Q.3 Evaluate the following integrals

(a) $\int_C |z|\cdot\bar{z}dz, \quad C=\left\{z(t)=e^{it}, \quad 0\le t\le\frac{\pi}{2}\right\}$ (5 marks)

Solution:

$$\int_C |z|\cdot\bar{z}\ dz=\int_0^{\pi/2}\left|e^{it}\right|e^{-it}ie^{it}dt=i\,t\Big|_0^{\pi/2}=\boxed{\frac{i\pi}{2}}$$

(b) $\int_C \frac{1}{z-(3+i)}\,dz, \quad C=\left\{z(t)=3e^{it}, \quad 0\le t\le 2\pi\right\}$ (5 marks)

Solution: The singular point of $f(z)=\dfrac{1}{z-(3+i)}$ is at $3+i$ with $|3+i|=\sqrt{10}>3$. It is out of the circle *C*. Thus

$$\int_C \frac{1}{z-(3+i)}\,dz=\boxed{0}$$

(c) Use the complex integration approach to compute $\displaystyle\int_0^{2\pi}\frac{1}{2\sin\theta+2.5}\,d\theta$ (7 marks)

Solution:

$$\int_0^{2\pi}\frac{1}{2\sin\theta+2.5}\,d\theta=\int_{|z|=1}\frac{1}{2\dfrac{1}{2i}\left(z-\dfrac{1}{z}\right)+2.5}\cdot\frac{1}{iz}dz=\int_{|z|=1}\frac{1}{z^2+2.5iz-1}dz$$

$$=\int_{|z|=1}\frac{1}{(z+0.5i)(z+2i)}dz=2\pi i\,\mathrm{Res}\left(f,-0.5i\right)=2\pi i\lim_{z\to-0.5i}\frac{1}{z+2i}=\boxed{\frac{4\pi}{3}}$$

(d) Use the complex integration approach to compute $\displaystyle\int_0^{2\pi}\frac{1}{1-2p\cos\theta+p^2}\,d\theta,\ |p|<1$

(8 marks)

Solution:

$$\int_0^{2\pi}\frac{1}{1-2p\cos\theta+p^2}\,d\theta=\int_{|z|=1}\frac{1}{1-2p\dfrac{1}{2}\left(z+\dfrac{1}{z}\right)+p^2}\cdot\frac{1}{iz}dz=\frac{1}{-i}\int_{|z|=1}\frac{dz}{pz^2-(p^2+1)z+p}$$

$$=i\int_{|z|=1}\frac{dz}{p\left(z-1/p\right)(z-p)}=i\cdot2\pi i\cdot\mathrm{Res}\left(f,p\right)=i\cdot2\pi i\cdot\lim_{z\to p}\frac{1}{p\left(z-1/p\right)}=\boxed{\frac{2\pi}{1-p^2}}$$

Q.4 Ah Beng owns a tiny food stall in the Engin Canteen, which hires two workers W_1 and W_2, and sells only two kinds of dishes, D_1 and D_2, the profit being \$3 and \$1, respectively. Worker W_1 prepares all cooking materials for D_1 in 2 minutes and for D_2 in 4 minutes. Worker W_2 cooks D_1 in 3 minutes and D_2 in 2 minutes. Assuming that all dishes made can be sold to hungry professors and students without difficulty, use the Simplex Method to determine production figures that maximize the profit for lunch daily that lasts from 12:00 noon to 2:00 pm. Based on your result, do you think Ah Beng can survive in the Engin Canteen if he only cares to maximize the profit?

(25 marks)

Solution:

If Ah Beng's food stall cooks x_1 sets of dish D_1 and x_2 sets of dish D_2 for lunch daily, the profit per daily lunch is

$$f(x_1, x_2) = 3x_1 + x_2$$

The constraints are

$$2x_1 + 4x_2 \leq 120 \quad \text{(resulting from Worker } W_1 \text{ in 2 hours)}$$

$$3x_1 + 2x_2 \leq 120 \quad \text{(resulting from Worker } W_2 \text{ in 2 hours)}$$

$$x_1 \geq 0, \quad x_2 \geq 0$$

We introduce slack variables w_1 and w_2. The constraints become

$$2x_1 + 4x_2 + w_1 = 120$$

$$3x_1 + 2x_2 + w_2 = 120$$

$$x_1, x_2, w_1, w_2 > 0$$

and the objective function $f - 3x_1 - x_2 = 0$. We construct the following Simplex Table:

basis	x_1	x_2	w_1	w_2	b	check
w_1	2	4	1	0	120	127
w_2	3	2	0	1	120	126
f	–3	–1	0	0	0	–4

basis	x_1	x_2	w_1	w_2	b	check
w_1	2	4	1	0	120	127
w_2	1	2/3	0	1/3	40	42
f	–3	–1	0	0	0	–4

basis	x_1	x_2	w_1	w_2	b	check
w_1	0	8/3	1	−2/3	40	43
x_1 $\cancel{w_2}$	1	2/3	0	1/3	40	42
f	0	1	0	1	120	122

We have

$$f - x_2 - w_2 = 120 \quad \Rightarrow \quad f = 120 + x_2 + w_2$$

Obviously, the maximum profit occurs when we set $x_2 = w_2 = 0$ and $x_1 = 40$. The maximum profit is $120 for lunch daily by selling 40 sets of dish D_1.

To maximize the profit, Ah Beng's food stall only needs to produce a single kind of food. Under normal circumstances, Ah Beng will be out of business very soon.