EE3331C Feedback Control Systems T4

Q.1 (a)

1. Consider the first-order system, $G(s) = \dfrac{2}{0.2s+1}$

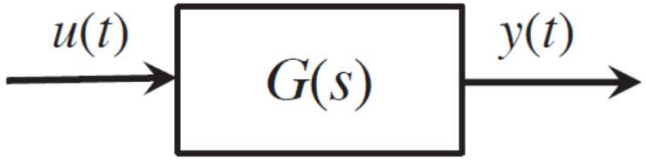


Figure 1: Open loop system, $G(s)$

(a) Suppose that the input is a sinusoidal signal $u(t) = \sin 3t$ (see Figure 1). Find the output of the system. Identify the steady-state response. Show that the amplitude ratio and phase shift of the steady-state response are equal to values given by $|G(j\omega)|$ and $\angle G(j\omega)$ where ω is frequency of the sinusoidal input.

A.1 (a)

$$U(s) = \mathcal{L}\{\sin 3t\} = \frac{3}{s^2+9}$$

The output of the first order system, $G(s)$, is

$$\begin{aligned}
Y(s) &= G(s)\;U(s) \\
&= \frac{2}{0.2s+1}\,\frac{3}{s^2+9} \\
&= \frac{0.176}{0.2s+1} + \frac{-0.88s+4.41}{s^2+9} \\
&= \frac{0.88}{s+5} - \frac{0.88s}{s^2+9} + \frac{4.41}{s^2+9} \\
y(t) &= \mathcal{L}^{-1}\left\{\frac{0.88}{s+5} - \frac{0.88s}{s^2+9} + 1.46\frac{3}{s^2+9}\right\} \\
&= 0.88e^{-5t} - 0.88\cos 3t + 1.47\sin 3t
\end{aligned}$$

When $t \to \infty$, $0.88e^{-5t} \to 0$. Therefore, the output of the first order system at steady-state is

$$\begin{aligned}
y_{ss}(t) &= -0.88\cos 3t + 1.47\sin 3t \\
&= A\sin(3t+\phi)
\end{aligned}$$

where $A = \sqrt{0.88^2 + 1.47^2} = 1.71$

$$\phi = \tan^{-1}\frac{-0.88}{1.47} = -0.54 \text{ rad}$$

Verify that $\frac{A}{1} = |G(j\omega)|$ and $\phi = \angle G(j\omega)$ where $\omega = 3$ rad/s

$$\begin{aligned}
|G(j\omega)|_{\omega=3} &= \left|\frac{2}{0.2\times 3j+1}\right| \\
&= \frac{2}{\sqrt{0.6^2+1^2}} = 1.71
\end{aligned}$$

$$\begin{aligned}
\angle G(j\omega)|_{\omega=3} &= \angle\frac{2}{0.2\times 3j+1} \\
&= -\tan^{-1}\frac{0.6}{1} \\
&= -0.54 \text{ rad}
\end{aligned}$$

Q.1 (b)

(b) The first order system in part(a) is placed under the control of a proportional controller with a gain of $K(s) = 5$ (see Figure 2). What is the steady-state output of the closed-loop system if the reference input, $r(t)$, is $\sin 3t$?

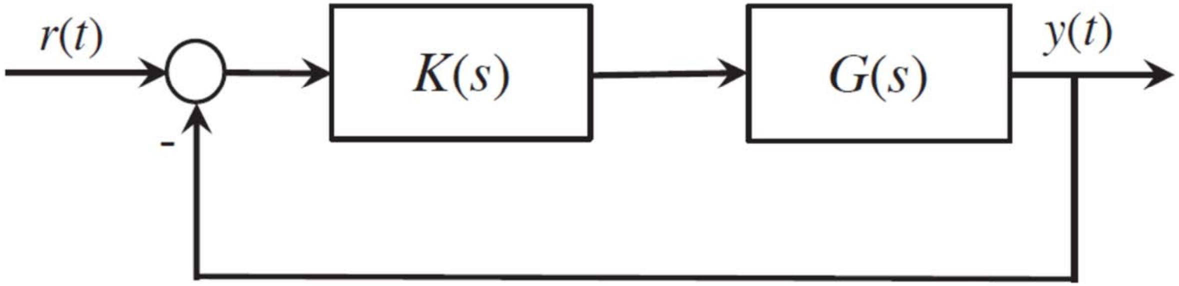
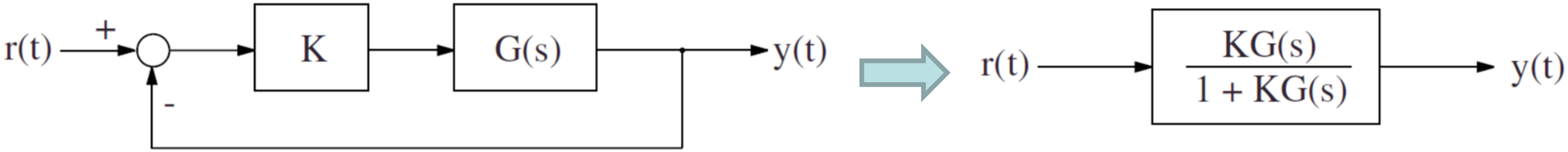


Figure 2: Closed loop system

A.1 (b)



Since the gain of the proportional controller is 5, the closed-loop transfer function is

$$H(s) = \frac{KG(s)}{1+KG(s)} = \frac{10}{0.2s+1+10} = \frac{10}{0.2s+11}$$

When $\omega = 3$ rad/s,

$$\left|H(j\omega)\right|_{\omega=3} = \left|\frac{10}{0.2\times 3j+11}\right| = \frac{10}{\sqrt{0.6^2+11^2}} = 0.91$$

$$\angle H(j\omega)|_{\omega=3} = \angle\frac{10}{0.2\times 3j+11} = -\tan^{-1}\frac{0.6}{11} = -0.054 \text{ rad}$$

The steady-state output of the closed-loop system is

$$y_{ss}(t) = 0.91\sin(3t-0.054)$$

Q.2

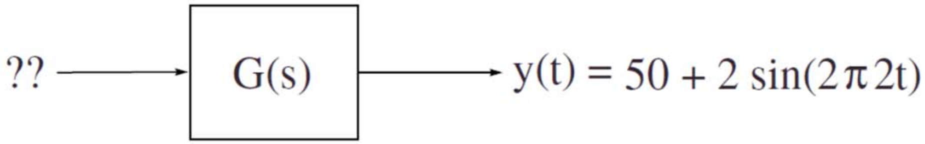
2. A high speed recorder monitors the temperature of an air stream as sensed by a thermocouple. The following observations were made:

   - The recorded temperature shows an essentially sinusoidal variation after about 1 second.
   - The maximum recorded temperature is about 52^oC and the minimum is 48^oC at 2 cycles per minute.

   The information indicates that the recorded steady-state temperature may be expressed as $50 + 2\sin(4\pi t)$. If the system (thermocouple and high speed recorder) has unity steady-state gain and first order dynamics with a time constant of approximately 1 minute under these conditions, estimate the actual maximum and minimum air temperatures.

A.2

In this problem, the output and the transfer function is known and the task is to determine the input.



Question states that the plant (thermocouple and recorder) may be represented by a first order system with unity steady-state gain ($K = 1$), time constant (τ) of approximately 1 minute and no dead time i.e.

$$G(s) = \frac{K}{\tau s + 1} = \frac{1}{s+1}$$

At steady state, the recorded temperature oscillates with a frequency of 2 cycles per minute between 52^oC and 48^oC i.e.

$$y(t) = 50 + 2\sin(2\pi f t) \qquad \text{where } f = 2 \text{ cycles per minute}$$

By the principle of superposition, the input will have two components because the output comprises of a dc component and a sinusoidal signal.

- The dc component of the output is 50. Since the transfer function has unity gain when $\omega = 0$ rad/s, the dc component of the input must also be 50.
- The frequency of the sinusoidal component is 2 cycles/min. The magnitude and phase of the plant, $G(s)$, at that frequency is

$$
\begin{aligned}
G(j\omega) &= \frac{1}{2\pi f j + 1} \\
&= \frac{1}{4\pi j + 1} \\
|G(j\omega)|_{\omega=4\pi} &= \frac{1}{\sqrt{16\pi^2 + 1}} \\
&= 0.0793 \\
\angle G(j\omega)|_{\omega=4\pi} &= -\tan^{-1} 4\pi
\end{aligned}
$$

It has been established that $|G(j\omega)|$ is the ratio of the output to the input and $\angle G(j\omega)$ is the phase shift. As the sinusoidal component in the ouput has an amplitude of 2, the input sinusoidal waveform is

$$
\frac{2}{0.0793} \sin(2\pi 2t + \tan^{-1} 4\pi) = 25.2 \sin(4\pi t + \tan^{-1} 4\pi)
$$

Hence, the actual air temperature is

$$
[50 + 25.2 \sin(4\pi t + \tan^{-1} 4\pi)]^o\mathrm{C}
$$

The input temperature oscillates between $50 - 25.2 = 24.8^o\mathrm{C}$ and $50 + 25.2 = 75.2^o\mathrm{C}$. Clearly, the recorder does not have sufficient bandwidth !

Q.3

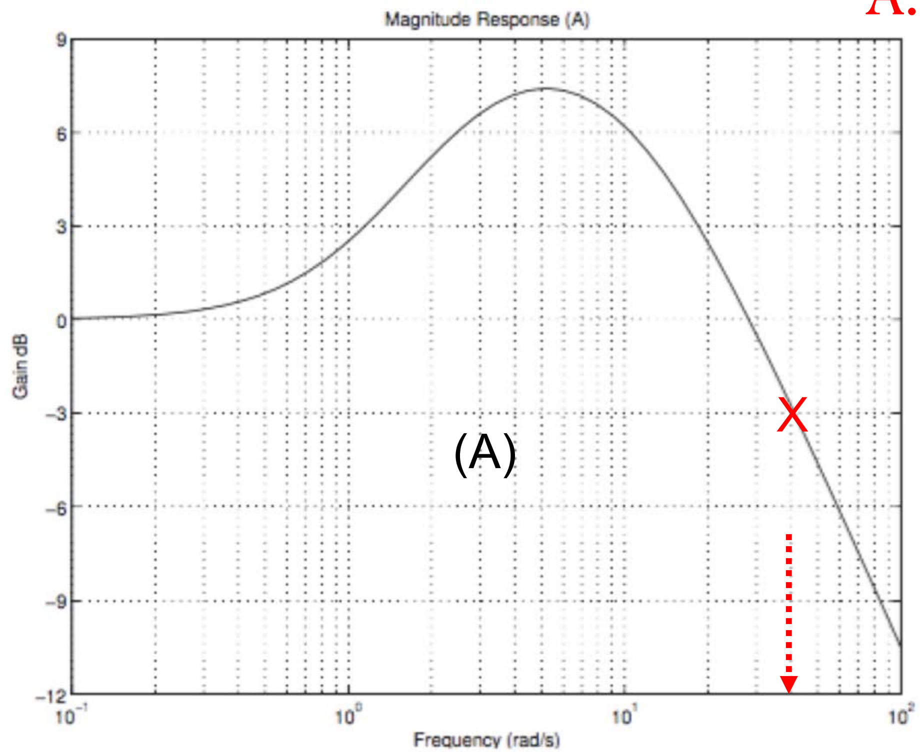
3. The magnitude responses for the following second-order systems are shown in Figure 3.

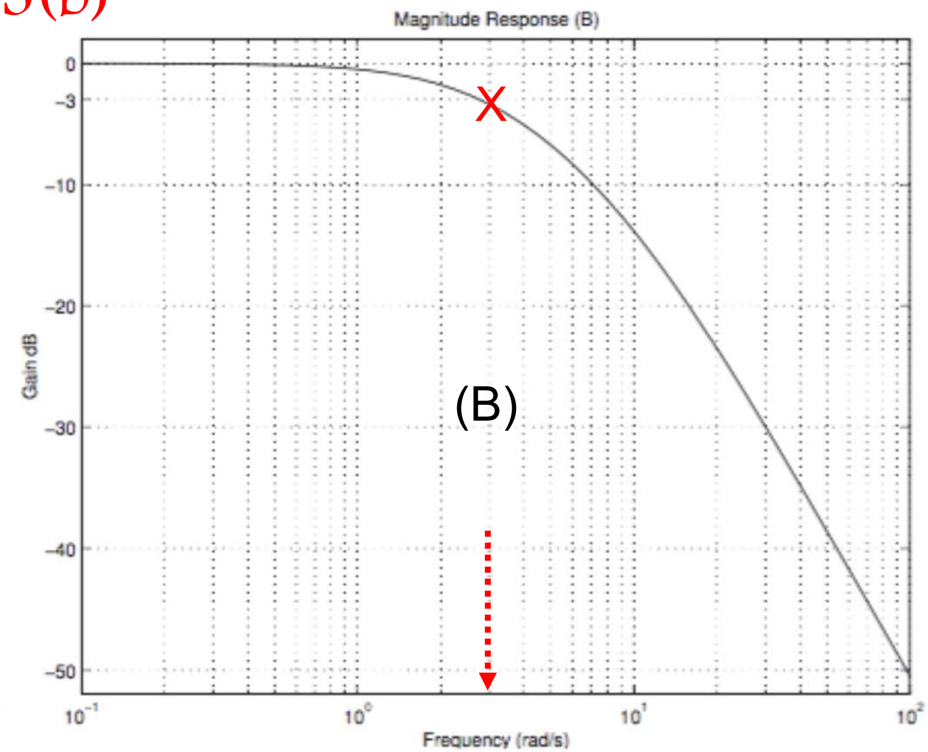
$$G_1 = \frac{30}{s^2+13s+30} \qquad G_2(s) = \frac{30(s+1)}{s^2+13s+30} \qquad G_3(s) = \frac{6(s+5)}{s^2+13s+30}$$

(a) Match the magnitude responses with the given transfer functions.

(b) What is the system bandwidth of each system?

A.3(b)





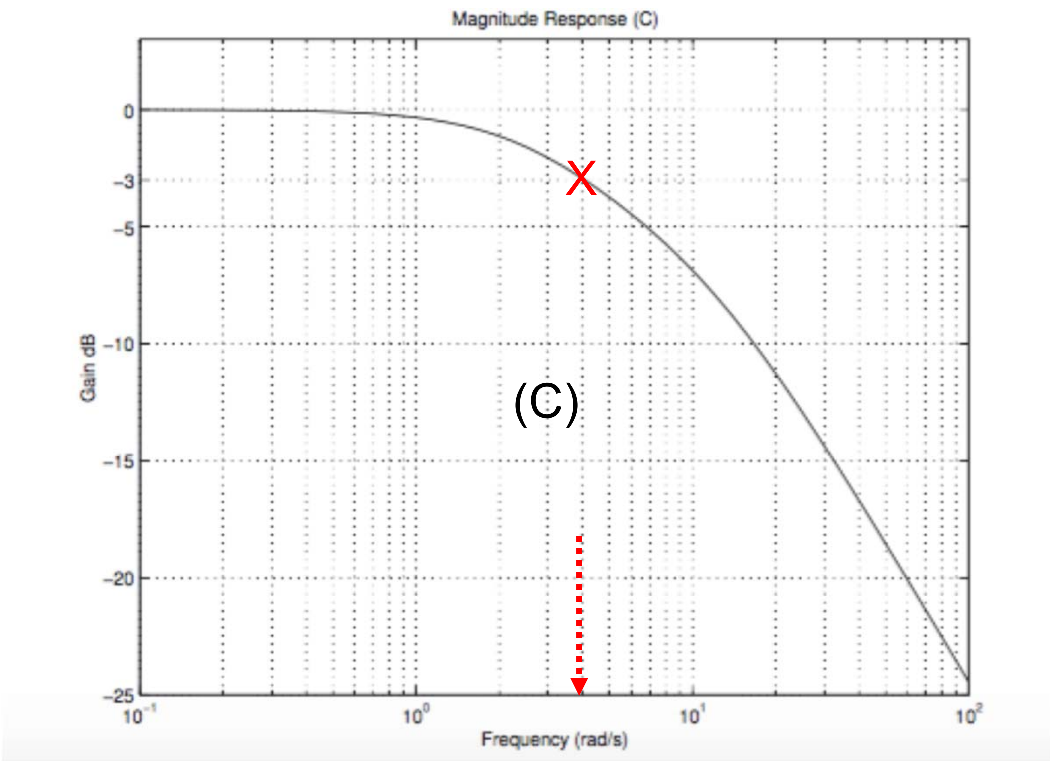


Figure 3: Magnitude response

A.3(a)

(a) Match the magnitude responses with the given transfer functions

$$G_1(s) = \frac{30}{s^2+13s+30}; \qquad G_2(s) = \frac{30(s+1)}{s^2+13s+30}; \qquad G_3(s) = \frac{6(s+5)}{s^2+13s+30}$$

- *Information that can be deduced by inspecting the transfer functions*
  - The poles of all the second order systems are located at $s = -3$ and $s = -10$. Since the poles lie on the real axis, the second order systems are overdamped i.e. $\zeta > 1$. Since $\zeta > \frac{1}{\sqrt{2}}$, the magnitude response should not have a resonant peak.
  - All systems have unity static gain (dc gain).
  - Pole excess of $G_1(s)$ is 2. $\Longrightarrow$ High frequency slope of the magnitude response should be -40 dB/decade.
  - Pole excess of $G_2(s)$ and $G_3(s)$ is 1. Hence, high frequency slope of the magnitude response should be -20 dB/decade.
  - $G_2(s)$ and $G_3(s)$ has a zero at $s = -1$ and $s = -5$ respectively.

- *Characteristics of the magnitude response*
  - High frequency slopes of magnitude responses (A) and (C) are -20 dB/decade while the slope of magnitude response (B) is -40 dB/decade when ω is large. Since pole excess of $G_1(s)$ is 2,
$$\text{Magnitude response (B)} \leftrightarrow G_1(s)$$
  - Slope of magnitude response (A) is 20 dB/decade when $\omega \approx 1$ rad/s. Indicates the existence of a zero at $s \approx -1$.
$$\text{Magnitude response (A)} \leftrightarrow G_2(s)$$
  - By elimination,
$$\text{Magnitude response (C)} \leftrightarrow G_3(s)$$

Q.4

4. Consider the closed-loop system in Figure 2, some open-loop frequency response data of the plant is tabulated in Table 1 while Figure 4 shows the Bode plot of the plant.

Freq in rads/min	Magnitude in dB	Phase in degrees
0.02	11.2	-112
0.04	6.4	-125
0.07	0.95	-144
0.11	-5.8	-168
1.00	-54.2	-253

Table 1: Frequency response of the plant.

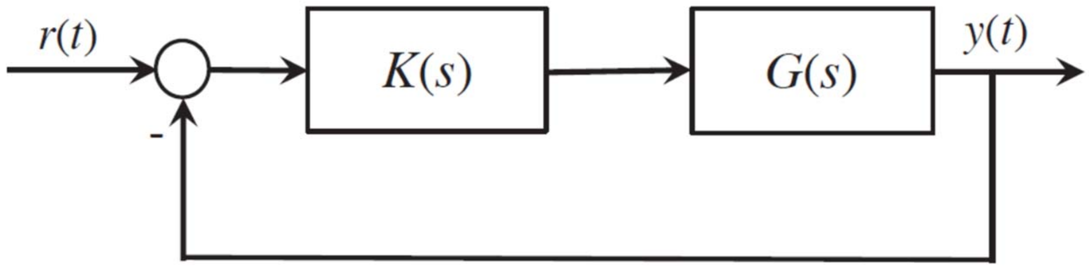


Figure 2: Closed loop system

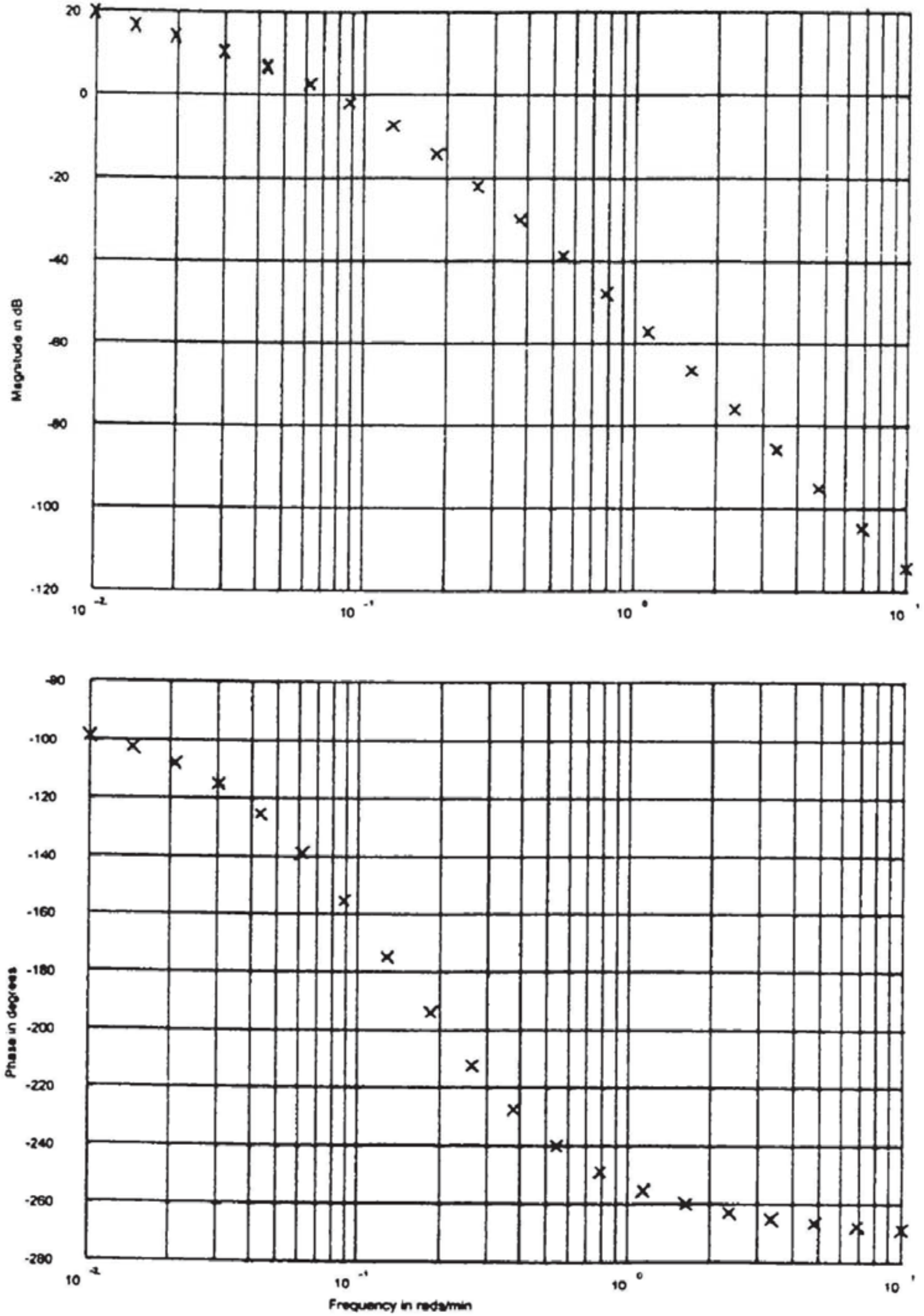


Figure 4: Bode plot of plant

A.4

(a) From the open-loop frequency response data, suggest the structure for a transfer function which may describe this plant (e.g. What is the order of the system i.e. how many poles does the system have? Is there an integrator? What is the dead time, if any?). State how you arrive at your conclusions.

(a) Suggest a structure for the plant

- High frequency slope is approximately -60 dB/decade.
  $\Longrightarrow$ Suggests the existance of 3 poles.
- Low frequency slope is approximately -20 dB/decade. In addition, low frequency phase shift is -90^o.
  $\Longrightarrow$ Suggests the presence of an integrator.
- High frequency phase asymptote is -270^o.
  $\Longrightarrow$ Suggests 3 poles and no dead time.
- Hence, structure of the plant is likely to be

$$\frac{K}{s(1+sT_1)(1+sT_2)}.$$

where T_1 and T_2 are the time constants of the first order factors measured in minutes.

- Plant is third order and Type 1.

(b) Assume that the transfer function of the controller is $K(s) = 2$. If the reference setpoint, $r(t)$, is a sinusoidal signal equal to $\sin(0.04t) + \sin(0.07t)$, determine the steady-state output of the closed-loop system?

(b) Find $y(t)$ when input, $r(t) = \sin(0.04t) + \sin(0.07t)$

- Given that the proportional controller has a gain of 2, the closed loop transfer function is

$$\frac{2G(s)}{1+2G(s)} \qquad \text{where } G(s) \text{ is the open-loop transfer function.}$$

- The input comprises of 2 sinusoidal waveform. Using the principle of superposition, it may be concluded that the output also contains 2 sinusoidal waveform.

- When $\omega = 0.04$ rad/min, the open-loop transfer function has a phase shift of -125^o while the magnitude is 6.4 dB i.e.

$$\begin{aligned} 20\log_{10}|G(j\omega)| &= 6.4 \\ |G(j\omega)| &= 2.09 \end{aligned}$$

$$\frac{(2)2.09\angle(-125^o)}{1+(2)2.09\angle(-125^o)} = \frac{4.18\angle(-125^o)}{1+4.18\angle(-125^o)} = \frac{4.18\angle(-125^o)}{1-2.4-3.4j} = 1.14\angle(-13^o)$$

The steady-state output corresponding to $\sin(0.04t)$ is $1.14\sin(0.04t - 13^o)$

- When $\omega = 0.07$ rad/min, the open-loop transfer function has a phase shift of -144^o while the magnitude is 0.95 dB i.e.

$$\begin{aligned} 20\log_{10}|G(j\omega)| &= 0.95 \\ |G(j\omega)| &= 1.12 \end{aligned}$$

Since

$$\begin{aligned} \frac{(2)1.12\angle(-144^o)}{1+(2)1.12\angle(-144^o)} &= \frac{2.24\angle(-144^o)}{1+2.24\angle(-144^o)} \\ &= \frac{2.24\angle(-144^o)}{1-1.81-1.32j} \\ &= \frac{2.24\angle(-144^o)}{1.55\angle(-121^o)} \\ &= 1.44\angle(-23^o) \end{aligned}$$

The steady-state output corresponding to $\sin(0.07t)$ is $1.44\sin(0.07t-23^o)$

- Hence, the steady-state output is

$$1.14\sin(0.04t-13^o)+1.44\sin(0.07t-23^o)$$

Q.5

5. The magnitude response of the closed-loop system in Figure 2 is shown in Figure 5. Given that
$$K(s)G(s) = \frac{K\omega_n^2}{s(s+2\zeta\omega_n)+\omega_n^2(1-K)}$$
Determine K, ζ and ω_n. Sketch the corresponding unit-step response of the closed-loop system, indicating the values of the maximum overshoot, the peak-time, the 2% settling time and the steady-state error, $\lim_{t\to\infty}[r(t)-y(t)]$.

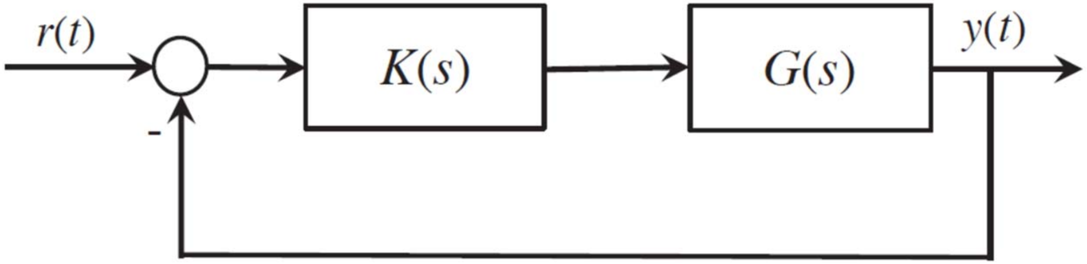


Figure 2: Closed loop system

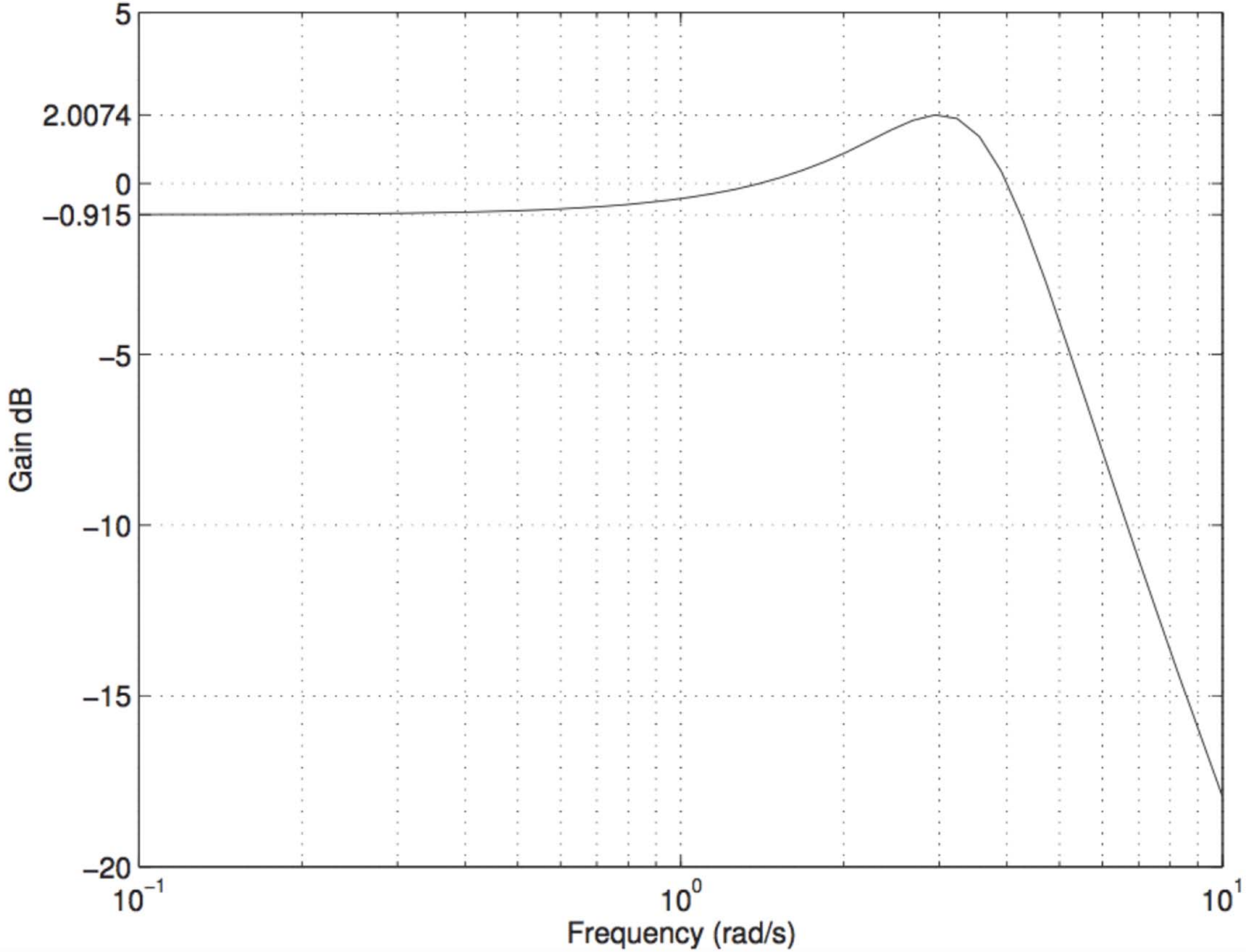


Figure 5: Closed loop magnitude response

A.5

Given that $G(s) = \dfrac{K\omega_n^2}{s(s+2\zeta\omega_n)+\omega_n^2(1-K)}$, the closed-loop transfer function is

$$\begin{aligned}\frac{G(s)}{1+G(s)} &= \frac{K\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2} \\ &= \frac{K}{1+2\zeta\frac{j\omega}{\omega_n}+\left(\frac{j\omega}{\omega_n}\right)^2}\end{aligned}$$

Hence, the magnitude response shown in Figure 1 comprises of the following factors :

- Static gain, K
- Quadratic factor, $\dfrac{1}{1+2\zeta\frac{j\omega}{\omega_n}+\left(\frac{j\omega}{\omega_n}\right)^2}$

The damping ratio, ζ, and the undamped natural frequency ω_n can be determined by comparing the magnitude response shown in Figure 5 with the magnitude response of a prototype second order factor.

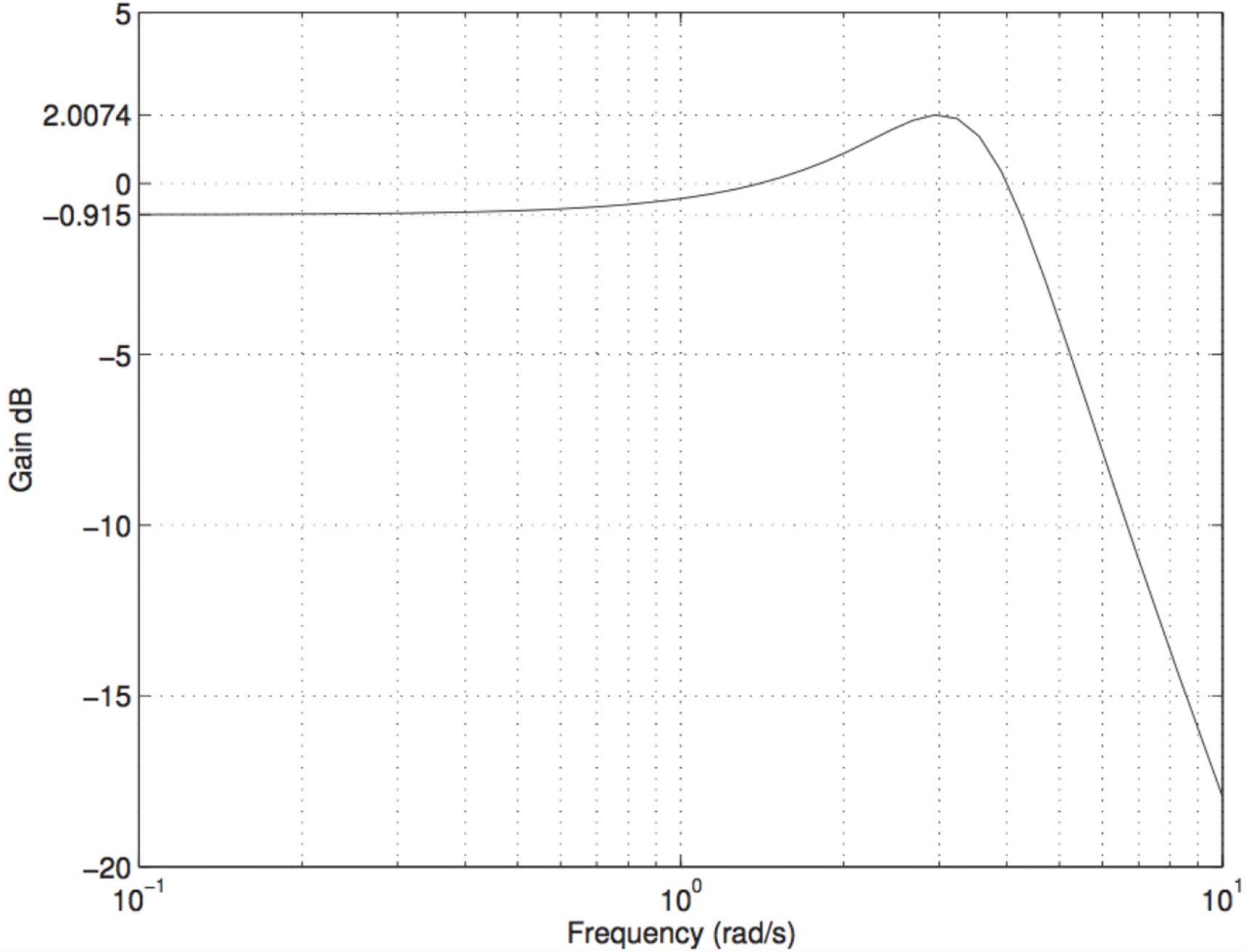


Figure 5: Closed loop magnitude response

From Figure 5, it may be deduced that

- Static gain, K = -0.915 dB = 0.9
- Maximum magnitude of system = 2.0074 dB = 1.26
- Resonant frequency, $\omega_r = 3$ rad/s
- $\zeta < \frac{1}{\sqrt{2}}$ because there is a hump in the magnitude response.

The maximum magnitude of the prototype second order system when $\zeta < \frac{1}{\sqrt{2}}$, is

$$M_r = \left| \frac{1}{1 + 2\zeta \frac{j\omega}{\omega_n} + \left(\frac{j\omega}{\omega_n}\right)^2} \right|_{max} = \frac{1}{2\zeta\sqrt{1-\zeta^2}}$$

Since the maximum magnitude of the system is 1.26, i.e.

$$|K| \left| \frac{1}{1 + 2\zeta \frac{j\omega}{\omega_n} + \left(\frac{j\omega}{\omega_n}\right)^2} \right|_{max} = 1.26 \quad \Rightarrow \quad \frac{1}{2\zeta\sqrt{1-\zeta^2}} = \frac{1.26}{0.9}$$

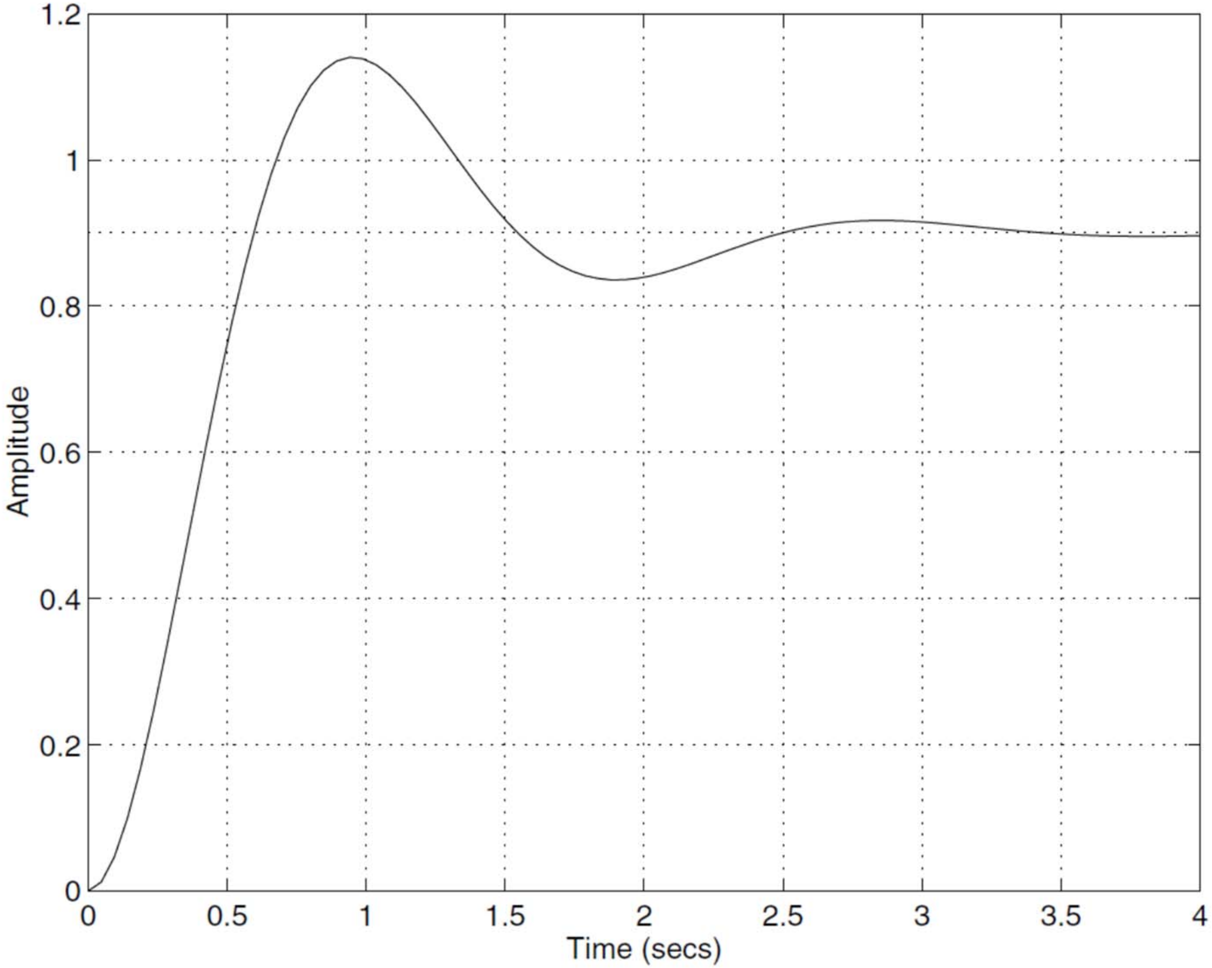
$$\Rightarrow \quad \zeta^4 - \zeta^2 + \frac{1}{2.8^2} \;=\; = 0 \qquad \Rightarrow \quad \zeta^2 \;=\; \frac{1 \pm \sqrt{1 - \frac{4}{2.8^2}}}{2} = 0.15 \text{ or } 0.85$$

$$\Rightarrow \quad \therefore \zeta \;=\; \sqrt{0.15} \qquad \because \zeta < \frac{1}{\sqrt{2}}$$

The resonant frequency, ω_r, of a prototype second-order system is $\omega_n\sqrt{1-2\zeta^2}$. Hence,

$$\omega_n\sqrt{1-2\zeta^2} \;=\; 3 \qquad \Rightarrow \quad \omega_n \;=\; \frac{3}{\sqrt{1-2(0.15)}} = \frac{3}{\sqrt{0.7}} \text{ rad/s}$$

- Maximum overshoot $= 0.9 \times e^{\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}}$
  $= 0.2405$
- Peak Time, $t_p = \dfrac{\pi}{\omega_d}$
  $= 0.9503$ seconds
- 1% settling time, $t_s = \dfrac{4.6}{\zeta\omega_n}$
  $= 3.3124$ seconds
- Steady state error of closed-loop system = 1 - K = 1 - 0.9 = 0.1



Step response of the closed-loop system

Q.6

6. Figure 6 shows the magnitude plot of $G(s) = \dfrac{K(s+\alpha)}{(s+\beta)(s+\eta)(s+\lambda)}$.

   (a) Using the approximate (straight line asymptotes) magnitude response, determine K, α, β, η and λ.

   (b) Calculate the magnitude and phase response of $G(j\omega)$ when ω is 3 rad/s, 25 rad/s and 67.5 rad/s. Hence, or otherwise, sketch the polar plot for $G(j\omega)$. The behavior of the polar plot when $\omega \to 0$ and $\omega \to \infty$ should be clearly shown.

   (c) Write down the transfer function of another plant that may have the same magnitdue response.

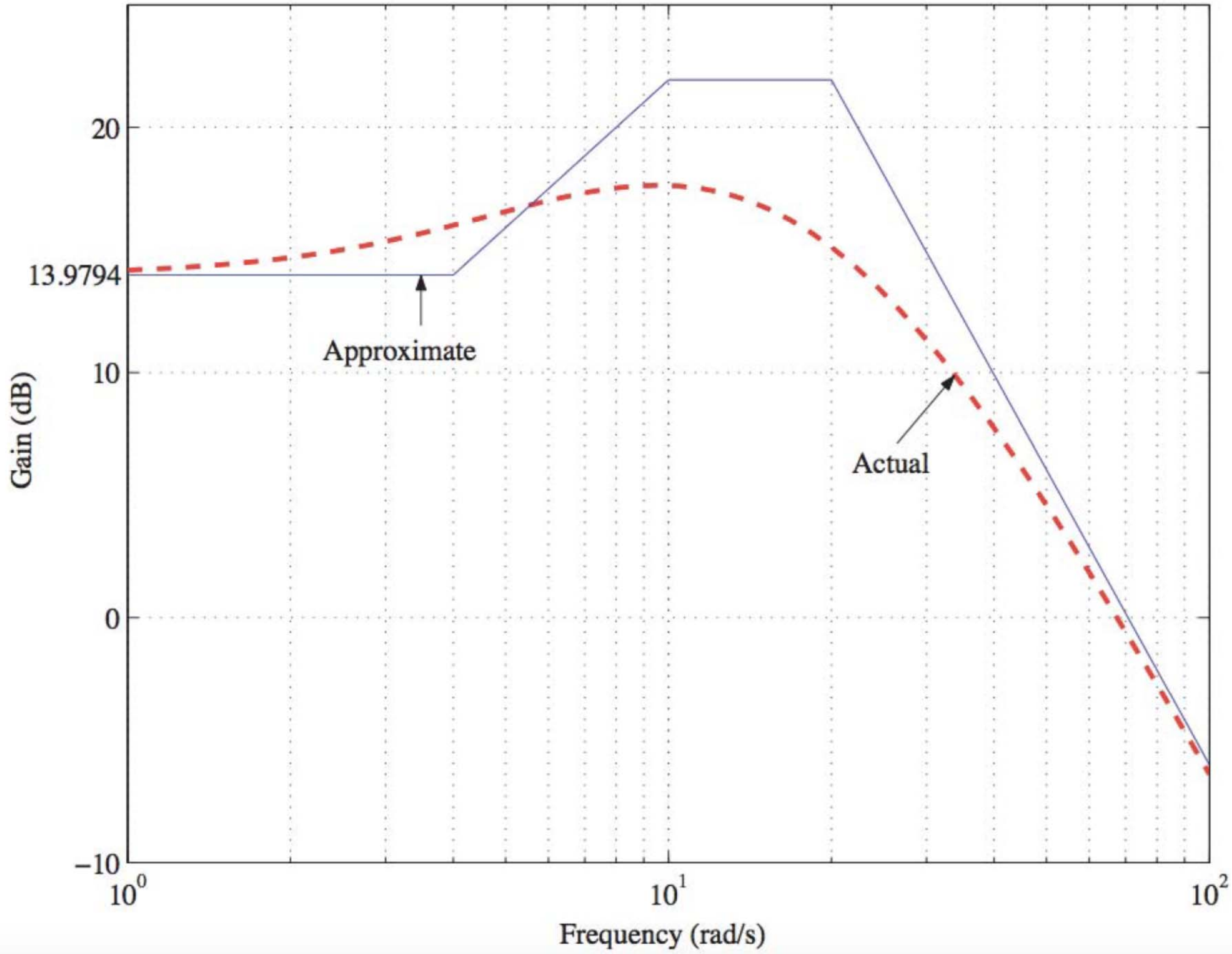


Figure 6: Magnitude response

A.6

(a)

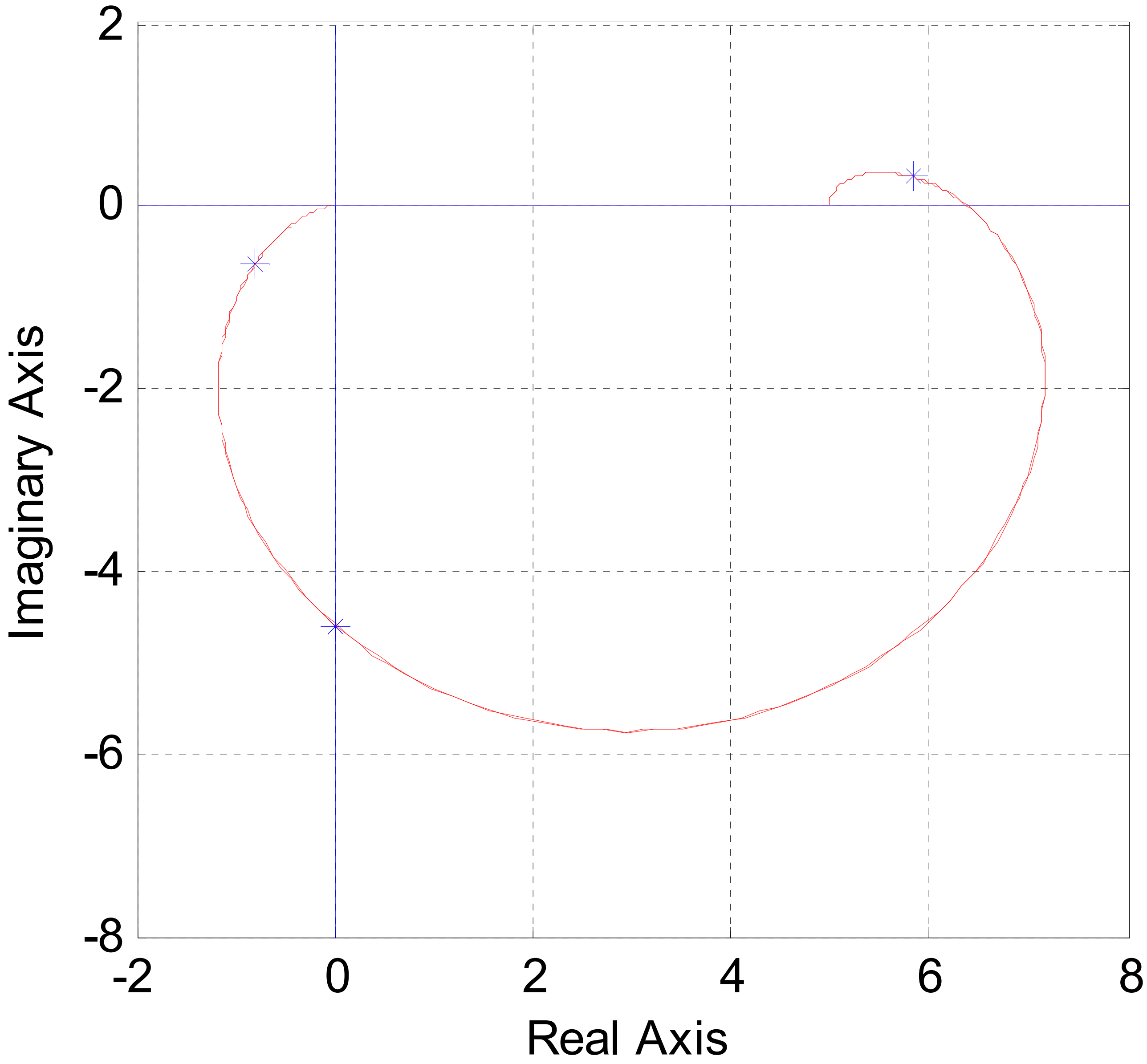
$$G(s)=\frac{k\cdot\left(1+\frac{s}{\alpha}\right)}{\left(1+\frac{s}{\beta}\right)\cdot\left(1+\frac{s}{\eta}\right)\cdot\left(1+\frac{s}{\lambda}\right)}=\frac{5\cdot\left(1+\frac{s}{4}\right)}{\left(1+\frac{s}{10}\right)\cdot\left(1+\frac{s}{20}\right)\cdot\left(1+\frac{s}{20}\right)}=\frac{5000\cdot(s+4)}{(s+10)\cdot(s+20)\cdot(s+20)}$$

(b)

$$G(j3)=\frac{5000\cdot(j3+4)}{(j3+10)\cdot(j3+20)\cdot(j3+20)}=5.8547\angle 3.1^\circ$$

$$G(j25)=\frac{5000\cdot(j25+4)}{(j25+10)\cdot(j25+20)\cdot(j25+20)}=4.5868\angle -90^\circ$$

$$G(j67.5)=\frac{5000\cdot(j67.5+4)}{(j67.5+10)\cdot(j67.5+20)\cdot(j67.5+20)}=1\angle -142^\circ$$



2
0
-2
-4
-6
-8
-2
0
2
4
6
8
Imaginary Axis
Real Axis

(c)

$$G(s)=\frac{5000\cdot(s-4)}{(s-10)\cdot(s-20)\cdot(s-20)}$$

$$G(s)=\frac{5000\cdot(s+4)}{(s-10)\cdot(s-20)\cdot(s-20)}$$

$$G(s)=\frac{5000\cdot(s+4)}{(s+10)\cdot(s-20)\cdot(s-20)}$$

$$G(s)=\frac{5000\cdot(s-4)}{(s+10)\cdot(s-20)\cdot(s+20)}$$

Q.7

7. The bode diagram of a first-order plant, $G(s)$, with unity steady-state gain is shown in Figure 7.

   (a) Draw the polar plot of $G(s)$

   (b) Does $G(s)$ have any transportation delay? If yes, determine the transportation lag.

   (c) Using the solution from part(a), sketch the polar plot of $\dfrac{Ke^{-sL}}{s(\tau s+1)}$.

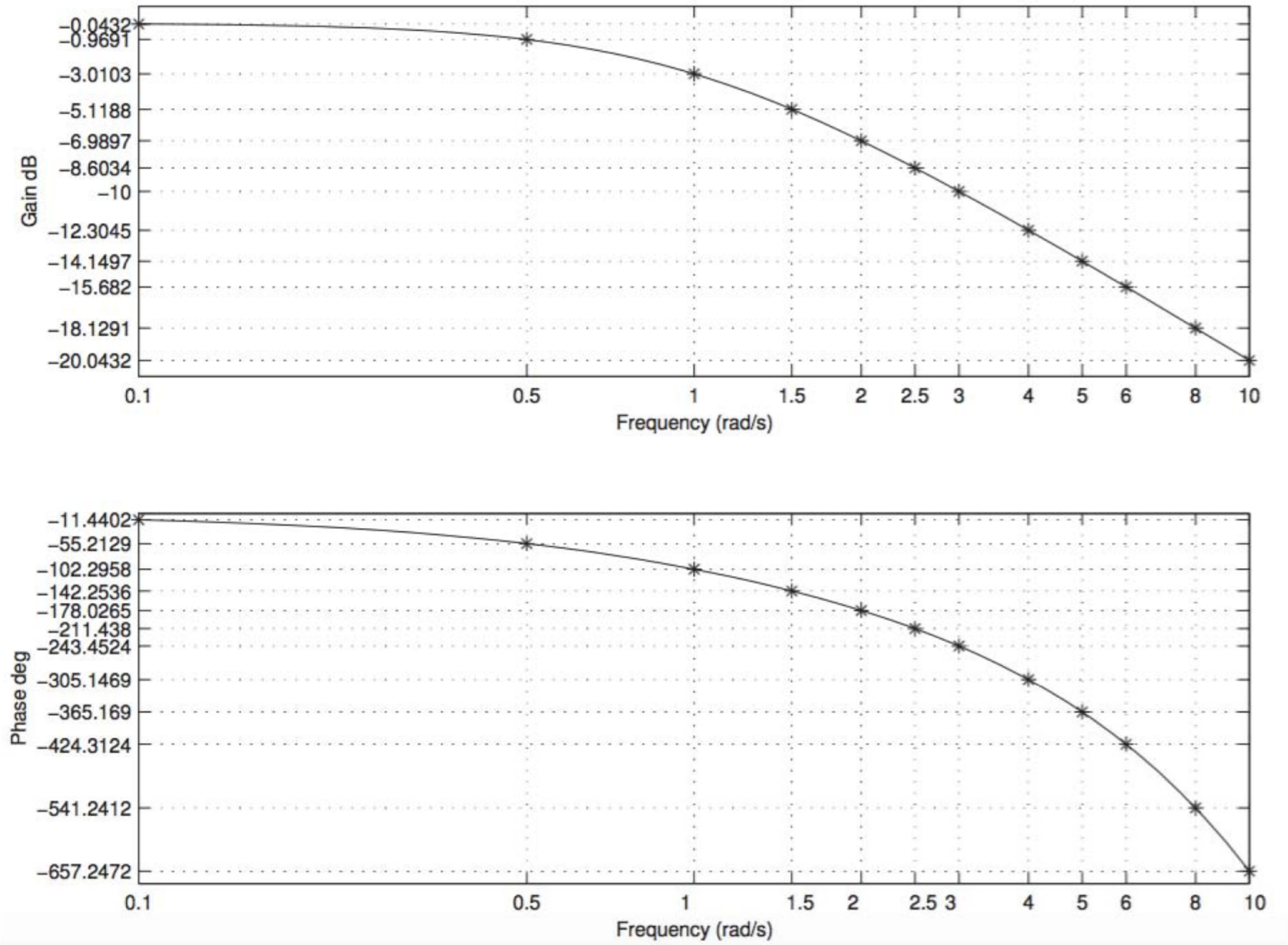


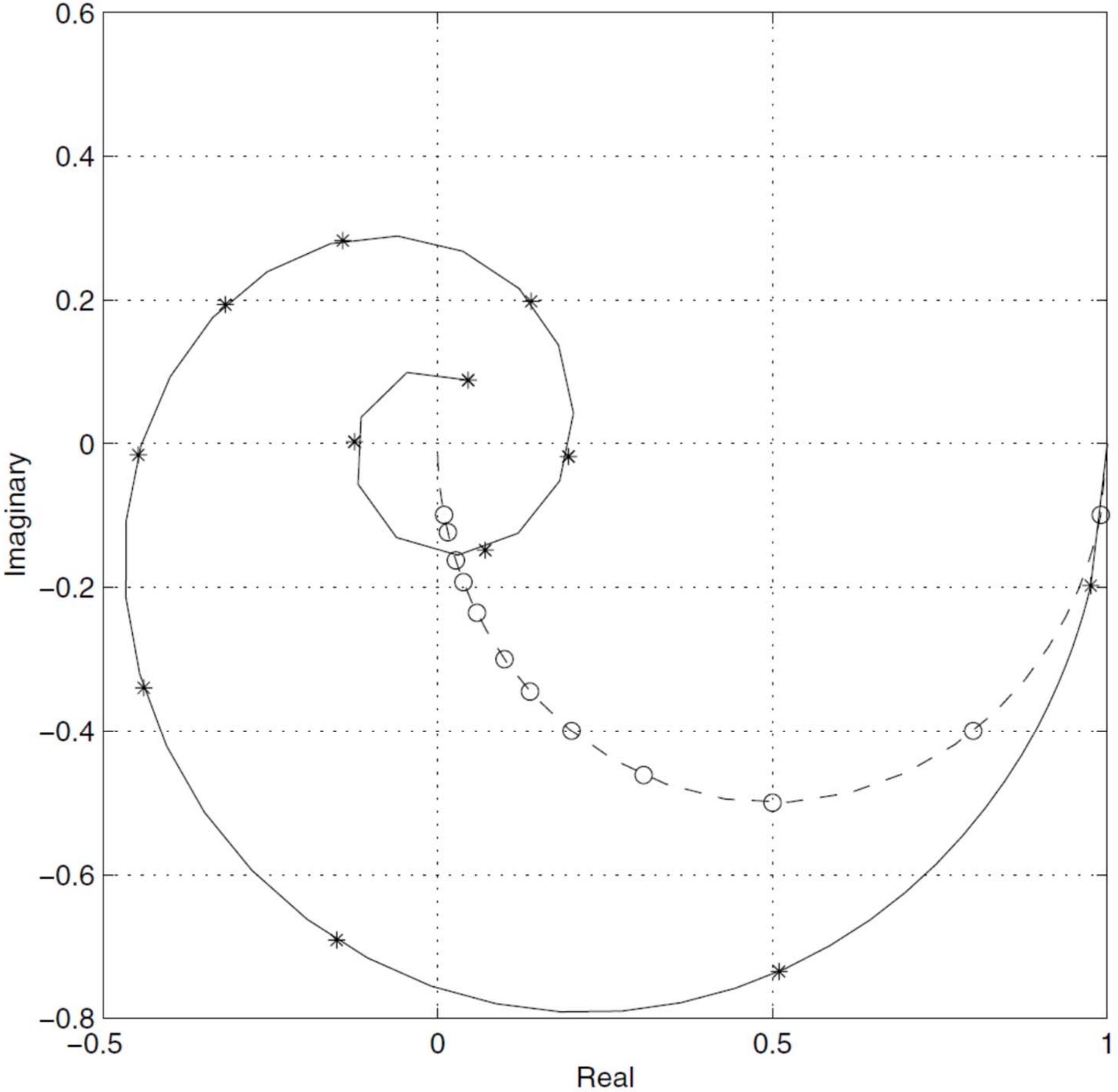
Figure 7: Bode plot

A.7

(a) Sketch the polar plot using open-loop frequency response data

From the Bode diagram, the following frequency response data is obtained

ω	$\lvert G_p(j\omega)\rvert$ in dB	$\lvert G_p(j\omega)\rvert$	$\angle G_p(j\omega)$
0.1	-0.04	0.99	-11.4^o
0.5	-0.97	0.89	-55^o
1	-3.01	0.71	-102^o
1.5	-5.12	0.55	-142^o
2	-6.99	0.45	-178^o
2.5	-8.6	0.37	-211^o
3.0	-10	0.31	-243.44^o
4.0	-12.30	0.24	-305.15^o
5.0	-14.15	0.20	-365.17^o
6.0	-15.68	0.16	-424.31^o
8.0	-18.13	0.12	-541.24^o
10.0	-20.04	0.10	-657.25^o



Polar plot of $G_p(s)$ and a first order system with zero deadtime

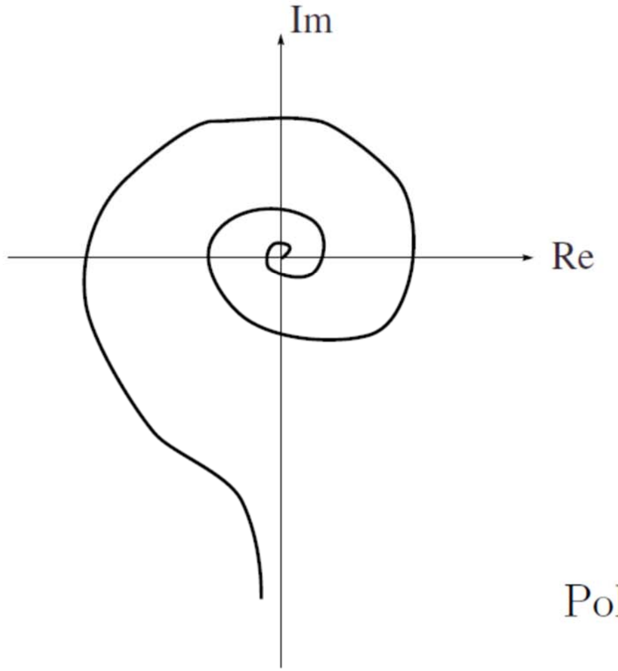
(b) Determine the transportation lag

- The corner frequency of a first order factor, $G(j\omega) = j\omega\tau + 1$, occurs at the point where $|G(j\omega)|$ is 3 dB below the static gain.
- Since the static gain is unity, corner frequency (ω_c)is the frequency where $|G(j\omega_c)| = -3$ dB. From the Bode diagram or the table on the previous page, $\omega_c = 1$ rad/s.
- When $\omega = 1$ rad/s, the phase response of the given plant is -102.2958.
- As the phase response of a pure first order system at the corner frequency is -45^o, the given plant has transportation lag.
- Phase lag contributed by the transportation delay term is $102.2958^o - 45^o = 57.2958$. Hence,

$$\begin{aligned} -\omega t_d &= -57.2958 \times \frac{\pi}{180} \qquad \omega = 1 \text{ rad/s} \\ \therefore\ t_d &= 1 \text{ sec} \end{aligned}$$

(c) Sketch the polar plot of the system resulting from the addition of an integrator to $\mathbf{G_p(s)}$.

- The resulting system, $G_1(s)$, is Type 1. As $\omega \to 0$, $|G_1(j\omega)| \to \infty|$ and $\angle G_1(j\omega) \approx -90^o$. Hence, the polar plot at low frequencies will be parallel to the negative imaginary axis.
- As the frequency incresase, the transportation delay term will force the polar plotfor $G_1(s)$ to spiral into the origin.
- A sketch of the polar plot for $G_1(s)$ is shown in



Polar plot for $G_1(s)$

EE3331C Feedback Control Systems T5

Q.1

1. Consider the following first-order compensators:

- $D_c(s) = 2.5\dfrac{s+2}{s+5}$
- $D_c(s) = 0.1\dfrac{s+10}{s+1}$
- $D_c(s) = \dfrac{2s+1}{s+1}$
- $D_c(s) = \dfrac{5s+1}{20s+1}$

For each of these compensators,

- Identify whether it is a Lead or Lag compensator
- Determine the frequency (ω_m) at which the compensator gives the largest phase shift
- Determine the maximum phase contribution ϕ_m
- Determine the gain as $\omega \to \infty$

A.1 (a)

$$2.5\frac{s+2}{s+5}$$

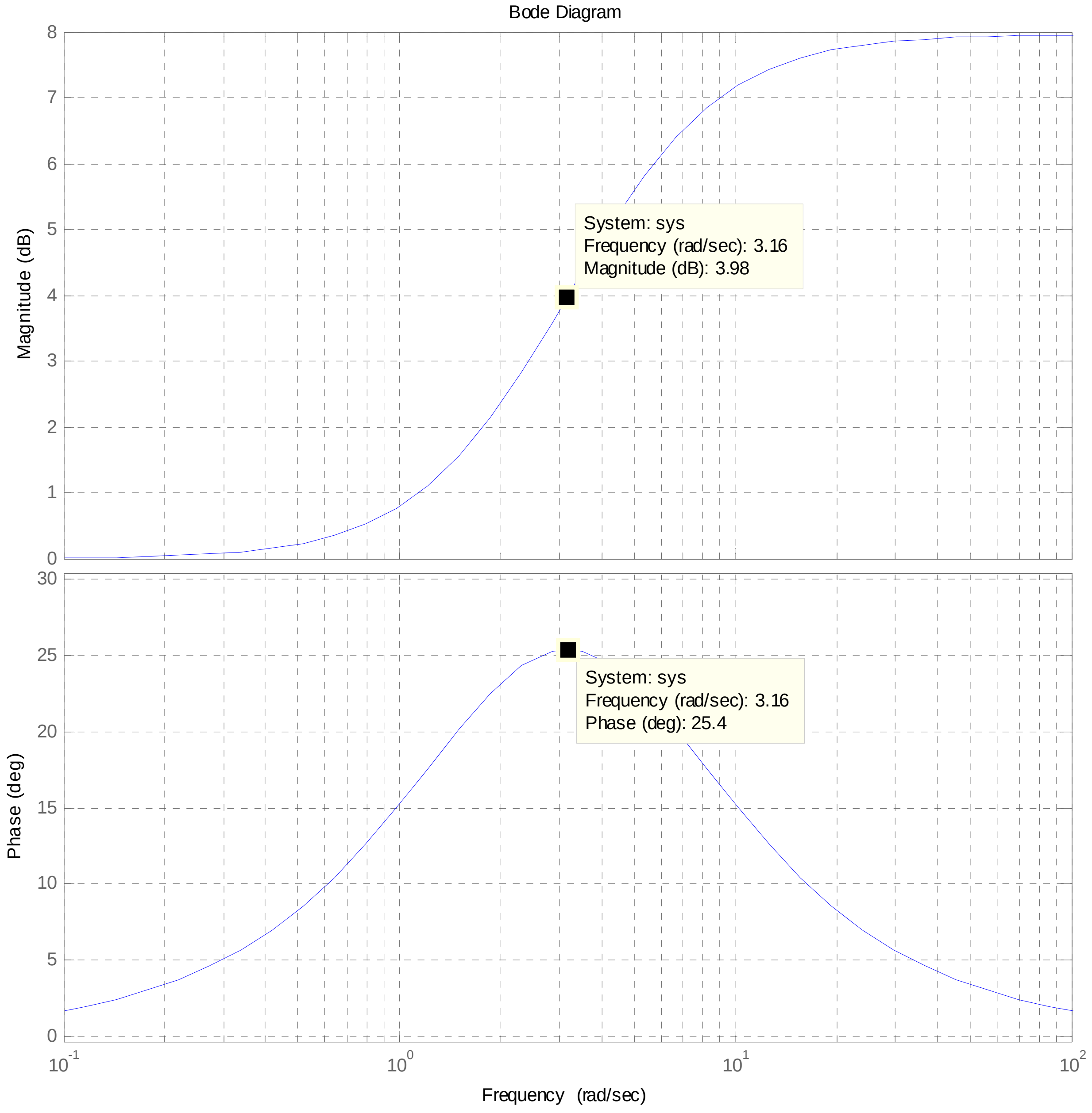
Lead,

$$\omega_m = \sqrt{2\times 5} = 3.16$$

$$\alpha = 0.4, \quad \sin\phi_m = \frac{1-0.4}{1+0.4}, \quad \phi_m = 25.4^\circ.$$

Gain with $\omega \to \infty$ is equal to

$$\frac{1}{\alpha} = \frac{1}{0.4} = 2.5 \quad \Rightarrow \quad 7.96dB.$$



Bode Diagram
Magnitude (dB)
Phase (deg)
Frequency (rad/sec)
System: sys
Frequency (rad/sec): 3.16
Magnitude (dB): 3.98
System: sys
Frequency (rad/sec): 3.16
Phase (deg): 25.4

A.1 (b)

$$0.1\frac{s+10}{s+1}$$

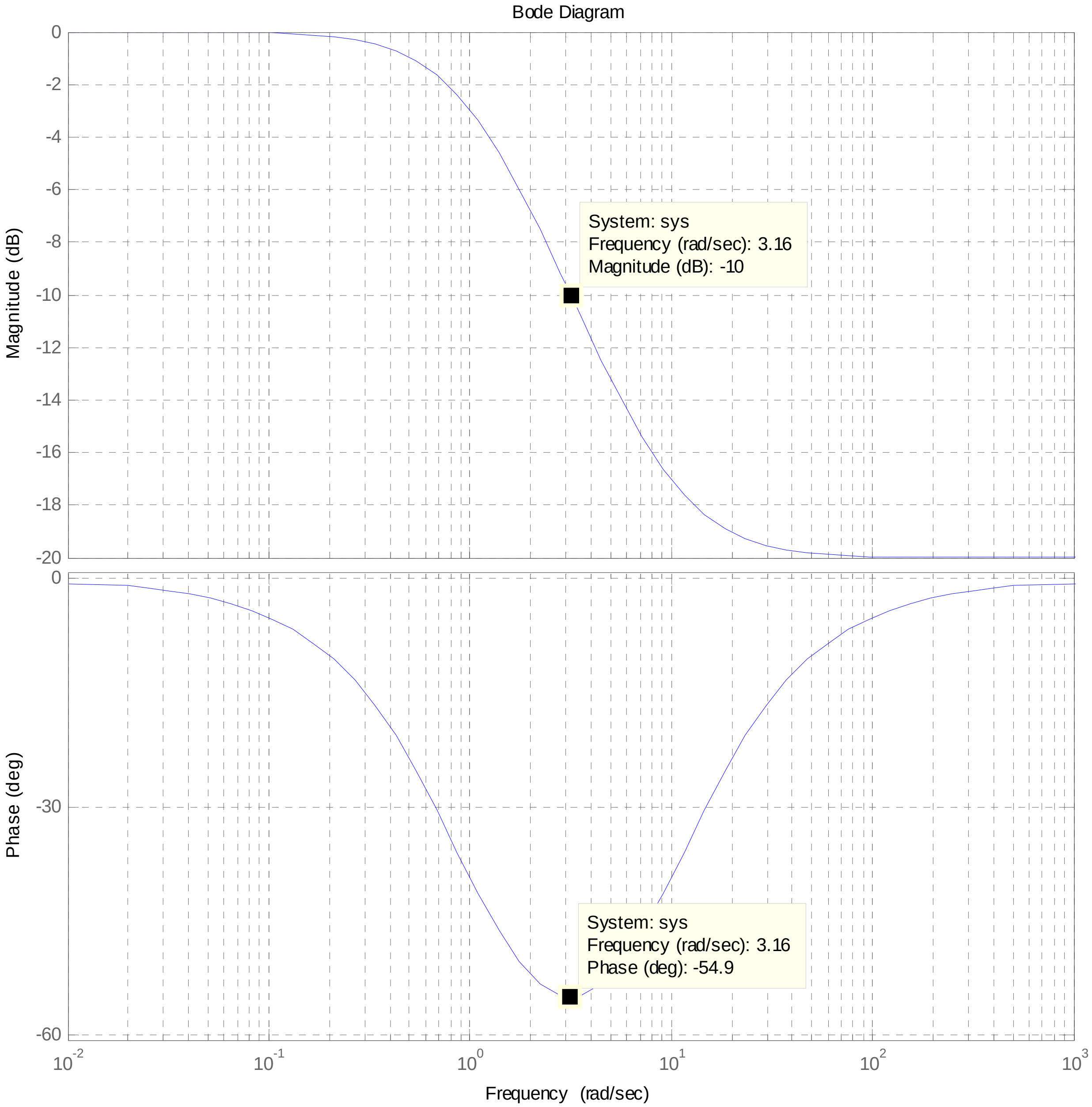
Lag,

$$\omega_m = \sqrt{10 \times 1} = 3.16$$

$$\alpha = 10, \quad \sin\phi_m = \frac{1-10}{1+10}, \quad \phi_m = -54.9^\circ.$$

Gain with $\omega \to \infty$ is equal to

$$\frac{1}{\alpha} = \frac{1}{10} = 0.1 \quad \Rightarrow \quad -20dB.$$


Bode Diagram
Magnitude (dB)
Phase (deg)
System: sys
Frequency (rad/sec): 3.16
Magnitude (dB): -10
System: sys
Frequency (rad/sec): 3.16
Phase (deg): -54.9
0
-2
-4
-6
-8
-10
-12
-14
-16
-18
-20
0
-30
-60
10^{-2}
10^{-1}
10^{0}
10^{1}
10^{2}
10^{3}
Frequency (rad/sec)

A.1 (c)

$$\frac{2s+1}{s+1}$$

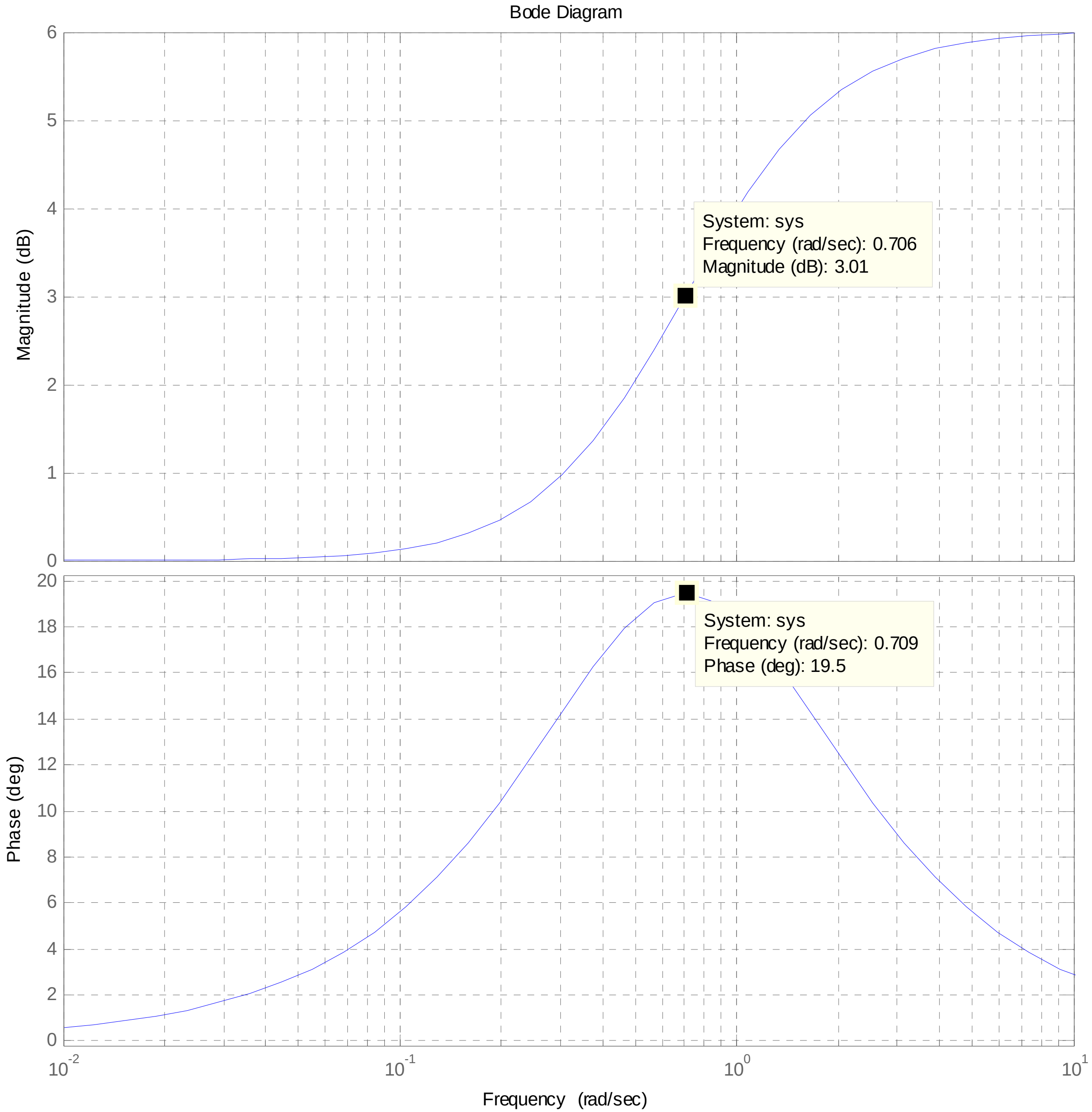
Lead,

$$\omega_m = \sqrt{0.5 \times 1} = 0.707$$

$$\alpha = 0.5, \quad \sin\phi_m = \frac{1-0.5}{1+0.5}, \quad \phi_m = 19.5^\circ.$$

Gain with $\omega \to \infty$ is equal to

$$\frac{1}{\alpha} = \frac{1}{0.5} = 2 \quad \Rightarrow \quad +6dB.$$


Bode Diagram
Magnitude (dB)
Phase (deg)
Frequency (rad/sec)
System: sys
Frequency (rad/sec): 0.706
Magnitude (dB): 3.01
System: sys
Frequency (rad/sec): 0.709
Phase (deg): 19.5

A.1 (d)

$$\frac{5s+1}{20s+1}$$

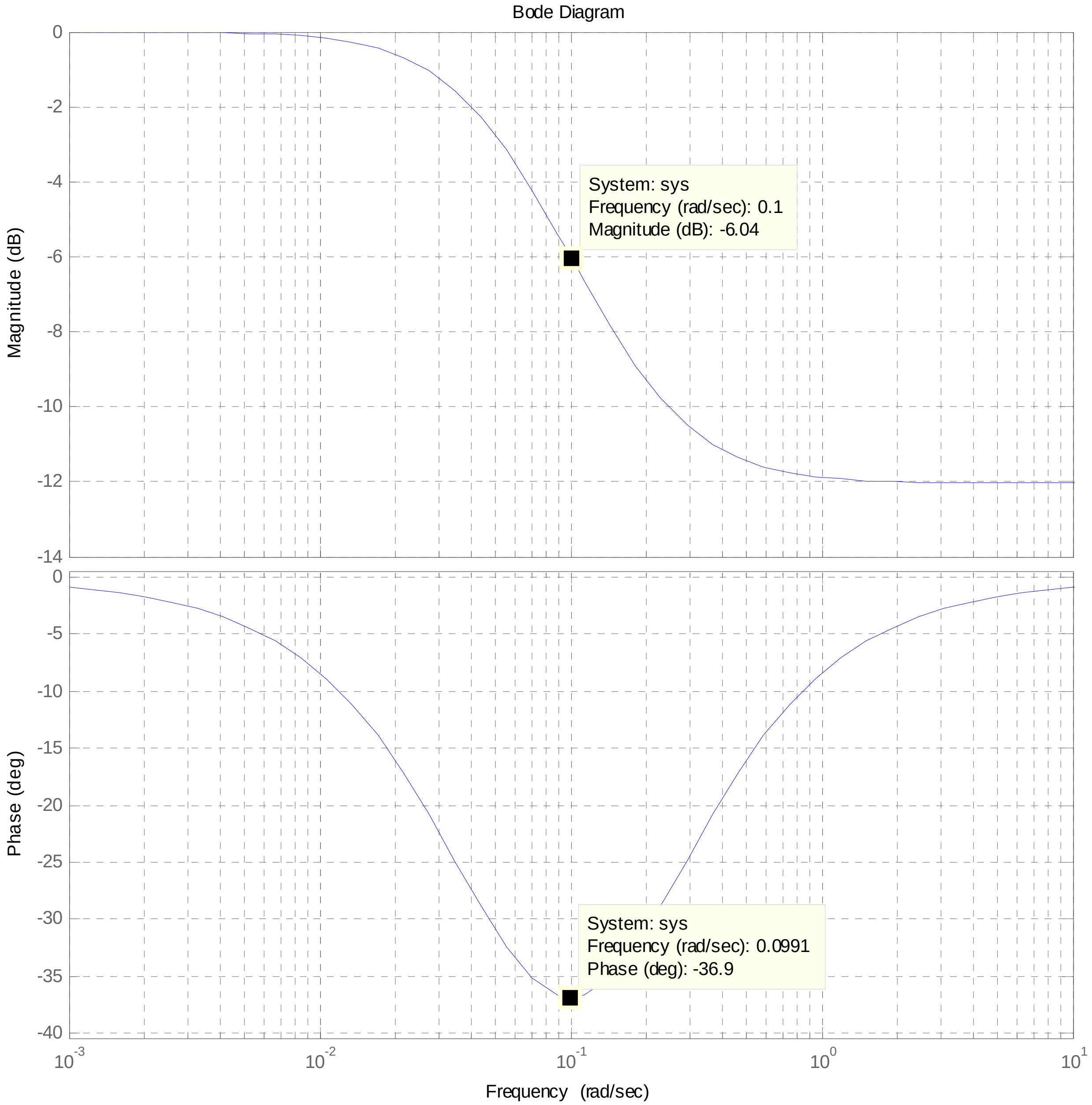
Lag,

$$\omega_m = \sqrt{0.2 \times 0.05} = 0.1$$

$$\alpha = 4, \quad \sin\phi_m = \frac{1-4}{1+4}, \quad \phi_m = -36.9^\circ.$$

Gain with $\omega \to \infty$ is equal to

$$\frac{1}{\alpha} = \frac{1}{4} = 0.25 \quad \Rightarrow \quad -12dB.$$


Bode Diagram
Magnitude (dB)
0
-2
-4
-6
-8
-10
-12
-14
System: sys
Frequency (rad/sec): 0.1
Magnitude (dB): -6.04
Phase (deg)
0
-5
-10
-15
-20
-25
-30
-35
-40
System: sys
Frequency (rad/sec): 0.0991
Phase (deg): -36.9
10^{-3}
10^{-2}
10^{-1}
10^{0}
10^{1}
Frequency (rad/sec)

Q.2

2. Write the transfer function of a first-order compensator that provides maximum $+45^o$ phase at 30 rad/sec.

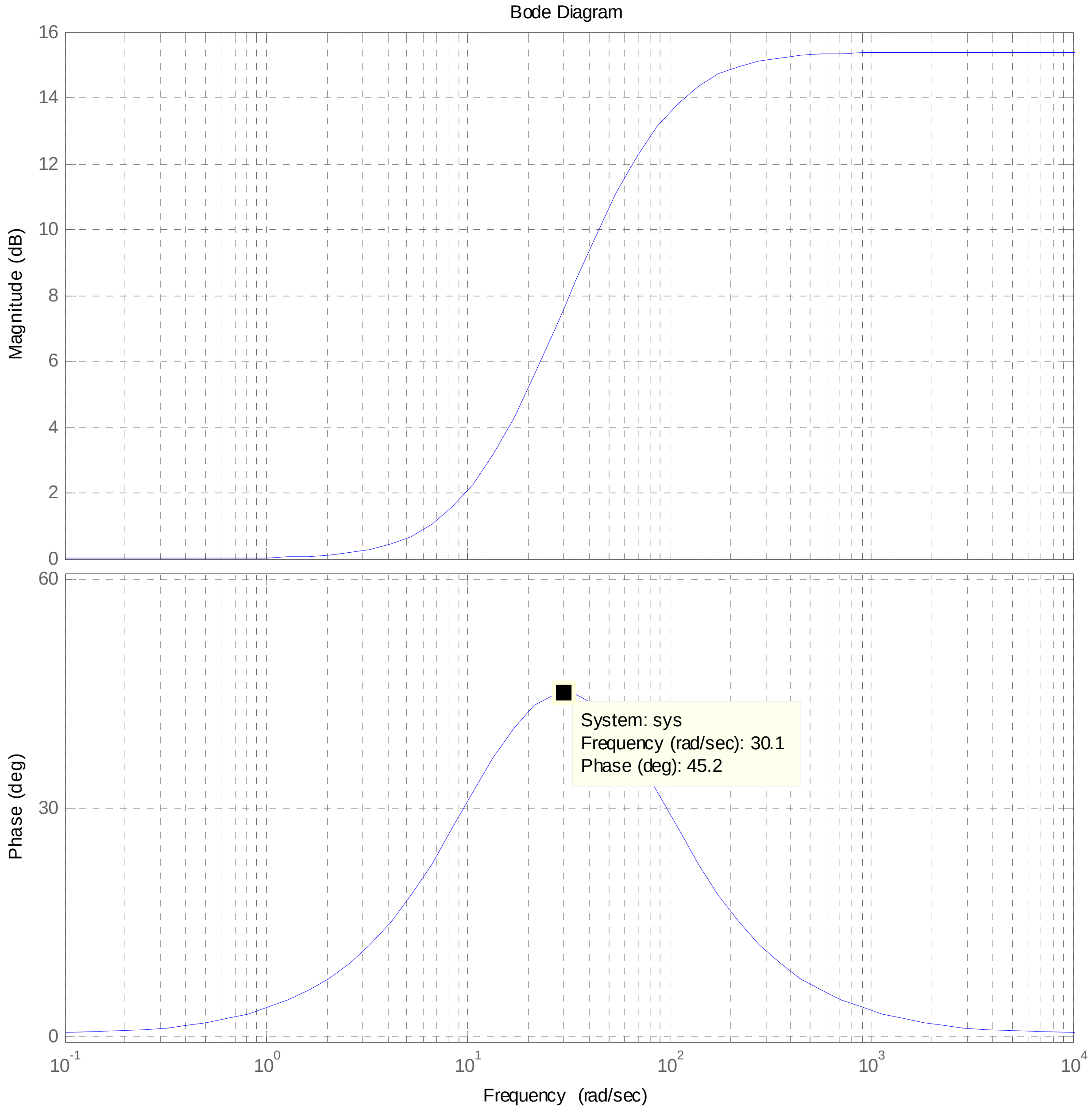
A.2

2. Write the transfer function of a first-order compensator that provides maximum $+45^\circ$ phase at 30 rad/s.

$$\sin\phi_m = \frac{1-\alpha}{1+\alpha} \quad \Rightarrow \quad \alpha = 0.17.$$

$$\omega_m = \frac{1}{T\sqrt{\alpha}} \quad \Rightarrow \quad T = \frac{1}{\omega_m\sqrt{\alpha}} = \frac{1}{30\times\sqrt{0.17}} = 0.08.$$

$$C(s) = \frac{0.08s+1}{0.17\times 0.08s+1}$$


Bode Diagram
Magnitude (dB)
Phase (deg)
System: sys
Frequency (rad/sec): 30.1
Phase (deg): 45.2
Frequency (rad/sec)

Q.3

3. Write the transfer function of a first-order compensator that can be used to provide 20 dB gain at 30 rad/sec with insignificant change in loop phase at that frequency. Calculate the actual gain and phase of the compensator at 30 rad/sec.

A.3

3. Write the transfer function of a first-order compensator that can be used to provide -20 dB gain at 30 rad/s with insignificant change in loop phase at that frequency. Calculate the actual gain and phase of the compensator at 30 rad/s. Answer:

$$|C(j\omega|_{\omega>>\omega_z} \approx \frac{1}{\alpha}.$$

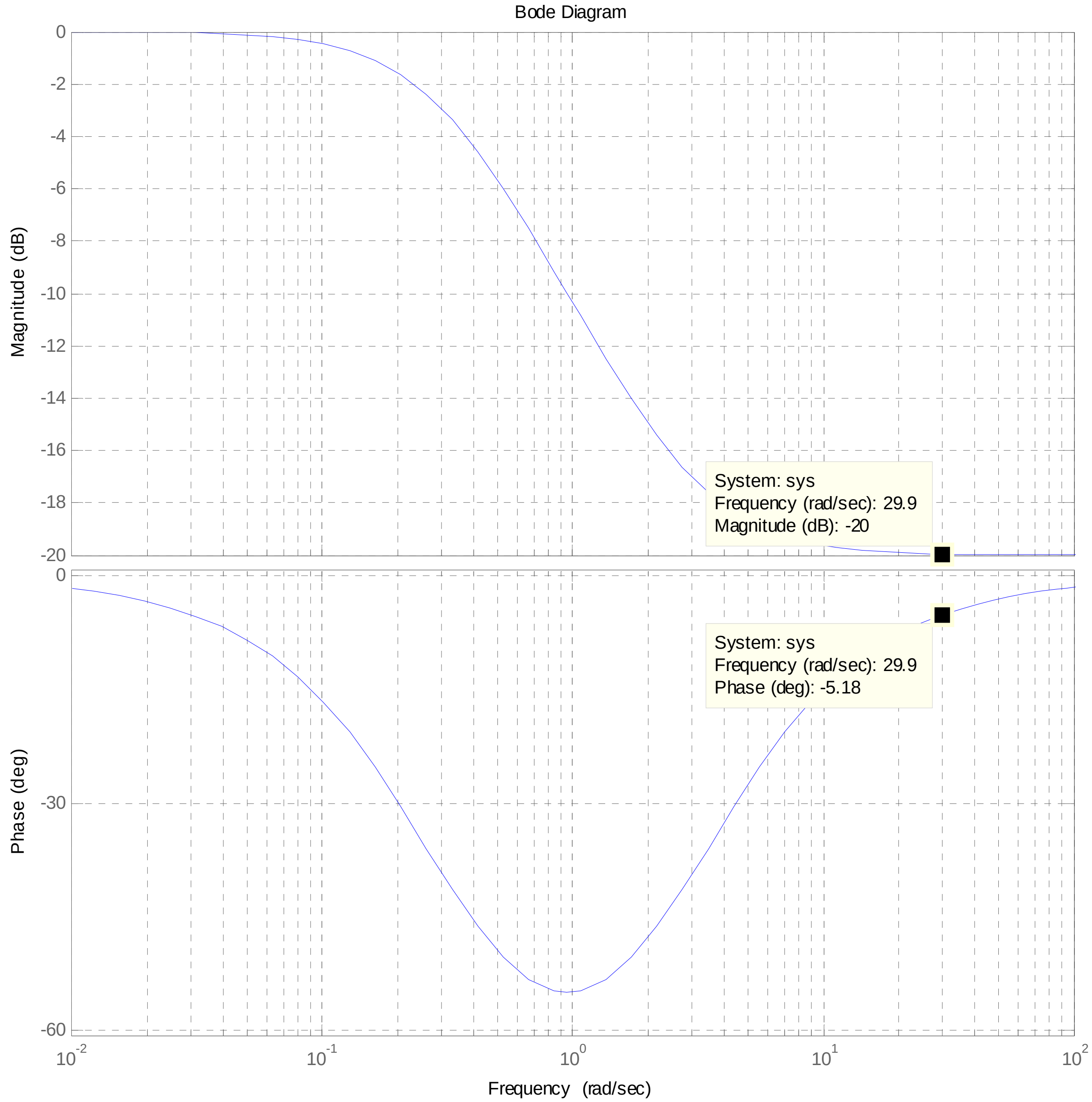
Required compensator gain is $-20dB$.

$$-20 = 20\log\frac{1}{\alpha} \quad \Rightarrow \quad \alpha = 10.$$

We choose the corner frequency of the compensator zero to be one-tenth of the frequency where attenuation is required.

$$\frac{1}{T} = \frac{\omega_x}{10} = \frac{30}{10} \quad \Rightarrow \quad T = \frac{1}{3}.$$

$$C(s) = \frac{\frac{1}{3}s+1}{\frac{10}{3}s+1}.$$



Calculation of the gain and phase at 30 rad/s can be done directly from the transfer function of the compensator.

Q.4

4. Consider the feedback control system shown in Figure 1 where $K(s) = K$ and $G(s) = \dfrac{1}{s(s+1)}$.

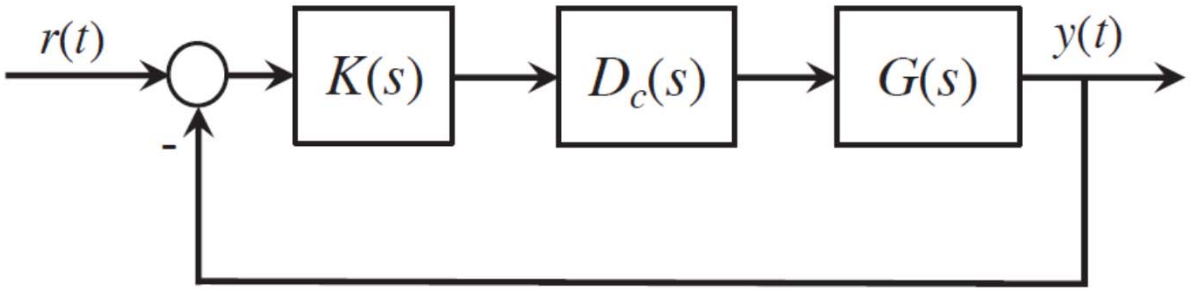


Figure 1: Feedback Control System

Design a controller that includes lead compensation to meet the following specifications:

- zero steady-state error for step input and 10% error for ramp input as $t \to \infty$,
- phase margin of 40^o.

Solution: (1) The velocity error constant is then given by

$$k_{\mathrm{v}} = \lim_{s\to 0} s \cdot C(s) \cdot G(s) = \lim_{s\to 0} C(s) \cdot \lim_{s\to 0} \frac{K}{s+1} = K \quad \Rightarrow \quad \frac{1}{k_{\mathrm{v}}} = 10\% \quad \Rightarrow \quad K = 10$$

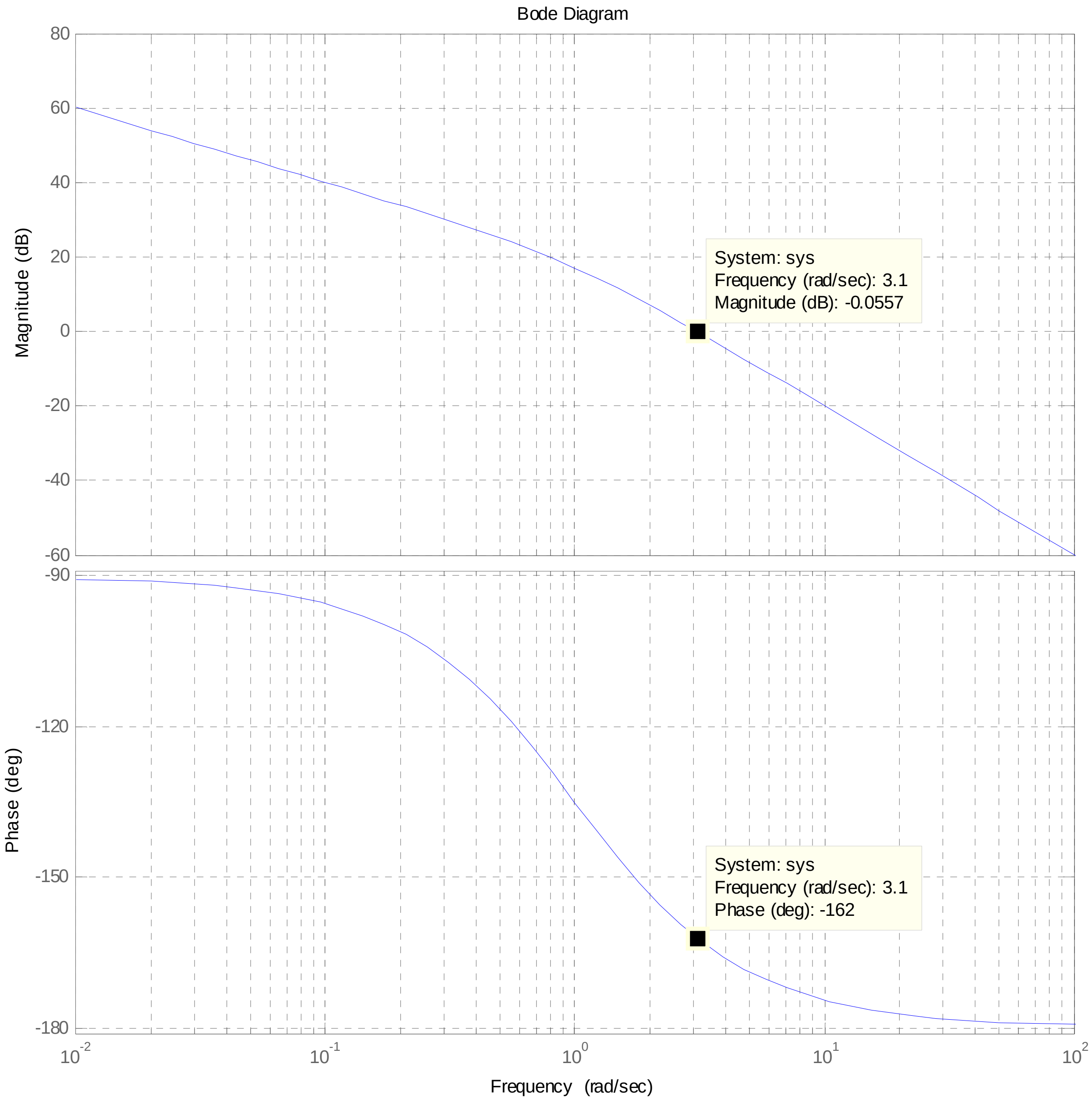
The resulting open loop transfer function

$$L_{\mathrm{u}}(s) = \frac{10}{s(s+1)}$$

i. Find the phase margin of the uncompensated system and compare it to the design specs to obtain the necessary phase needed (say, ϕ_{req}) to be added to the loop. To be safe, we usually choose $\phi_m > \phi_{req}$ to start with the design.

From the Bode plot of the uncompensated open loop transfer function given on the next page, we have

PM = 180° – 162° = 18° (at frequency around 3.1 rad/s)



To be safe, let us choose to add an additional phase of

$$\phi_{req} = 22^\circ + 5^\circ = 27^\circ$$

Alternatively, we can find the phase margin of the uncompensated loop

$$|L_u(j\omega_{cg})| = 1 \quad \Rightarrow \quad \frac{10}{\omega_{cg}(\sqrt{\omega_{cg}^2 + 1})} = 1.$$

$$\omega_{cg} = 3.1.$$

The phase margin:

$$PM_u = -90^0 - \tan^{-1}(3.1) - (-180^\circ) = 17.9^\circ.$$

ii. From $\sin\phi_m = \dfrac{1-\alpha}{1+\alpha}$, we find α.

$$\frac{1-\alpha}{1+\alpha} = \sin 27^\circ = 0.454 \quad \Rightarrow \quad \alpha = 0.3755$$

iii. We look for a frequency (ω_m) that has an uncompensated gain of $\sqrt{\alpha}$.

$$\sqrt{\alpha} = \sqrt{0.3755} = -4.25\,\mathrm{dB} \quad \Rightarrow \quad \omega_\mathrm{m} = 4\ \mathrm{rad/s}$$

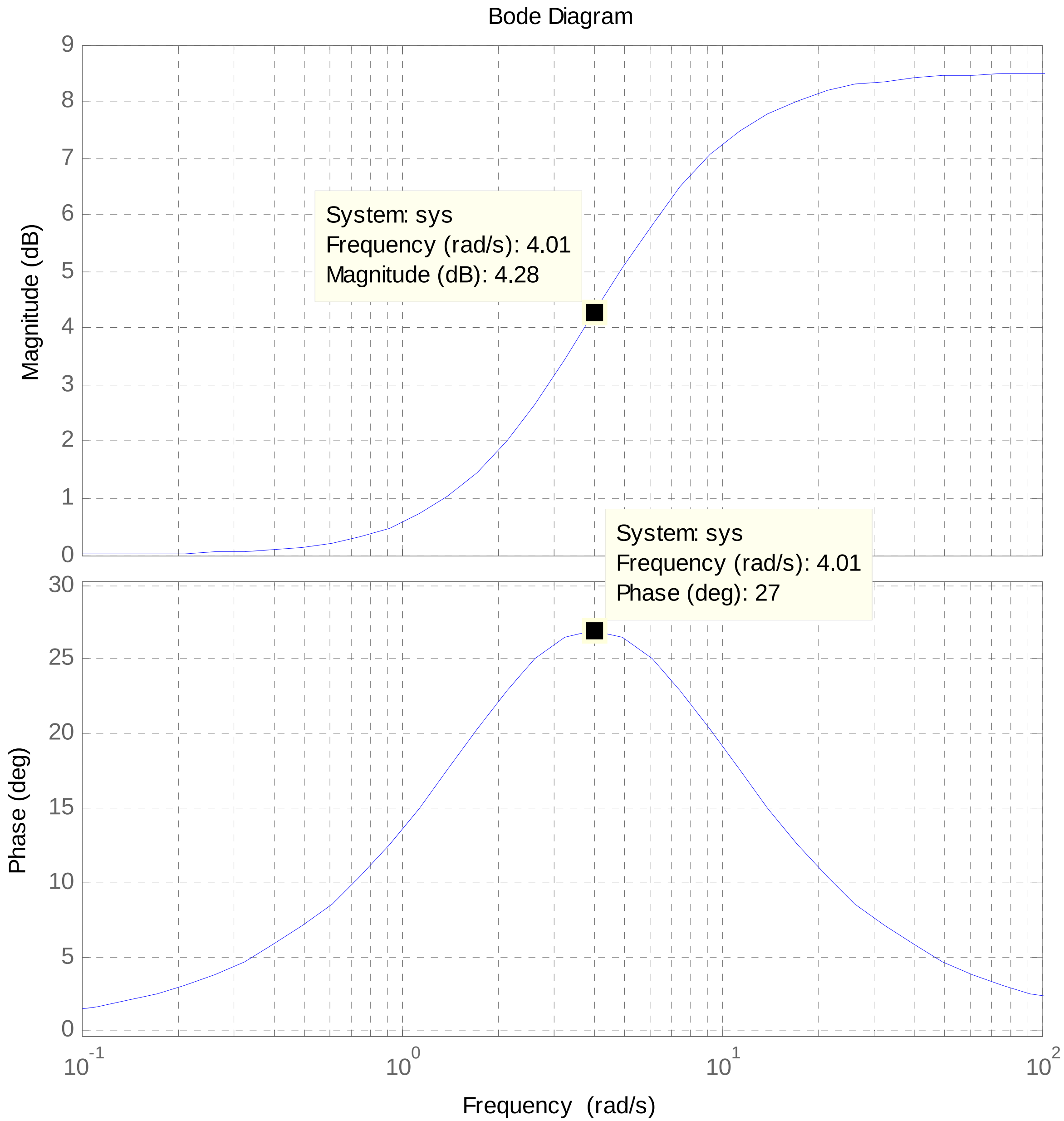
iv. Computer $T = \dfrac{1}{\omega_m\sqrt{\alpha}}$

$$T = \frac{1}{\omega_\mathrm{m}\sqrt{\alpha}} = \frac{1}{4\sqrt{0.3755}} = 0.41$$

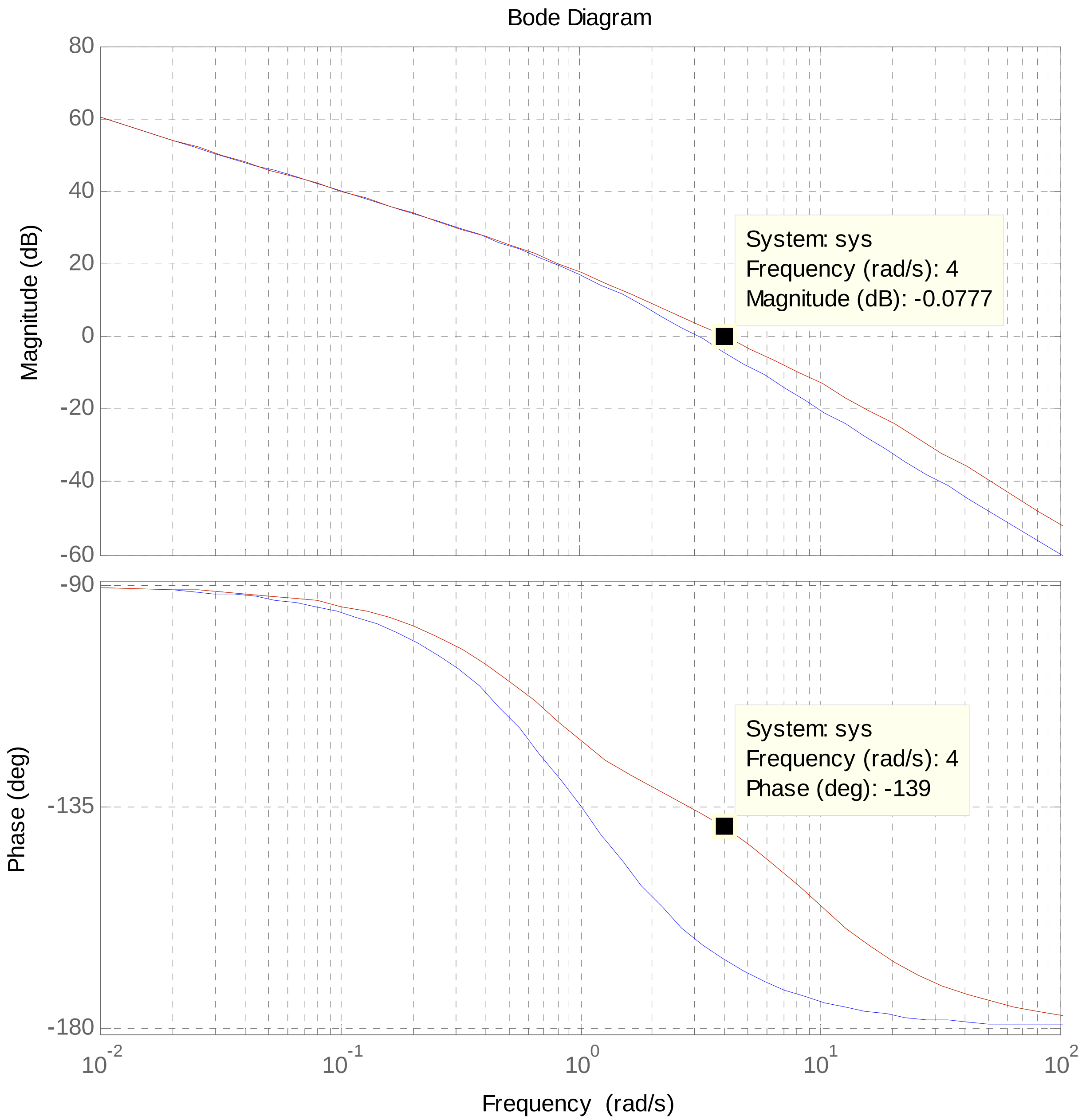
v. The required lead compensator is given as

$$C(s) = \frac{Ts+1}{\alpha Ts+1} = \frac{0.41s+1}{0.3755 \times 0.41s+1} = \frac{0.41s+1}{0.154s+1}$$

vi. Verify your result. Repeat the above steps if necessary.



$$C(s) = \frac{0.41s + 1}{0.154s + 1}$$



The new gain cross-over frequency is

ω_{cg} = 4 rad/s

PM = 41°

✓

Q.5

5. Consider the feedback control system shown in Figure 1 where $K(s) = K$ and

$$G(s) = \frac{1}{s(s+10)^2}$$

- Determine the value of K such that the steady-state error is 5% for ramp input. What is the corresponding phase margin?
- Design a lag compensator to increase the phase margin to 65^o.

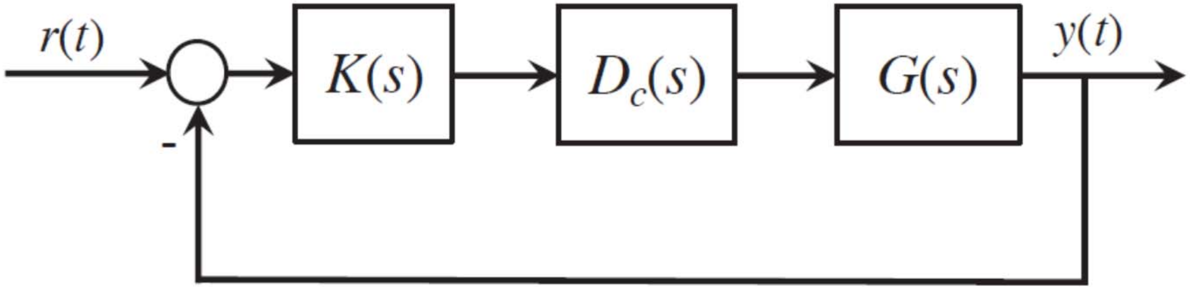


Figure 1: Feedback Control System

Solution: (1) The velocity error constant is then given by

$$k_{\mathrm{v}} = \lim_{s\to 0} s \cdot C(s) \cdot G(s) = \lim_{s\to 0} C(s) \cdot \lim_{s\to 0} \frac{K}{(s+10)^2} = \frac{K}{100} \quad \Rightarrow \quad \frac{1}{k_{\mathrm{v}}} = \frac{5}{100} \quad \Rightarrow \quad K = 2000$$

The resulting open loop transfer function $L_{\mathrm{u}}(s) = \dfrac{2000}{s(s+10)^2}$

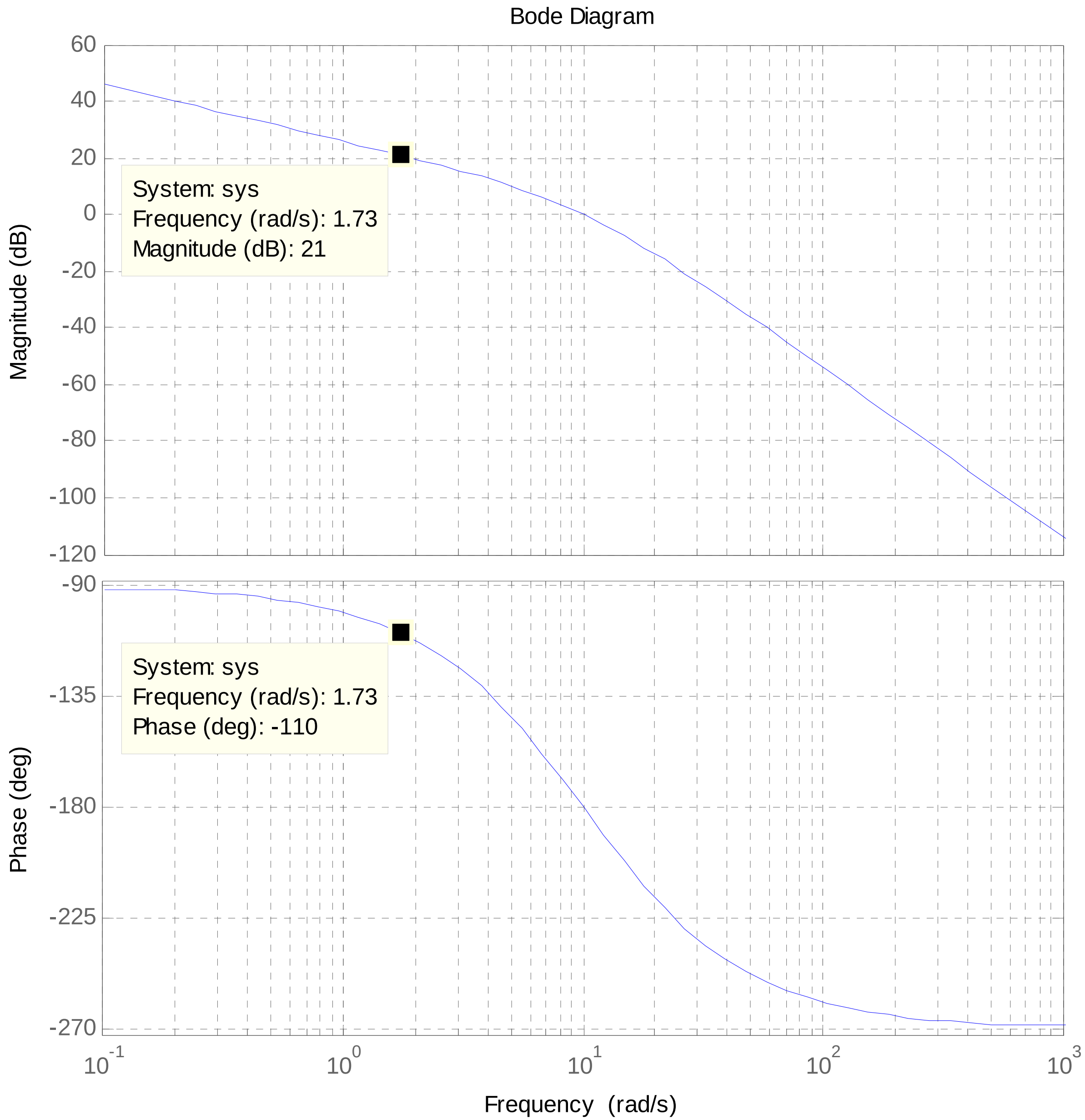
i. Find the phase margin of the uncompensated system and compare it to the design specs on phase margin (say, ϕ_{req}). To be safe, we usually choose a value $\phi_m > \phi_{req}$ to start with the design.

To be safe, we choose

$$\phi_m = 65° + 5° = 70°$$

ii. Find the frequency ω_x such that the uncompensated loop at which has a phase response of $-180° + \phi_m$.

$$\omega_x = 1.73 \text{ rad/s}$$



Looking for a frequency that gives us the required ϕ_m.

iii. Find the corresponding gain value (say, G_x) of the uncompensated loop at ω_x.

G_x = 21 dB ⇨ G_x = 11.22

iv. Compute $\alpha = G_x$ and $T = \dfrac{10}{\omega_x}$.

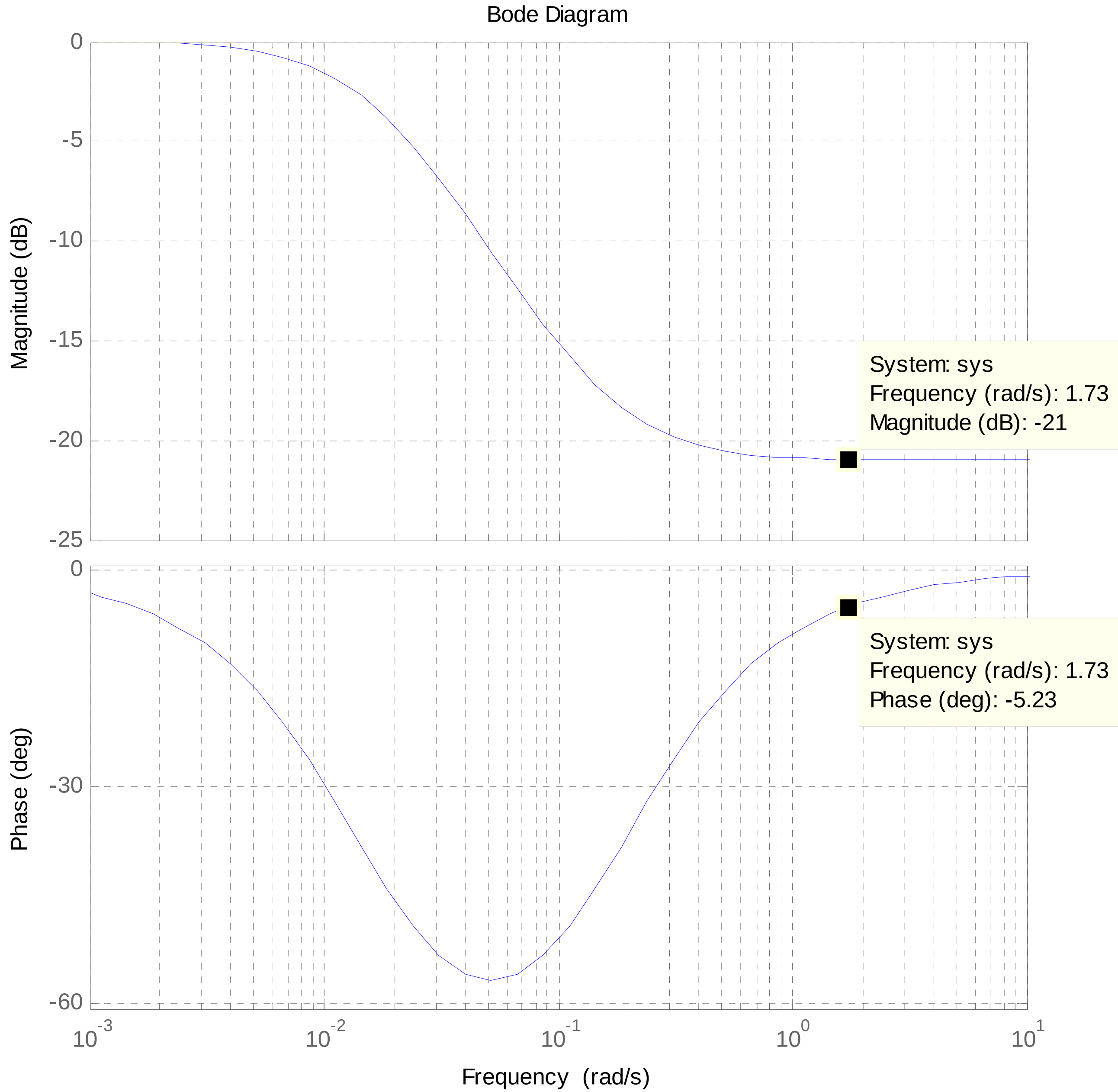
α = 11.22 and T = 5.78

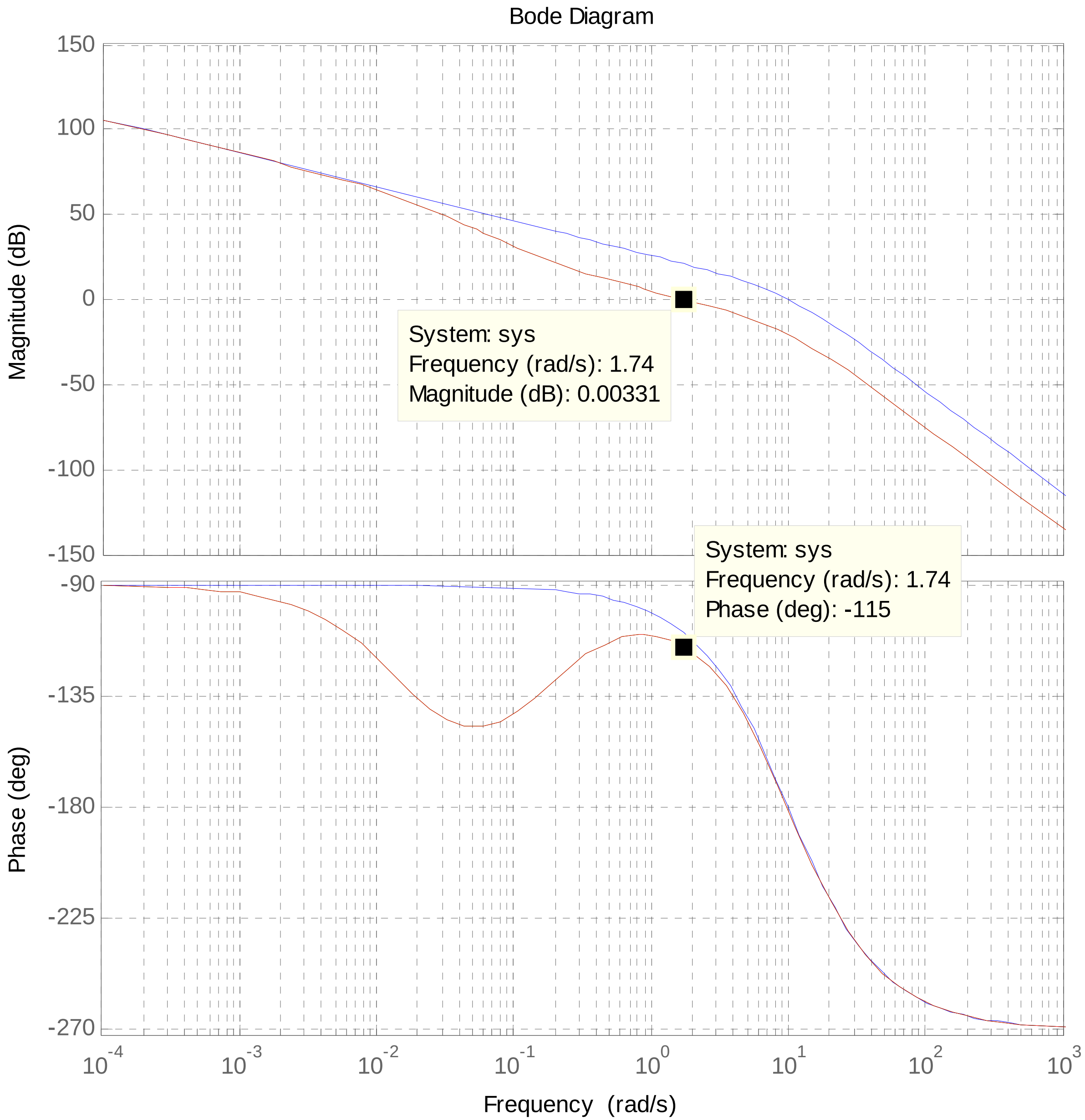
v. The required lag compensator is given as

$$C(s) = \frac{Ts+1}{\alpha Ts+1} = \frac{5.78s+1}{11.22 \times 5.78s+1} = \frac{5.78s+1}{64.85s+1}$$

vi. Verify your result. Repeat the above steps if necessary.

$$C(s) = \frac{5.78s + 1}{64.85s + 1}$$





The new gain cross-over frequency is about

ω_{cg} = 1.75 rad/s

PM = 65°

✓