Q.3 (a) Give an upper bound for the following integral

$$\left|\int_C (z^2+1)\,dz\right|, \quad C=\{z(t)=(1+i)\,t, \quad 0\le t\le 1\}$$

(7 marks)

<u>Solution:</u>

$$\left|z^2+1\right| \le |z|^2+1=(x^2+y^2)+1\le 3=M$$

$$L=\int_0^1 |z'(t)|\,dt=\int_0^1 |1+i|\,dt=\sqrt{2}$$

Thus,

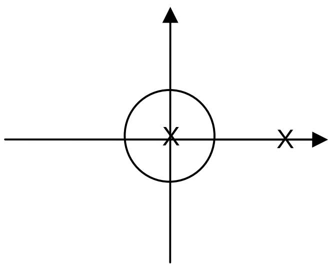
$$\left|\int_C (z^2+1)\,dz\right| \le ML = 3\sqrt{2}$$

(b) Evaluate the following integral

$$\int_C \frac{1}{z(z-1)}\,dz, \quad C=\left\{z(t)=\frac{e^{it}}{2012}, \quad 0\le t\le 2\pi\right\}$$

(5 marks)

<u>Solution:</u>



$$\int_C \frac{1}{z(z-1)}\,dz=\int_C \frac{1/(z-1)}{z}\,dz=2\pi i\,\frac{1}{0-1}=-2\pi i$$

(c) Evaluate the following integral

$$\int_{-\infty}^{\infty} \frac{(\sin x)(x+1)}{x^2+1}\, dx$$

(5 marks)

Solution:

$$f(z) = \frac{z+1}{z^2+1}$$

$$\text{Res}(f(z)e^{iz}, i) = \lim_{z\to i}(z-i) f(z)e^{iz} = \lim_{z\to i}(z-i)\frac{(z+1)e^{iz}}{z^2+1}$$

$$= \lim_{z\to i}\frac{(z+1)e^{iz}}{z+i} = e^{-1}\frac{i+1}{i2}$$

$$\int_{-\infty}^{\infty} \frac{(\sin x)(x+1)}{x^2+1}\, dx = \text{Im}\left[2\pi i\ \text{Res}(f(z)e^{iz}, i)\right] = \text{Im}\left[2\pi i e^{-1}\frac{i+1}{i2}\right] = \text{Im}\left[\pi e^{-1}(i+1)\right] = \frac{\pi}{e}$$

(d) Evaluate the following integral

$$\int_{2}^{\infty}\left[\frac{1}{(x-2)^2+4}\right]^2 dx$$

(8 marks)

Solution: Let $y = x-2$. We have

$$\int_{2}^{\infty}\left[\frac{1}{(x-2)^2+4}\right]^2 dx = \int_{0}^{\infty}\left[\frac{1}{y^2+4}\right]^2 dy$$

$$= \frac{1}{2}\int_{-\infty}^{\infty}\left[\frac{1}{x^2+4}\right]^2 dx = \pi i\ \text{Res}(f, i2)$$

$$= \pi i \lim_{z\to i2}\frac{d}{dz}(z-i2)^2\frac{1}{(z+i2)^2(z-i2)^2}$$

$$= \pi i \lim_{z\to i2}\frac{-2}{(z+i2)^3}$$

$$= \pi i\frac{-2}{(i4)^3}$$

$$= \frac{\pi}{32}$$

Q.4 (a) Find and plot the image of the lens-shaped region as shown in Fig. Q.4 (a) under the transformation $w=\dfrac{z-1}{z-i}e^{-i3\pi/4}$. Clearly indicate the images of boundaries B_1 and B_2. Note that both circles have a unity radius.

(15 marks)

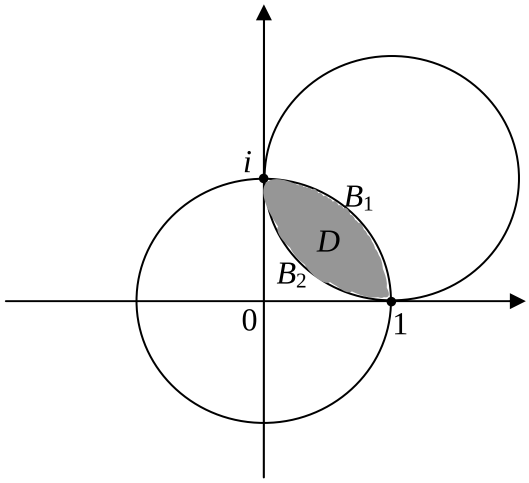


Fig. Q.4 (a)

<u>Solution:</u> From what we have learnt in the class, any linear fractional transformation is mapping a circle in z-domain into a circle or a straight line in w-domain in this case. Noting that

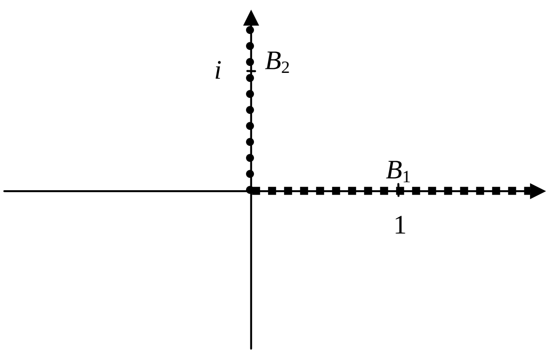
$$z=1 \quad \Rightarrow \quad w=\frac{z-1}{z-i}e^{-i3\pi/4}=\frac{0}{0-i}e^{-i3\pi/4}=0$$

$$z=i \quad \Rightarrow \quad w=\frac{z-1}{z-i}e^{-i3\pi/4}=\frac{i-1}{i-i}e^{-i3\pi/4}=\infty$$

Since both circles have a unity radius, the central point of B_1 is thus given by $\dfrac{\sqrt{2}}{2}+i\dfrac{\sqrt{2}}{2}$ and the central point of B_2 is given by $\left(1-\dfrac{\sqrt{2}}{2}\right)+i\left(1-\dfrac{\sqrt{2}}{2}\right)$. We have

$$z=\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2} \quad \Rightarrow \quad w=\frac{z-1}{z-i}e^{-i3\pi/4}=1$$

$$z=\left(1-\frac{\sqrt{2}}{2}\right)+i\left(1-\frac{\sqrt{2}}{2}\right) \quad \Rightarrow \quad w=\frac{z-1}{z-i}e^{-i3\pi/4}=i$$



(c) Calculate the electric potential everywhere on the region *D* if boundaries B_1 and B_2 are kept at potentials of $\Phi_1 = -10\ \text{V}$ and $\Phi_2 = 10\ \text{V}$, respectively.

(10 marks)

Solution: It clear that the problem belongs to Example 3 of the simple geometries, and thus the solution can be expressed as

$$F(w) = -iA\ln w + B$$

with

$$A = \frac{\Phi(\pi/2) - \Phi(0)}{\pi/2} = \frac{10-(-10)}{\pi/2} = \frac{40}{\pi}, \quad B = \Phi(0) = -10$$

$$F(w) = -\frac{i40}{\pi}\ln w - 10 = \left[\frac{40}{\pi}\arg(w) - 10\right] - \frac{i40}{\pi}\ln|z|$$

Thus,

$$\Phi(x,y) = \text{Re}\left[F\left(w = \frac{z-1}{z-i}e^{-i3\pi/4}\right)\right] = \frac{40}{\pi}\arg\left(\frac{z-1}{z-i}e^{-i3\pi/4}\right) - 10$$