Complex Analysis

Flow Chart of Material in Complex Analysis

Starting

$$z = x + iy$$

$$z(t) = x(t) + iy(t)$$

Complex function of a real variable
~ define a ***curve*** on a complex plane

$$f(z) = u + iv = u(x, y) + iv(x, y)$$

Complex function of a complex variable
~ define a complex mapping from one complex plane to another

$$\int_C f(z)\,dz = \int_\alpha^\beta f[z(t)]\,z'(t)\,dt$$

Complex integral
~ integration of a complex function (of a complex variable) on a complex curve
~ ***upper bound of a complex integration***

$$\oint_C f(z)\,dz = 0$$

Cauchy's Integral theorem
~ valid when $f(z)$ is ***analytical*** for every point encircled by a closed curve C
~ ***Cauchy-Riemann equations, singularities***

to be continued on the next page...

Flow Chart of Material in Complex Analysis (cont.)

$$\oint_C \frac{f(z)}{z - z_0}\, dz = 2\pi i\, f(z_0)$$

Cauchy's integral formula
~ C is a closed curve encloses z_0 and $f(z)$ is analytic for every point inside C

$$\oint_C \frac{f(z)}{(z - z_0)^{n+1}}\, dz = 2\pi i\, \frac{f^{(n)}(z_0)}{n!}$$

Another integral formula
~ $n \ge 0$, C is a closed curve encloses z_0 and $f(z)$ is analytic inside C
~ ***power series of an analytic function***

$$\oint_C f(z)\, dz = 2\pi i\, \mathbf{Res}(f, z_0)$$

complex integral in terms of residues
~ ***Taylor series, Laurent series, order of singularities, residues***
~ The most general result for complex integral !

Applications of complex integral

to be continued on the next page…

Flow Chart of Material in Complex Analysis (cont.)

Applications of complex integral

$$\int_0^{2\pi} f(\cos\theta, \sin\theta)\, d\theta$$

$$\int_{-\infty}^{\infty} f(x)\, dx = \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}\, dx$$

$$\int_{-\infty}^{\infty} f(x) \cos mx dx$$

$$\int_{-\infty}^{\infty} f(x) \sin mx dx$$

Laplace transform

argument principle

Rouche's theorem

Mission completed

Basic Operations of Complex Numbers

Cartesian and Polar Coordinates:

$$z = x + iy = |z| e^{i \arg z} = \sqrt{x^2 + y^2}\, e^{i \tan^{-1}\left(\frac{y}{x}\right)} = |z| \left[\cos(\arg z) + i \sin(\arg z)\right]$$

Euler's Formula: $e^{i\theta} = \cos(\theta) + i\sin(\theta)$

Additions: It is easy to do additions (subtractions) in Cartesian coordinate, i.e.,

$$(a + ib) + (v + iw) = (a + v) + i(b + w)$$

Multiplication: It is easy to do multiplication (division) in Polar coordinate, i.e.,

$$re^{i\theta} \cdot ue^{i\omega} = (ru)e^{i(\theta+\omega)}$$

$$\frac{re^{i\theta}}{ue^{i\omega}} = \frac{r}{u} e^{i(\theta-\omega)}$$

Complex Functions of a Real Variable

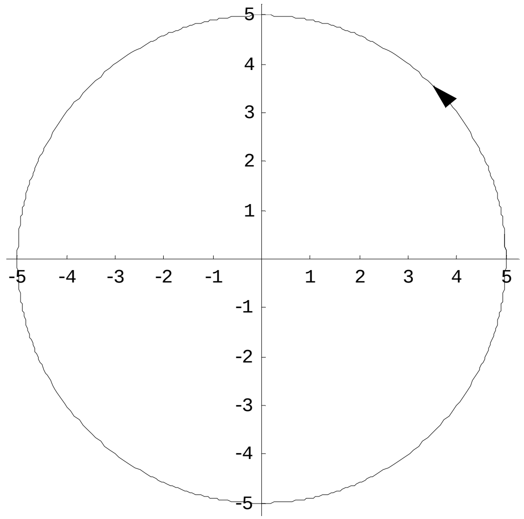
Complex functions of a real variable are needed to represent paths or contours in the complex plane.

$$z(t) = x(t) + i\,y(t), \qquad t \in [a,b]$$

Example 1

$$z(t) = 5e^{it}, \qquad t \in [0, 2\pi]$$
$$= 5\cos t + i\,5\sin t$$

$$\Rightarrow \quad x(t) = 5\cos t$$
$$y(t) = 5\sin t, \quad t \in [0, 2\pi]$$



Properties of Complex Function of Real Variable

- $\lim_{t \to a} z(t) = \lim_{t \to a} x(t) + i \lim_{t \to a} y(t)$

- z is continuous if x and y are continuous, i.e. $\lim_{t \to a} x(t) = x(a)$, $\lim_{t \to a} y(t) = y(a)$

- $z'(t) = x'(t) + i\, y'(t)$

- $z(t)$ is smooth if $z'(t)$ is continuous, i.e. if $x'(t)$ and $y'(t)$ are continuous.

- $z(t)$ is piecewise smooth if $z(t)$ is smooth everywhere except for a finite number of discontinuities.

Properties of Complex Function of Real Variable (cont.)

- Normal differentiation and integration rules are applicable:

$$(c_1 z_1 + c_2 z_2)' = c_1 z_1' + c_2 z_2'$$

$$\int_a^b (c_1 z_1 + c_2 z_2)\, dt = c_1 \int_a^b z_1\, dt + c_2 \int_a^b z_2\, dt$$

$$\int_a^b z'\, dt = z(b) - z(a)$$

Curves

- The set of images

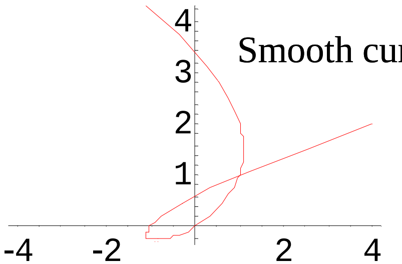
  $$C = \{z(t) \mid t \in [a,b]\}$$

  is called a **curve** in the complex plane

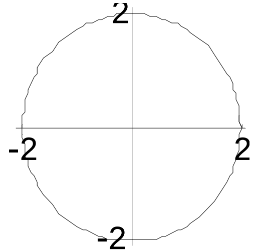
- The curve is smooth if

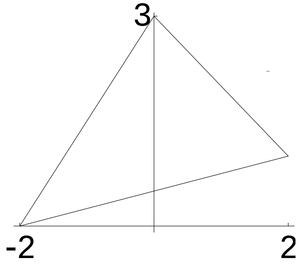
  $z'(t)$ is continuous

Smooth closed curve



Smooth curve





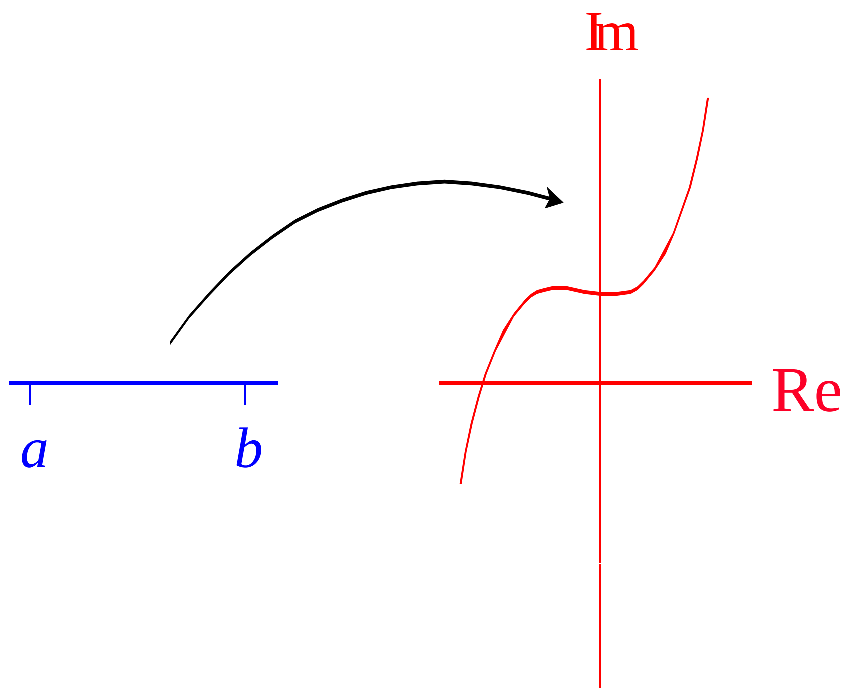
Piecewise smooth (pws) curve

Curves (cont.)

- The **length of a curve** is given by

$$L = \int_a^b |z'(t)|\, dt$$

- A curve is thus a mapping of the real number line onto the complex plane

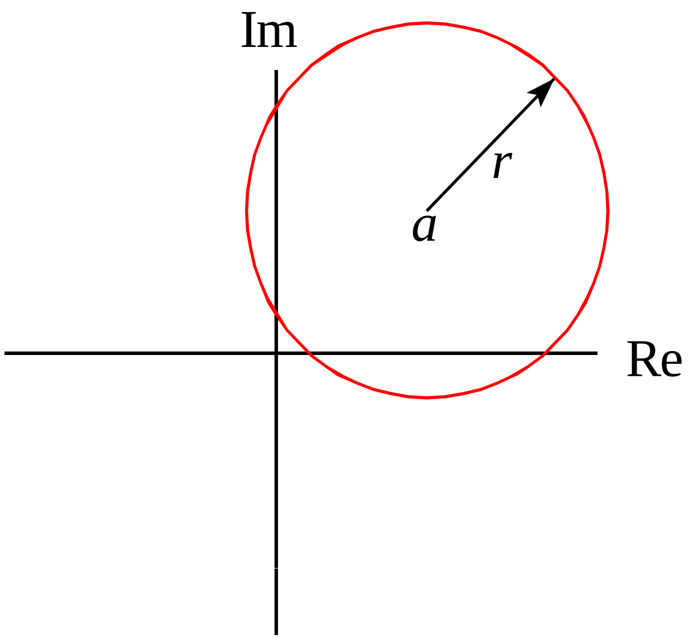


Two Special Curves

- **Circle**

The parametric description for a **circle** centred at complex point *a* and with a radius *r* is

$$z(t) = a + re^{it}, \quad t \in [0, 2\pi]$$

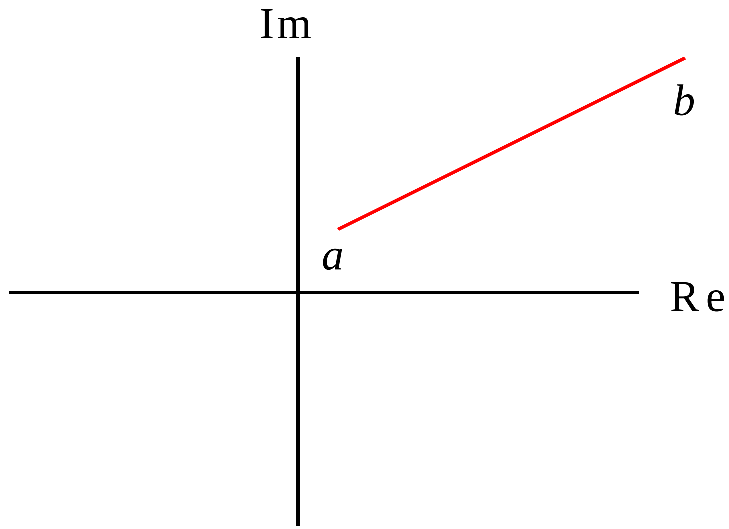


Two Special Curves

- **Straight Line**

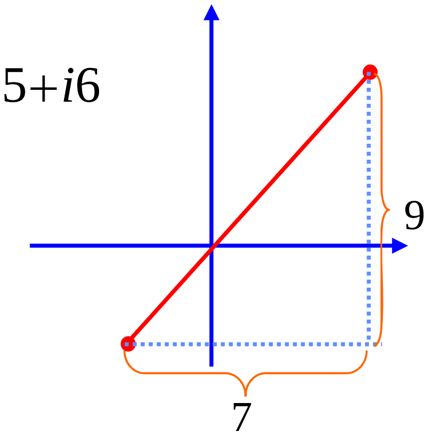
  The parametric description of a **straight line** segment with starting point a and endpoint b is

$$z(t)=(b-a)t+a, \quad t\in[0,1]$$



Example 2 : Parametric Representation and Length of Curves

a) The line segment that connects the points $-2-i3$ and $5+i6$



$$z(t)=(7+i9)\,t+(-2-i3), \quad t\in[0,1]$$

$$L=\int_a^b |z'(t)|dt=\int_0^1 |7+i9|dt=\int_0^1 \sqrt{7^2+9^2}\,dt=\sqrt{130}$$

b) The circle with radius 2 and centre $1-i$

$$z(t)=(1-i)+2e^{it}, \quad t\in[0,2\pi]$$

$$L=\int_a^b |z'(t)|\,dt=\int_0^{2\pi} |2i\,e^{it}|\,dt=\int_0^{2\pi} |2i|\times|e^{it}|\,dt=\int_0^{2\pi} 2\times 1dt=4\pi$$

Example 2 (cont.)

c) $y = x + 2, \quad 2 \le x \le 3$

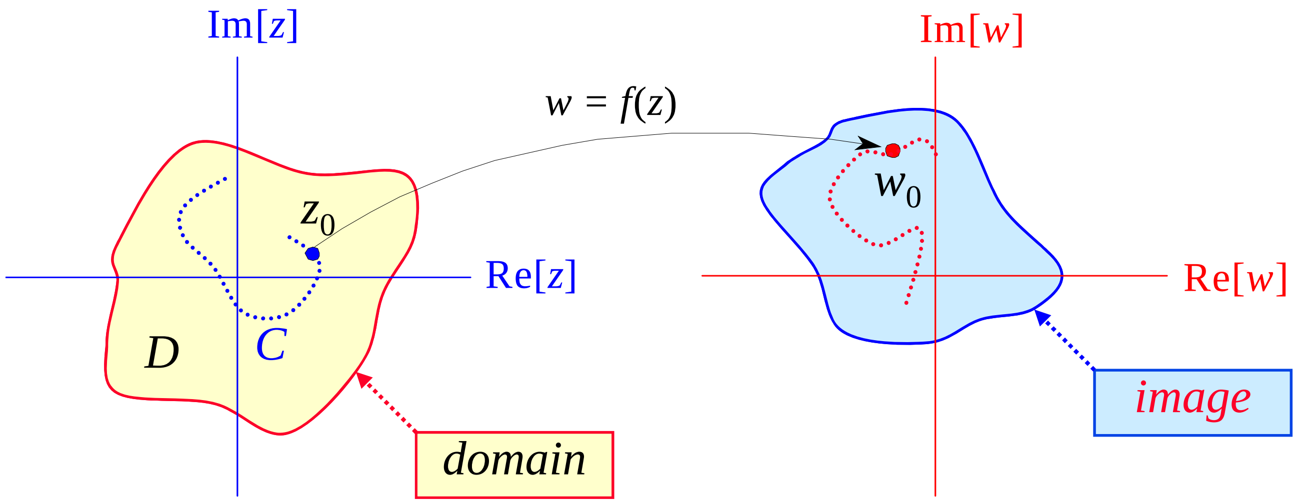
Let $x = t, \quad 2 \le t \le 3 \quad \Rightarrow \quad y = t + 2$

$$\Rightarrow \quad z(t) = t + i(t+2), \quad 2 \le t \le 3$$

$$L = \int_a^b |z'(t)|\,dt = \int_2^3 |1+i|\,dt = \int_2^3 \sqrt{2}\,dt = \sqrt{2}$$

Complex Functions of a Complex Variable

A complex function of a complex variable maps one plane to another plane.



These functions are of the form $f(z) = w \;\Rightarrow\; f(x+iy) = u + iv = u(x,y) + iv(x,y)$.

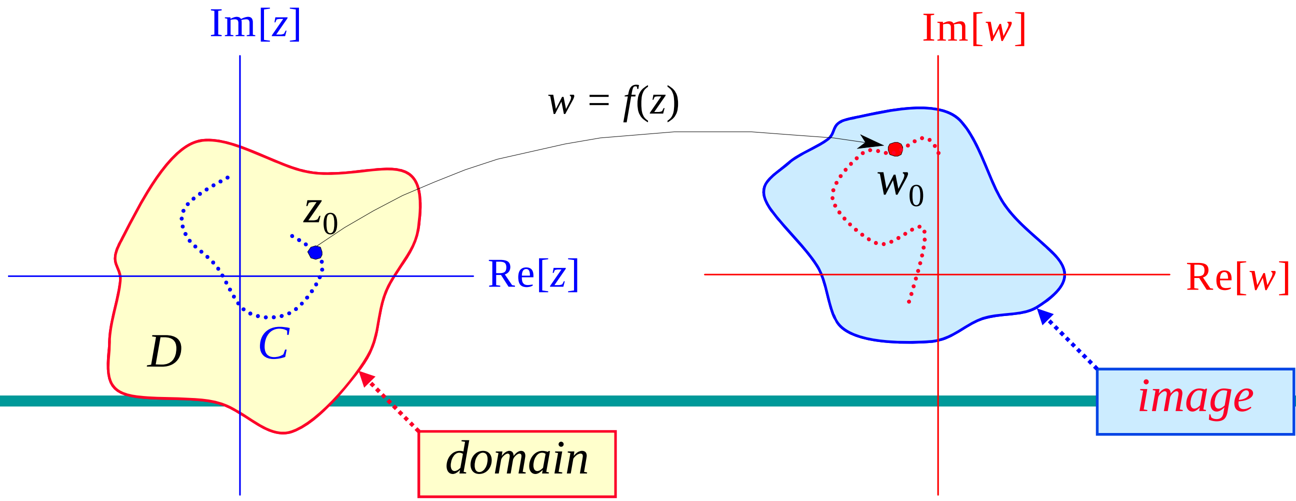
Complex Mapping

A complex-valued function

$$w = f(z) = f(x + i\,y) = u(x, y) + i\,v(x, y), \qquad z = x + i\,y$$

defines a **mapping** of a **domain**, D, onto its **image** the w-plane.

For any point z_0 in D, we call the point $w_0 = f(z_0)$ the image of z_0. Similarly, the points of a curve C are mapped onto a curve on the w-plane.



Example: z^2

a) $w = f(z) = z^2 = (x+iy)^2$

$$= x^2 - y^2 + i2xy$$

$$= u + iv$$

The function $f(z) = z^2$ would for example map the complex number $1 + i$ to $i2$ and the number $2 - i$ to $3 - i\,4$.

The line segment $x \in [0,2],\ y = 0$ or $x = t,\ t \in [0,2],\ y = 0$ is mapped to the line segment $u = t^2,\ t \in [0,2],\ v = 0$, since

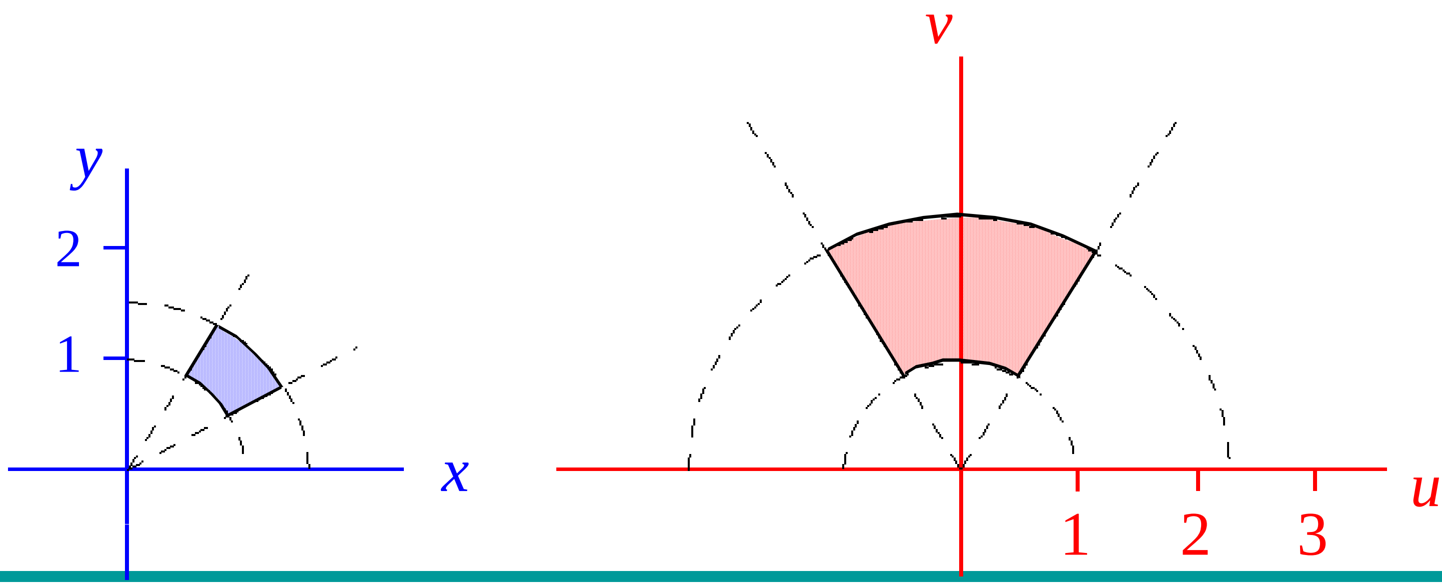
$$u + iv = (x+iy)^2 = (t+i0)^2 = t^2$$

Example: z^2 (cont.)

In polar coordinate: $w = f(z) = R e^{i\theta} = z^2 = (r e^{i\phi})^2 = r^2 e^{i2\phi}$

For example, the image of the region $1 \le r \le 3/2, \ \pi/6 \le \phi \le \pi/3$

under the mapping $w = z^2$ is $1 \le R \le 9/4, \ \pi/3 \le \theta \le 2\pi/3$



Example: e^z

b) $w = f(z) = e^z$

$$= e^{x+iy}$$

$$= e^x(\cos y + i\sin y)$$

$$= e^x \cos y + i\, e^x \sin y = u + iv$$

For example, $f(1+i) = e\cos 1 + i\, e\sin 1$

Example: e^z (cont.)

In terms of polar coordinates

$$w = f(z) = R\,e^{i\theta} = e^{z} = e^{x+iy} = e^{x}\,e^{iy}$$

Therefore

$$R = e^{x}, \quad \theta = y$$

Note that $w = e^{z} \neq 0 \;\; \forall\, z$

Example: e^z (cont.)

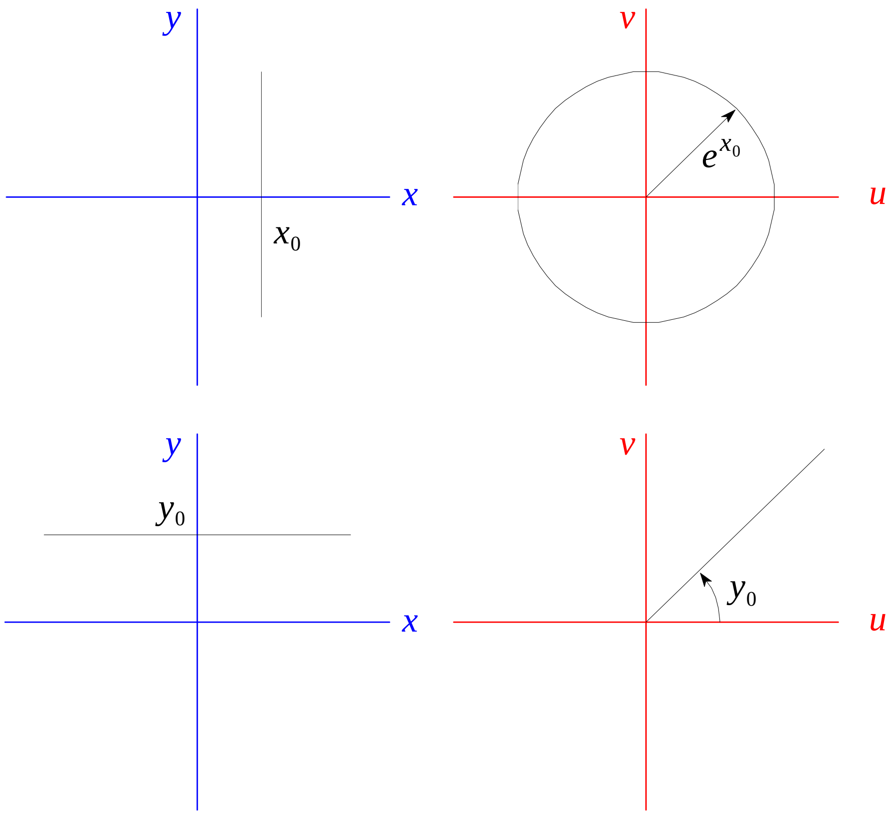
For $w = e^z$, consider the images of:

1. Straight lines $x = x_0 = \text{const}$

   and $y = y_0 = \text{const}$

   From $R = e^x, \quad \theta = y$, we see that $x = x_0$ is mapped onto the circle $|w| = e^{x_0}$ and $y = y_0$ is mapped onto the ray

   $\arg(w) = y_0$



$$R = e^x, \quad \theta = y$$

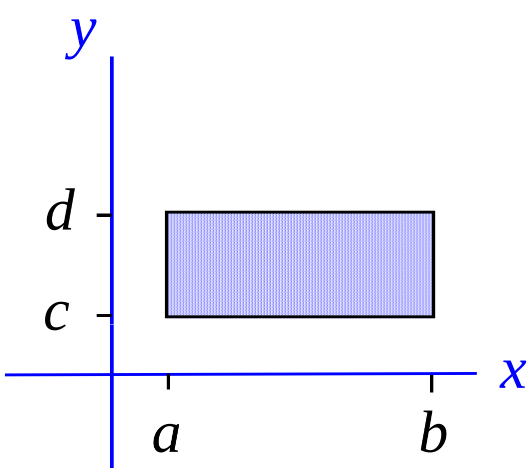
Example: e^z (cont.)

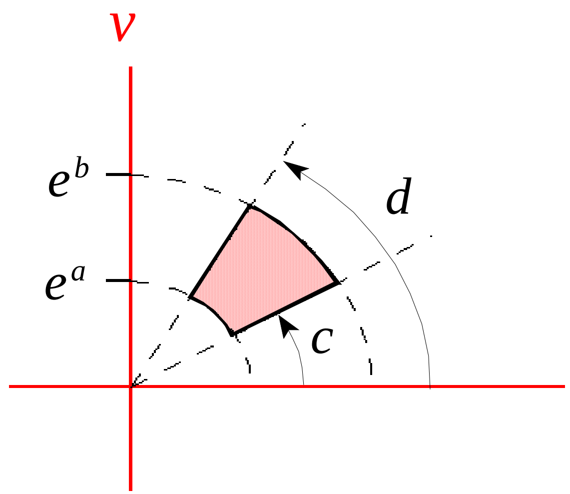
$$R = e^x, \quad \theta = y$$

2. Rectangle $D = \{ z = x + iy \,/\, a \le x \le b,\ c \le y \le d \}$:

From (a), we can conclude that any rectangle with side parallel to the coordinate axes is mapped onto a region bounded by portions of rays and circles. Therefore the image of D is

$$D' = \{ w = Re^{i\theta} \,/\, e^a \le R \le e^b, c \le \theta \le d \}$$

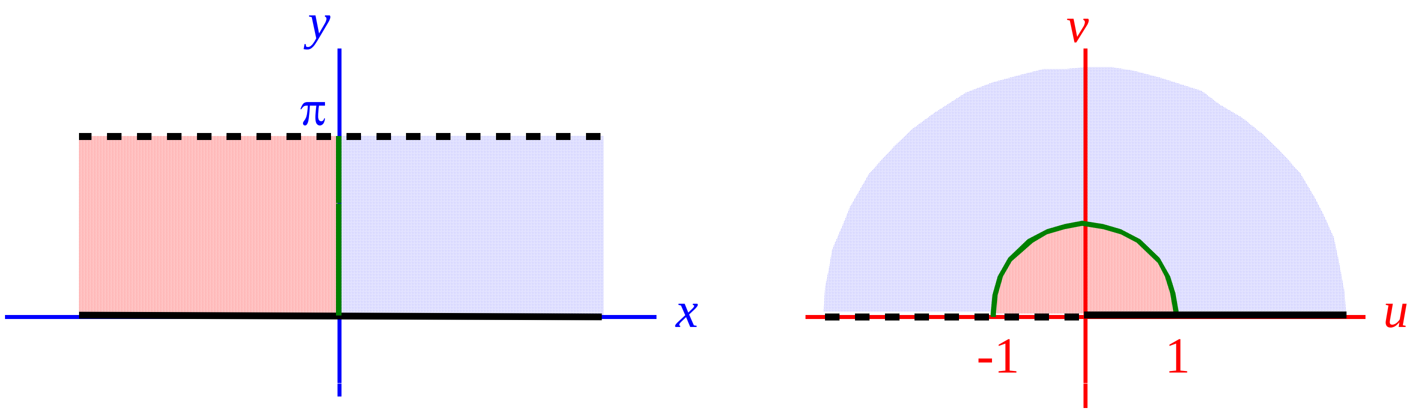




Example: e^z (cont.)

3. The fundamental region $-\pi \le y \le \pi$:

The fundamental region is mapped onto the entire w-plane, excluding the origin. The strip $0 \le y \le \pi$ is mapped onto the upper half-plane



More generally, every horizontal strip $c \le y \le c + 2\pi$ is mapped onto the full w-plane excluding the origin.

$$R = e^x, \quad \theta = y$$

Example: ln z

The natural logarithm $w = \ln(z)$ is the inverse relation of the exponential function e^z, i.e.,

$$\ln e^z = z, \qquad e^{\ln z} = z$$

It follows

$$\ln z = \ln\left(|z| \cdot e^{i \arg z}\right) = \ln\left(e^{\ln|z| + i \arg z}\right) = \ln|z| + i \arg z$$

More Example

c) $$f(z)=\frac{e^{iz}+e^{-iz}}{2}$$

$$=\frac{e^{i(x+iy)}+e^{-i(x+iy)}}{2}=\frac{e^{ix-y}+e^{-ix+y}}{2}$$

$$=\frac{e^{-y}(\cos x+i\sin x)+e^{y}(\cos x-i\sin x)}{2}$$

$$=\cos x\frac{e^{y}+e^{-y}}{2}+i\sin x\frac{e^{-y}-e^{y}}{2}$$

$$=\cos x\cosh y-i\sin x\sinh y$$

$$=u+iv$$

More Example

Alternatively, $f(z)=\cos z=\cos(x+iy)$

$$=\cos(x)\cos(iy)-\sin(x)\sin(iy)$$

$$=\cos(x)\cosh(y)-i\sin(x)\sinh(y)$$

$$=u+iv$$

since $\sin(ix)=i\sinh(x),\quad \cos(ix)=\cosh(x),\quad \tan(ix)=i\tanh(x).$

Keep in mind that

$$\sin z=\frac{e^{iz}-e^{-iz}}{2i}\qquad \cos z=\frac{e^{iz}+e^{-iz}}{2}$$

$$\sinh z=\frac{e^{z}-e^{-z}}{2}\qquad \cosh z=\frac{e^{z}+e^{-z}}{2}$$

Observations

Identities for trigonometric functions also hold for complex arguments, e.g.

$$\cos^2 z + \sin^2 z = 1$$

$$\sin(z_1 \pm z_2) = \sin z_1 \cos z_2 \pm \cos z_1 \sin z_2$$

$$\cos(z_1 \pm z_2) = \cos z_1 \cos z_2 \mp \sin z_1 \sin z_2$$

$$\sin(\tfrac{\pi}{2} \pm z) = \cos z \quad , \quad \cos(\tfrac{\pi}{2} \pm z) = \mp \sin z$$

Also note that

$$\sin z = \sin x \cosh y + i \cos x \sinh y,$$

$$\cos z = \cos x \cosh y - i \sin x \sinh y$$

Observations

- $f(z)=w$ is continuous at the point $z = x_0 + iy_0$ if $u(x,y)$ and $v(x,y)$ are continuous at $x_0 + iy_0$.

  For example, $f(z) = z^2$ is continuous everywhere, because $u = x^2 - y^2$ and $v = 2xy$ are continuous everywhere.

- It can be shown that the regular rules of differentiation and integration are still valid, e.g.

$$\frac{d}{dz} z^n = n z^{n-1}$$

$$\frac{d}{dz}\sin z = \cos z$$

Complex Integral

Integration is an important and useful concept in elementary calculus. The two-dimensional nature of the complex plane suggests the consideration of integrals along arbitrary curves in $\mathbb{C}$ instead of only on segments of the real axis. These "line integrals" have interesting and unusual properties when the function being integrated is analytic. Complex integration is one of the most beautiful and elegant theories in mathematics.

Consider the curve $C: t \to z(t),\ t \in [\alpha, \beta]$ and the complex function f which is continuous on C. The complex integral of f along C is then defined as

$$\int_C f(z)\,dz = \int_\alpha^\beta f\left[z(t)\right] z'(t)\,dt$$

Example 4

a) Let $C: z(t) = 2t + i3t, \quad 1 \le t \le 2$

and $f(z) = z^2$

$$\int_C z^2\, dz = \int_1^2 (2t + i3t)^2 (2 + i3)\, dt$$

$$= (2 + i3)^3 \int_1^2 t^2\, dt$$

$$= (2 + i3)^3 \tfrac{7}{3}$$

$$= -\tfrac{322}{3} + i21$$

b) Let C be a circle with radius r and centred at the origin and

$$f(z) = 1/z$$

$$C: \; z(t) = re^{it}, \quad t \in [0, 2\pi]$$

$$\int_C 1/z\, dz = \int_0^{2\pi} \frac{ire^{it}}{re^{it}}\, dt$$

$$= i \int_0^{2\pi} dt = 2\pi i$$

Properties of Complex Integrals

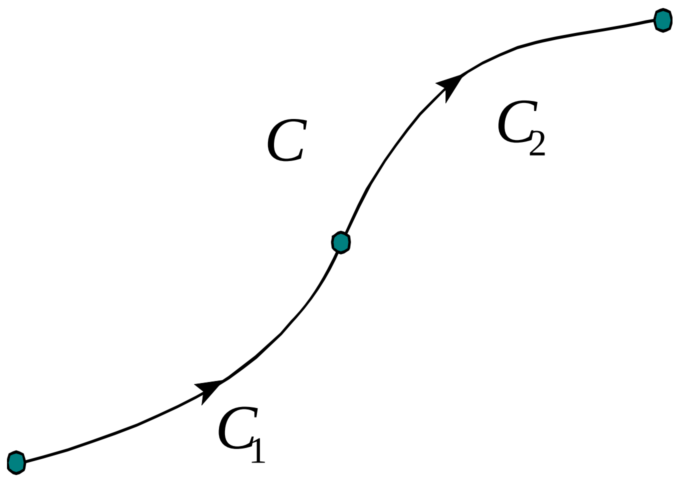
As for real integrals, the following rules apply:

1. $$\int_C [f(z) + g(z)]\,dz = \int_C f(z)\,dz + \int_C g(z)\,dz$$

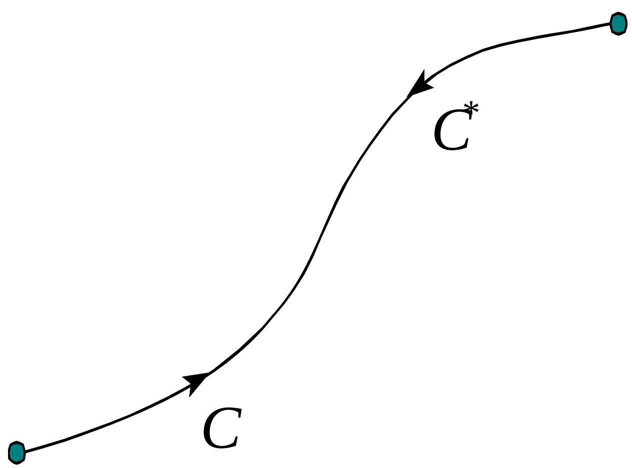
2. $$\int_C k\, f(z)\,dz = k \int_C f(z)\,dz, \quad k \text{ complex}$$

Properties of Complex Integrals

3. $\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz$



4. $\int_C f(z)\,dz = -\int_{C^*} f(z)\,dz$



Estimation of a Complex Integral

- Let $f(z)$ be continuous on $C: t \to z(t),\ t \in [\alpha, \beta]$. If $|f(z)| \le M$ on C, then

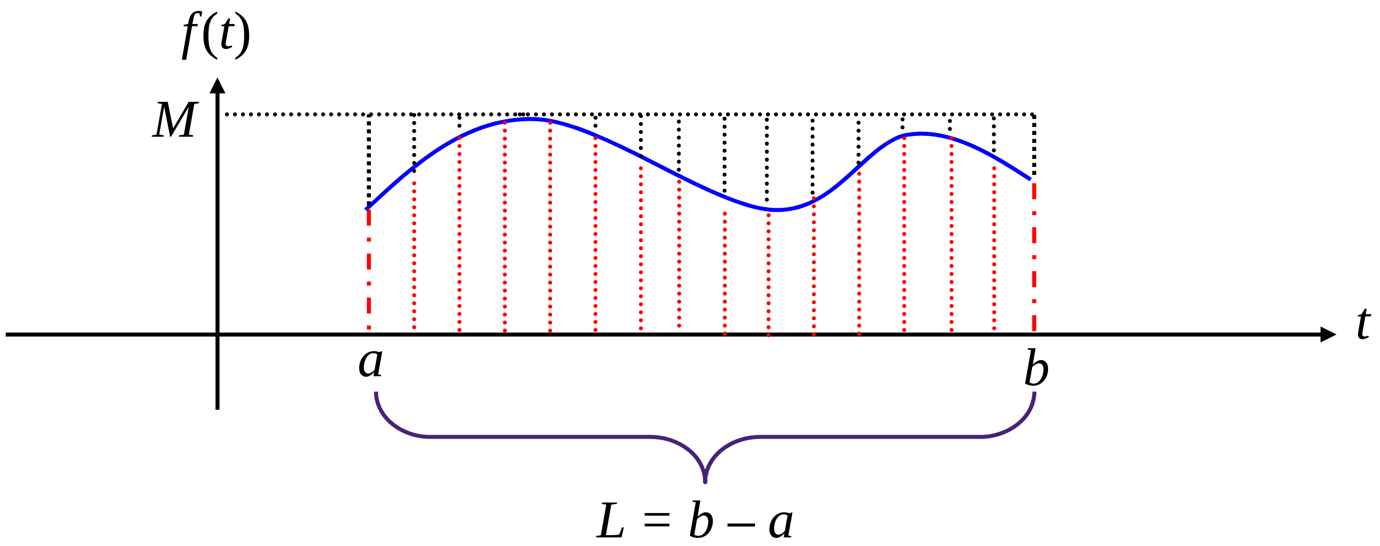
$$\left| \int_C f(z)\, dz \right| \le M L$$

where L is the length of the curve C, i.e.

$$L = \int_\alpha^\beta |z'(t)|\, dt = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Estimation of Complex Integral – An Illustration

Graphically, take real integration as an example,

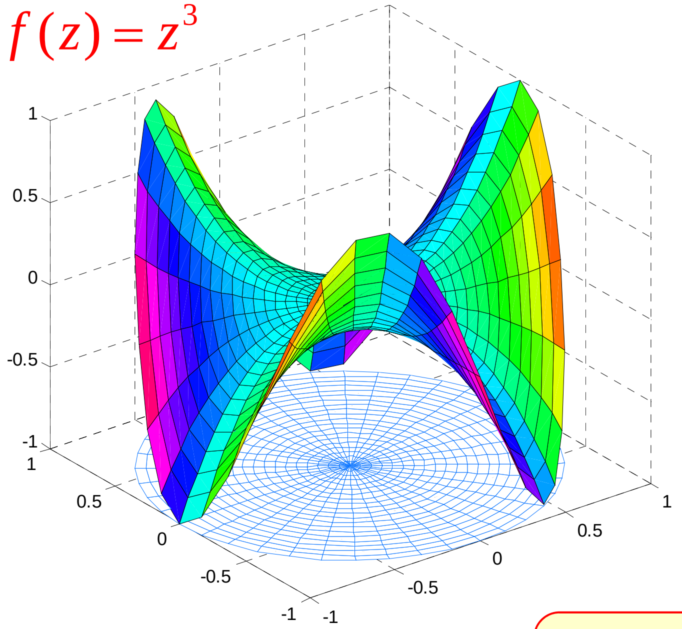


$$\left|\int_a^b f(t)dt\right| = \text{shaded area with red lines} \le M \cdot L$$

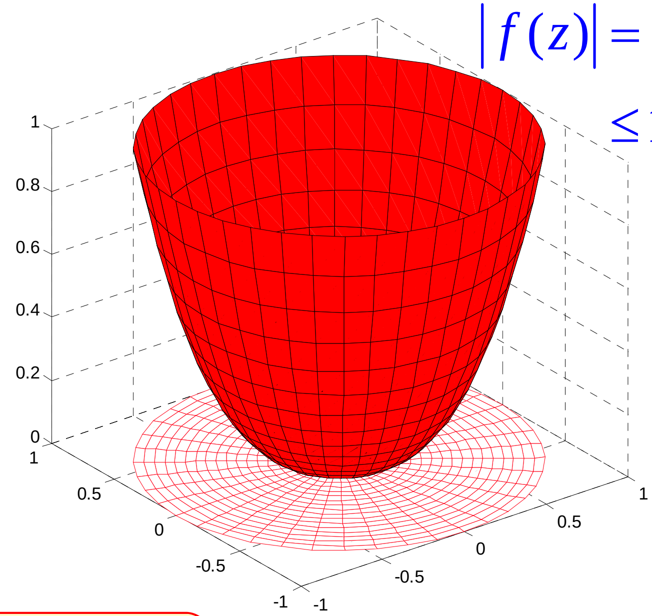
Estimation of Complex Integral – An Illustration

For complex cases, for example, we take $f(z) = z^3$ and C to be a unit circle

$f(z) = z^3$

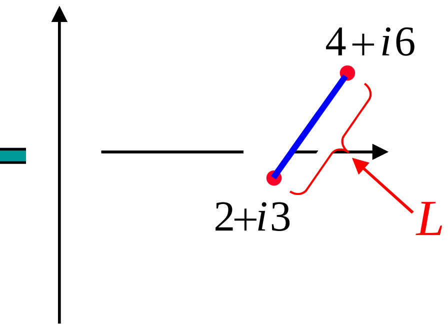


$|f(z)| = |z^3| \le 1 = M$



$$\left|\int_C f(z)dz\right| \le ML = 2\pi$$

Example 5



Find the upper bound for the absolute value of $\int_C e^z\,dz$

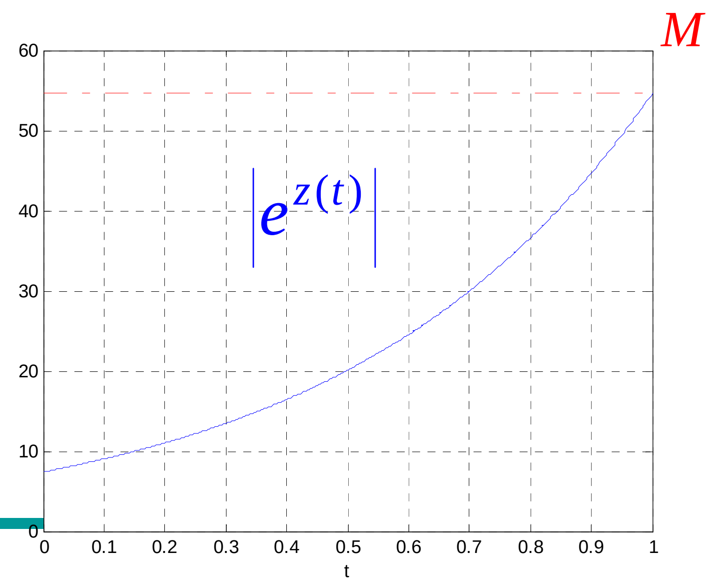
where C is the line connecting the points $2+i3$ and $4+i6$

$$L = \int_0^1 \sqrt{2^2+3^2}\,dt = \sqrt{13}$$

$$z(t) = (2+i3)t + (2+i3), \qquad t \in [0,1]$$
$$= (2t+2) + i(3t+3), \qquad t \in [0,1]$$
$$= x(t) + i\,y(t)$$

$$\left|e^z\right| = \left|e^{x+iy}\right| = \left|e^x e^{iy}\right| = \left|e^x\right| \cdot \left|e^{iy}\right| = e^x$$

$$\Rightarrow \quad M = e^x\Big|_{x=\text{largest}} = e^{2+2t}\Big|_{t=1} = e^4 = 54.5982$$



Thus, $\left|\int_C e^z\,dz\right| \le M\,L = \sqrt{13}e^4 = 196.8566$

Example 6

$$|a+b| \le |a| + |b|$$

Show that $\left|\int_{|z|=2} \frac{1}{z^2+1}\,dz\right| \le \frac{4\pi}{3}$

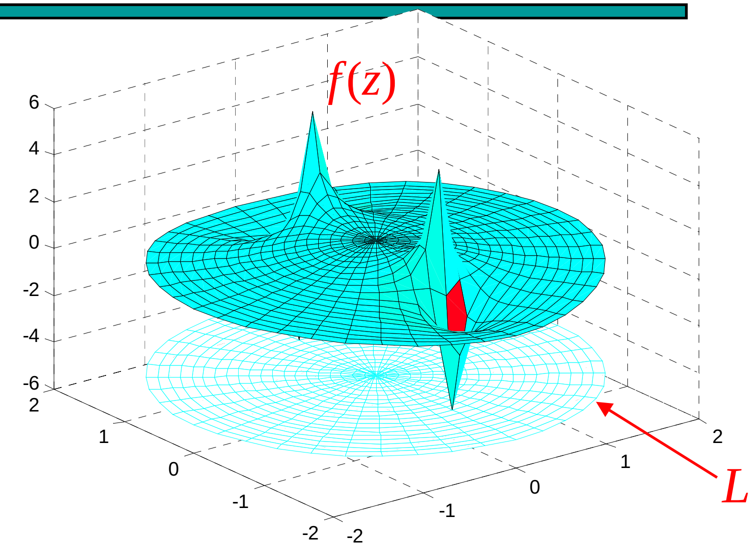
Noting that

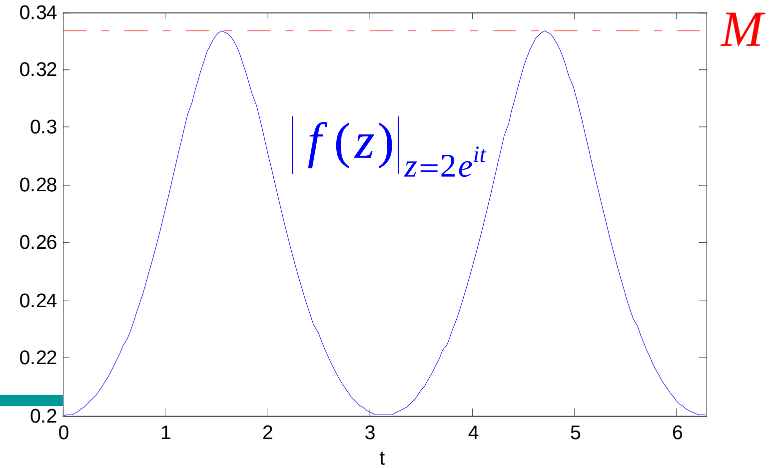
$$|z^2| = |(z^2+1)+(-1)| \le |z^2+1| + |-1| = |z^2+1| + 1$$

and thus on the circle $|z| = 2$, we have

$$|z^2+1| \ge |z^2| - 1 = 3 \quad \Rightarrow \quad \left|\frac{1}{z^2+1}\right| \le \frac{1}{3} = M$$

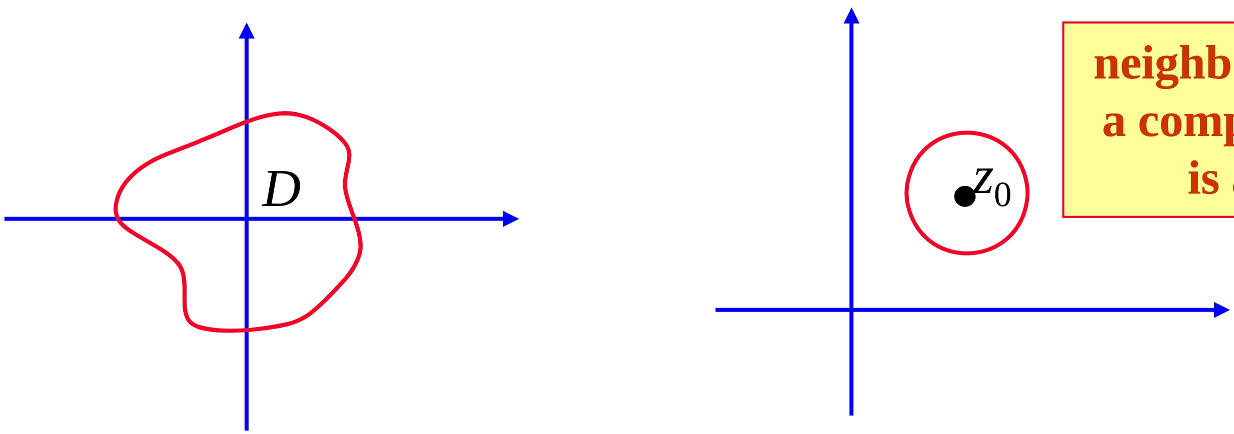
$$\Rightarrow \quad \left|\int_{|z|=2} \frac{1}{z^2+1}\,dz\right| \le ML = \frac{4\pi}{3}$$





Analytical Functions

A function $f(z)$ defined in domain D is said to be an analytic function if it is differentiable with a continuous first order derivative in all points of D. The function $f(z)$ is said to be analytic at a point z_0 in D if f is analytic in a neighbourhood of z_0.



neighborhood on a complex plane is a disc

Analytical Functions

Discrepancy of Definitions of Analytic Functions…

Definition given by most popular texts (e.g., the reference text):

A function $f(z)$ defined in domain D is said to be analytic if it is differentiable at every point of D.

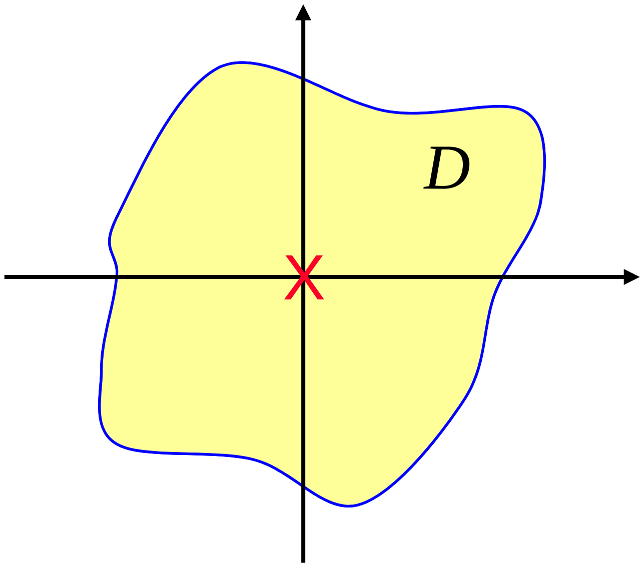
Definition given in some odd books or the text by Garg et al.:

A function $f(z)$ defined in domain D is said to be analytic if it is differentiable with a continuous first order derivative at every all point of D.

Nevertheless, we will use the definition given by Garg et al. throughout…

Singularities

Points where a function is not analytic are called **singular points** or **singularities** or **poles** sometimes.



Example:

$f(z) = \dfrac{1}{z}$ is analytic everywhere in D except $z = 0$, which is thus the singular point or pole of the function.

Note that a function is either analytic or singular at any given point…

Analytical Functions

Observations:

It can be shown that the existence of continuous first derivatives implies the existence of a continuous second derivative, etc. This also implies the existence of a Taylor series.

$$f(z_0)+f'(z_0)(z-z_0)+f''(z_0)\frac{(z-z_0)^2}{2!}+\cdots\cdots$$

An analytic function can thus also be defined as a function for which a Taylor series expansion exists.

Theorem

If a function $f(z) = u + i\,v$ is analytic in D, then the following Cauchy-Riemann equations hold, i.e.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Alternatively, if the Cauchy-Riemann equations hold for a function $f(z) = u + i\,v$ and the function has continuous first order partial derivatives, then $f(z)$ is analytic in D.

Proof.

In spite of these similarities, there is a fundamental difference between differentiation for functions of real variables and differentiation for functions of a complex variable. Let $z = (x, y)$ and suppose that h is real. Then

$$f'(z) = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h} = \frac{\partial f}{\partial x}(z) = f_x(z).$$

But if $h = ik$ is purely imaginary, then

$$f'(z) = \lim_{k \to 0} \frac{f(x, y+k) - f(x, y)}{ik} = \frac{1}{i}\frac{\partial f}{\partial y}(z) = -if_y(z).$$

Thus, the existence of a complex derivative forces the function to satisfy the partial differential equation

$$f'(z) = f_x = -if_y .$$

Writing $f(z) = u(z) + iv(z)$, where u and v are real-valued functions of a complex variable, and equating the real parts and imaginary parts of

$$u_x + iv_x = f_x = -if_y = v_y - iu_y ,$$

we obtain the **Cauchy-Riemann** differential equations

$$u_x = v_y , \qquad v_x = -u_y .$$

Example

$$f(z) = z^2 = x^2 - y^2 + i\,2xy = u + iv$$

$$\frac{\partial u}{\partial x} = 2x = \frac{\partial v}{\partial y} \quad , \quad \frac{\partial u}{\partial y} = -2y = -\frac{\partial v}{\partial x}$$

and the partial derivatives are continuous $\forall\, z$.

Consequently, $f(z)$ is analytic $\forall\, z$.

Example

$$f(z)=\frac{z^*}{|z|^2}=\frac{x}{x^2+y^2}-i\frac{y}{x^2+y^2}=u+iv$$

$$\frac{\partial u}{\partial x}=\frac{y^2-x^2}{(x^2+y^2)^2}=\frac{\partial v}{\partial y}\quad,\quad\frac{\partial u}{\partial y}=\frac{-2xy}{(x^2+y^2)^2}=-\frac{\partial v}{\partial x}$$

$\Rightarrow$ $f(z)$ is analytic everywhere, except where $x^2+y^2=0$ i.e. at the origin.

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Example

$$f(z) = z^* = x - iy = u + iv$$

$$\frac{\partial u}{\partial x} = 1 \quad , \quad \frac{\partial v}{\partial y} = -1 \quad , \quad \frac{\partial u}{\partial y} = 0 = \frac{\partial v}{\partial x}$$

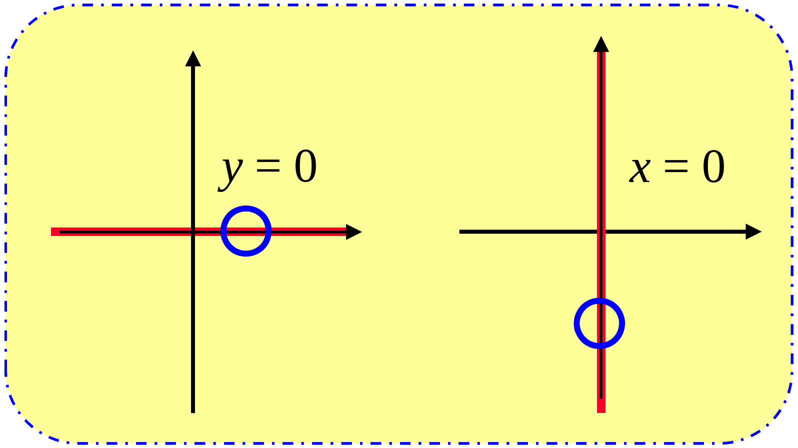
$\Rightarrow \quad f(z)$ is not analytic anywhere

Example

$$f(z) = x^2 y^2 + i\, 2x^2 y^2 = u + iv$$

$$\frac{\partial u}{\partial x} = 2xy^2 \quad , \quad \frac{\partial v}{\partial y} = 4x^2 y \quad , \quad \frac{\partial u}{\partial y} = 2x^2 y \quad , \quad \frac{\partial v}{\partial x} = 4xy^2$$

The Cauchy-Riemann equations only hold for $x = 0$ and/or $y = 0$. Since the function is not analytic in a neighbourhood of $x = 0$ or $y = 0$, $f(z)$ is not analytic anywhere.



Observations

1. The sum or product of analytic functions is analytic.

2. All polynomials are analytic.

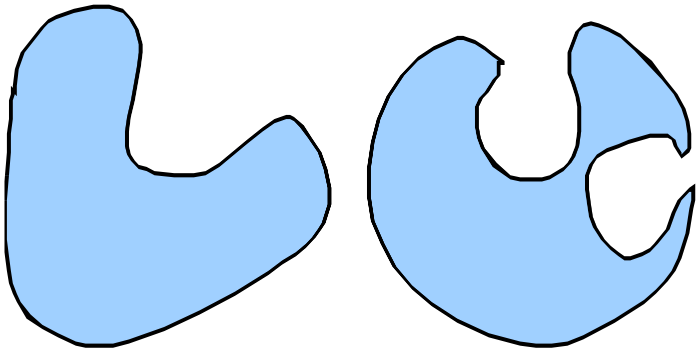
3. A rational function (the quotient of two polynomials) is analytic, except at zeroes of the denominator.

4. An analytic function of an analytic function is analytic.

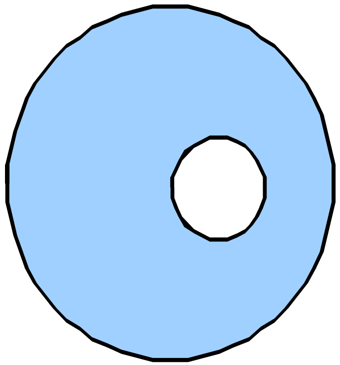
5. Functions $e^z, \ \sin z, \ \cos z, \ \sinh z, \ \cosh z$ are analytic everywhere.

Cauchy's Integral Theorem

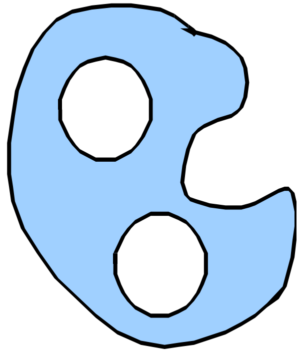
A domain D in the complex plane is called **simply connected** if every closed curve in D only encloses points in D. A domain that is not simply connected is called **multiply connected**.



Simply connected



Doubly connected

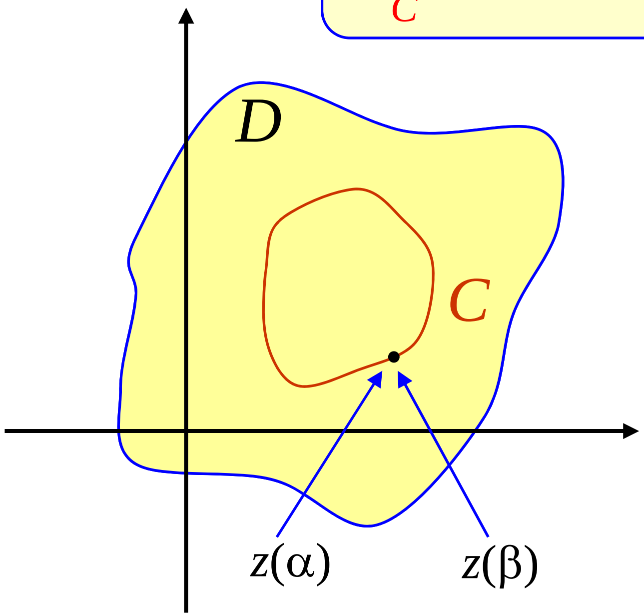


Triply connected

Cauchy's Integral Theorem

If $f(z)$ is analytic in a simply connected domain D, then for every closed path C in D

$$\oint_C f(z)\,dz = 0$$



Fundamental Theorem of Calculus

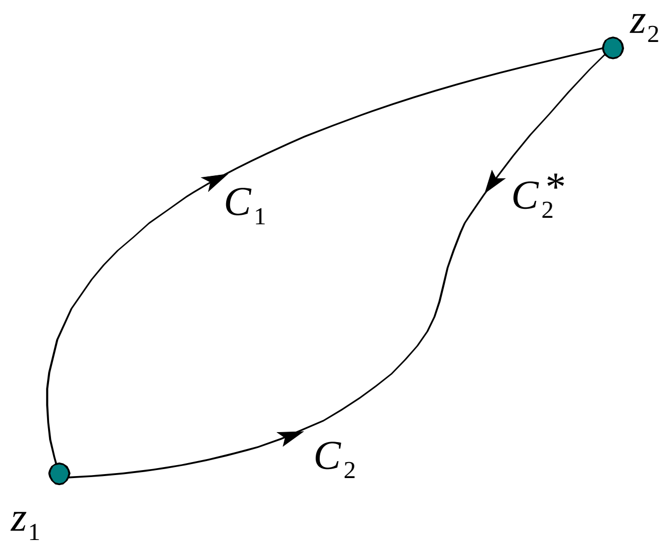
If $F(z)$ is an analytic function with a continuous derivative $f(z) = F'(z)$ in a region D containing a piecewise smooth (pws) arc $\gamma : z = z(t),\ \alpha \le t \le \beta$

$$\int_\gamma f(z)dz = F(z(\beta)) - F(z(\alpha))$$

Applications of Cauchy's Theorem

Applications:

1. If $f(z)$ is analytic in a simply connected domain D, then the integral of $f(z)$ is independent of path in D.



$$\int_{C_1} f(z)\,dz + \int_{C_2^*} f(z)\,dz = 0$$

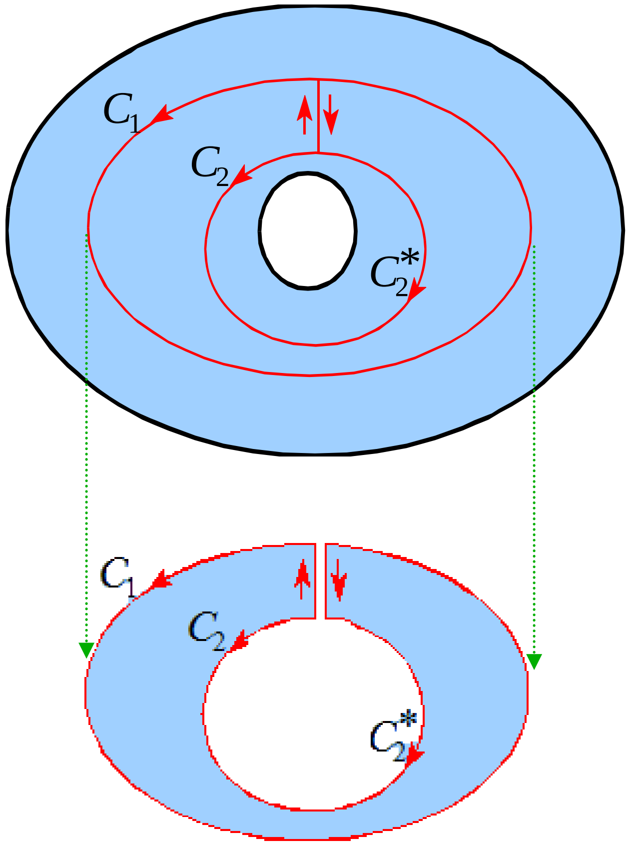
$$\Rightarrow \quad \int_{C_1} f(z)\,dz = -\int_{C_2^*} f(z)\,dz = \int_{C_2} f(z)\,dz$$

Applications (cont.)

2. Consider a doubly connected domain D. If the function $f(z)$ is analytic in D, then the integral of $f(z)$ is the same around any closed path that encircles the opening.

$$\int_{C_1} f(z)\,dz + \int_{C_2^*} f(z)\,dz = 0$$

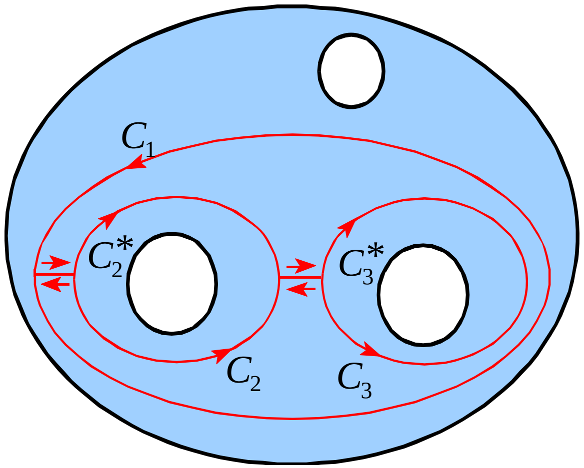
$$\Rightarrow \quad \int_{C_1} f(z)\,dz = -\int_{C_2^*} f(z)\,dz = \int_{C_2} f(z)\,dz$$



Applications (cont.)

3. The integral along a closed path C_1 of the function $f(z)$ which is analytic in the multiply connected domain D, is given by the sum of the integrals around paths which encircle all openings within the region bounded by C_1, e.g.

$$\int_{C_1} f(z)\,dz + \int_{C_2^*} f(z)\,dz + \int_{C_3^*} f(z)\,dz = 0$$



Thus
$$\int_{C_1} f(z)\,dz = -\int_{C_2^*} f(z)\,dz - \int_{C_3^*} f(z)\,dz$$
$$= \int_{C_2} f(z)\,dz + \int_{C_3} f(z)\,dz$$

Applications (cont.)

4. In general, it can be shown that if the path C encloses the point z_0, then

$$\oint_C (z-z_0)^n dz = \begin{cases} 0 & n \neq -1 \\ 2\pi i & n = -1 \end{cases}$$

Proof. Without loss of any generality, we assume $z_0 = 0$. For $n \geq 0$, z^n is analytic anywhere on the whole complex plane. By Cauchy's theorem, its integration over any closed curve is 0. For $n = -1$, By Application 2, the integration over any path enclosed z_0 is the same. It was shown earlier that the integration of $1/z$ over a circle $C: \; z(t) = re^{it}, \quad t \in [0, 2\pi]$

$$\int_C \frac{1}{z} dz = \int_0^{2\pi} \frac{ire^{it}}{re^{it}} dt = i \int_0^{2\pi} dt = 2\pi i$$

Proof of Application 4 (cont.)

For $n < -1$, let $m = -n > 1$ and integrate over $C: \ z(t) = re^{it}, \quad t \in [0, 2\pi]$. We have

$$\int_C \frac{1}{z^m} dz = \int_0^{2\pi} \frac{ire^{it}}{r^m e^{imt}} dt = \frac{i}{r^{m-1}} \int_0^{2\pi} e^{-i(m-1)t} dt$$

$$= \frac{i}{r^{m-1}} \cdot \frac{1}{-i(m-1)} \cdot e^{-i(m-1)t} \Big|_0^{2\pi}$$

$$= -\frac{1}{(m-1)r^{m-1}} \left[e^{-i2(m-1)\pi} - 1 \right]$$

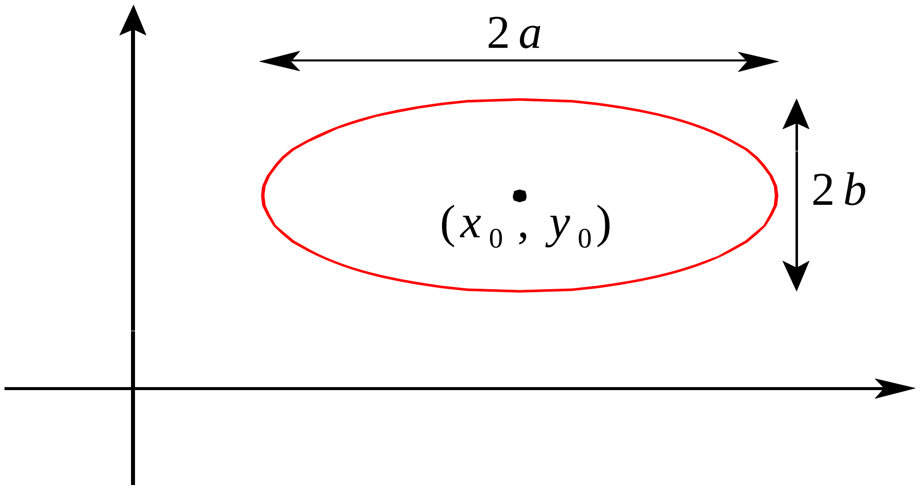
$$= -\frac{1}{(m-1)r^{m-1}} \left[\cos\left(2(m-1)\pi\right) - i \sin\left(2(m-1)\pi\right) - 1 \right]$$

$$= -\frac{1}{(m-1)r^{m-1}} \left[1 - 0 - 1 \right] = 0$$

Example

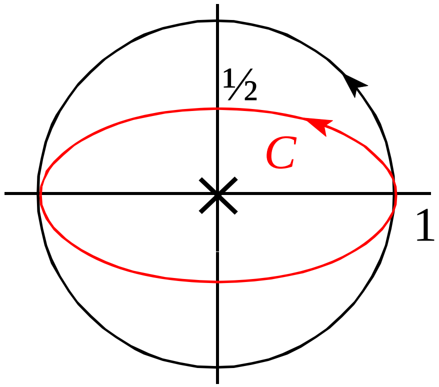
(a) $\oint_C 1/z\,dz$ with C the ellipse $x^2 + 4y^2 = 1$

The equation of the ellipse is given by $\frac{(x-x_0)^2}{a^2} + \frac{(y-y_0)^2}{b^2} = 1$



Example (cont.)

Consequently, the ellipse $x^2 + 4y^2 = 1$ is centred at the origin. The function $f(z) = 1/z$ is differentiable everywhere except at $z = 0$. The integral of $f(z)$ is therefore the same for any path which encloses the origin. C may thus be replaced with a circular path of radius 1, i.e., $z(t) = e^{it}, \quad t \in [0, 2\pi]$

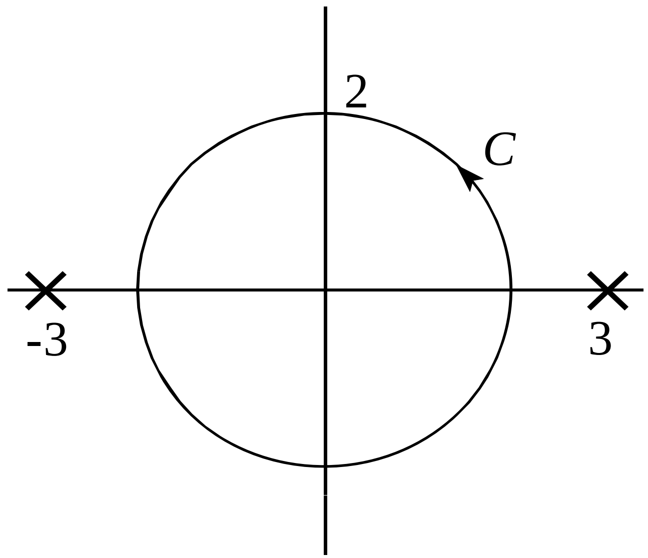


$$\oint_C \frac{1}{z} dz = \int_0^{2\pi} \frac{ie^{it}}{e^{it}} dt = 2\pi i$$

Example

$$\oint_{|z|=2} \frac{e^z}{z^2-9}\, dz = 0$$

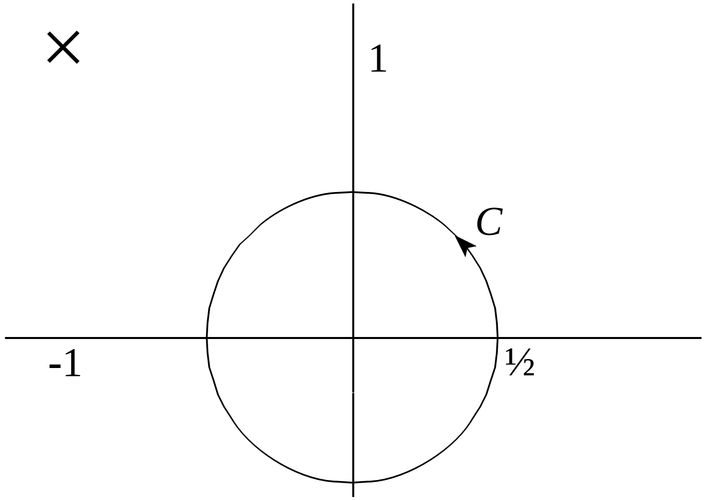
The function $f(z) = e^z/(z^2-9)$ is analytic inside the region enclosed by the curve.



Example

$$\oint_{|z|=1/2} \frac{1}{z+1-i}\,dz = \oint_{|z|=1/2} \frac{1}{z-(-1+i)}\,dz = 0$$

The function is analytic inside the region enclosed by the path of integration.



Cauchy's Integral Formula

Let D be a simply connected domain with z_0 a fixed point in D. Let $f(z)$ be analytic in D.

Then $\dfrac{f(z)}{z-z_0}$ is not analytic in D.

Consequently, $\oint_C \dfrac{f(z)}{z-z_0}\,dz \neq 0$ if C is a closed curve enclosing z_0

However, the integral $\oint_C \dfrac{f(z)}{z-z_0}\,dz \neq 0$ will be the same for any path enclosing z_0

Cauchy's Integral Formula

Consider a circle $z = z_0 + Re^{it}$, $t \in [0, 2\pi]$ with centre z_0. Then

$$\oint_C \frac{f(z)}{z - z_0} dz = \int_0^{2\pi} \frac{f(z_0 + Re^{it})}{Re^{it}} iRe^{it} dt = i \int_0^{2\pi} f(z_0 + Re^{it}) dt$$

Since $f(z)$ is continuous and the integral will have the same value for all values of R, it follows that

$$\oint_C \frac{f(z)}{z - z_0} dz = \lim_{R \to 0} i \int_0^{2\pi} f(z_0 + Re^{it}) dt = i \int_0^{2\pi} f(z_0) dt = i f(z_0) \int_0^{2\pi} dt = 2\pi i f(z_0)$$

Cauchy's Integral Formula

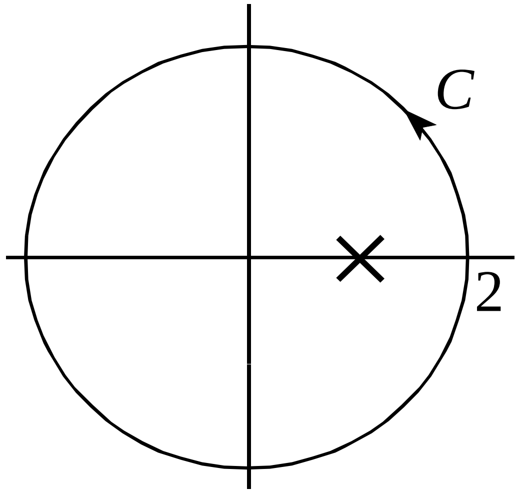
This leads to **Cauchy's integral formula**, stating the following:

Let $f(z)$ be analytic in D. Let C be a closed curve in D which encloses z_0. Then

$$\oint_C \frac{f(z)}{z - z_0}\, dz = 2\pi i\, f(z_0)$$

Examples

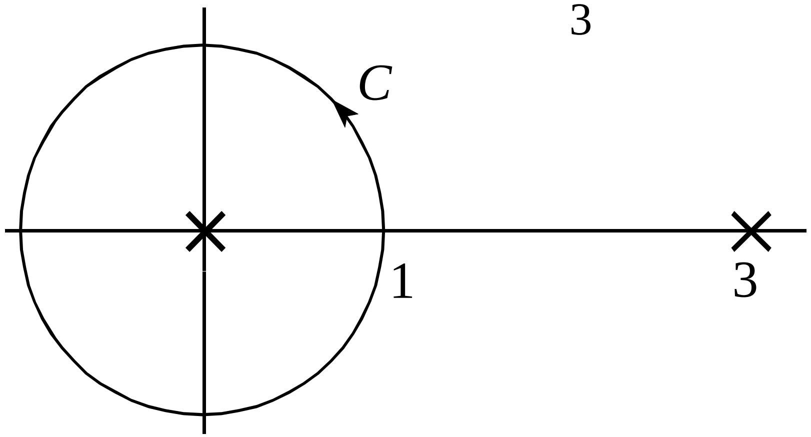
(a) $$\oint_{|z|=2} \frac{\sin z}{z-1} dz = 2\pi i \sin 1$$



(b) $$\oint_{|z|=1} \frac{z^2+1}{z(z-3)} dz = \oint_{|z|=1} \frac{\frac{z^2+1}{z-3}}{z} dz$$

$$= 2\pi i \left(\frac{0^2+1}{0-3} \right) = 2\pi i \left(\frac{1}{-3} \right)$$

$$= -\frac{2\pi i}{3}$$



Power Series as Analytic Function

A power series is of the form

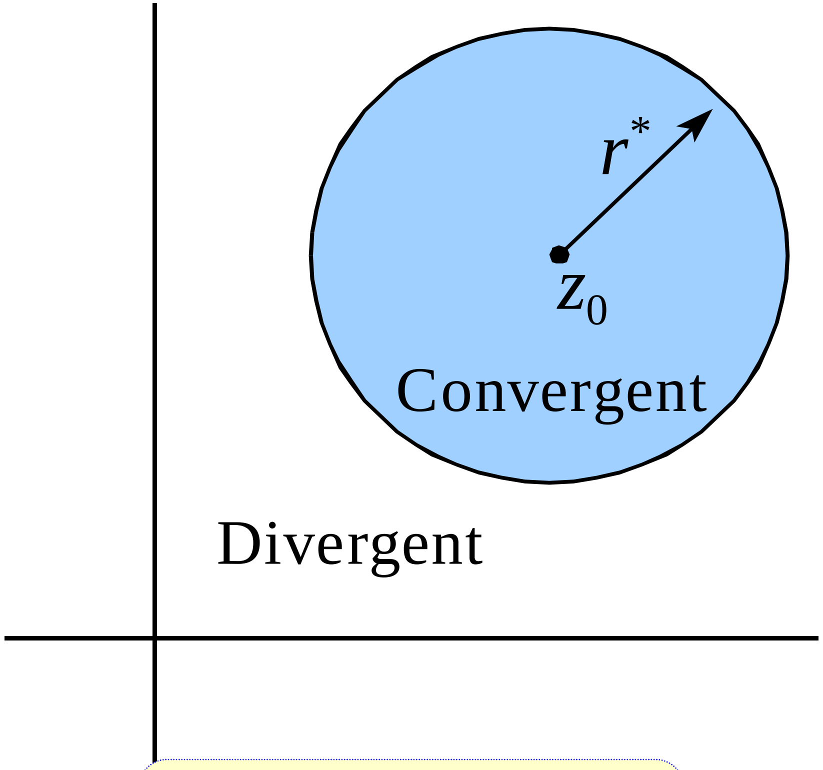
$$\sum_{n=0}^{\infty} c_n (z-z_0)^n = c_0 + c_1(z-z_0) + c_2(z-z_0)^2 + \cdots$$

Convergence: Every power series $\sum_{n=0}^{\infty} c_n (z-z_0)^n$ has a radius of convergence r^* such that the series is absolutely convergent for $|z-z_0| < r^*$ and divergent for $|z-z_0| > r^*$.

Example: The following well known geometric series has an $r^* = 1$:

$$\frac{1}{1-z} = 1 + z + z^2 + \cdots$$

Power Series as Analytic Functions



The radius of convergence is given by

$$r^* = \lim_{n\to\infty} \left| \frac{c_n}{c_{n+1}} \right|$$

Each power series defines a function which is analytic inside the radius of convergence.

Example: $\dfrac{1}{1-z} = 1 + z + z^2 + \cdots$

Integration and Differentiation of a Power Series

Inside the radius of convergence, the power series

$$f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n$$

can be integrated and differentiated on a term-by-term basis, i.e.

$$\int_{z_1}^{z_2} f(z)\, dz = \sum_{n=0}^{\infty} c_n \int_{z_1}^{z_2} (z - z_0)^n\, dz$$

$$\frac{d}{dz}[f(z)] = \sum_{n=0}^{\infty} n c_n (z - z_0)^{n-1}, \quad |z - z_0| < r^*$$

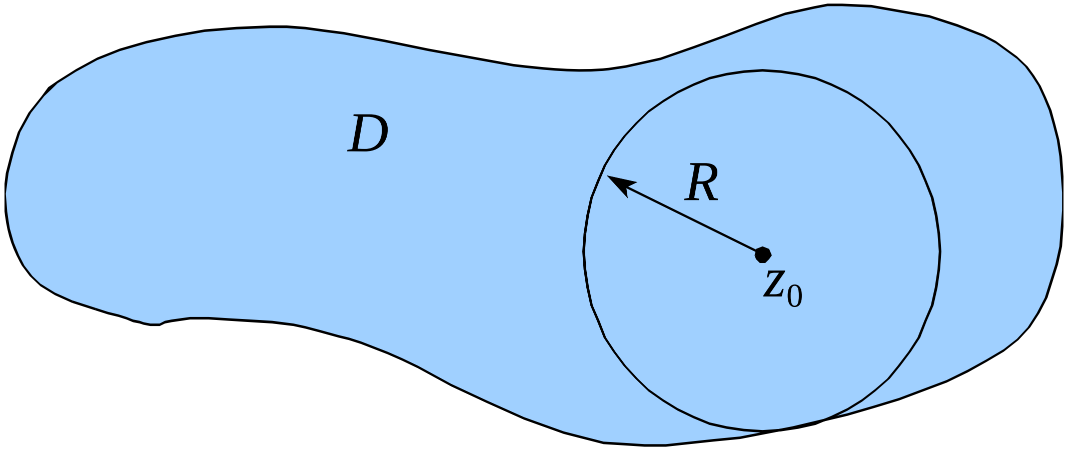
Analytic Functions as Power Series

Theorem: Let $f(z)$ be an analytic function in domain D. Let z_0 be a point in D and R be the radius of the largest circle with centre z_0 lying inside D. Then there is a power series $\sum_{n=0}^{\infty} c_n (z-z_0)^n$ which converges to $f(z)$ for $|z-z_0|<R$. Furthermore

$$c_n = \frac{f^{(n)}(z_0)}{n!} = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\, dz$$

where C is the closed circle which encloses z_0.

Analytic Functions as Power Series



From the last equation, also note that

$$\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz = 2\pi i\,\frac{f^{(n)}(z_0)}{n!}$$

Examples

(a) $$\oint_{|z|=1} \frac{\sin z}{z^4}\, dz = \frac{2\pi i}{3!} \frac{d^3}{dz^3}[\sin z]_{z=0} = -\frac{2\pi i}{6}$$

(b) $$\oint_{|z|=2} \frac{z\, e^z}{(z-1)^4}\, dz = \frac{2\pi i}{3!} \frac{d^3}{dz^3}\left[z\, e^z\right]_{z=1} = \frac{\pi i}{3}\left[3e^z + ze^z\right]_{z=1} = \frac{4e\pi i}{3}$$

Laurent and Taylor Series Expansions of Complex Functions

If a function $f(z)$ is analytic for $|z - z_0| < R$,

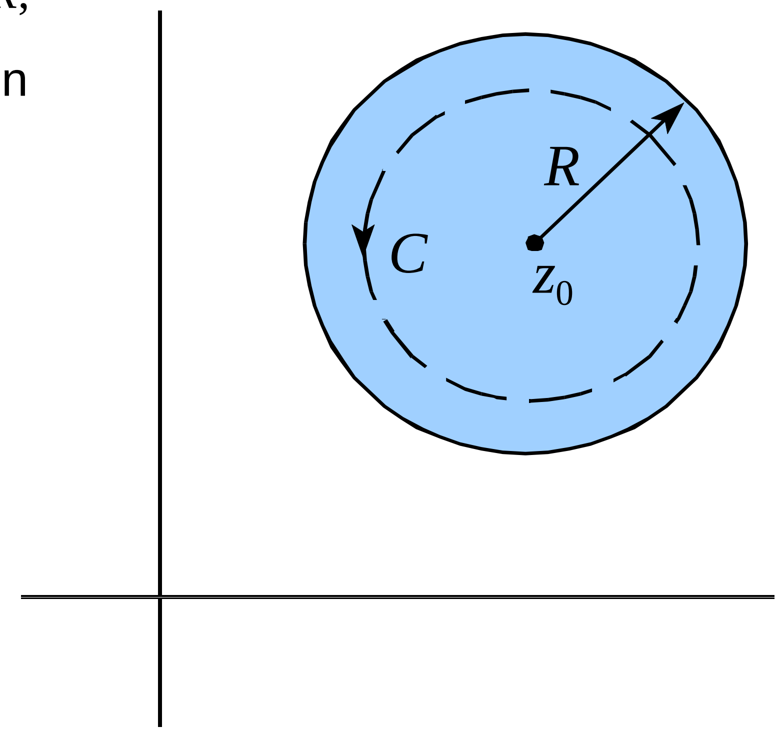
then $f(z)$ has a **Taylor series** expansion

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

where

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}} dz$$

$$= \frac{f^{(n)}(z_0)}{n!}$$



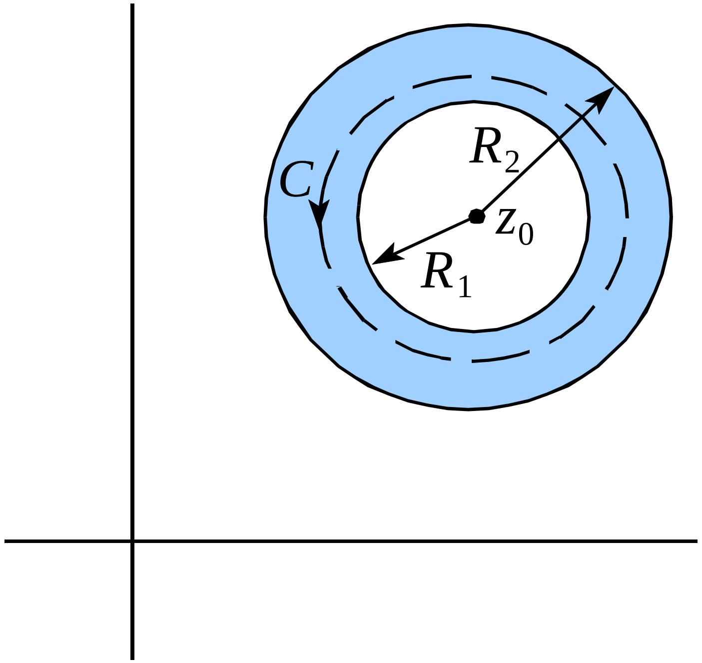
Laurent and Taylor Series Expansions of Complex Functions

If a function $f(z)$ is analytic in the ring area $R_1 < |z - z_0| < R_2$, then $f(z)$ has a **Laurent series** expansion

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

where

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}} dz$$



Points where a function is not analytic are called **singularities**.

Example 10

(a) The functions e^z, $\sin z$ and $\cos z$ are analytic functions, and have Taylor series expansions with a centre $z_0 = 0$ of

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots \quad \Rightarrow \quad c_n = \frac{1}{n!} \quad \Rightarrow \quad r^* = \lim_{n\to\infty}\left|\frac{c_n}{c_{n+1}}\right|$$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots$$

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots$$

$$= \lim_{n\to\infty}\frac{\dfrac{1}{n!}}{\dfrac{1}{(n+1)!}}$$

$$= \lim_{n\to\infty}\frac{(n+1)!}{n!}$$

$$= \lim_{n\to\infty}(n+1) = \infty$$

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Example 10 (cont.)

(b) The functions $\frac{e^z}{z}$ and $\frac{\sin z}{z^3}$ are not analytic in the point $z = 0$. In the region excluding the point $z_0 = 0$, these functions have Laurent series expansions of

$$\frac{e^z}{z} = \frac{1}{z} + 1 + \frac{z}{2!} + \frac{z^2}{3!} + \cdots \qquad \frac{\sin z}{z^3} = \frac{1}{z^2} - \frac{1}{3!} + \frac{z^2}{5!} - \frac{z^4}{7!} + \cdots$$

Example 10 (cont.)

(c) The function $\frac{1}{1-z}$ is analytic for $|z|<1$. The Taylor series expansion with centre $z_0 = 0$ of this function is the geometric series, i.e.,

$$\frac{1}{1-z} = 1 + z + z^2 + \cdots$$

(d)* Find Taylor series expansion of $f(z) = \frac{1}{z}$ at $z_0 = 3$ & its convergence radius.

$$f(z) = \frac{1}{z} = \frac{\frac{1}{3}}{1-\left(\frac{3-z}{3}\right)} = \frac{1}{3}\left[1 + \frac{3-z}{3} + \left(\frac{3-z}{3}\right)^2 + \left(\frac{3-z}{3}\right)^3 + \cdots\right] = \frac{1}{3} - \frac{z-3}{3^2} + \frac{(z-3)^2}{3^3} - \frac{(z-3)^3}{3^4} + \cdots$$

The series converges for all $\left|\frac{3-z}{3}\right| < 1 \quad \Rightarrow \quad |z-3| < 3$. Thus, its $r^* = 3$.

Classification of Singularities

Poles

Consider the Laurent expansion of different functions:

1. No negative powers of z in the expansion. For example $\dfrac{\sin z}{z}$ has a singularity at $z_0 = 0$.

$$\frac{\sin z}{z} = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \frac{z^6}{7!} + \cdots\cdots$$, so that its Laurent expansion

has no negative powers of $(z - z_0)$. The function is said to have a ***removable singularity*** at $z_0 = 0$.

Poles (cont.)

2. A finite number of negative powers of z in the expansion, e.g.,

$$\frac{e^z}{z^3} = \frac{1}{z^3} + \frac{1}{z^2} + \frac{1}{2!z} + \frac{1}{3!} + \frac{z}{4!} \cdots$$

The highest negative power is 3. This function is said to have a **3rd order pole** at $z_0 = 0$.

3. An infinite number of negative powers of z in the expansion, e.g.,

$$e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \cdots$$

This function is said to have an ***essential singularity*** at $z_0 = 0$.

Example 11

(a) $$f(z)=\frac{\cos z-1}{z}=\frac{1}{z}\left[\left(1-\frac{z^2}{2!}+\frac{z^4}{4!}-\frac{z^6}{6!}+\cdots\right)-1\right]=-\frac{z}{2!}+\frac{z^3}{4!}-\frac{z^5}{6!}+\cdots$$

The function $f(z)$ has a removable at $z_0=0$.

(b) $$f(z)=\frac{\sin z}{z^5}=\frac{1}{z^5}\left[z-\frac{z^3}{3!}+\frac{z^5}{5!}-\frac{z^7}{7!}+\cdots\right]=\frac{1}{z^4}-\frac{1}{3!z^2}+\frac{1}{5!}-\frac{z^2}{7!}+\cdots$$

The function $f(z)$ has a **4th order pole** at $z_0=0$.

Example 11 (cont.)

(c) $$f(z) = z^2 e^{1/z} = z^2\left[1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \cdots\right] = z^2 + z + \frac{1}{2!} + \frac{1}{3!z} + \cdots$$

The function $f(z)$ has an essential singularity at $z_0 = 0$.

(d) $$f(z) = \frac{z^2 - 2}{(z+1)^2} = \frac{z^2 + 2z + 1 - 2z - 1 - 2}{(z+1)^2} = \frac{(z+1)^2 - 2(z+1) - 1}{(z+1)^2}$$

$$= -\frac{1}{(z+1)^2} - \frac{2}{(z+1)} + 1$$

Thus, $f(z)$ has 2nd order pole at $z_0 = -1$.

Example 11 (cont.)

(e) $$f(z)=\frac{z^2-2}{z(z+1)}=\frac{z+2}{z+1}-\frac{2}{z}=1+\frac{1}{z+1}-\frac{2}{z}$$

It is clear that $f(z)$ has 2 singular points at $z_0=0$ & $z_0=-1$, respectively.

For $z_0=0$, we have the following Laurent series of $f(z)$ centered at $z_0=0$

$$f(z)=\frac{z^2-2}{z(z+1)}=1+\frac{1}{z+1}-\frac{2}{z}=1-\frac{2}{z}+\frac{1}{1-(-z)}$$

$$=-\frac{2}{z}+1+\left[1+(-z)+(-z)^2+(-z)^3+\cdots\right]$$

$$=-\frac{2}{z^1}+2-z+z^2-z^3+\cdots$$

Thus, the order of singularity of $f(z)$ at $z_0=0$ is 1.

Example 11 (cont.)

For $z_0 = -1$, we have the following Laurent series of $f(z)$ centered at $z_0 = -1$

$$f(z) = \frac{z^2 - 2}{z(z+1)} = 1 + \frac{1}{z+1} - \frac{2}{z} = 1 + \frac{1}{z+1} + \frac{2}{1-(z+1)}$$

$$= \frac{1}{z+1} + 1 + 2 \cdot \left[1 + (z+1) + (z+1)^2 + (z+1)^3 + \cdots \right]$$

$$= \frac{1}{(z+1)^1} + 3 + 2(z+1) + 2(z+1)^2 + 2(z+1)^3 + \cdots$$

Thus, the order of singularity of $f(z)$ at $z_0 = -1$ is again equal to 1.

1st order poles** are also called **simple poles** or **simple singularities.

Zeros

If $g(z_0) = 0$, then the function $g(z)$ is said to have a zero or root in $z = z_0$.

If $g(z_0) = g'(z_0) = g''(z_0) = \ldots g^{(n-1)}(z_0) = 0$ and $g^{(n)}(z_0) \neq 0$, then the function is said to have an nth order zero in $z = z_0$.

Theorem:

If the function $g(z)$ has an nth order zero in $z = z_0$, then $f(z) = \dfrac{1}{g(z)}$ has an n-th order pole in $z = z_0$.

Example 12

a) Consider the function $f(z)=\dfrac{1}{(z-1)(e^z-e)}$, which has a singularity at $z_0=1$

Let $g(z)=(z-1)(e^z-e)$

Then $g'(z)=e^z-e+(z-1)e^z \qquad g'(1)=0$

$$g''(z)=e^z+(z-1)e^z+e^z \qquad g''(1)=2e\neq 0$$

Therefore $g(z)$ has a 2nd order zero in $z_0=1$, and $f(z)$ has a 2nd order pole in $z_0=1$.

Example 12 (cont.)

b) Consider $f(z)=\dfrac{1}{z-\sin z}$, which has a singularity at $z_0=0$

Let $g(z)=z-\sin z$.

Then $g'(z)=1-\cos z \qquad g'(0)=0$

$g''(z)=\sin z \qquad g''(0)=0$

$g'''(z)=\cos z \qquad g'''(0)=1\neq 0$

Therefore $g(z)$ has a 3rd order zero in $z_0=0$, and $f(z)$ has a 3rd order pole at $z_0=0$.

Order of Singularities

The following method is modified from a suggestion made by Ang Zhi Ping, a student taking EE2012 in Semester 1 of Year 2007/08.

Given a function $f(z) = h(z) / g(z)$ and $g(z_0) = 0$, the order of the pole at $z = z_0$ can be determined without finding Laurent series as follows:

1. Find the order (say n) of zero of $g(z)$ at $z = z_0$.

2. Find the order (say m) of zero of $h(z)$ at $z = z_0$, if it is a zero of $h(z)$; Otherwise, $m = 0$.

The order of the pole of $f(z)$ at $z = z_0$ is given by $n - m$. Note that there are m pole-zero cancellations between the numerator and denominator.

Example

An alternative solution to Q.2.2.1 in Tutorial 2.2, i.e., finding the order of poles for

$$f(z) = \frac{h(z)}{g(z)} = \frac{e^z - \sin z - 1}{z^2}$$

1. It is obvious that $g(z)$ has a zero of order 2 at $z = 0$, i.e., $n = 2$.

2. Noting $\left\{ \begin{aligned} h(0) &= \left[e^z - \sin z - 1 \right]_{z=0} = 0 \\ h'(0) &= \left[e^z - \cos z \right]_{z=0} = 0 \\ h''(0) &= \left[e^z + \sin z \right]_{z=0} = 1 \neq 0 \end{aligned} \right\}$, we have $m = 2$.

3. The order of the pole of $f(z)$ at $z_0 = 0$ is given $n - m = 0$. It is removable.

Residues

We know that if $f(z)$ is analytic in domain D except at the point z_0, it has a Laurent series expansion of

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

with

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}} dz$$

and C any closed curve in D which encloses z_0.

Residues

From last expression of a_n, it follows that

$$a_{-1} = \frac{1}{2\pi i} \oint_C f(z)\, dz = \mathrm{Res}(f, z_0)$$

where $\mathrm{Res}(f, z_0)$ is known as the residue of f at z_0. Thus

$$\oint_C f(z)\, dz = 2\pi i \,\mathrm{Res}(f, z_0)$$

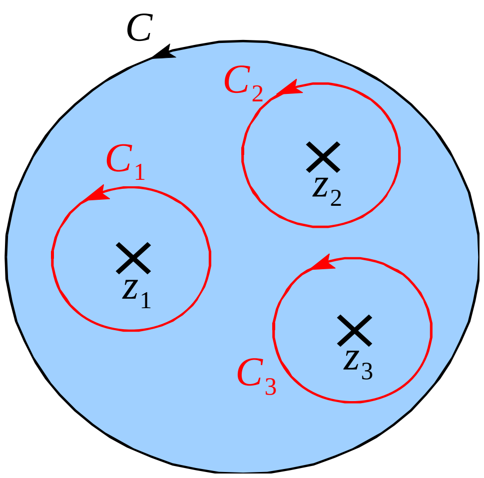
Residues

If $f(z)$ is not analytic in several points $z_1, z_2, \ldots z_n$, then

$$\oint_C f(z)\,dz = \oint_{C_1} f(z)\,dz + \oint_{C_2} f(z)\,dz + \cdots + \oint_{C_n} f(z)\,dz$$

$$= 2\pi i \,\text{Res}(f, z_1) + 2\pi i \,\text{Res}(f, z_2) + \cdots + 2\pi i \,\text{Res}(f, z_n)$$

$$= 2\pi i \sum_{i=1}^{n} \text{Res}(f, z_i)$$



Calculation of Residues

1. $f(z)$ has a simple pole at $z = z_0$: $\text{Res}(f, z_0) = \lim_{z \to z_0} (z - z_0) f(z)$

2. $f(z)$ has an nth order pole at $z = z_0$:

$$\text{Res}(f, z_0) = \frac{1}{(n-1)!} \lim_{z \to z_0} \left[\frac{d^{n-1}}{dz^{n-1}} \left[(z - z_0)^n f(z) \right] \right]$$

3. $f(z) = \dfrac{A(z)}{B(z)}$ where $B(z)$ has a simple zero at $z = z_0$, while $A(z_0) \neq 0$

and both A and B are differentiable at $z = z_0$: $\text{Res}(f, z_0) = \dfrac{A(z_0)}{B'(z_0)}$

Calculation of Residues

2'. The 2nd formula of the previous page can be modified as follows:

Assuming that $f(z)$ has an n-th order pole at $z = z_0$, then

$$\text{Res}(f, z_0) = \frac{1}{(m-1)!} \lim_{z \to z_0} \left[\frac{d^{m-1}}{dz^{m-1}} \left[(z - z_0)^m f(z) \right] \right]$$

where m is any integer with $m \geq n$.

This formula was proposed and derived by Phang Swee King, a student taking EE2012 in Semester 2 of Year 2007/08. It may yield a simpler way in computing the residue for certain situations.

Example using Formula 2′

Example: The extra freedom in selecting m in Formula 2′ can simplify some problems in computing residues and thus complex integrals. We consider the following example (which was solved earlier),

$$\oint_{|z|=1} \frac{\sin z}{z^4} dz$$

It was shown that the function has a 3rd order pole at $z_0 = 0$. However, if we use $m = 4$ instead of 3, it is much easier to compute the associated residue compared with that using the original formula 2, i.e.,

$$\oint_{|z|=1} \frac{\sin z}{z^4} dz = 2\pi i \frac{1}{3!} \lim_{z\to 0} \left[\frac{d^3}{dz^3} \left(z^4 \cdot \frac{\sin z}{z^4} \right) \right] = 2\pi i \frac{1}{3!} \lim_{z\to 0} \left[\frac{d^3}{dz^3} (\sin z) \right] = 2\pi i \frac{1}{3!} (-\cos 0) = -\frac{\pi i}{3}$$

Example 13

(a) Calculate $\oint_{|z|=2} \frac{4-3z}{z^2-z}\,dz.$

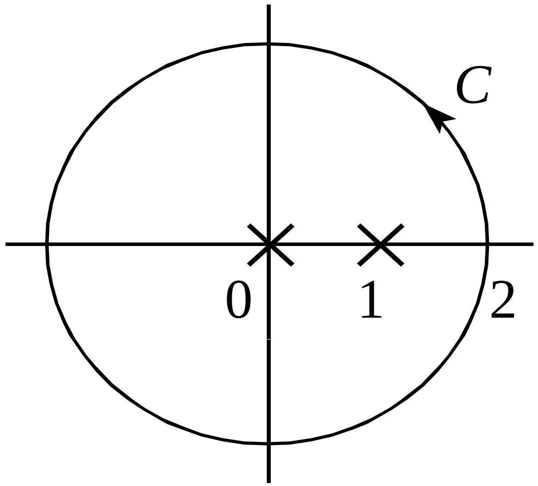
$$\text{Res}(f, z_0) = \lim_{z \to z_0} (z - z_0)\, f(z)$$

$f(z) = \frac{4-3z}{z^2-z} = \frac{4-3z}{z(z-1)}$ has simple poles in $z_0 = 0$ and $z_0 = 1$.

$$\oint_{|z|=2} \frac{4-3z}{z^2-z}\,dz = 2\pi i\left[\text{Res}(f,0) + \text{Res}(f,1)\right]$$

$$= 2\pi i\left[\lim_{z\to 0}\frac{4-3z}{z-1} + \lim_{z\to 1}\frac{4-3z}{z}\right]$$

$$= 2\pi i[-4+1] \quad = -6\pi i$$



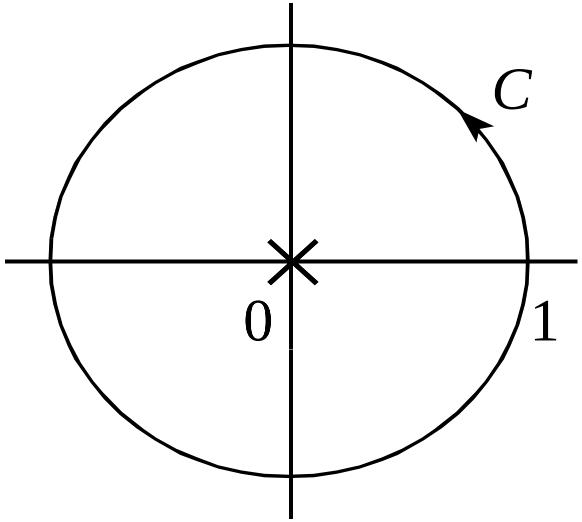
Example 13

(b) Compute $\oint_{|z|=1} \frac{e^z}{z}\, dz$

$$\oint_{|z|=1} \frac{e^z}{z}\, dz = 2\pi i \,\mathrm{Res}(f,0)$$

$$= 2\pi i \lim_{z\to 0} e^z$$

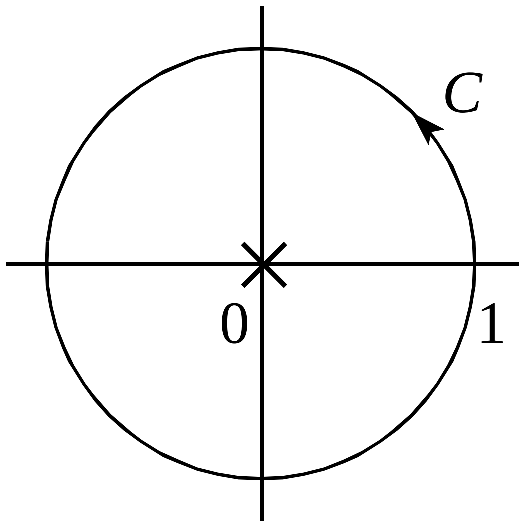
$$= 2\pi i$$



$$\mathrm{Res}(f, z_0) = \lim_{z\to z_0} (z - z_0)\, f(z)$$

Example 13

(c) Compute $\displaystyle\oint_{|z|=1} \frac{\sin z}{z^2}\, dz$



$f(z) = \dfrac{\sin z}{z^2}$ has a simple pole at $z_0 = 0$.

Thus

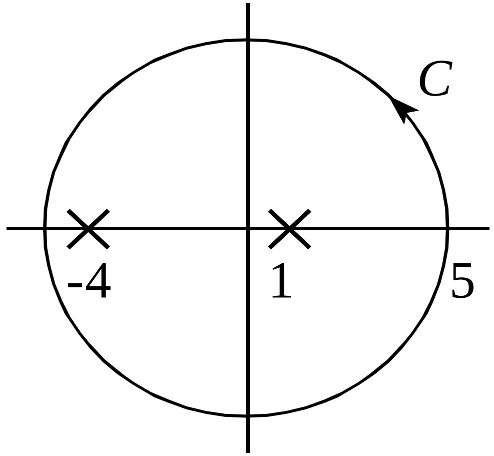
$$\oint_{|z|=1} \frac{\sin z}{z^2}\, dz = \oint_{|z|=1} \frac{\frac{\sin z}{z}}{z}\, dz = 2\pi i \lim_{z\to 0} \frac{\sin z}{z} = 2\pi i$$

$$\mathrm{Res}(f, z_0) = \lim_{z\to z_0} (z - z_0)\, f(z)$$

Example 13

$$\text{Res}(f, z_0) = \frac{1}{(n-1)!} \lim_{z \to z_0} \left[\frac{d^{n-1}}{dz^{n-1}} \left[(z - z_0)^n f(z) \right] \right]$$

(d) Compute $\displaystyle\oint_{|z|=5} \frac{2z}{(z+4)(z-1)^2} dz$



The function $f(z) = \dfrac{2z}{(z+4)(z-1)^2}$ has a simple pole at $z_0 = -4$ and a 2nd order pole at $z_0 = 1$.

$$\text{Res}(f, -4) = \lim_{z \to -4} \frac{2z}{(z-1)^2} = -\frac{8}{25}$$

$$\text{Res}(f, 1) = \frac{1}{1!} \lim_{z \to 1} \frac{d}{dz} \left[\frac{2z}{z+4} \right]$$

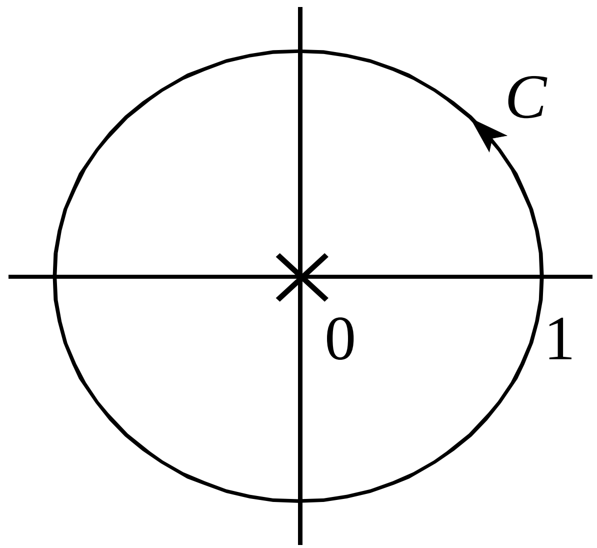
$$= \lim_{z \to 1} \frac{2(z+4) - 2z}{(z+4)^2} = \frac{8}{25}$$

$$\oint_{|z|=5} \frac{2z}{(z+4)(z-1)^2} dz = 2\pi i \left\{ \text{Res}(f, -4) + \text{Res}(f, 1) \right\} = 0$$

$$\text{Res}(f, z_0) = \lim_{z \to z_0} (z - z_0) f(z)$$

Example 13

(e) Compute $\displaystyle\oint_{|z|=1} \frac{1}{1-e^z}\,dz$



$$\oint_{|z|=1} \frac{1}{1-e^z}\,dz = 2\pi i\,\mathrm{Res}(f,0)$$

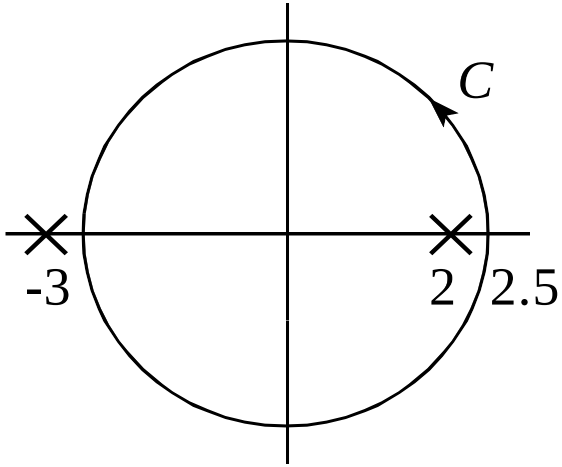
$$= 2\pi i \left[\frac{1}{-e^z}\right]_{z=0}$$

$$= -2\pi i$$

$$\mathrm{Res}(f,z_0) = \frac{A(z_0)}{B'(z_0)}$$

Example 13

(f) Compute $\oint_{|z|=2.5} \frac{2z+4}{z^2+z-6}\,dz$



$$f(z) = \frac{2z+4}{z^2+z-6} = \frac{2z+4}{(z+3)(z-2)}$$

has a simple pole at $z_0 = 2$ enclosed by C. Thus,

$$\oint_{|z|=2.5} \frac{2z+4}{z^2+z-6}\,dz = 2\pi i \,\mathrm{Res}(f,2) = 2\pi i \lim_{z\to 2}\left[\frac{2z+4}{z+3}\right] = \frac{16\pi i}{5}$$

$$\mathrm{Res}(f,z_0) = \lim_{z\to z_0} (z-z_0)\, f(z)$$

Applications

Real Integral of the form $\int_0^{2\pi} f(\cos\theta, \sin\theta)\, d\theta$

These integrals can be transformed to an integral of a complex function along the circle $|z| = 1$.

The circle can be described by $z = e^{i\theta}, \quad 0 \le \theta \le 2\pi$, Then

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{1}{2}\left(z + \frac{1}{z}\right)$$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} = \frac{1}{2i}\left(z - \frac{1}{z}\right)$$

$$dz = \frac{dz}{d\theta} d\theta = i e^{i\theta} d\theta$$

Real Integrals

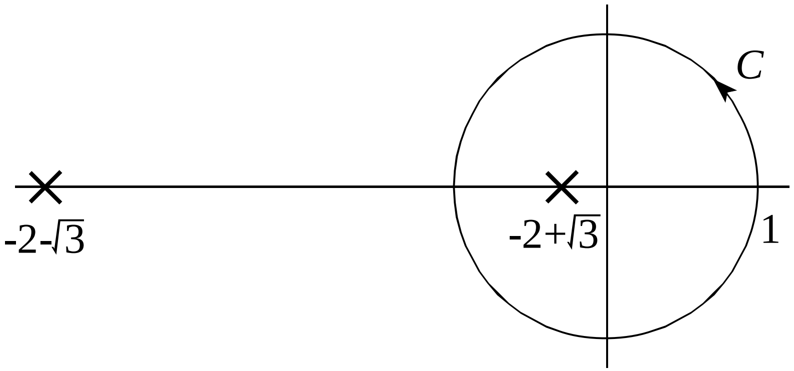
Consequently, we have

$$\int_0^{2\pi} f(\cos\theta, \sin\theta)\, d\theta = \oint_{|z|=1} f\left[\frac{1}{2}\left(z+\frac{1}{z}\right), \frac{1}{2i}\left(z-\frac{1}{z}\right)\right]\frac{ie^{i\theta}}{ie^{i\theta}}\, d\theta$$

$$= \oint_{|z|=1} f\left[\frac{1}{2}\left(z+\frac{1}{z}\right), \frac{1}{2i}\left(z-\frac{1}{z}\right)\right]\frac{1}{iz}\, dz$$

Example 14

(a) Evaluate $\displaystyle\int_0^{2\pi} \frac{1}{\cos\theta + 2}\, d\theta$



$$\int_0^{2\pi} \frac{1}{\cos\theta + 2}\, d\theta = \oint_{|z|=1} \frac{1}{\frac{1}{2}\left(z + \frac{1}{z}\right) + 2} \frac{1}{i\,z}\, dz$$

$$= -2i \oint_{|z|=1} \frac{1}{z^2 + 4z + 1}\, dz$$

$$= -2i \oint_{|z|=1} \frac{1}{\left(z - (-2 + \sqrt{3})\right)\left(z - (-2 - \sqrt{3})\right)}\, dz$$

$$= -2i \;\; 2\pi i \,\mathrm{Res}(f, -2 + \sqrt{3}) = \frac{2\pi}{\sqrt{3}}$$

$$\mathrm{Res}(f, z_0) = \lim_{z \to z_0} (z - z_0)\, f(z)$$

Example 14

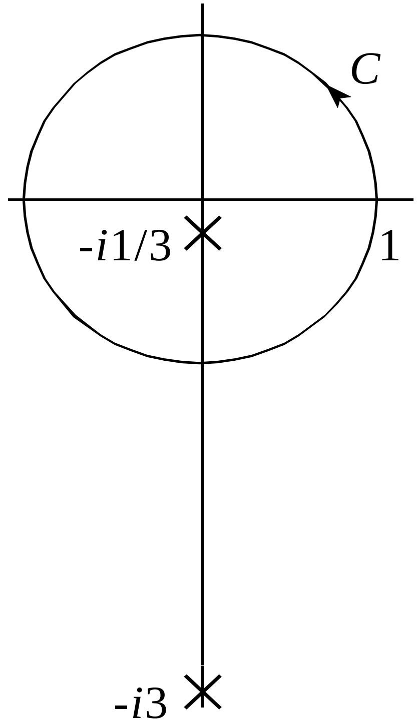
(b) Compute $\int_0^{2\pi} \frac{1}{5+3\sin\theta} d\theta$

$$\int_0^{2\pi} \frac{1}{5+3\sin\theta} d\theta = \oint_{|z|=1} \frac{1}{5+3\left[\frac{1}{2i}\left(z-\frac{1}{z}\right)\right]} \frac{1}{iz} dz$$

$$= 2\oint_{|z|=1} \frac{1}{3z^2+10iz-3} dz = \frac{2}{3}\oint_{|z|=1} \frac{1}{z^2+\frac{10}{3}iz-1} dz$$

$$= \frac{2}{3}\oint_{|z|=1} \frac{1}{\left(z+\frac{i}{3}\right)(z+i3)} dz = \frac{2}{3} 2\pi i \,\mathrm{Res}\left(f,-\frac{i}{3}\right)$$

$$= \frac{4\pi i}{3} \lim_{z\to -\frac{i}{3}} \frac{1}{(z+i3)} = \pi/2$$



$$z^2 + i(a-b)z + ab = (z-ib)(z+ia)$$

Improper Integrals of Rational Functions

We now consider real integrals for which the interval of integration is not finite. These are called improper integrals, and are defined by

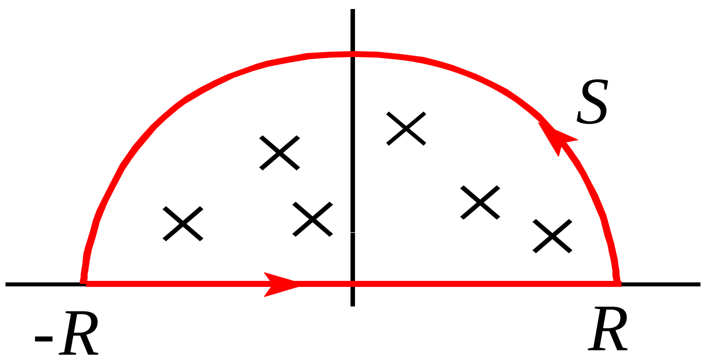
$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{R\to\infty}\int_{-R}^{R} f(x)\,dx$$

Assume that the $f(x) = \dfrac{P(x)}{Q(x)}$ is a real rational function with

- $Q(x) \neq 0$ for all real x (i.e. no real poles)
- $\text{degree}[Q(x)] \geq \text{degree}[P(x)] + 2$

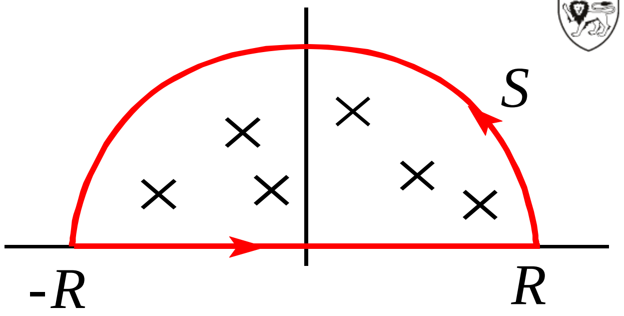
Improper Integrals of Rational Functions (cont.)

Consider the complex integral $\oint_C f(z)\,dz$ with C as indicated in the figure below. Since $f(x)$ is rational, $f(z)$ will have a finite number of poles in the upper half-plane, and if we choose R large enough, C encloses all these poles.



Note that C consists of a straight path from $-R$ to R and a half circle S on the upper plane.

Improper Integrals…



Then

$$\oint_C f(z)\,dz = \int_{-R}^{R} f(z)\,dz + \int_S f(z)\,dz \qquad (*)$$

The 2nd condition, i.e., degree $[Q(x)] \geq$ degree $[P(x)] + 2$, implies if

$$Q(z) = q_n z^n + q_{n-1} z^{n-1} + \cdots + q_1, \quad P(z) = p_m z^m + p_{m-1} z^{m-1} + \cdots + p_1$$

then, $n \geq m + 2$. Thus, for z on S when R is large

$$\left|z^2 f(z)\right| = \left|\frac{z^2 P(z)}{Q(z)}\right| = \left|\frac{p_m z^{m+2} + p_{m-1} z^{m+1} + \cdots}{q_n z^n + q_{n-1} z^{n-1} + \cdots}\right| \cong \left|\frac{p_m}{q_n}\right| \cdot \frac{1}{|z|^{n-m-2}} \leq K \quad \Rightarrow \quad |f(z)| \leq \frac{K}{|z^2|} = \frac{K}{|z|^2} = \frac{K}{R^2}$$

$$\Rightarrow \quad \left|\int_S f(z)\,dz\right| \leq ML = \frac{K}{R^2}\pi R = \frac{K\pi}{R} \to 0 \quad \text{as} \quad R \to \infty$$

(Result S)

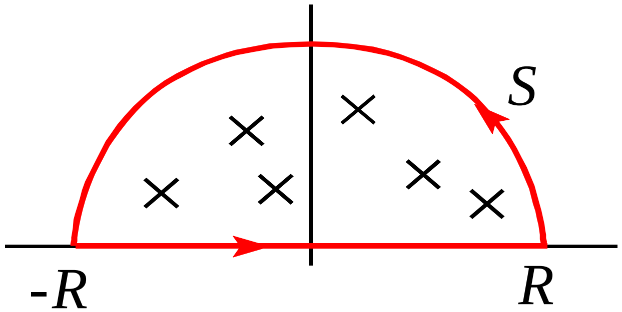
Improper Integrals of Rational Functions (cont.)

And consequently $\lim_{R\to\infty}\int_S f(z)\,dz = 0$. From the equation (*) on last slide, we therefore have that

$$\lim_{R\to\infty}\int_{-R}^{R} f(z)\,dz = \oint_C f(z)\,dz$$

or

$$\int_{-\infty}^{\infty} f(x)\,dx = 2\pi i \sum_j \mathrm{Res}(f, a_j)$$

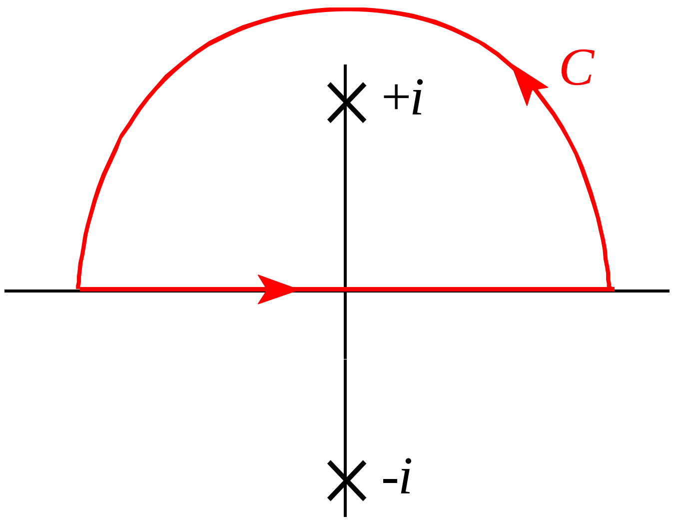


where the sum is taken over all the poles in the upper half-plane.

Example 15

(a) Calculate $\displaystyle\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$

Let $\displaystyle f(z) = \frac{1}{1+z^2} = \frac{1}{(z+i)(z-i)}$



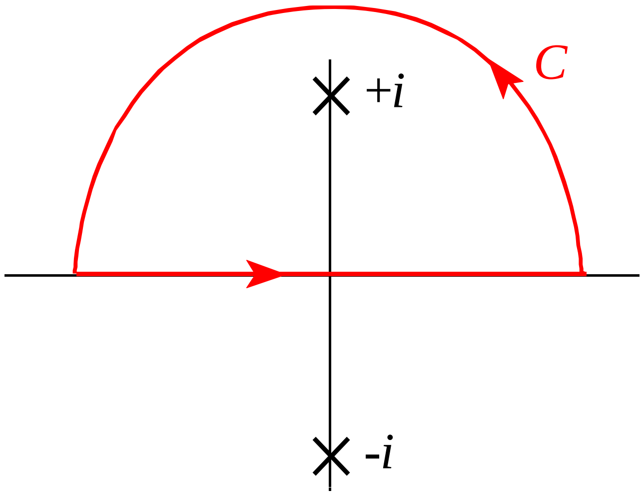
$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = 2\pi i \,\mathrm{Res}(f,i) = 2\pi i \frac{1}{2i} = \pi$$

$$\mathrm{Res}(f,z_0) = \lim_{z\to z_0} (z-z_0)\, f(z)$$

Example 15

(b) Calculate $\displaystyle\int_{-\infty}^{\infty} \frac{1}{(1+x^2)^3}\,dx$

Let $\displaystyle f(z) = \frac{1}{(1+z^2)^3} = \frac{1}{(z+i)^3(z-i)^3}$



$$\int_{-\infty}^{\infty} \frac{1}{(1+x^2)^3}\,dx = 2\pi i \,\mathrm{Res}(f, i) = \pi i \lim_{z\to i} \frac{12}{(z+i)^5} = \pi i \left(\frac{-6i}{16}\right) = \frac{3\pi}{8}$$

$$\mathrm{Res}(f, z_0) = \frac{1}{(n-1)!} \lim_{z\to z_0} \left[\frac{d^{n-1}}{dz^{n-1}} \left[(z-z_0)^n f(z) \right] \right]$$

Example 15

(c) Calculate $\displaystyle\int_{-\infty}^{\infty} \frac{1}{4+x^4}\,dx$. Let $f(z)=\dfrac{1}{z^4+4}$ and its poles are given by

$$z^4+4=0 \quad\Rightarrow\quad z^4=-4=4\,e^{i(2n+1)\pi} \quad\Rightarrow\quad z=(4)^{1/4}\,e^{i(2n+1)\pi/4}$$

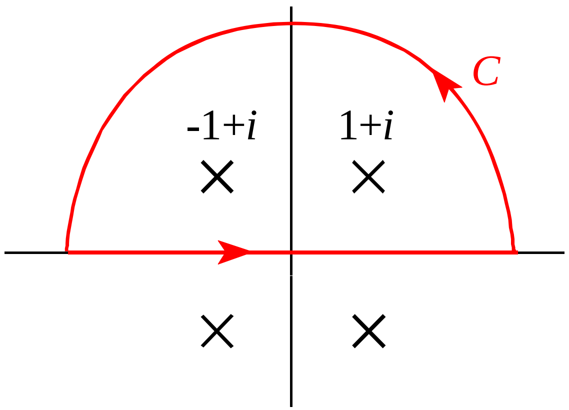
$$\Rightarrow\quad z=\sqrt{2}\,e^{i(2n+1)\pi/4},\qquad n=0,\,1,\,2,\,3,\cdots$$

$$\Rightarrow\quad z_1=\sqrt{2}\,e^{i(2\times0+1)\pi/4}=\sqrt{2}\,e^{i\pi/4}=1+i,\qquad n=0$$

$$z_2=\sqrt{2}\,e^{i(2\times1+1)\pi/4}=\sqrt{2}\,e^{i3\pi/4}=-1+i,\quad n=1$$

$$z_3=\sqrt{2}\,e^{i(2\times2+1)\pi/4}=\sqrt{2}\,e^{i5\pi/4}=-1-i,\quad n=2$$

$$z_4=\sqrt{2}\,e^{i(2\times3+1)\pi/4}=\sqrt{2}\,e^{i7\pi/4}=1-i,\qquad n=3$$



Example 15

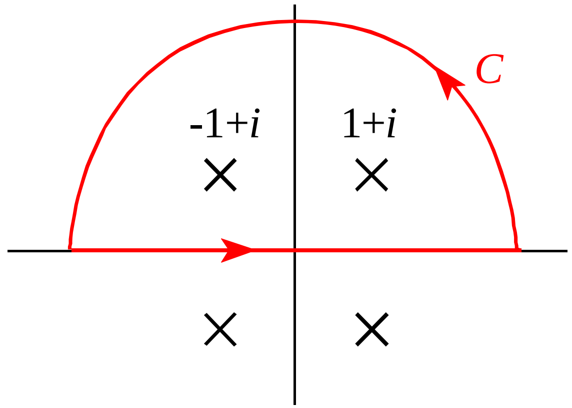
Then

$$\int_{-\infty}^{\infty} \frac{1}{4+x^4}\,dx = 2\pi i\left[\mathrm{Res}(f,1+i)+\mathrm{Res}(f,-1+i)\right]$$

$$= 2\pi i\left\{\left[\frac{1}{4z^3}\right]_{z=1+i} + \left[\frac{1}{4z^3}\right]_{z=-1+i}\right\}$$

$$= 2\pi i\left[\frac{-1}{16}(1+i)+\frac{1}{16}(1-i)\right]$$

$$= 2\pi i\left(\frac{-i}{8}\right) = \frac{\pi}{4}$$



$$\mathrm{Res}(f,z_0) = \frac{A(z_0)}{B'(z_0)}$$

Improper Integrals of Fourier-type

Consider integrals of the form

$$\int_{-\infty}^{+\infty} f(x)\cos mx\,dx \qquad \text{or} \qquad \int_{-\infty}^{+\infty} f(x)\sin mx\,dx$$

Assume that $f(x) = P(x) / Q(x)$ is a real rational function with

- $Q(x) \neq 0$ for all real x (i.e. no real poles)
- $\text{degree } [Q(x)] \geq \text{degree } [P(x)] + 1$
- $m > 0$

Improper Integrals of Fourier-type

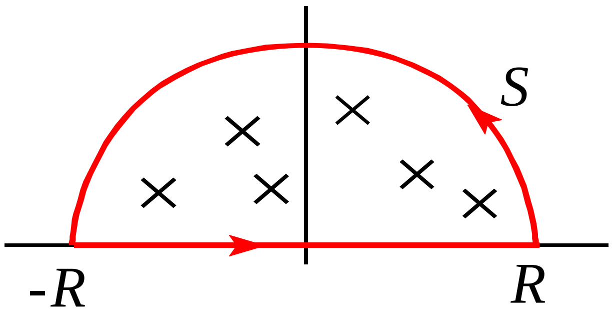
Consider the complex integral

$$\oint_C f(z)e^{imz}\,dz = \int_{-R}^{R} f(z)e^{imz}dz + \int_S f(z)e^{imz}dz = 2\pi i \sum_j \mathrm{Res}(f\,e^{imz}, a_j)$$

We will show (i.e., Theorem X next page) that under the conditions $m > 0$ and degree $[Q(x)] \geq$ degree $[P(x)] + 1$, we have $\oint_S f(z)e^{imz}\,dz = 0$ as $R \to \infty$. Noting that $e^{imz} = \cos mz + i \sin mz$, it can be shown

$$\int_{-\infty}^{\infty} f(x)\cos mx\,dx = \mathrm{Re}\left[2\pi i \sum_j \mathrm{Res}(f\,e^{imz}, a_j)\right]$$

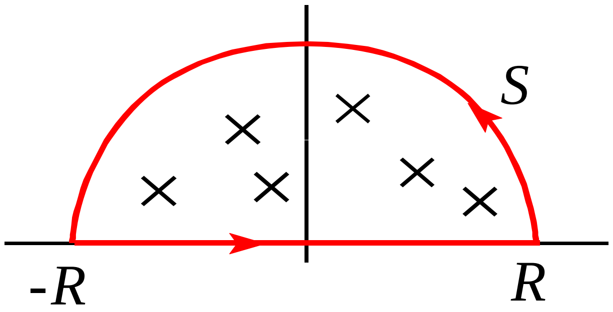
$$\int_{-\infty}^{\infty} f(x)\sin mx\,dx = \mathrm{Im}\left[2\pi i \sum_j \mathrm{Res}(f\,e^{imz}, a_j)\right]$$



Theorem X

If $g(z) = \dfrac{P(z)}{Q(z)}$ with $\text{degree}[Q(x)] \geq \text{degree}[P(x)] + 1$ and $m > 0$, then

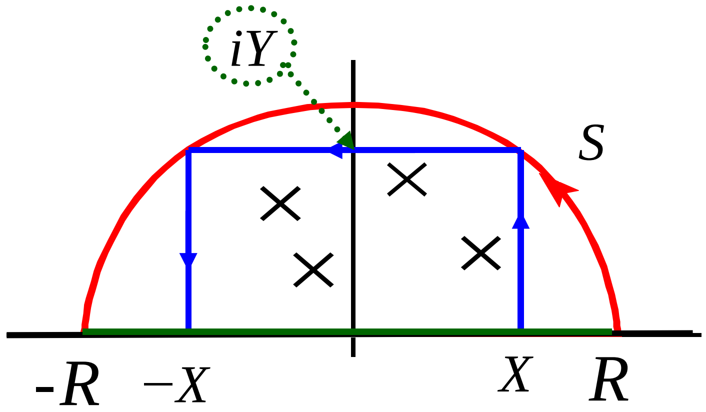
$$\lim_{R \to \infty} \int_S g(z) e^{imz} \, dz = 0$$



There is no name for this theorem. As such, for easy references, we call it Theorem X. The result has been used earlier in deriving improper integrals of Fourier-type. It will be used later few more times.

Proof of Theorem X (self-study)

Observing the curves on the right, if we let $X \to \infty$ ($\Rightarrow Y \to \infty$ and $R \to \infty$), the integral of the function along S is the same as it along the blue straight lines. Under that



$$\text{degree } [Q(x)] \geq \text{degree } [P(x)] + 1$$

and along the straight line from $X + i0$ to $X + iY$, we have

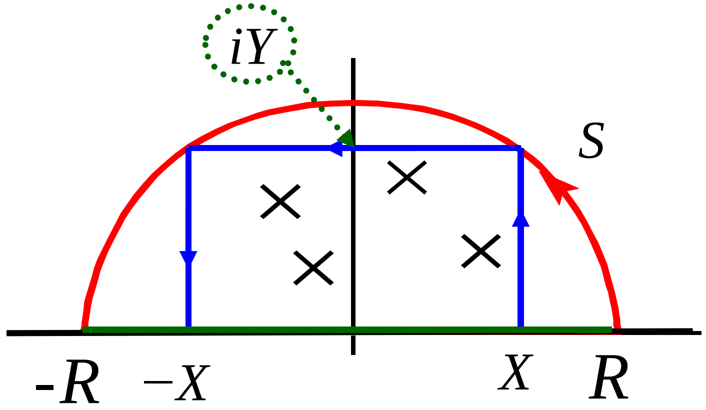
$$|zg(z)| \leq K \quad \Rightarrow \quad |g(z)| \leq \frac{K}{|z|} = \frac{K}{|X+iy|} \leq \frac{K}{X}, \quad |e^{imz}| = \left|e^{im(X+iy)}\right| = |e^{imX}| \cdot |e^{-my}| = e^{-my}$$

and thus

$$\left| \int_{X+i0}^{X+iY} g(z) e^{imz} dz \right| \leq \frac{K}{X} \int_0^Y e^{-my} dy = \frac{K}{X} \cdot \frac{1}{m} \left(1 - e^{-mY}\right) < \frac{K}{mX} \to 0 \text{ as } X \to \infty$$

Proof of Theorem X (cont.) (self-study)

Similarly, we can show the integral along the straight line from $-X+iY$ to $-X+i0$ has a same bound, i.e.,



$$\left|\int_{-X+iY}^{-X+i0} g(z)e^{imz}dz\right| < \frac{K}{mX} \to 0 \text{ as } X\to\infty$$

The integral along the line from $X+iY$ to $-X+iY$, we have

$$|zg(z)|\le K \quad\Rightarrow\quad |g(z)|\le\frac{K}{|z|}=\frac{K}{|x+iY|}\le\frac{K}{Y},\qquad |e^{imz}|=\left|e^{im(x+iY)}\right|=|e^{imx}|\cdot|e^{-mY}|=e^{-mY}$$

and thus

$$\left|\int_{X+iY}^{-X+iY} g(z)e^{imz}dz\right| \le \frac{Ke^{-mY}}{Y}\int_{-X}^{X}dx = 2K\left(\frac{X}{Y}e^{-mY}\right)\to 0 \text{ as } X\to\infty, Y\to\infty$$

QED

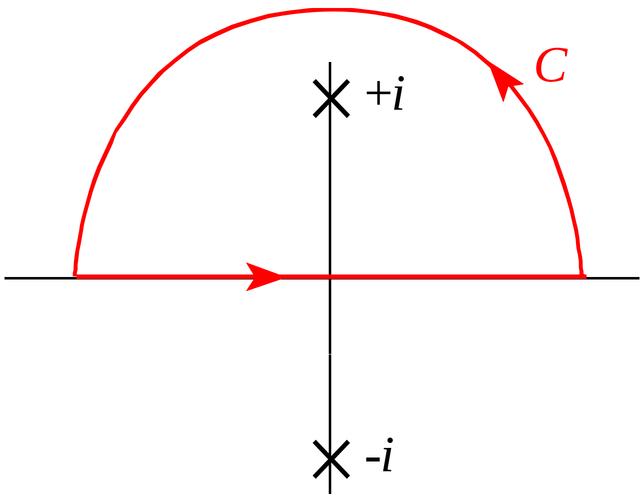
Example 16

(a) $$\int_{-\infty}^{\infty} \frac{x\cos x}{x^2+1}dx = \text{Re}\left[2\pi i \text{ Res}\left(\frac{z e^{iz}}{z^2+1}, i\right)\right]$$

$$= \text{Re}\left[2\pi i \left(\frac{z e^{iz}}{2z}\right)_{z=i}\right]$$

$$= \text{Re}\left[\frac{\pi i}{e}\right]$$

$$= 0$$



$$\text{Res}(f, z_0) = \frac{A(z_0)}{B'(z_0)}$$

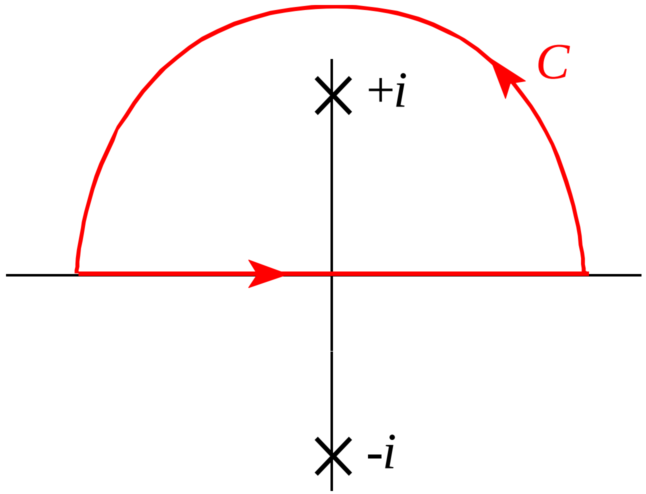
Example 16

(b) $$\int_{-\infty}^{\infty} \frac{x \sin x}{x^2+1} dx = \text{Im}\left[2\pi i \,\text{Res}\left(\frac{z\,e^{iz}}{z^2+1}, i \right) \right]$$

$$= \text{Im}\left[2\pi i \left[\frac{ze^{iz}}{2z} \right]_{z=i} \right]$$

$$= \text{Im}\left[\frac{\pi i}{e} \right]$$

$$= \pi / e$$



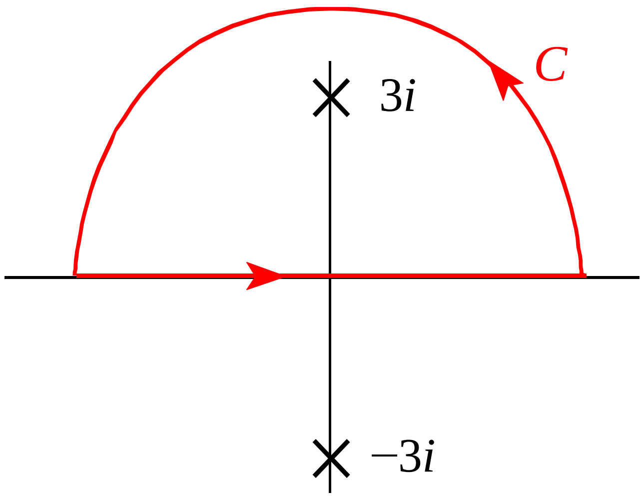
$$\text{Res}(f, z_0) = \frac{A(z_0)}{B'(z_0)}$$

Example 16

(c) $$\int_{-\infty}^{\infty} \frac{\cos 2x}{9+x^2}\,dx = \mathrm{Re}\left[2\pi i \,\mathrm{Res}\left(\frac{e^{i2z}}{9+z^2}, 3i\right)\right]$$

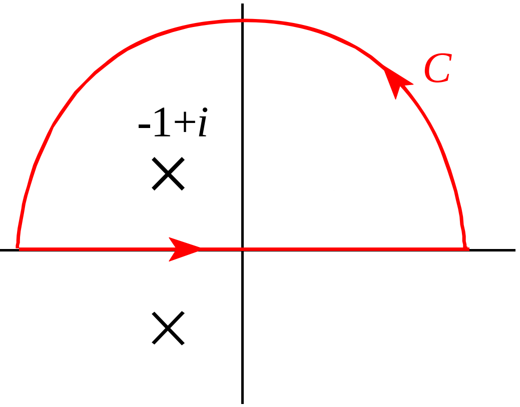
$$= \mathrm{Re}\left[2\pi i \left(\frac{e^{i2z}}{2z}\right)_{z=3i}\right]$$

$$= \mathrm{Re}\left[\frac{\pi e^{-6}}{3}\right] = \frac{\pi e^{-6}}{3}$$



$$\mathrm{Res}(f, z_0) = \frac{A(z_0)}{B'(z_0)}$$

Example 16



(d) $$\int_{-\infty}^{\infty} \frac{\sin x}{x^2+2x+2}\,dx = \mathrm{Im}\left[2\pi i \,\mathrm{Res}\left(\frac{e^{iz}}{z^2+2z+2}, -1+i\right)\right]$$

$$= \mathrm{Im}\left[2\pi i \left(\frac{e^{iz}}{2z+2}\right)_{z=-1+i}\right] = \mathrm{Im}\left[2\pi i \left(\frac{e^{-i}e^{-1}}{2i}\right)\right]$$

$$= \mathrm{Im}\left[\frac{\pi e^{-i}}{e}\right] = \mathrm{Im}\left[\frac{\pi(\cos 1 - i \sin 1)}{e}\right]$$

$$= -\frac{\pi}{e}\sin 1$$

$$\mathrm{Res}(f, z_0) = \frac{A(z_0)}{B'(z_0)}$$

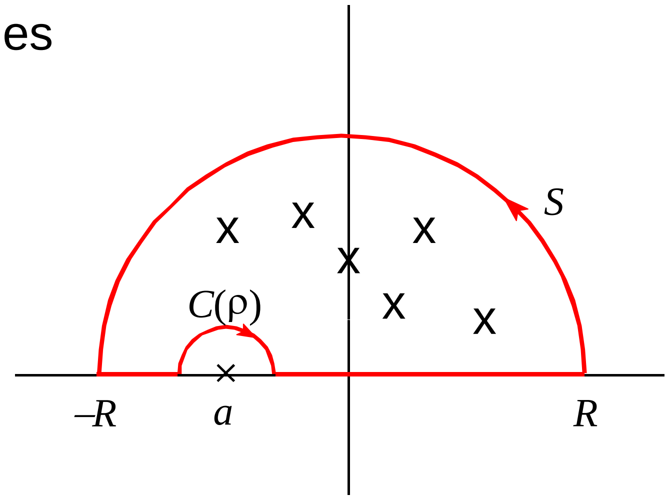
Simple Poles on the Real Axis

Up to now, we have only considered functions which do not have poles on the real axis. We can avoid this pole by sidestepping it with a small half-circle of which the radius tends to zero.

Consider a function $f(z)$ which has a simple pole at the point $z = a$ on the real axis. The Laurent series of this function is then given by

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-a)^n = \frac{a_{-1}}{z-a} + g(z)$$

where the function $g(z)$ is differentiable in the neighbourhood of the point $z = a$.



Simple Poles on the Real Axis

Let $C(\rho)$ be the circle segment with parametric description

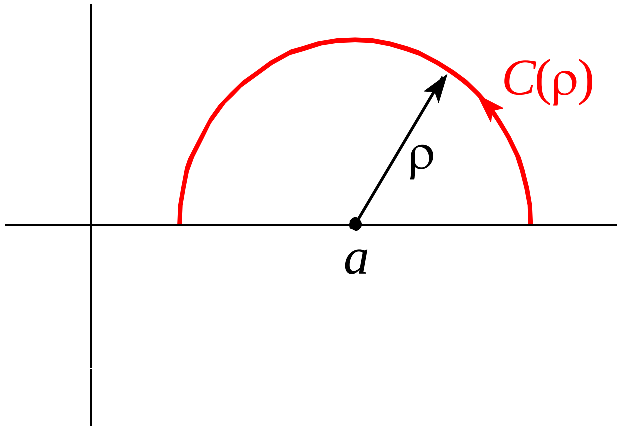
$$C(\rho): \quad z = a + \rho e^{i\theta}, \; 0 \le \theta \le \pi$$

Then

$$\int_{C(\rho)} f(z)\,dz = \int_{C(\rho)} \frac{a_{-1}}{z-a}\,dz + \int_{C(\rho)} g(z)\,dz$$

$$= a_{-1}\int_0^{\pi} \frac{i\rho e^{i\theta}}{\rho e^{i\theta}}\,d\theta + \int_{C(\rho)} g(z)\,dz$$

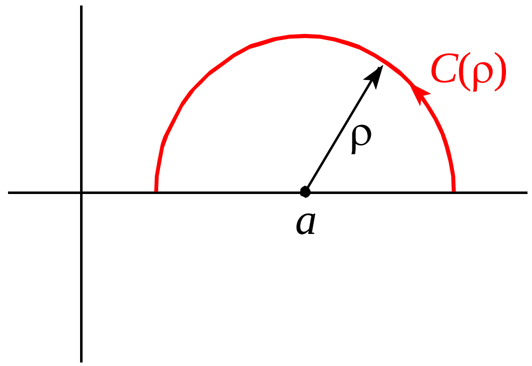
$$= \pi i a_{-1} + \int_{C(\rho)} g(z)\,dz$$



Simple Poles on the Real Axis

Since $g(z)$ is differentiable, it is continuous and bounded, and therefore

$$\left| \int_{C(\rho)} g(z)\, dz \right| \le M L = M(\pi\rho) \xrightarrow[0]{\rho} 0$$



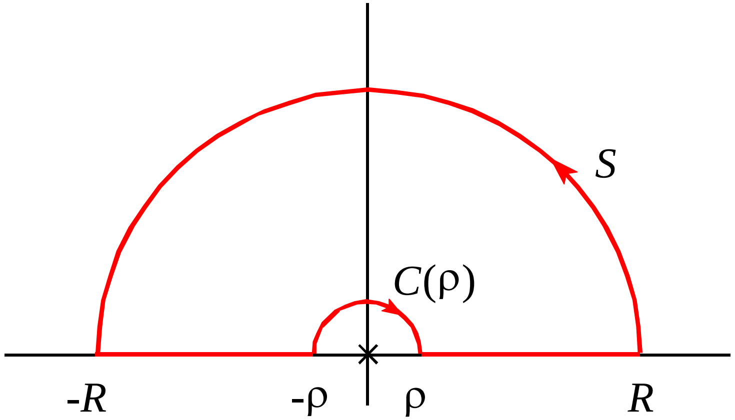
This results in Jordan's lemma, stating that

$$\lim_{\rho \to 0} \int_{C(\rho)} f(z)\, dz = \pi i\, a_{-1} = \pi i \,\mathrm{Res}(f, a)$$

Example 17

(a) Calculate $\int_{-\infty}^{\infty} \frac{\sin x}{x} dx$

Consider $\oint_C \frac{e^{iz}}{z} dz$, with C the path as indicated in the illustration.



Because the function $f(z) = \frac{e^{iz}}{z}$ is analytic in the domain enclosed by C, and according to Cauchy's integral theorem, the integral must be zero, i.e.,

$$\oint_C \frac{e^{iz}}{z} dz = \left[\int_{-R}^{-\rho} + \int_{C(\rho)} + \int_{\rho}^{R} + \int_{S} \right] \frac{e^{iz}}{z} dz = 0$$

Example 17

Let $\rho \to 0$ and $R \to \infty$. Then, by Jordan's lemma

$$\lim_{\rho\to 0} \int_{C(\rho)} f(z)\,dz = -\pi i \,\text{Res}(f,0) = -\pi i \lim_{z\to 0} e^{iz} = -\pi i$$

By Theorem X, we have $\lim_{R\to\infty} \int_S f(z)\,dz = 0$.

$$\oint_C \frac{e^{iz}}{z} dz = \left[\int_{-R}^{-\rho} + \int_{C(\rho)} + \int_{\rho}^{R} + \int_S \right] \frac{e^{iz}}{z} dz = \int_{-\infty}^{0} \frac{e^{iz}}{z} dz + (-i\pi) + \int_{0}^{\infty} \frac{e^{iz}}{z} dz + 0 = 0$$

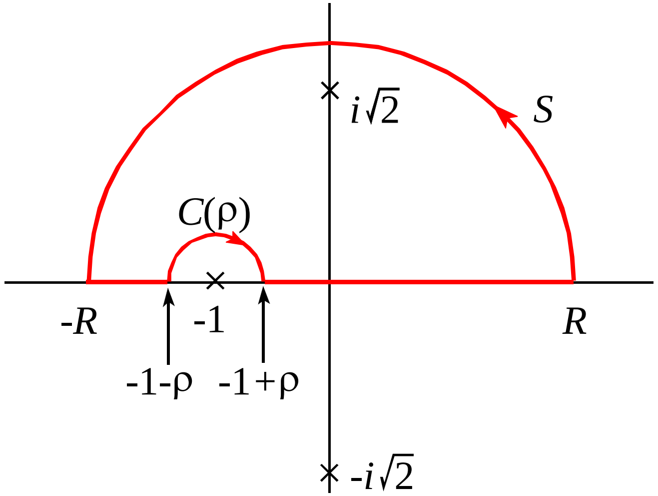
$$\Rightarrow \int_{-\infty}^{\infty} \frac{e^{iz}}{z} dz = i\pi$$

$$\int_{-\infty}^{\infty} \frac{\cos x}{x} dx = 0$$

$$\int_{-\infty}^{\infty} \frac{\sin x}{x} dx = \pi$$

Example 17

(b) Calculate $\displaystyle\int_{-\infty}^{\infty} \frac{1}{(x+1)(x^2+2)}\,dx$



Consider

$$\oint_C \frac{1}{(z+1)(z^2+2)}\,dx = \left[\int_{-R}^{-1-\rho} + \int_{C(\rho)} + \int_{-1+\rho}^{R} + \int_S\right] \frac{1}{(z+1)(z^2+2)}\,dz = 2\pi i\,\mathrm{Res}(f, i\sqrt{2})$$

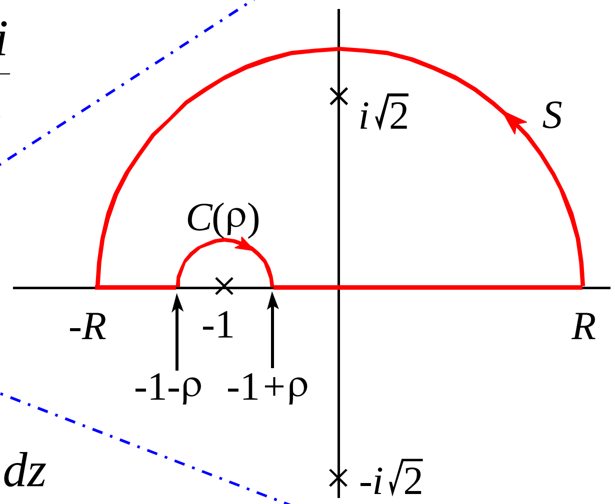
Example 17

Note

$$2\pi i \,\mathrm{Res}(f, i\sqrt{2}) = 2\pi i \lim_{z\to i\sqrt{2}} \frac{1}{(z+1)(z+i\sqrt{2})} = \frac{\pi}{\sqrt{2}}\left(\frac{1-i\sqrt{2}}{3}\right)$$

$$\lim_{\rho\to 0}\int_{C(\rho)} f(z)\,dz = -\pi i \,\mathrm{Res}(f,-1) = -\pi i \lim_{z\to -1}\frac{1}{z^2+2} = -\frac{\pi i}{3}$$

$$\lim_{R\to\infty}\int_S f(z)\,dz = 0$$ (Result S)



and

$$\oint_C \frac{1}{(z+1)(z^2+2)}dx = \left[\int_{-R}^{-1-\rho} + \int_{C(\rho)} + \int_{-1+\rho}^{R} + \int_S\right]\frac{1}{(z+1)(z^2+2)}dz$$

$$= 2\pi i \,\mathrm{Res}(f, i\sqrt{2})$$

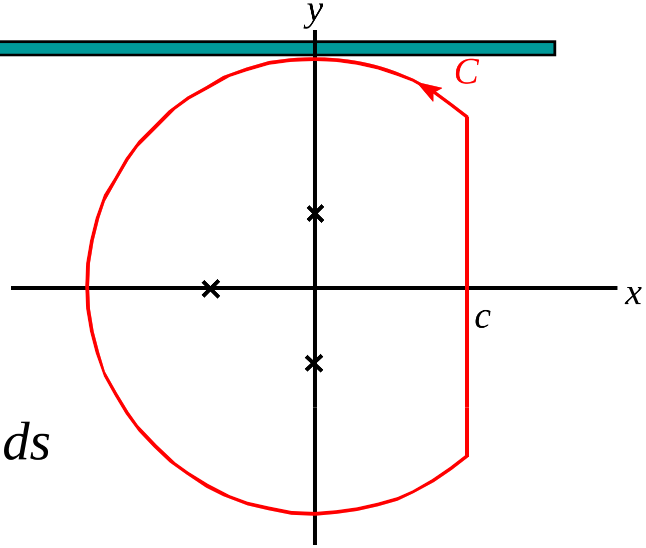
$$\int_{-\infty}^{\infty}\frac{1}{(z+1)(z^2+2)}dz - \frac{\pi i}{3} = \frac{\pi}{\sqrt{2}}\left(\frac{1-i\sqrt{2}}{3}\right) \Rightarrow \int_{-\infty}^{\infty}\frac{1}{(x+1)(x^2+2)}dx = \frac{\pi}{3\sqrt{2}}$$

Calculation of Inverse Laplace Transforms

Laplace transform of a function $f(t)$ is defined as

$$F(s) = L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$$

Inverse transform: $f(t) = L^{-1}[F(s)] = \dfrac{1}{2\pi i} \oint_C e^{st} F(s)\,ds$



Conditions for the existence of an inverse Laplace transform of $F(s)$:

$$\lim_{s\to\infty} F(s) = 0 \quad \text{and} \quad \lim_{s\to\infty} s \cdot F(s) = \text{finite}$$

In terms of the complex variable z: $f(t) = \dfrac{1}{2\pi i} \oint_C e^{zt} F(z)\,dz = \sum_i \mathrm{Res}[F(z)e^{zt}, s_i]$

Example 18

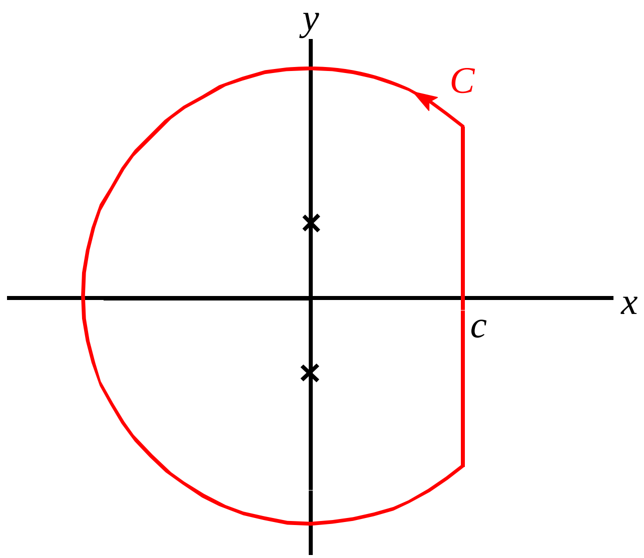
Find the inverse Laplace transform of $F(s)=\dfrac{1}{s^2+\omega^2}$

Solution:

$$f(t)=\operatorname{Res}\left[\frac{e^{zt}}{z^2+\omega^2}, i\omega\right]+\operatorname{Res}\left[\frac{e^{zt}}{z^2+\omega^2}, -i\omega\right]$$

$$=\left[\frac{e^{zt}}{z+i\omega}\right]_{z=i\omega}+\left[\frac{e^{zt}}{z-i\omega}\right]_{z=-i\omega}$$

$$=\frac{e^{i\omega t}-e^{-i\omega t}}{2i\omega}=\frac{\sin\omega t}{\omega}$$

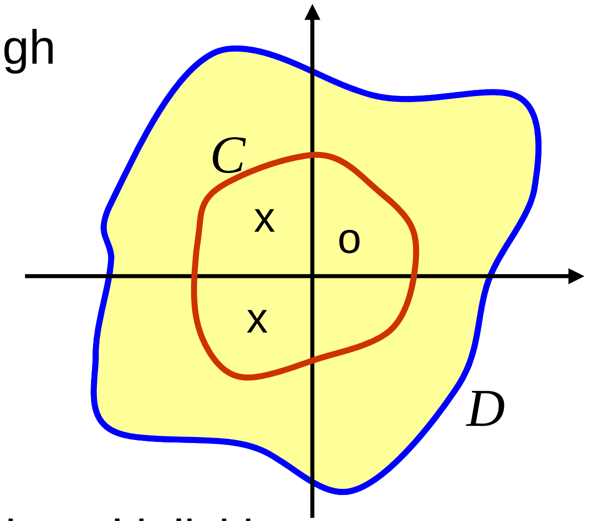


$$\operatorname{Res}(f,z_0)=\lim_{z\to z_0}(z-z_0)f(z)$$

Argument Principle

Argument theorem (the proof can be found in the reference text):

Let $f(z)$ be analytic in a domain D except at a finite number of poles. Let C be a simple closed path in D not passing through any of the zeroes or poles of $f(z)$. Then



$$\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}\,dz = n - p$$

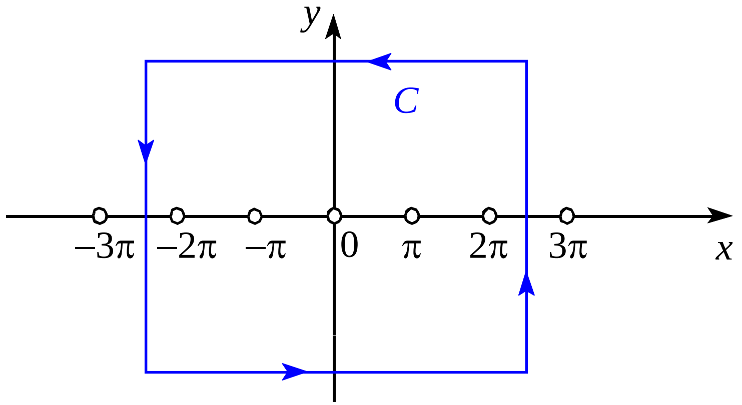
where

n = number of zeroes of $f(z)$ inside C, counting their multiplicities,

p = number of poles of $f(z)$ inside C, counting multiplicities.

Example 19

Calculate $\oint_C \cot z \, dz$



Note that $\cot z = \dfrac{f'(z)}{f(z)}$ with $f(z) = \sin z$

Inside C, $f(z)$ has zeros of order 1 at $-2\pi, -\pi, 0, \pi, 2\pi$ and no poles.

Then, according to the argument theorem,

$$\oint_C \cot z \, dz = \oint_C \frac{f'(z)}{f(z)} dz = 2\pi i (5 - 0) = 10\pi i$$

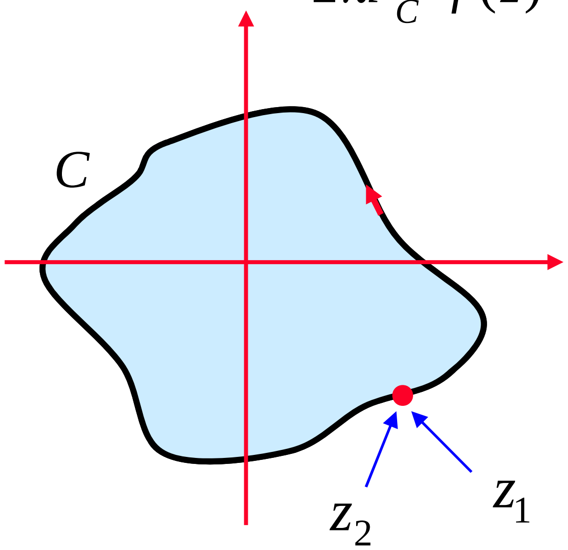
Application of Argument Principle

Let us focus on the case when $f(z)$ is a rational function, which is analytic except at possibly a finite number of points. The argument principle implies

$$\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}dz = \frac{1}{2\pi i}\oint_C d\ln[f(z)] = \frac{1}{2\pi i}\oint_C d\{\ln|f(z)| + i\arg[f(z)]\}$$

$$= \frac{1}{2\pi i}\left[\int_{z_1}^{z_2} d\ln|f(z)| + \int_{z_1}^{z_2} d\{i\arg[f(z)]\}\right]$$

$$= \frac{1}{2\pi i}\left\{0 + i\arg[f(z)]\Big|_{z_1}^{z_2}\right\} = \frac{1}{2\pi}\arg[f(z)]\Big|_{z_1}^{z_2} = n - p$$



Note that the $\ln|f(z)|$ is a real function.

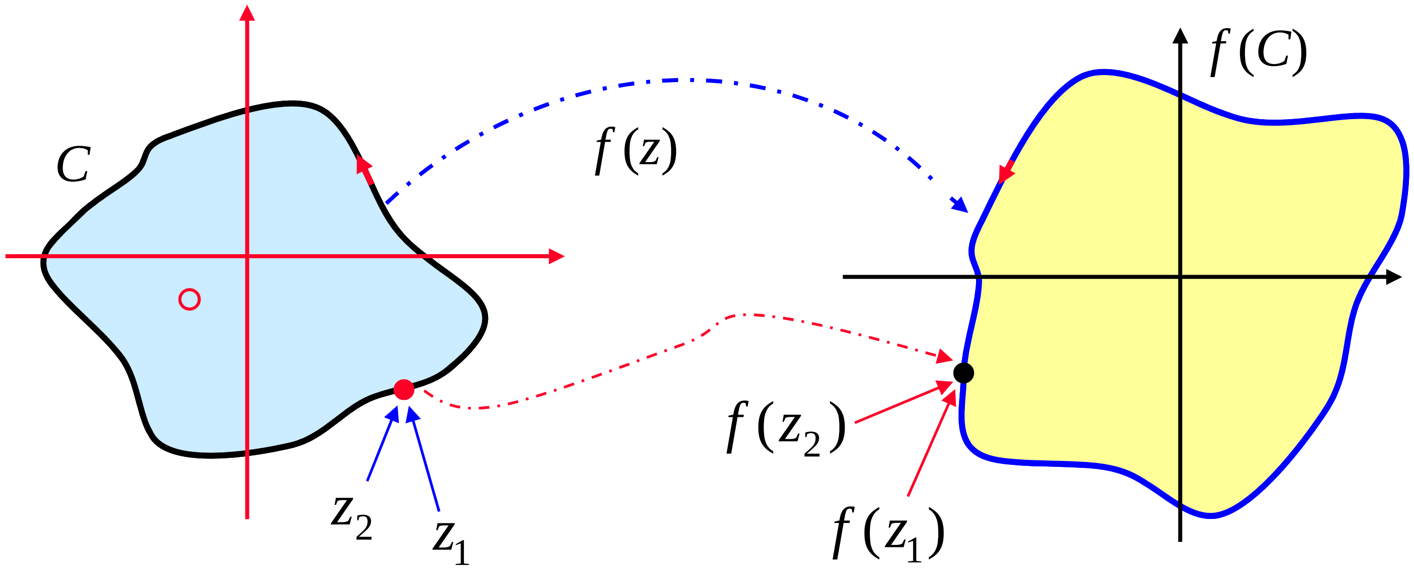
Application of Argument Principle

$$\frac{1}{2\pi}\arg[f(z)]\Big|_{z_1}^{z_2} = n - p$$

Case 1: If there is one zero and no pole of $f(z)$ encircled by C on z-plane, then

$$\arg[f(z)]\Big|_{z_1}^{z_2} = 2\pi(n-p) = 2\pi(1-0) = 2\pi$$

i.e., $f(C)$ will encircle the origin on the image plane once anti-clockwise.



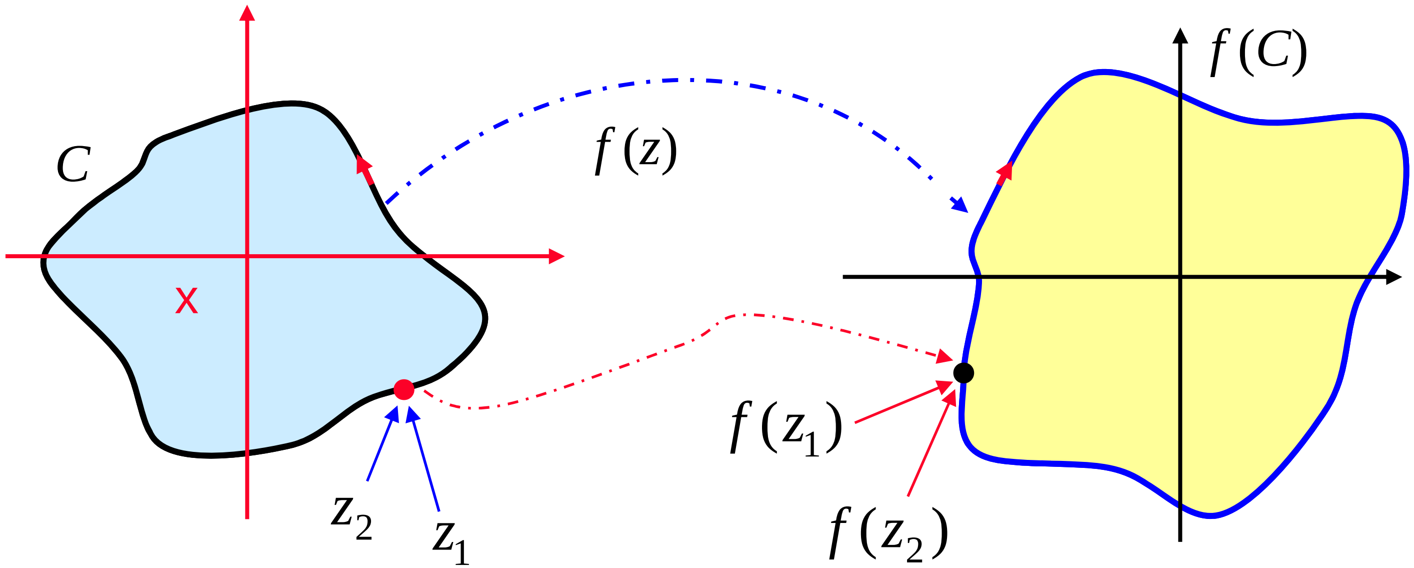
Application of Argument Principle

$$\frac{1}{2\pi}\arg[f(z)]\Big|_{z_1}^{z_2} = n - p$$

Case 2: If there is no zero and one pole of $f(z)$ encircled by C on z-plane, then

$$\arg[f(z)]\Big|_{z_1}^{z_2} = 2\pi(n-p) = 2\pi(0-1) = -2\pi$$

i.e., $f(C)$ will encircle the origin on the image plane once clockwise.



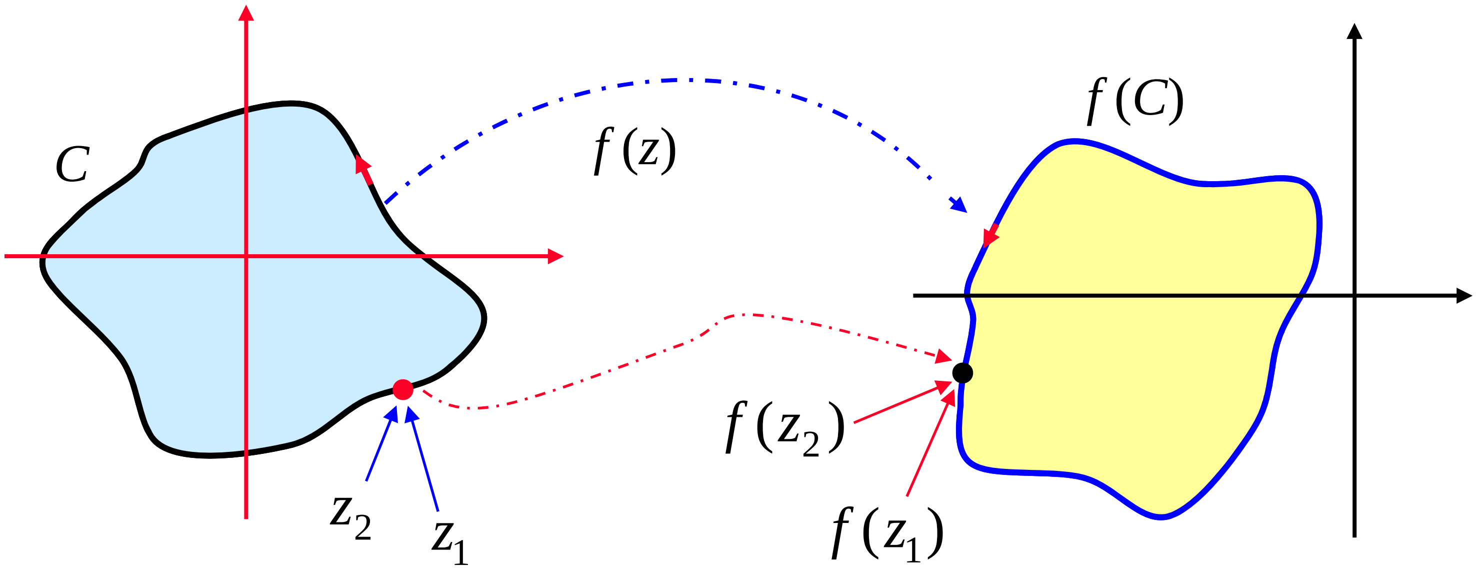
Application of Argument Principle

$$\frac{1}{2\pi}\arg[f(z)]\Big|_{z_1}^{z_2} = n - p$$

Case 3: If there is no zero and no pole (or equal numbers of poles and zeros) of $f(z)$ encircled by C on z-plane, then

$$\arg[f(z)]\Big|_{z_1}^{z_2} = 2\pi(n-p) = 2\pi\cdot 0 = 0$$

i.e., $f(C)$ will not encircle the origin on the image plane at all.

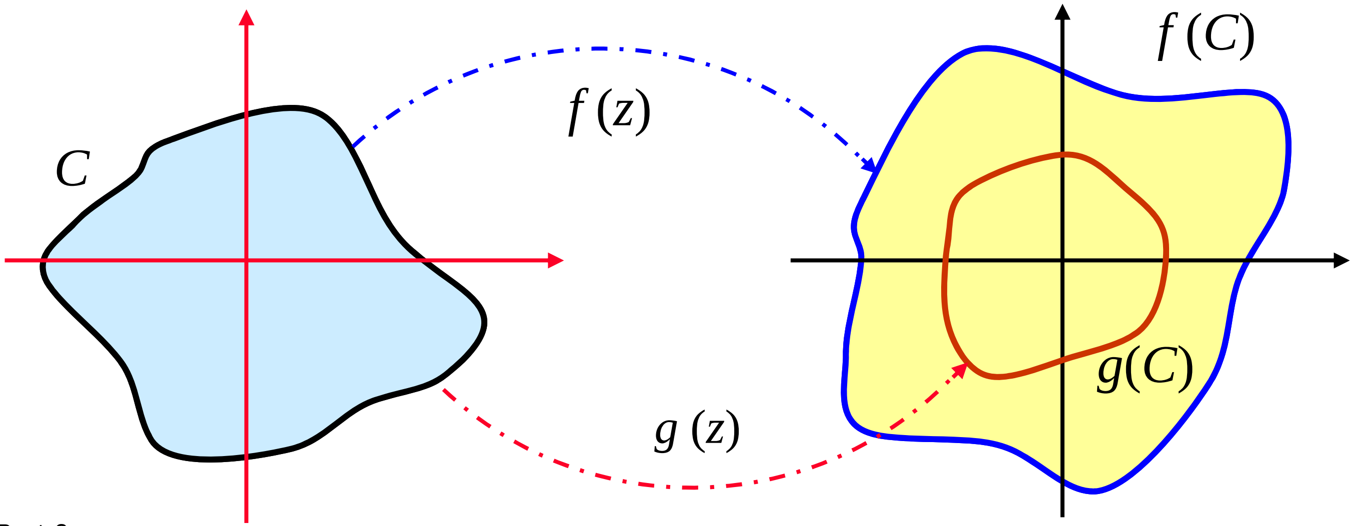


Rouche's Theorem

Let functions $f(z)$ and $g(z)$ be analytic everywhere on and inside C. If

$$|f(z)| > |g(z)|$$

on C, then the total number of zeros of $p(z) = f(z) + g(z)$ inside C is equal to the number of zeros of $f(z)$ inside C.



Proof of Rouche's Theorem (self-study)

Let $t \in [0, 1]$. Since $f(z)$ and $g(z)$ are analytic everywhere on and inside C and since $|f(z)| > |g(z)|$ on C, we have $f(z) + t\,g(z) \neq 0$ for any $z \in C$ (why?). Let

$$\varphi(t) = \frac{1}{2\pi i}\int_C \frac{f'(z) + tg'(z)}{f(z) + tg(z)}\,dz$$

By Argument Principle $\varphi(t)$ = **number of zeros** of $f(z) + t\,g(z)$ inside C (why?), i.e., $\varphi(t)$ is integer-valued. It can also be shown (pretty complicated!) that $\varphi(t)$ is a continuous function of t. Thus, $\varphi(t)$ can only be a constant, which implies that the number of zeros of $f(z) + t\,g(z)$ inside C is constant for all $t \in [0, 1]$. The result of Rouche's theorem follows by letting $t = 0$ and $t = 1$, respectively.

Example 20

Determine the number of roots (zeros) of $p(z) = z^9 - 2z^6 + z^2 - 8z - 2$ that lie within the circle $C: |z| = 1$.

Choose $f(z) = -8z$, $g(z) = z^9 - 2z^6 + z^2 - 2$. Then on the circle $C: |z| = 1$

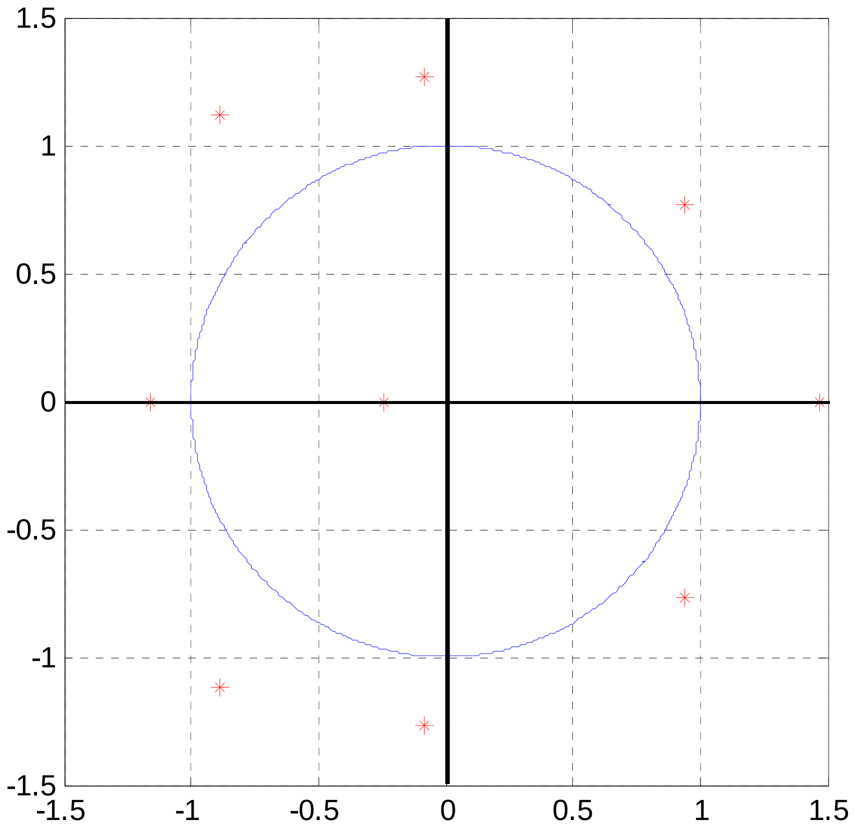
$$|f(z)| = |-8z| = |8| \cdot |z| = 8$$

$$|g(z)| = |z^9 - 2z^6 + z^2 - 2| \le |z^9| + 2|z^6| + |z^2| + 2 = 1 + 2 + 1 + 2 = 6$$

Thus, $|f(z)| = 8 > 6 = |g(z)|$ on C and $f(z)$ and $g(z)$ are analytic on and within C.

As $f(z)$ has only one root inside C, $p(z) = f(z) + g(z)$ also has one root inside C.

Example 20 (cont.)



9 roots of $f(z)$ are marked in *; and Indeed only one of them is inside the unit circle.

Example 21

Show that all the roots (zeros) of $p(z) = z^6 + az^5 + 0.1z^4 - 0.2z^2 - 0.3z + 0.1$, where a is unknown, lie within the unit circle, i.e., $C: |z| = 1$, if $|a| < 0.3$.

Let $f(z) = z^6$, $g(z) = az^5 + 0.1z^4 - 0.2z^2 - 0.3z + 0.1$. Then on the unit circle:

$$|f(z)| = |z^6| = 1$$

$$|g(z)| = |az^5 + 0.1z^4 - 0.2z^2 - 0.3z + 0.1| \le |a| \cdot |z^5| + 0.1|z^4| + 0.2|z^2| + 0.3|z| + 0.1$$

$$= |a| + 0.1 + 0.2 + 0.3 + 0.1 = |a| + 0.7 < 1$$

Thus, $|f(z)| > |g(z)|$ on C and $f(z)$ and $g(z)$ are analytic on and inside C. As $f(z)$ has 6 roots inside C, $p(z) = f(z) + g(z)$ also has its 6 roots inside C.

Example 21 (cont.)