Q.1. An input-output relationship of a very badly made thermometer is characterized by the following differential equation:

$$\frac{d^2 y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = u(t)$$

where $u(t)$ is the temperature of the environment in which the thermometer is placed, and $y(t)$ is the measured temperature.

The thermometer is to be placed inside a passenger car under the sun at noon in Singapore, where the temperature is considered to reach the daily highest (a constant temperature at steady state).

(a) Assume the thermometer is placed for a long time at an office where its room temperature has been kept at 20°C all the time, i.e., $u(t)$ = 20°C. What is the actual reading in the thermometer?

(7 marks)

Solution: If the thermometer is placed for a long time at a constant room temperature, its reading can be assumed to reach the steady state, i.e., $y(t)$ is a constant. Then,

$$\frac{d^2 y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = u(t)$$

Implies that $2y(t) = u(t) = 20$, and hence $y(t) = 10$. The actual reading of the thermometer is 10°C.

(b) After thermometer is taking out from the office (you can assume $y'(0)=0$ and $y(0)$ is the reading of the thermometer in Part (a) in such a situation), it is immediately put into the passenger car, where the temperature is known to be 40°C. Determine the reading of the thermometer, i.e., find the solution for $y(t)$ using the time-domain approach.

(12 marks)

Solution: We are looking for two types of the solutions, i.e., the steady state solution and the transient response for the ODE,

$$\frac{d^2y(t)}{dt^2}+3\frac{dy(t)}{dt}+2y(t)=u(t)$$

For the steady state solution, we take $y_{ss}=k$ and

$$\frac{d^2y(t)}{dt^2}+3\frac{dy(t)}{dt}+2y(t)=\frac{d^2y_{ss}}{dt^2}+3\frac{dy_{ss}}{dt}+2y_{ss}=2y_{ss}=u(t)=40 \quad\Rightarrow\quad y_{ss}=20$$

For the transient response, we find the characteristic polynomial and its roots

$$z^2+3z+2=0 \quad\Rightarrow\quad z_1=-1,\quad z_2=-2$$

$$\Rightarrow\quad y_{tr}(t)=k_1e^{-t}+k_2e^{-2t}$$

The complete solution of the ODE is then given by

$$y(t)=y_{ss}+y_{tr}(t)=20+k_1e^{-t}+k_2e^{-2t}$$

$$y(0)=20+k_1+k_2=10 \quad\Rightarrow\quad k_1+k_2=-10$$

$$y'(t)=-k_1e^{-t}-2k_2e^{-2t} \quad\Rightarrow\quad y'(0)=-k_1-2k_2=0 \quad\Rightarrow\quad k_1+2k_2=0$$

Solving the above equations, we obtain $k_1=-20,\quad k_2=10$ and

$$y(t)=20-20e^{-t}+10e^{-2t},\quad t\ge 0$$

(c) What is the reading of the thermometer in Part (b) after 10 minutes?

(6 marks)

Solution: For t = 10 x 60 = 600, we have

$$y(600) = 20 - 20e^{-600} + 10e^{-1200} = 20$$

The reading of the thermometer is 20°C.

Q.2. As the thermometer in Q.1 is not designed properly, a group of BTech students in the EE2010E Class has come out with a solution to redesign it such that it is characterized by the following new differential equation instead:

$$\frac{d^2 y(t)}{dt^2} + \frac{dy(t)}{dt} + y(t) = u(t)$$

where $u(t)$ is again the temperature of the environment in which the thermometer is placed, and $y(t)$ is the measured temperature.

(a) The new thermometer is again placed for a long time at the office where its room temperature has been kept at 20°C all the time, i.e., $u(t)$ = 20°C. What is the actual reading in the new thermometer?

(5 marks)

Solution: If the new thermometer is put for a long time at a constant room temperature, its reading can be assumed to reach the steady state, i.e., $y(t)$ is a constant. Then,

$$\frac{d^2 y(t)}{dt^2} + \frac{dy(t)}{dt} + y(t) = u(t)$$

Implies that $y(t) = u(t) = 20$, and hence $y(t) = 20$. The actual reading of the thermometer is 20°C.

(b) What is the transfer function of the new thermometer?

(5 marks)

Solution: Taking Laplace transform on both sides of the ODE, i.e.,

$$L\left\{\frac{d^2y(t)}{dt^2}+\frac{dy(t)}{dt}+y(t)\right\}=L\{u(t)\}$$

$$\Downarrow$$

$$s^2Y(s)+sY(s)+Y(s)=U(s)$$

$$\Downarrow$$

$$H(s)=\frac{Y(s)}{U(s)}=\frac{1}{s^2+s+1}$$

(c) What is the overshoot in its step response? What is the 2% settling time?

(10 marks)

Solution: Rewriting the transfer function as

$$H(s)=\frac{1^2}{s^2+2\times 0.5\times 1\times s+1^2}=\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$

we obtain $\zeta=0.5, \omega_n=1$. Thus,

$$M_p=17\% \quad (16.3\%) \quad \text{and} \quad t_s=\frac{4}{\zeta\omega_n}=8 \quad \text{seconds}$$

(d) If the new thermometer is used to measure the temperature inside a passenger car, which is known to be 40°C, what is the peak value of the reading from the thermometer?

(5 marks)

Solution: The peak of the reading happens at the time when the overshoot occurs, it is given by

$$y_{\text{peak}} = y_{ss}(1+M_p) = 40(1+0.17) = 46.8°\text{C} \quad (\text{or } 46.52°\text{C})$$