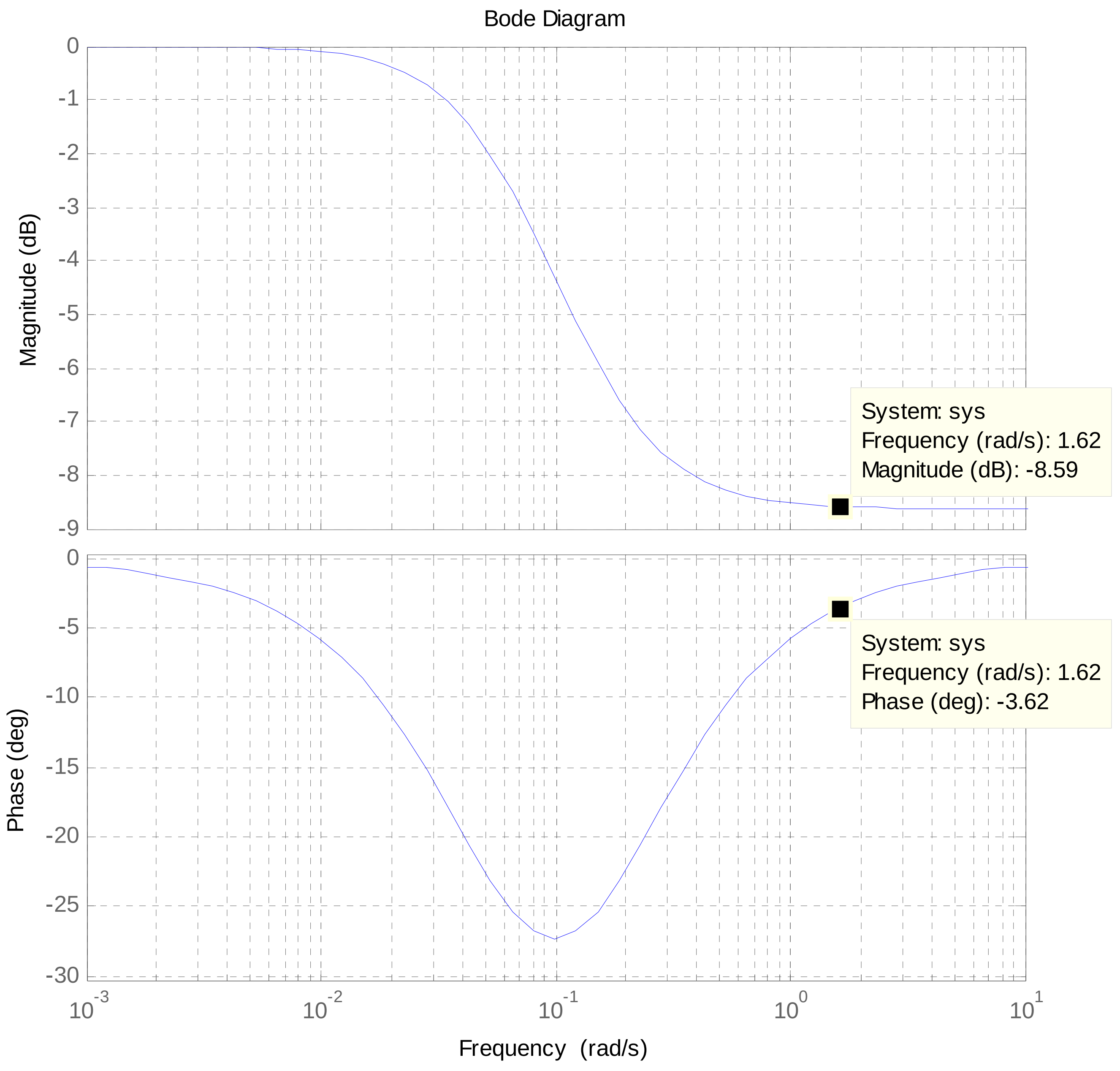
$\alpha = 2.7$ and $T = 6.135$

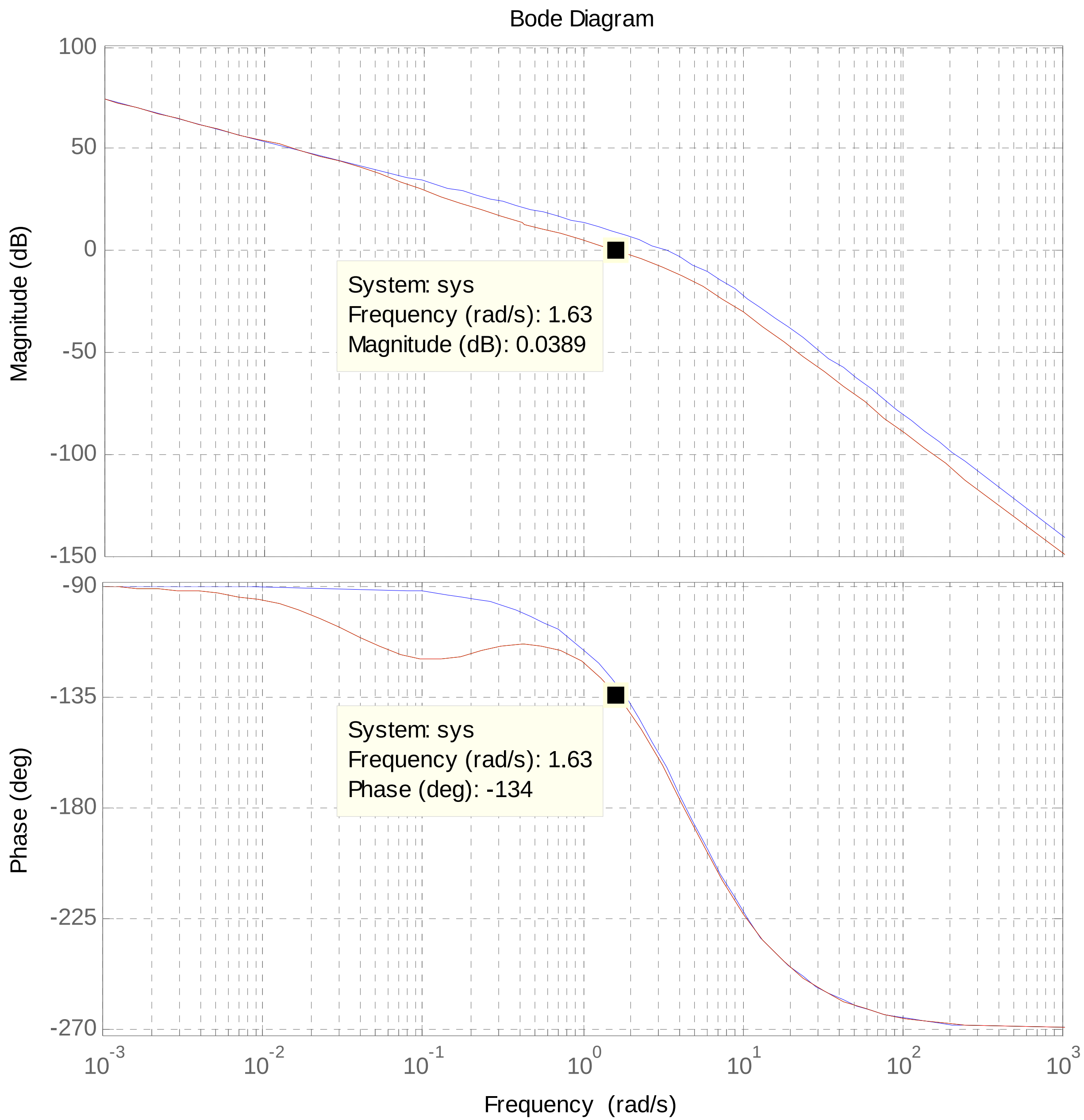
v. The required lag compensator is given as

$$C(s) = \frac{Ts+1}{\alpha Ts+1} = \frac{6.135s+1}{2.7\times 6.135s+1} = \frac{6.135s+1}{16.56s+1}$$

vi. Verify your result. Repeat the above steps if necessary.

$$C(s) = \frac{6.135s+1}{16.56s+1}$$





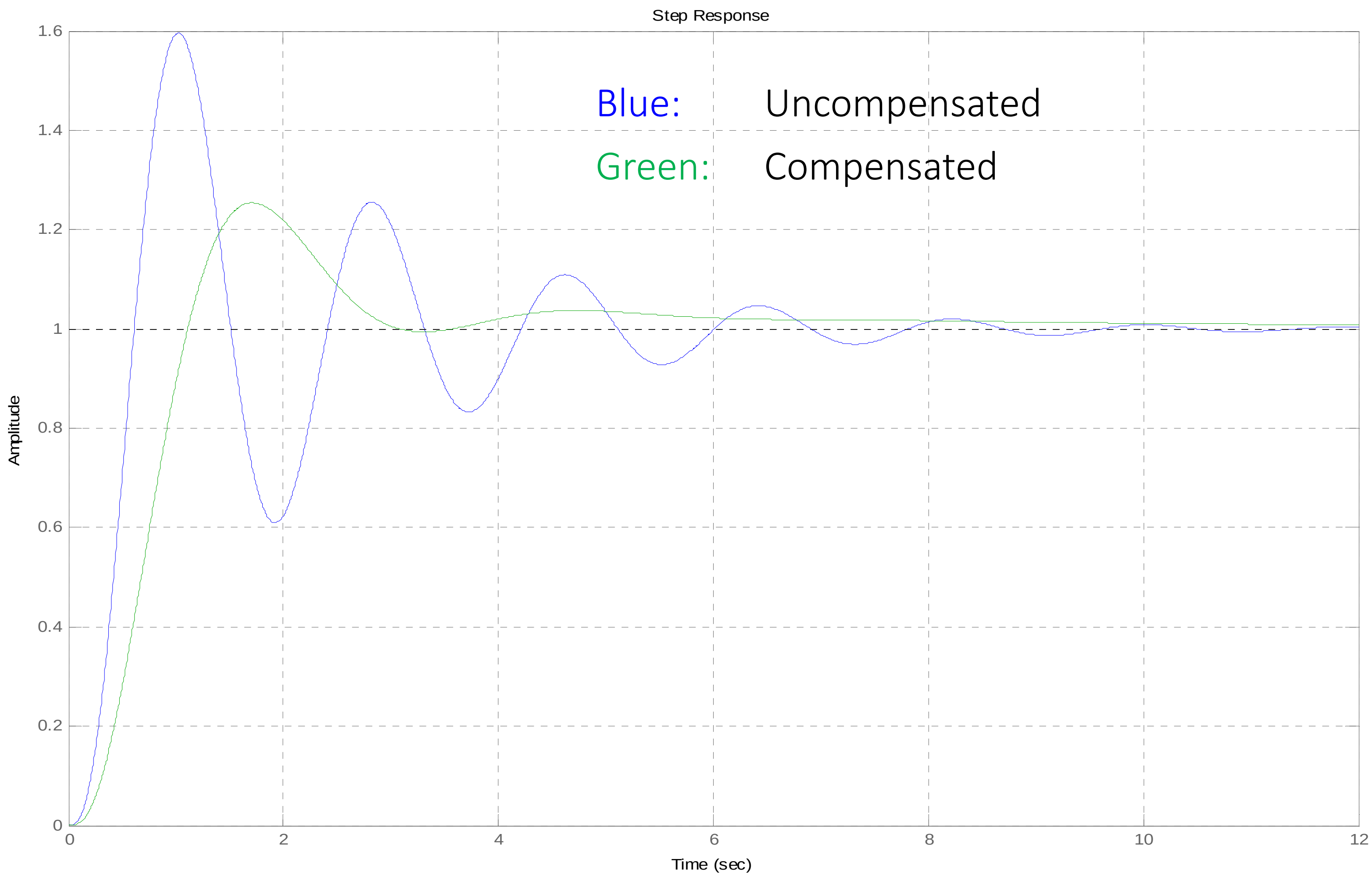
The new gain cross-over frequency is about

$$\omega_{cg,new} = 1.64 \text{ rad/s}$$

$$\text{PM} = 46^\circ$$

✓

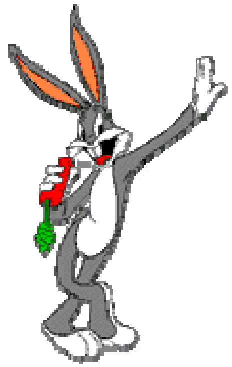
(2) Obtain the step response of the uncompensated and compensated systems.



Exercise A.2: Solve the problem in Example A.2 using a lead compensator.

That's all, folks!

Thank You!



Bugs Bunny
1940–