P39. 1. $f(z) = e^{x}(\cos y + i \sin y)$

SOLUTION: $u(x,y) = e^{x}\cos y$ and $v(x,y) = e^{x}\sin y$

$$\frac{\partial u(x,y)}{\partial x} = e^{x}\cos y \qquad \frac{\partial v(x,y)}{\partial x} = e^{x}\sin y$$

2

$$\frac{\partial u(x,y)}{\partial y} = -e^{x}\sin y \qquad \frac{\partial v(x,y)}{\partial y} = e^{x}\cos y$$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = e^{x}\cos y, \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = e^{x}\sin y$$

$f(z)$ satisfies the Cauchy-Riemann equations.

2. $f(z) = \cos x \cosh y - i \sin x \sinh y$

SOLUTION:

$$u(x,y) = \cos x \cdot \cosh y = \cos x \cdot (e^{y} + e^{-y})/2$$

$$v(x,y) = -\sin x \cdot \sinh y = \sin x \cdot (e^{-y} - e^{y})/2$$

$$\frac{\partial u}{\partial x} = -\sin x \cdot (e^{y} + e^{-y})/2$$

2

$$\frac{\partial u}{\partial y} = \cos x \cdot (e^{y} - e^{-y})/2$$

$$\frac{\partial v}{\partial x} = \cos x \cdot (e^{-y} - e^{y})/2 = -\cos \cdot (e^{y} - e^{-y})/2$$

$$\frac{\partial v}{\partial y} = \sin x \, (-e^{-y} - e^{y})/2 = -\sin x \, (e^{-y} + e^{y})/2$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

SO, $F(z)$ satisfies the Cauchy-Riemann equations.

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P39. 3. $f(z) = \sin x \cosh y + i \cos x \cdot \sinh y$

SOLUTION:

$$u(x,y) = \sin x \cdot \cosh y = \sin x \cdot (e^{y}+e^{-y})/2$$

$$V(x,y) = \cos x \cdot \sinh y = \cos x \cdot (e^{y}-e^{-y})/2$$

$$\frac{\partial u}{\partial x} = \cos x \cdot \cosh y$$

$$\frac{\partial u}{\partial y} = \sin x \ (e^{-y}-e^{-y})/2 = \sin x \sinh y$$

V

$$\frac{\partial V}{\partial x} = -\sin x \cdot \sinh y$$

$$\frac{\partial V}{\partial y} = \cos x \cdot (e^{y}+e^{-y})/2 = \cos x \cdot \cosh y$$

$$\frac{\partial u}{\partial x} = \frac{\partial V}{\partial y} \quad \text{and} \quad \frac{\partial V}{\partial x} = -\frac{\partial u}{\partial y}$$

SO, $f(z)$ stastifies the Cauchy-Riemann equations.

4. $f(z) = e^{x^2-y^2}(\cos 2xy + i \sin 2xy)$

SOLUTION:

$$u(x,y) = e^{x^2-y^2}\cos 2xy$$

$$V(x,y) = e^{x^2-y^2}\sin 2xy$$

$$\frac{\partial u}{\partial x} = e^{x^2-y^2}\cdot 2x\cos 2xy - e^{x^2-y^2}\cdot\sin 2xy \cdot 2y$$

V

$$\frac{\partial u}{\partial y} = e^{x^2-y^2}\cdot(-2y)\cos 2xy - e^{x^2-y^2}\cdot\sin 2xy \cdot 2x$$

$$\frac{\partial V}{\partial x} = e^{x^2-y^2}\ 2x\cdot\sin 2xy + e^{x^2-y^2}\cos 2xy \cdot 2y$$

$$\frac{\partial V}{\partial y} = e^{x^2-y^2}(-2y)\sin 2xy + e^{x^2-y^2}\cos 2xy \cdot 2x$$

$$\frac{\partial u}{\partial x} = \frac{\partial V}{\partial y} \quad \text{and} \quad \frac{\partial V}{\partial x} = -\frac{\partial u}{\partial y}$$

SO, $f(z)$ stastifies the Cauchy-Riemann equations.

IN EXERCISES 1 – 7, EXPRESS EACH NUMBER IN THE FORM $x+iy$

2. $e^{(1+\pi i)/2} = e^{\frac{1}{2}} \cdot e^{i\frac{\pi}{2}} = e^{\frac{1}{2}} \cdot (\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}) = 0 + i \cdot e^{\frac{1}{2}}$ Z

4. $e^{(-1+\pi i)/4} = e^{-\frac{1}{4}} \cdot e^{i\frac{\pi}{4}} = e^{-\frac{1}{4}} \cdot (\cos\frac{\pi}{4} + i\cdot\sin\frac{\pi}{4}) = \frac{\sqrt{2}}{2}e^{-\frac{1}{4}} + i\frac{\sqrt{2}}{2}e^{-\frac{1}{4}}$ Z

6. $e^{-i\pi/2} = \cos(-\frac{\pi}{2}) + i\cdot\sin(-\frac{\pi}{2}) = 0 - i$ Z

IN EXERCISES 8 – 10, FIND ALL THE COMPLEX NUMBERS z THAT SATISFY THE GIVEN CONDITIONS.

8. $e^{2z} = -1$, Let $z = x+iy$

$e^{2\cdot z} = e^{2(x+iy)} = e^{2x}\cdot(\cos 2y + i\sin 2y) = -1$

$|e^{2z}| = e^{2x} = |-1| = 1 \quad \Rightarrow \quad x = 0$

$\therefore \quad \cos 2y + i\cdot\sin 2y = -1 + i\cdot 0$

$\Rightarrow \begin{cases} \cos 2y = -1 \\ \sin 2y = 0 \end{cases} \Rightarrow 2y = 2k\pi + \pi, \quad k = 0, \pm1, \pm2, \cdots$

$\therefore \; y = k\pi + \frac{\pi}{2}, \quad k = 0, \pm1, \pm2, \cdots$ Z

$z = (0, \; k\pi + \frac{\pi}{2}) \quad k = 0, \pm1, \pm2, \cdots$

10. $e^{iz} = -1$, Let $z = x+iy$

$e^{iz} = e^{i(x+iy)} = e^{-y+ix} = e^{-y}(\cos x + i\sin x) = -1$

$|e^{iz}| = e^{-y} = 1 \quad \Rightarrow \quad y = 0$ Z

$\cos x + i\sin x = -1 + i\cdot 0$

$\Rightarrow \begin{cases} \cos x = -1 \\ \sin x = 0 \end{cases} \Rightarrow x = (2k+1)\pi, \quad k = 0, \pm1, \pm2, \cdots$

$\therefore \; z = (2k\pi + \pi, \; 0), \quad k = 0, \pm1, \pm2, \cdots$

12. SHOW THAT $\overline{(e^{z})} = e^{\bar{z}}$; LET $z = x+iy$ Z

$\overline{(e^{z})} = \overline{(e^{x+iy})} = \overline{e^{x}\cdot(\cos y + i\sin y)} = e^{x}(\cos y - i\sin y)$

$= e^{x}(\cos(-y) + i\sin(-y)) = e^{x}\cdot e^{-iy} = e^{(x-iy)} = e^{\bar{z}}$.

IN EXERCISES 13-20, CALCULATE EACH NUMBER USING De Moivre's THEOREM.

14. $(-1+i)^{17}$

$$(-1+i) = \sqrt{2}\left(\cos\frac{3\pi}{4} + i\cdot\sin\frac{3\pi}{4}\right)$$

$$(-1+i)^{17} = (\sqrt{2})^{17}\cdot\left(\cos\frac{3\times17}{4}\pi + i\sin\frac{3\times17}{4}\pi\right)$$

$$= 256\sqrt{2}\left(\cos\left(12\pi+\frac{3\pi}{4}\right) + i\sin\left(12\pi+\frac{3\pi}{4}\right)\right)$$

$$= 256\sqrt{2}\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = -256 + i256$$

2

16. $(2+2i)^{12}$

2

$$2+2i = 2\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

$$(2+2i)^{12} = (2\sqrt{2})^{12}\left(\cos\frac{\pi}{4}\times12 + i\sin\frac{\pi}{4}\times12\right) = -2^{12}\cdot2^{6} = -2^{18}$$

18. $(-\sqrt{3}+i)^{13}$

$$(-\sqrt{3}+i) = 2\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$$

2

$$(-\sqrt{3}+i)^{13} = 2^{13}\left(\cos\frac{5\pi}{6}\times13 + i\sin\frac{5\pi}{6}\times13\right)$$

$$= 2^{13}\left(\cos\left(10\pi+\frac{5\pi}{6}\right) + i\sin\left(10\pi+\frac{5\pi}{6}\right)\right)$$

$$= 2^{13}\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right) = 2^{13}\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right) = -2^{12}\sqrt{3} + i2^{12}$$

20. $$\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right)^{19} = \left(\cos\left(-\frac{\pi}{4}\right), \sin\left(-\frac{\pi}{4}\right)\right)^{19}$$

$$= \left(\cos\left(-\frac{\pi}{4}\times19\right), \sin\left(-\frac{\pi}{4}\times19\right)\right)$$

2

$$= \left(\cos\left(-4\pi-\frac{3\pi}{4}\right), \sin\left(-4\pi-\frac{3\pi}{4}\right)\right)$$

$$= \left(\cos\left(\frac{-3\pi}{4}\right), \sin\left(-\frac{3\pi}{4}\right)\right)$$

$$= \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\cdot i$$

IN EXERCISES 1-8, EXPRESS EACH OF THE NUMBERS IN THE FORM $x+iy$.

2. $\cos(-i) = \frac{1}{2}(e^{i(-i)} + e^{-i(-i)}) = \frac{1}{2}(e + e^{-1}) + 0\cdot i$ 2

4. $\sinh \pi i = \frac{1}{2}(e^{\pi i} - e^{-\pi i}) = i\sin\pi = 0$ 2

6. $\tan 2i = \sin 2i / \cos 2i = (e^{i(2i)} - e^{-i(2i)}) / i(e^{i(2i)} + e^{-i(2i)})$ 2

$= (e^{-2} - e^{2}) / i(e^{-2} + e^{2})$

$= 0 - i \cdot \dfrac{1-e^4}{1+e^4}$

8. $\cosh(\pi i/4) = \frac{1}{2}(e^{i\cdot\frac{\pi}{4}} + e^{-i\frac{\pi}{4}}) = \frac{1}{2}(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4} + \cos\frac{\pi}{4} - i\sin\frac{\pi}{4})$ 2

$= \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

IN EXERCISES 9-12, FIND ALL COMPLEX NUMBERS z SUCH THAT THE GIVEN CONDITIONS ARE MET.

10. $\cos z = -i\sin z$, LET $z = x+iy$.

$\cos z = \frac{1}{2}(e^{iz} + e^{-iz}) = \frac{1}{2}(e^{i(x+iy)} + e^{-i(x+iy)})$

$= \frac{1}{2}(e^{(-y+ix)} + e^{(y-ix)})$

$= \frac{1}{2}[e^{-y}(\cos x + i\sin x) + e^{y}(\cos x - i\sin x)]$ 2

$= \frac{1}{2}[(e^{-y}+e^{y})\cos x + i\cdot(e^{-y}-e^{y})\sin x]$

$-i\sin z = -\frac{1}{2}[e^{iz} - e^{-iz}] = -\frac{1}{2}[e^{i(x+iy)} - e^{-i(x+iy)}]$

$= -\frac{1}{2}[e^{(-y+ix)} - e^{(y-ix)}]$

$= -\frac{1}{2}[(e^{-y}-e^{y})\cos x + i(e^{-y}+e^{y})\sin x]$

$\therefore \Rightarrow \begin{cases} (e^{-y}+e^{y})\cos x = (e^{y}-e^{-y})\cos x \\ (e^{-y}-e^{y})\sin x = +(e^{y}+e^{-y})\sin x \end{cases}$

$\Rightarrow \begin{cases} 2e^{-y}\cos x = 0 \\ 2e^{-y}\sin x = 0 \end{cases} \Rightarrow \begin{cases} \cos x = 0 \\ \sin x = 0 \end{cases}$ no exists

12. $\cosh z = i$, LET $z = x+iy$

$$\cosh z = \tfrac{1}{2}(e^{z}+e^{-z}) = \tfrac{1}{2}(e^{(x+iy)}+e^{(-x-iy)})$$

$$= \tfrac{1}{2}\left(e^{x}(\cos y + i\sin y) + e^{-x}(\cos y - i\sin y)\right)$$

$$= \tfrac{1}{2}\left[(e^{x}+e^{-x})\cos y + i(e^{x}-e^{-x})\sin y\right] = i$$

$$\Rightarrow \begin{cases} \tfrac{1}{2}(e^{x}+e^{-x})\cos y = 0 \\ \tfrac{1}{2}(e^{x}-e^{-x})\sin y = 1 \end{cases} \Rightarrow \begin{cases} \cos y = 0 \\ (e^{x}-e^{-x})\sin y = 2 \end{cases}$$

$\cos y = 0 \quad \Rightarrow \quad y = 2k\pi \pm \frac{\pi}{2}$, $k = 0, \pm 1, \pm 2, \cdots$

$\Rightarrow \sin y = \pm 1$

$\therefore \quad e^{x} - e^{-x} = \pm 2$

$(e^{x})^{2} - 1 = \pm 2e^{x}$

still, very good work!

Let $u = e^{x}$

$\therefore \quad u^{2} \pm 2u - 1 = 0$

$u_1 = 0.618 \quad \Rightarrow \quad x_1 = -0.481$

$u_2 = 1.618 \quad \Rightarrow \quad x_2 = 0.481$

$\therefore \quad z_1 = (-0.481,\ 2k\pi + \frac{\pi}{2})$

$z_2 = (0.481,\ 2k\pi + \frac{\pi}{2})$, $k = 0, \pm 1, \pm 2, \cdots$

14. SHOW THAT $\overline{\sin z} = \sin \bar{z}$

According to Homework #8. Problem 12. $(\overline{e^{z}}) = e^{\bar{z}}$,

2

$$\overline{\sin z} = \overline{\tfrac{1}{2i}(e^{iz}-e^{-iz})} = \overline{\tfrac{1}{2i}}\cdot(\overline{e^{iz}} - \overline{e^{-iz}})$$

$$= (\overline{-\tfrac{1}{2}i})\cdot(e^{\bar{i}\cdot\bar{z}} - e^{\overline{-i}\cdot\bar{z}})$$

$$= -\tfrac{1}{2i}\cdot(e^{-i\cdot\bar{z}} - e^{i\bar{z}}) = \tfrac{1}{2i}(e^{i\bar{z}} - e^{-i\bar{z}})$$

$$= \sin \bar{z}.$$

IN EXERCISES 16 − 21, PROVE THE IDENTITIES.

16. $\sin(z_1 \pm z_2) = \sin z_1 \cos z_2 \pm \cos z_1 \cdot \sin z_2$

$$\sin z_1 \cos z_2 \pm \cos z_1 \cdot \sin z_2 = \frac{1}{2i}(e^{iz_1} - e^{-iz_1}) \cdot \frac{1}{2}(e^{iz_2} + e^{-iz_2})$$

$$\pm \frac{1}{2}(e^{iz_1} + e^{-iz_1}) \cdot \frac{1}{2i}(e^{iz_2} - e^{-iz_2})$$

✓

$$= \frac{1}{2i} \cdot \Big[\frac{1}{2}\left(e^{i(z_1+z_2)} + e^{i(z_1-z_2)} - e^{-i(z_1-z_2)} - e^{-i(z_1+z_2)}\right)$$

$$\pm \frac{1}{2}\left(e^{i(z_1+z_2)} - e^{i(z_1-z_2)} + e^{-i(z_1-z_2)} - e^{-i(z_1+z_2)}\right)\Big]$$

19/20

$$\therefore \sin z_1 \cos z_2 + \cos z_1 \sin z_2 = \frac{1}{2i}\left[e^{i(z_1+z_2)} - e^{-i(z_1+z_2)}\right] = \sin(z_1+z_2)$$

$$\sin z_1 \cos z_2 - \cos z_2 \sin z_1 = \frac{1}{2i}\left[e^{i(z_1-z_2)} - e^{-i(z_1-z_2)}\right] = \sin(z_1 - z_2)$$

SO, $\sin(z_1 \pm z_2) = \sin z_1 \cos z_2 \pm \cos z_1 \sin z_2$.

18. $\sin(-z) = -\sin z$, $\cos(-z) = \cos z$

$$\sin(-z) = \frac{1}{2i}\left(e^{i(-z)} - e^{-i(-z)}\right) = \frac{1}{2i}\left(e^{-iz} - e^{iz}\right)$$

✓

$$= -\frac{1}{2i}\left(e^{iz} - e^{-iz}\right) = -\sin z$$

$$\cos(-z) = \frac{1}{2}\left(e^{i(-z)} + e^{-i(-z)}\right) = \frac{1}{2}\left(e^{iz} + e^{-iz}\right)$$

✓

$$= \cos z.$$

Homework # 10 | Complex Var. | Benmei Chen

IN EXERCISES 1-6, FIND ALL THE VALUES OF THE GIVEN EXPRESSIONS.

2. $\log(1+i) = \log|1+i| + i\cdot\arg(1+i)$ 2

$= \log\sqrt{2} + i\cdot(\frac{\pi}{4} + 2k\pi)$, $k=0,\pm1,\pm2,\cdots$ 2

4. $1^i = e^{i\log 1} = e^{i\cdot(\log 1 + i\cdot\arg(1))} = e^{i\cdot i\cdot(2k\pi)} = e^{-2k\pi}$, $k=0,\pm1,\pm2,\cdots$

6. $(1+i)^{1+i} = e^{(1+i)\log(1+i)} = e^{(1+i)(\log\sqrt{2} + i(\frac{\pi}{4}+2k\pi))}$ $k=0,\pm1,\pm2,\cdots$ 2

$= e^{\log\sqrt{2} - (\frac{\pi}{4}+2k\pi)}\cdot\cos(\log\sqrt{2} + \frac{\pi}{4}) + i\cdot e^{\log\sqrt{2} - (\frac{\pi}{4}+2k\pi)}\cdot\sin(\log\sqrt{2} + \frac{\pi}{4})$

IN EXERCISES 7-10, FIND THE PRINCIPAL VALUES OF THE GIVEN EXPRESSIONS.

8. $\log(1+i)$ 2

$\text{Log}(1+i) = \log|1+i| + i\,\text{Arg}(1+i) = \log\sqrt{2} + i\frac{\pi}{4}$

10. $(1+i)^{1+i} = e^{(1+i)\text{Log}(1+i)}$ 2

$= e^{(1+i)(\log\sqrt{2} + i\cdot\frac{\pi}{4})}$

$= e^{\log\sqrt{2} - \frac{\pi}{4}}\cdot\cos(\log\sqrt{2} + \frac{\pi}{4}) + i\cdot e^{\log\sqrt{2} - \frac{\pi}{4}}\cdot\sin(\log\sqrt{2} + \frac{\pi}{4})$

18 SHOW THAT $\text{Log}(i^3) \neq 3\,\text{Log}\, i$ $-\pi/2$ ~~2~~ 1

SHOW: $\text{Log}(i^3) = \text{Log}(-i) = \log|-i| + i(\frac{\pi}{2}) = i(\frac{\pi}{2})$

$3\cdot\text{Log}\, i = 3\cdot(\log|i| + i\cdot\frac{\pi}{2}) = i\cdot\frac{3\pi}{2}$ Why?

$\therefore \text{Log}(i^3) \neq 3\,\text{Log}\, i$ $-\frac{\pi}{2}$ + $\frac{3\pi}{2}$ are not on the same branch

20. Is 1 raised to any power always equal to 1?

NO. 2

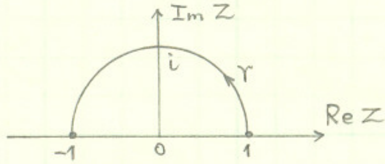
SEE PROBLEM # 4, $1^i = e^{-2k\pi}$, $k=0,\pm1,\pm2,\ldots$

WHEN $k=-2$, $e^{-2k\pi} = e^{4\pi} = 286751.3148$.

MAT 417   H.W. # 12   Complex Var.   Benmei Chen

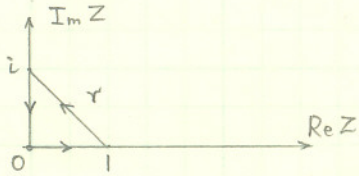
IN EXERCISES 2–5, DETERMINE PWS PARAMETRIZATIONS FOR THE INDICATED ARCS OR CURVES.

2. SEMICIRCLE FROM 1 to −1   2



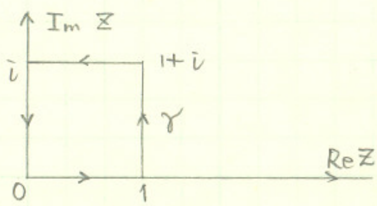
$$\gamma:\ z(t) = \cos t + i\sin t \qquad 0 \le t \le \pi$$

3. TRIANGLE   2



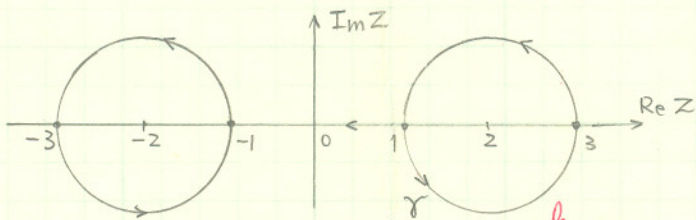
$$\gamma:\ z(t) = \begin{cases} i\cdot(1-t) & 0 \le t \le 1 \\ t-1 & 1 \le t \le 2 \\ (3-t)+i(t-2) & 2 \le t \le 3 \end{cases}$$

4. SQUARE   2



$$\gamma:\ z(t) = \begin{cases} i(1-t) & 0 \le t \le 1 \\ t-1 & 1 \le t \le 2 \\ 1+i(t-2) & 2 \le t \le 3 \\ 4-t+i & 3 \le t \le 4 \end{cases}$$

5. BARBELL BEGINNING AT 1   2

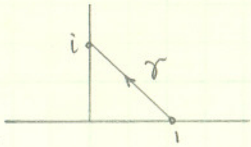


ok

$$\gamma:\ z(t) = \begin{cases} (\cos \pi t + 2) + i\sin \pi t & -1 \le t \le 1, \\ 2-t & 1 \le t \le 3 \\ \cos\pi(t-3)-2+i\sin\pi(t-3) & 3 \le t \le 5 \end{cases}$$

$\cancel{\cos\pi(t+1) - 2 + i\sin\pi(t+1)}$

10. EVALUATE $\int_\gamma y\,dz$, WHERE γ IS THE STRAIGHT LINE JOINING 1 TO i



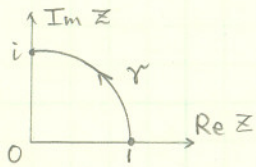
$$\gamma: z(t) = (1-t) + i\cdot t \qquad 0 \le t \le 1$$

$$dz(t) = -1 + i$$

2

$$\int_\gamma y\,dz = \int_0^1 t\cdot(-1+i)\,dt = -\tfrac{1}{2}t^2\Big|_0^1 + i\cdot\tfrac{1}{2}t^2\Big|_0^1 = -\tfrac{1}{2} + i\cdot\tfrac{1}{2}$$

11. EVALUATE $\int_\gamma y\,dz$, WHERE γ IS THE ARC IN THE FIRST QUADRANT ALONG $|z|=1$ JOINING 1 TO i



$$\gamma: \quad z(t) = \cos t + i\sin t \qquad 0 \le t \le \tfrac{\pi}{2}$$

$$dz(t) = -\sin t + i\cos t$$

$$\int_\gamma y\,dz = \int_0^{\frac{\pi}{2}} \sin t\,(-\sin t + i\cos t)\,dt$$

2

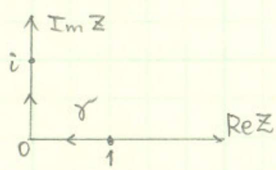
$$= \int_0^{\frac{\pi}{2}} -\sin^2 t\,dt + i\cdot\int_0^{\frac{\pi}{2}} \sin t\cdot\cos t\,dt$$

$$= -\tfrac{1}{2}\cdot\int_0^{\frac{\pi}{2}} (1-\cos 2t)\,dt + i\cdot\tfrac{1}{2}\int_0^{\frac{\pi}{2}} \sin 2t\,dt$$

$$= -\tfrac{1}{2}\left(t - \tfrac{1}{2}\sin 2t\right)\Big|_0^{\frac{\pi}{2}} + \tfrac{1}{2}i\left(-\tfrac{1}{2}\cos 2t\right)\Big|_0^{\frac{\pi}{2}}$$

$$= -\tfrac{\pi}{4} + i\cdot\tfrac{1}{2}$$

12. EVALUATE $\int_\gamma y\,dz$, WHERE γ IS THE ARC ALONG THE COORDINATE AXES JOINING 1 TO i.



$$\gamma: \quad z(t) = \begin{cases} 1-t & 0 \le t \le 1 \\ i(t-1) & 1 \le t \le 2 \end{cases}$$

$$dz(t) = \begin{cases} -1 & 0 \le t \le 1 \\ i & 1 \le t \le 2 \end{cases}$$

2

$$\int_\gamma y\,dz = \int_{\gamma'} y\,dz + \int_{\gamma''} y\,dz$$

$$= \int_0^1 0\cdot(-1)\,dt + \int_1^2 (t-1)\cdot i\,dt$$

$$= i\int_1^2 (t-1)\,dt$$

$$= i\left(\tfrac{t^2}{2} - t\right)\Big|_1^2 = i\cdot\tfrac{1}{2}$$

14. Evaluate the integral $\int (z-a)^n \, dz$, n: an integer, around the circle $|z-a| = R$.

$$F(z) = \frac{1}{n+1}(z-a)^{n+1} \qquad n \neq -1$$

$$F'(z) = (z-a)^n$$

$F(z)$ is an analytic function with a continuous derivative $F'(z)$.

$$\int_{|z-a|=R} (z-a)^n \, dz = 0$$

2

when $n = -1$,

$$F(z) = \log(z-a) = \log|z-a| + i \arg(z-a)$$

1

$$\therefore \int_{|z-a|=R} (z-a)^{-1} \, dz = i \cdot 2k\pi \qquad (k = 0, \pm 1, \pm 2, \ldots)$$

$$\int_{|z-a|=R} (z-a)^{-1} \, dz = 2\pi i$$

can't use log() as the antiderivative $F(z) = \log()$ is not defined everywhere on the arc $|z-a|=R$ (not defined at the branch cut)

15. Evaluate $\int_\gamma e^z \, dz$, where γ is the straight-line path joining 1 to i

Because $(e^z)' = e^z$,

$$\int_\gamma e^z \, dz = -e^1 + e^i = -e + (\cos 1 + i \sin 1)$$

2

$$= (\cos 1 - e) + i(\sin 1)$$

MAT 417 H.W. #13 Complex Variable Benmei Chen

P86. USE GREEN'S THEOREM FOR Exercises 2-4, WHERE A EQUALS THE AREA OF G AND ∂G IS THE BOUNDARY OF G.

2. SHOW THAT $\int_{\partial G} x\,dz = iA$

$f(z) = x$; $u(x,y) = x$, $v(x,y) = 0$, $\frac{\partial u}{\partial x} = 1$, $\frac{\partial u}{\partial y} = 0$, $\frac{\partial v}{\partial x} = 0$, $\frac{\partial v}{\partial y} = 0$

ALL ARE CONTINUOUS ON THE COMPLEX PLANE.

2

$$\int_{\partial G} x\,dz = -\iint_G \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)dx\,dy + i\iint_G \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)dx\,dy$$

$$= i\iint_G dx\,dy = iA$$

3. SHOW THAT $\int_{\partial G} y\,dz = -A$

$u(x,y) = y$, $v(x,y) = 0$, $\frac{\partial u}{\partial x} = 0$, $\frac{\partial u}{\partial y} = 1$, $\frac{\partial v}{\partial x} = 0$, $\frac{\partial v}{\partial y} = 0$

ALL ARE CONTINUOUS ON THE COMPLEX PLANE.

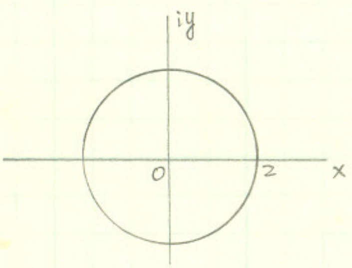
2

$$\int_{\partial G} y\,dz = -\iint_G \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)dx\,dy + i\iint_G \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)dx\,dy$$

$$= -\iint_G (0+1)\,dx\,dy = -A$$

P96. 15. Without computing the integral, show that

$$\left|\int_{|z|=2} \frac{dz}{z^2+1}\right| \leq \frac{4\pi}{3}$$



$$|z^2+1| \leq |z^2|+1 \leq |z|^2+1 \leq 5$$

2

$$\therefore \left|\frac{1}{z^2+1}\right| \geq \frac{1}{5}$$

$$|dz| = 2\pi$$

$$\left|\int_{|z|=2} dz/(z^2+1)\right| \leq \int_{|z|=2} \left|\frac{1}{z^2+1}\right| \cdot |dz|$$

$$\leq \frac{1}{5} \cdot 2\pi = \frac{2\pi}{5} < \frac{4\pi}{3}$$

IN EXERCISES 1–3, EVALUATE THE INTEGRAL

$$\int_\gamma \frac{dz}{(z-a)(z-b)}$$

BY DECOMPOSING THE INTEGRAND INTO PARTIAL FRACTIONS.

$$\int_\gamma \frac{dz}{(z-a)(z-b)} = \frac{1}{a-b}\int_\gamma \frac{dz}{z-a} - \frac{1}{a-b}\int_\gamma \frac{dz}{z-b}$$

1. IF a AND b lie inside γ.

$$\int_\gamma \frac{dz}{(z-a)(z-b)} = \frac{1}{a-b}\cdot 2\pi i - \frac{1}{a-b}\cdot 2\pi i = 0$$

2

2. IF a lies inside and b outside γ.

$$\int_\gamma \frac{dz}{(z-a)(z-b)} = \frac{1}{a-b}\cdot 2\pi i$$

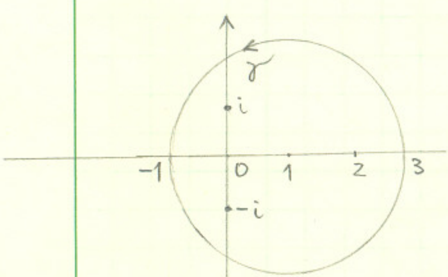
2

3. IF b lies inside and a outside γ.

$$\int_\gamma \frac{dz}{(z-a)(z-b)} = 0 - \frac{2\pi i}{a-b} = \frac{2\pi i}{b-a}$$

2

LET γ: $z(t) = 2e^{it}+1$, $0 \le t \le 2\pi$, EVALUATE THE INTEGRALS IN EX. 4–7.



4. $\int_\gamma \frac{e^z}{z}\,dz = 2\pi i\cdot e^0 = 2\pi i$ 2

5. $\int_\gamma \frac{\cos z}{z-1}\,dz = \cos 1\cdot 2\pi i = 2\pi i\cdot\cos 1$ 2

6. $\int_\gamma \frac{\sin z}{z^2+1}\,dz = \int_\gamma \frac{\sin z}{(z+i)(z-i)}\,dz$

oh Ok sorry!

$$= \frac{1}{2i}\left[\int_\gamma \frac{\sin z}{z-i}\,dz - \int_\gamma \frac{\sin z}{z+i}\,dz\right]$$

2

$-\frac{1}{2i}$ or $\frac{i}{2}$

$$= \frac{2\pi i}{2i}[\sin i - \sin(-i)] = 2\pi i\cdot\frac{e^{-1}-e^{1}}{i\cdot 2i} = \pi i(e-e^{-1})$$

$= 2\pi i \sinh 1$

$\cancel{\frac{2\pi(i)(i)}{2}} = -\pi$

7. $\int_\gamma \frac{\sin z}{z^2-z}\,dz = \int_\gamma \frac{\sin z}{z(z-1)}\,dz$

$$= \int_\gamma \frac{\sin z}{z-1}\,dz - \int_\gamma \frac{\sin z}{z}\,dz$$

$$= 2\pi i\cdot\sin 1 - 2\pi i\cdot\sin 0$$

2

$$= 2\pi i\cdot\sin 1$$

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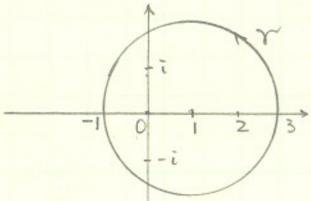
Let $\gamma: z(t) = 2e^{it}+1, \ 0 \le t \le 2\pi$. Evaluate the integrals in Exercises 8 – 11

8. $\displaystyle\int_\gamma \frac{e^z}{z^2}\,dz = \frac{2\pi i}{1!}(e^z)'\Big|_{z=0} = 2\pi i$ 2



9. $\displaystyle\int_\gamma \frac{\cos z}{(z-1)^2}\,dz = \frac{2\pi i}{1!}(\cos z)'\Big|_{z=1} = 2\pi i\cdot(-\sin z)\Big|_{z=1} = -2\pi \sin 1\cdot i$ 2

* 10. $\displaystyle\int_\gamma \frac{\sin z}{(z^2+1)^2}\,dz = \frac{1}{2}i\cdot\left[i\int_\gamma \frac{\sin z\,dz}{(z+i)^2} + i\int_\gamma \frac{\sin z}{(z-i)^2}dz + \int_\gamma \frac{\sin z}{z+i}dz - \int_\gamma \frac{\sin z}{z-i}dz\right.$ 1

$\displaystyle = \frac{1}{2}i\left[i\cdot 2\pi i\cdot\cos(-i) + i\cdot 2\pi i\cos i + 2\pi i\cdot\sin(-i) - 2\pi i\sin i\right] = \cancel{2}\pi i(\sinh 1 - \cosh 1)$

$\frac{1}{4}$

11. $\displaystyle\int_\gamma \frac{\sin z}{(z-1)^3}\,dz = \frac{2\pi i}{2!}(\sin z)''\Big|_{z=1} = \pi i\cdot(-\sin z)\Big|_{z=1} = -\pi i\cdot\sin 1$ 2

18. Let $f(z)$ be analytic and bounded by M in $|z| \le R$. Prove that

$$|f^{(n)}(z)| \le \frac{MRn!}{(R-|z|)^{n+1}}, \qquad |z| < R$$

PROOF: By Cauchy's theorem for derivatives:

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_\gamma \frac{f(w)}{(w-z)^{n+1}}dw$$

2

$$|f^{(n)}(z)| = \frac{n!}{2\pi}\left|\int_\gamma \frac{f(w)}{(w-z)^{n+1}}dw\right|$$

$$\le \frac{n!}{2\pi}\int_\gamma \frac{|f(w)|}{|(w-z)^{n+1}|}|dw|$$

$$= \frac{n!}{2\pi}\int_\gamma \frac{|f(w)|}{|w-z|^{n+1}}|dw|$$

$$|w-z| \ge \big||w|-|z|\big|$$

$$\therefore \frac{1}{|w-z|} \le \frac{1}{\big||w|-|z|\big|}, \qquad \frac{1}{|w-z|^{n+1}} \le \frac{1}{\big||w|-|z|\big|^{n+1}}$$

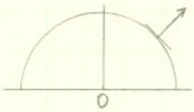
$$\therefore |f^{n}(z)| \le \frac{n!}{2\pi}\cdot\frac{M\cdot 2\pi R}{(R-|z|)^{n+1}} = \frac{MRn!}{(R-|z|)^{n+1}}$$

$$*.\ \frac{i}{(z+i)^2} + \frac{i}{(z-i)^2} + \frac{1}{z+i} - \frac{1}{z-i} = \frac{i(z^2-2iz+1+z^2+2iz-1)-2i(z^2+1)}{(z^2+1)^2}$$

$$= \frac{2iz^2-2iz^2-2i}{(z^2+1)^2} = \frac{-2i}{(z^2+1)^2}$$

PROBLEM SET #6

(2) $C(t) = (\cos t, \sin t)$ $\quad 0 \le t \le \pi$ $\quad F = (x, y)$ $\quad$ Compute $\int_C F \cdot N ds$



$$N(t) = \left(\frac{dy}{ds}, -\frac{dx}{ds}\right)$$

$$\int_C F \cdot N ds = \int_C (x, y)\left(\frac{dy}{ds}, -\frac{dx}{ds}\right) ds$$

$$= \int_C (x, y)(dy, -dx)$$

$$= \int_C -y dx + x dy \qquad 2$$

$$= \int_0^{\pi} -\sin t \cdot (-\sin t) dt + \cos t \cdot \cos t \, dt$$

$$= \int_0^{\pi} dt = \pi$$

(3) $C(t) = (\cos t, \sin t)$ $\quad 0 \le t \le \pi$ $\quad F = (-y, x)$ $\quad$ Compute $\int_C F \cdot N ds$

$$\int_C F \cdot N ds = \int_C (-y, x)(dy, -dx)$$

$$= \int_C -x \, dx - y \, dy$$

$$= -\frac{1}{2} \int_C dx^2 + dy^2 \qquad 2$$

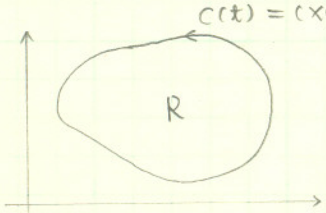
$$= -\frac{1}{2} \int_C d(x^2 + y^2)$$

$$= -\frac{1}{2} \int_C d(1) = 0$$

PROBLEM SET #7

(1) $\quad$ LET $F(x, y) = (P(x, y), Q(x, y))$

$C(t) = (x(t), y(t))$



$$N(t) = \left(\frac{dy}{ds}, -\frac{dx}{ds}\right)$$

$$\int_C F \cdot N ds = \int_C (P, Q) \cdot (dy, -dx) \qquad 2$$

$$= \int_C P dy - Q dx$$

$$= \iint_R \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) dx dy \qquad \text{(Green's Thm)}$$

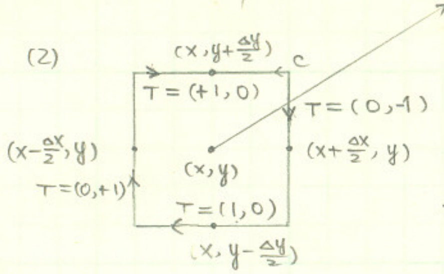
$$= \iint_R \operatorname{div} F \, dx dy$$

PROBLEM SET #7

$F = (P(x,y), Q(x,y))$

2

(2)



$$\int_C F\cdot T\,ds = -P\left(x, y-\frac{\Delta y}{2}\right)\Delta x - Q\left(x+\frac{\Delta x}{2}, y\right)\Delta y + P\left(x, y+\frac{\Delta y}{2}\right)\Delta x + Q\left(x-\frac{\Delta x}{2}, y\right)\Delta y$$

$$\text{Curl } F \triangleq \lim_{\substack{\Delta x\to 0\\ \Delta y \to 0}} \frac{1}{\Delta x\cdot\Delta y}\int_C F\cdot T\,ds$$

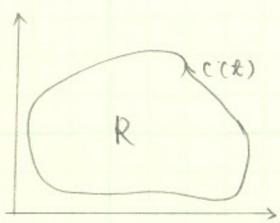
$$= \lim_{\substack{\Delta y\to 0\\ \Delta x \to 0}} \frac{-\left[P(x, y-\frac{\Delta y}{2}) - P(x, y+\Delta y/2)\right]\Delta x + \left[Q(x-\frac{\Delta x}{2}, y) - Q(x+\frac{\Delta x}{2}, y)\right]\Delta y}{\Delta x\,\Delta y}$$

$$= \lim_{\Delta y\to 0} -\frac{P(x, y-\frac{\Delta y}{2}) - P(x, y+\frac{\Delta y}{2})}{\Delta y} + \lim_{\Delta x\to 0}\frac{Q(x-\frac{\Delta x}{2}, y) - Q(x+\frac{\Delta x}{2}, y)}{\Delta x}$$

$$= -\frac{\partial P(x,y)}{\partial y} + \frac{\partial Q(x,y)}{\partial x}$$

$$= \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$$

(3)



$$\int_C F\cdot T\,ds$$

2

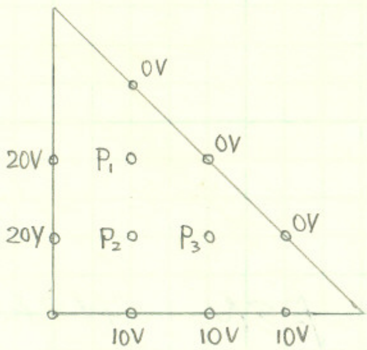
$$= \int_C (P, Q)(dx, dy)$$

$$= \int_C P\,dx + Q\,dy$$

$$= \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy \quad \text{(Green's Thm)}$$

$$= \iint_R \text{Curl } F \;dx\,dy$$

(1)



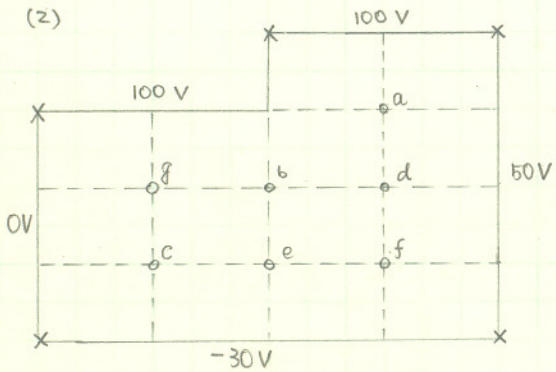
$$\begin{cases} P_1 = \frac{1}{4}(0+0+20+P_2) \\ P_2 = \frac{1}{4}(P_3+P_1+20+10) \\ P_3 = \frac{1}{4}(0+0+P_2+10) \end{cases}$$

$$\begin{cases} P_1 - 0.25 P_2 + 0 \cdot P_3 = 5 \\ -P_1 + 4 \cdot P_2 - P_3 = 30 \\ 0P_1 - P_2 + 4P_3 = 10 \end{cases}$$

$$\begin{cases} P_1 = 7.6786 \text{ V} \\ P_2 = 10.7143 \text{ V} \\ P_3 = 5.1786 \text{ V} \end{cases}$$

2

(2)



$$\begin{cases} 4a = 100+100+d+50 \\ 4b = 100+g+e+d \\ 4c = g+0-30+e \\ 4d = a+b+f+50 \\ 4e = b+c-30+f \\ 4f = d+e-30+50 \\ 4g = 100+0+c+b \end{cases}$$

2.

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P_{112} , 2

SHOW $\int_0^{\pi/2} \cos^{2n}\theta \, d\theta = (2n)!\,\pi / 2\cdot(2^n n!)^2$

LET $f(z) = (z+1/z)^{2n}/z$, $z = (\cos\theta, \sin\theta)$

$\therefore f(z)dz = (z+1/z)^{2n}/z = (z^2+1)^{2n}/z^{2n+1} = \cos^{2n}\theta \, d\theta / i$

$$\int_{|z|=1} f(z)dz = \int_{|z|=1} \frac{(z^2+1)^{2n}}{z^{2n+1}} dz = \int_{|z|=1} \frac{(z^2+1)^{2n}}{(z-0)^{2n+1}} dz$$

$$= 2\pi i \cdot g^{(2n)}(0) / (2n)!$$

WHERE $g(z) = (z^2+1)^{2n} \triangleq (h(z))^{2n}$

2

By USING THE EQUATION , IF $f(z) = x(y(z))$

$$f^{(2n)}(z) = x^{(2n)}(y(z))\cdot y'(z) + C_{2n}^2 x^{(2n-1)}(y(z))\cdot y''(z) + C_{2n}^3 x^{(2n-2)}(y(z))y^{(3)}(z)$$

$$+ \cdots + x'(y(z))y^{(2n)}(z)$$

IN THIS PROBLEM. $y(z) = z^2+1$

$\therefore$ $y'(z) = 2z$, $y'(0) = 0$

$y''(z) = 2$

$y'''(z) = 0$ $\cdots$, $y^{(2n)}(z) = 0$

SO, WE HAVE $g^{(2n)}(0) = (2n!)^2/(2^n n!)^2$

$$\therefore \int_{|z|=1} f(z)dz = 2\pi i \cdot (2n)! / (2^n \cdot n!)^2$$

$$\int_0^{\pi/2} \cos^{2n}\theta \, d\theta = \frac{1}{4i}\int_{|z|=1} f(z)dz = \frac{(2n)!}{(2^n \cdot n!)^2}\cdot\frac{\pi}{2}$$

Ex. 2 , p. 83 ?

MAT417 H.W. # ~~LAST~~ 15 GRD Complex Var. Benmei Chen.

Obtain the Maclaurin series given in Exercises 3-7

3. $\sin z = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{z^{2n-1}}{(2n-1)!}$, $|z| < \infty$

$(\sin z)' = \cos z \,|_{z=0} = 1$ $(\sin z)'' = -\sin z\,|_{z=0} = 0$

$(\sin z)''' = -\cos z\,|_{z=0} = -1$ $\sin^{(4)} z = \sin z\,|_{z=0} = 0$

2

$$\sin^{(k)} z\Big|_{z=0} = \begin{cases} 1 & k=4n+1 \\ 0 & k=4n+2 \\ -1 & k=4n+3 \\ 0 & k=4n+4 \end{cases} \qquad n=0,1,2,\cdots$$

$\therefore \sin z = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{z^{2n-1}}{(2n-1)!}$, $|z| < \infty$

4. $\cos z = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{z^{2n}}{(2n)!}$, $|z| < \infty$

$\cos' z = -\sin z\,|_{z=0} = 0$ $\cos'' z = -\cos z\,|_{z=0} = -1$

$\cos''' z = \sin z\,|_{z=0} = 0$ $\cos^{(4)} z = \cos z\,|_{z=0} = 1$

2

$$\cos^{(k)} z\Big|_{z=0} = \begin{cases} 0 & k=4n+1 \\ -1 & k=4n+2 \\ 0 & k=4n+3 \\ 1 & k=4n+4 \end{cases} \qquad n=0,1,2,\cdots$$

$\therefore \cos z = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{z^{2n}}{(2n)!}$ $|z| < \infty$

5. $\sinh z = \sum_{n=1}^{\infty} \frac{z^{2n-1}}{(2n-1)!}$, $|z| < \infty$

$\sinh' z = \left(\frac{e^z - e^{-z}}{2}\right)' = \cosh z\,|_{z=0} = 1$ $\sinh'' z = \sinh z\,|_{z=0} = 0$

$$\therefore \sinh^{(k)} z\Big|_{z=0} = \begin{cases} 1 & k=2n+1 \\ 0 & k=2n \end{cases}$$

$\sinh = \sum_{n=1}^{\infty} \frac{z^{2n-1}}{(2n-1)!}$, $|z| < \infty$

2

6. $\cosh z = \sum_{n=0}^{\infty} z^{2n}/(2n)!$, $|z| < \infty$

$\cosh' z = \sinh z\,|_{z=0} = 0$ $\cosh'' z = \sinh' z = \cosh z\,|_{z=0} = 1$

$$\cosh^{(k)} z\Big|_{z=0} = \begin{cases} 0 & k=2n \\ 1 & k=2n+1 \end{cases}$$

2

$\therefore \cosh z = \sum_{n=0}^{\infty} z^{2n}/(2n)!$, $|z| < \infty$

7. $\dfrac{1}{1-z^2} = \sum_{n=0}^{\infty} z^{2n}, \quad |z|<1$

USING THE RESULT IN EXPLE 2, WE HAVE

$$\frac{1}{1-\omega} = \sum_{n=0}^{\infty} \omega^n, \quad |\omega|<1$$

LET $\omega = z^2$, SO

$$\frac{1}{1-z^2} = \sum_{n=0}^{\infty} z^{2n}, \quad |z|<1$$

2

9. $f(z) = \dfrac{1}{1-z}, \quad z_0 = i$

$$f(z) = \frac{1}{1-z_0-(z-z_0)} = \frac{1}{1-\left(\frac{z-i}{1-i}\right)} \cdot \frac{1}{1-i}$$

$$= \frac{1}{1-i}\cdot\left(1+\frac{z-i}{1-i}+\frac{(z-i)^2}{(1-i)^2}+\cdots\right)$$

$$\frac{|z-i|}{|1-i|}<1 \quad \Rightarrow \quad |z-i|<\sqrt{2}.$$

2

10. $f(z) = \cos z, \quad z_0 = \frac{\pi}{2}$

$f'(z) = \cos' z = -\sin z\,\big|_{z=\frac{\pi}{2}} = -1$ $\qquad f''(z) = -\cos z\,\big|_{z=\frac{\pi}{2}} = 0$

$f'''(z) = \sin z\,\big|_{z=\frac{\pi}{2}} = 1$ $\qquad f^{(4)}(z) = \cos z\,\big|_{z=\frac{\pi}{2}} = 0$

$$f(z) = \cos z = -\left(z-\tfrac{\pi}{2}\right) + \frac{1}{3!}\left(z-\tfrac{\pi}{2}\right)^3 - \frac{1}{5!}\left(z-\tfrac{\pi}{2}\right)^5 + \frac{1}{7!}\left(z-\tfrac{\pi}{2}\right)^7 - \cdots$$

$$\left|z-\tfrac{\pi}{2}\right| < \infty$$

2

FIND THE LAURENT SERIES OF THE FUNCTION $(z^2+z)^{-1}$ IN THE REGIONS GIVEN IN EXERCISES 1–3

1. $0<|z|<1$

$$f(z)=\frac{1}{z^2+z}=\frac{1}{z(z+1)}=\frac{1}{z}-\frac{1}{z+1}=\frac{1}{z}-\frac{1}{1-(-z)} \qquad |-z|<1$$

$$=\frac{1}{z}-\left[1+(-z)+(-z)^2+(-z)^3+\cdots\right]$$

$$=\frac{1}{z}-1+z-z^2+z^3-z^4+\cdots\cdots=\sum_{n=-1}^{\infty}(-1)^{n+1}\cdot z^n$$

2. $0<|z-1|<1$

$$f(z)=\frac{1}{z}-\frac{1}{1+z}=\frac{1}{1+(z-1)}-\frac{\frac{1}{2}}{1+\frac{z-1}{2}} \qquad |z-1|<1$$

$$=1-(z-1)+(z-1)^2-(z-1)^3+\cdots$$

$$-\frac{1}{2}\left[1-\frac{z-1}{2}+\left(\frac{z-1}{2}\right)^2-\left(\frac{z-1}{2}\right)^3+\cdots\right]$$

$$=\frac{1}{2}-\frac{3}{4}(z-1)+\frac{7}{8}(z-1)^2-\frac{15}{16}(z-1)^3+\cdots=\sum_{n=0}^{\infty}\frac{2^{n+1}-1}{2^{n+1}}(-1)^n\cdot(z-1)^n$$

3. $1<|z-1|<2$

$$f(z)=\frac{1}{z}-\frac{1}{1+z}=\frac{1}{1-(-(z-1))}-\frac{1}{2-(-(z-1))}$$

$$=-\frac{1}{z-1}\left(\frac{1}{-1+\frac{-1}{z-1}}\right)-\frac{\frac{1}{2}}{1-\frac{[-(z-1)]}{2}}$$

$$=\frac{1}{z-1}\left[\frac{1}{1-\frac{-1}{z-1}}\right]-\frac{1}{2}\left[\frac{1}{1-\frac{[-(z-1)]}{2}}\right]$$

$$=\frac{1}{z-1}\left[1+\left(\frac{-1}{z-1}\right)+\left(\frac{-1}{z-1}\right)^2+\left(\frac{-1}{z-1}\right)^3+\cdots\right]-\frac{1}{2}\left[1-\frac{(z-1)}{2}+\frac{(z-1)^2}{4}+\cdots\right]$$

$$=\frac{1}{z-1}-\frac{1}{(z-1)^2}+\frac{1}{(z-1)^3}+\cdots\cdots-\frac{1}{2}+\frac{1}{4}(z-1)-\frac{1}{8}(z-1)^2+\frac{1}{16}(z-1)^3+\cdots\cdots$$

Represent the function $(z^3 - z)^{-1}$ as a Laurent series in the regions given in Exercises 4 – 7

4. $0 < |z| < 1$

$$f(z) = \frac{1}{z^3 - z} = \frac{1}{z(z+1)(z-1)} = -\frac{1}{z} + \frac{1/2}{z+1} + \frac{1/2}{z-1}$$

$$= -\frac{1}{z} + \frac{1}{2}\frac{1}{1-(-z)} - \frac{1}{2}\frac{1}{1-z}$$

$$= -\frac{1}{z} + \frac{1}{2}\left[1 - z + z^2 - z^3 + \cdots\right] - \frac{1}{2}\left[1 + z + z^2 + z^3 + \cdots\right]$$

$$= -\frac{1}{z} - z - z^3 - z^3 - \cdots = \sum_{n=1}^{\infty} - z^{2n-1}$$

5. $1 < |z|$

$$f(z) = -\frac{1}{z} + \frac{1}{2}\cdot\frac{1}{z}\cdot\frac{1}{1+\frac{1}{z}} + \frac{1}{2}\cdot\frac{1}{z}\cdot\frac{1}{1-\frac{1}{z}}$$

$$= -\frac{1}{z} + \frac{1}{2z}\left[1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} + \cdots\right] + \frac{1}{2z}\left[1 + \frac{1}{z} + \frac{1}{z^2} + \cdots\right]$$

$$= -\frac{1}{z} + \frac{1}{z} + \frac{1}{z^3} + \frac{1}{z^5} + \cdots$$

$$= \frac{1}{z^3} + \frac{1}{z^5} + \frac{1}{z^7} + \cdots = \sum_{n=1}^{\infty} z^{-2n-1}$$

6. $0 < |z-1| < 1$

$$f(z) = -\frac{1}{z-1+1} + \frac{1}{2}\cdot\frac{1}{z-1+2} + \frac{1}{2}\frac{1}{z-1}$$

$$= -\left[1 - (z-1) + (z-1)^2 - (z-1)^3 + \cdots\right] + \frac{1}{4}\left[1 - \frac{z-1}{2} + \left(\frac{z-1}{2}\right)^2 - \left(\frac{z-1}{2}\right)^3 + \cdots\right] + \frac{1}{2}\frac{1}{z-1}$$

$$= \frac{1}{2}\cdot\frac{1}{z-1} - \frac{3}{4} + \frac{7}{8}(z-1) - \frac{15}{16}(z-1)^2 + \cdots$$

$$= \sum_{n=0}^{\infty} (-1)^n \cdot \frac{2^{n+1}-1}{2^{n+1}} \cdot (z-1)^{n-1}$$

7. $1 < |z-1| < 2$

$$f(z) = \frac{1}{2}\cdot\frac{1}{z-1} + \frac{1}{4}\cdot\frac{1}{1 - \frac{-(z-1)}{2}} - \frac{1}{z-1}\cdot\frac{1}{1 - \frac{-1}{z-1}}$$

$$= \frac{1}{2}\cdot\frac{1}{z-1} + \frac{1}{4}\left[1 - \frac{z-1}{2} + \left(\frac{z-1}{2}\right)^2 - \left(\frac{z-1}{2}\right)^5 + \cdots\right] - \frac{1}{z-1}\left[1 - \frac{1}{z-1} + \frac{1}{(z-1)^2} - \cdots\right]$$

$$= \frac{1}{4} - \frac{1}{8}(z-1) + \frac{1}{16}(z-1)^2 + \cdots - \frac{1}{2}\cdot\frac{1}{z-1} + \frac{1}{(z-1)^2} - \frac{1}{(z-1)^3} + \frac{1}{(z-1)^5} - \cdots$$