EE2010E: Systems and Control

Part 1: Systems

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Course Outlines

Topics to be covered in Part 1…

1. **Introduction to systems**

   Properties of dynamic systems – causality, stability, time invariance, linearity.

2. **Time domain models of linear time invariant systems**

   Differential equation models of linear systems, Dynamic responses (natural, force and complete responses), First order transients – RC and RL circuits, Second order RLC circuits, State representation.

3. **Frequency domain description of systems**

   Review of Laplace transforms, transfer functions – poles and zeros, Response to sinusoidal inputs, Frequency response, Bode and polar plots.

4. **Properties of linear time invariant systems**

   Steady state versus transient response, impulse response, step response, convolution, Relationship between poles and natural response, Input-output stability, Stability analysis via poles.

Textbook

A. B. Carlson, *Circuits*, PWS Publishing Company, New York, 1999.

References

T. S. ElAli and M. A. Karim, *Continuous Signals and Systems with MATLAB*, CRC Press, New York, 2001.

C. W. de Silva, Modeling and Control of Engineering Systems, CRC Press, 2009

L. Qiu and K. Zhou, Introduction to Feedback Control, Pearson, 2010

Online Materials

- Control Tutorials at http://www.ece.ualberta.ca/~tchen/ctm/index.html
- Lecture Notes of 2nd Reference: http://www.mech.ubc.ca/~ial/ialweb/courses.htm

Lectures

Attendance is essential.

Ask any question at any time during the lecture.

Tutorials

There will be a total of 6 tutorial hours for this part. Another 6 more for the second part.

You should make an effort to attempt each question before the tutorial.

Some of the questions are straightforward, but quite a few are difficult and meant to serve as a platform for the introduction of new concepts.

Ask your tutor any question related to the tutorials and the course.

Labs and Final Examination

There will be two lab experiments for this module. Your two lab report marks will be counted as 30%. Hence, your final grade for this module will be computed as follows:

Your Final Grade = 15% of Lab Experiment 1

+ 15% of Lab Experiment 2

+ 70% of Final Examination

Introduction to Systems

What is a system?

Definition 1 (from a dictionary):

A system is a functionally related group of elements.

Definition 2 (from a dictionary):

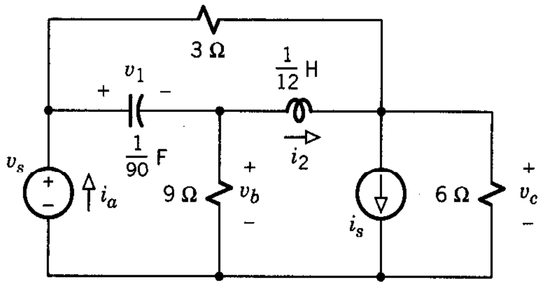
A system is a group of things working together as a whole.

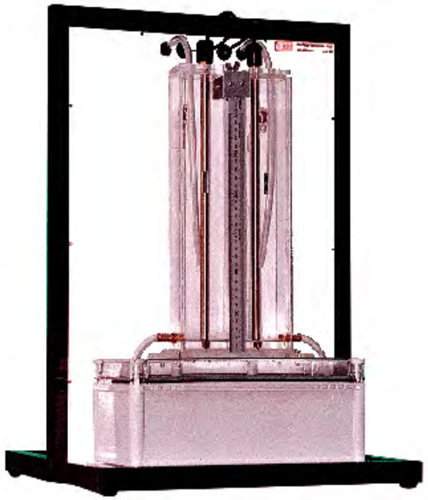
Definition 3 (from the reference by EiAli and Karim):

A system is an assemblage of things that are combined to form a complex whole.

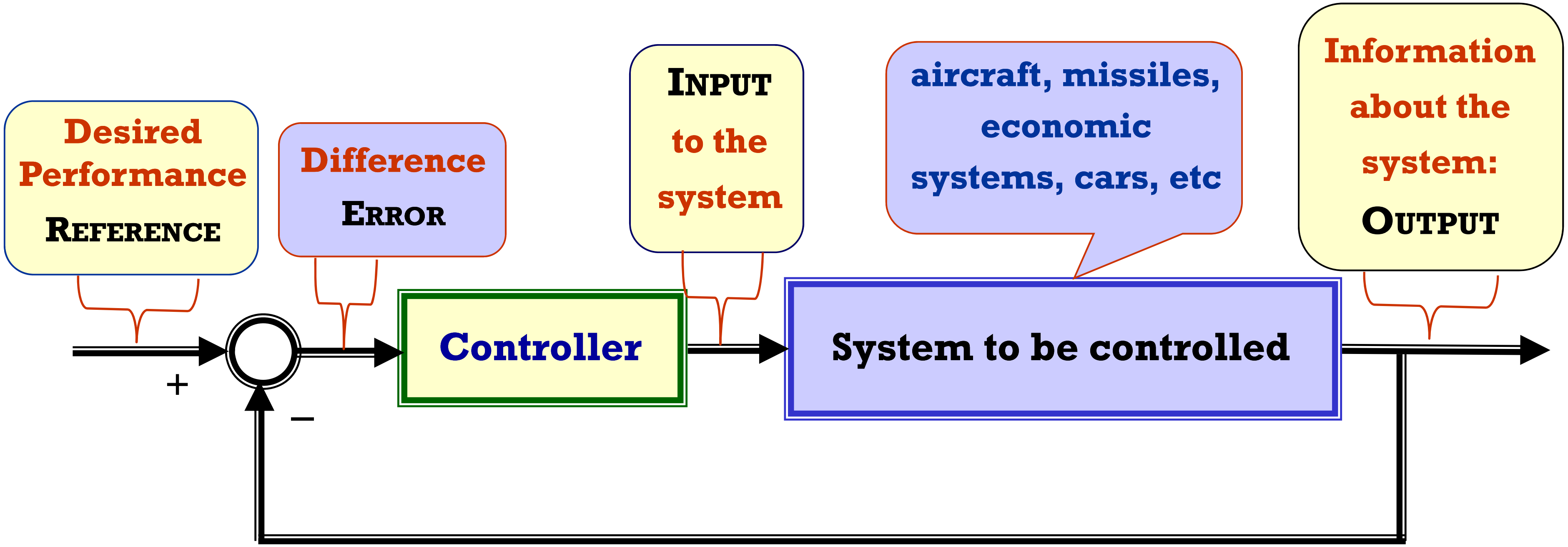
Examples include educational systems such as NUS, financial systems such as stock market, social systems such as government, weather, the human body, electrical systems such as electric circuits, mechanical systems, etc…

Examples on some systems of interest…





What is a control system?

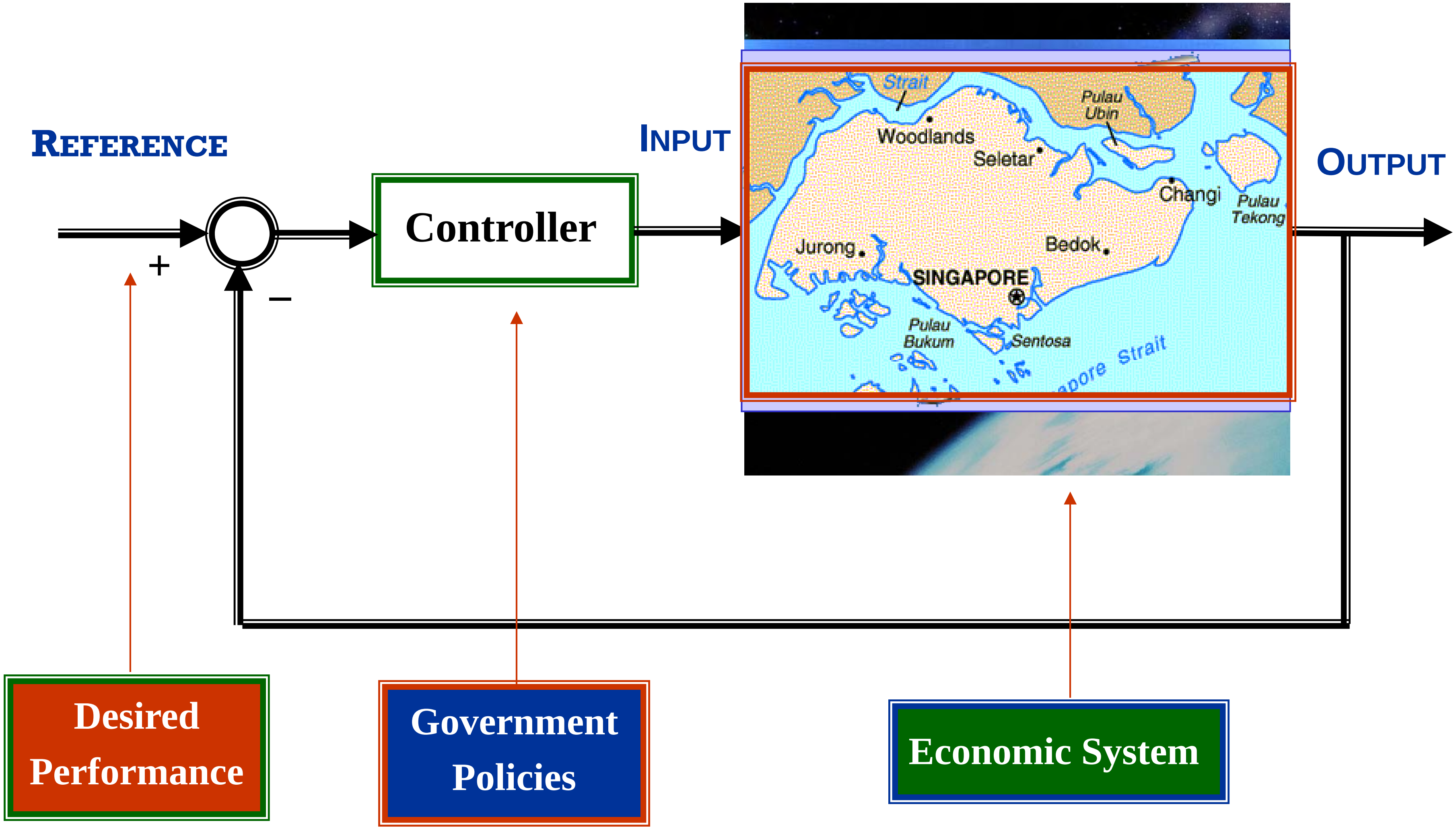


Objective: To make the system **OUTPUT** and the desired **REFERENCE** as close as possible, i.e., to make the **ERROR** as small as possible.

Key Issues: 1) How to describe the system to be controlled? **(Modeling)**

2) How to design the controller? **(Control)**

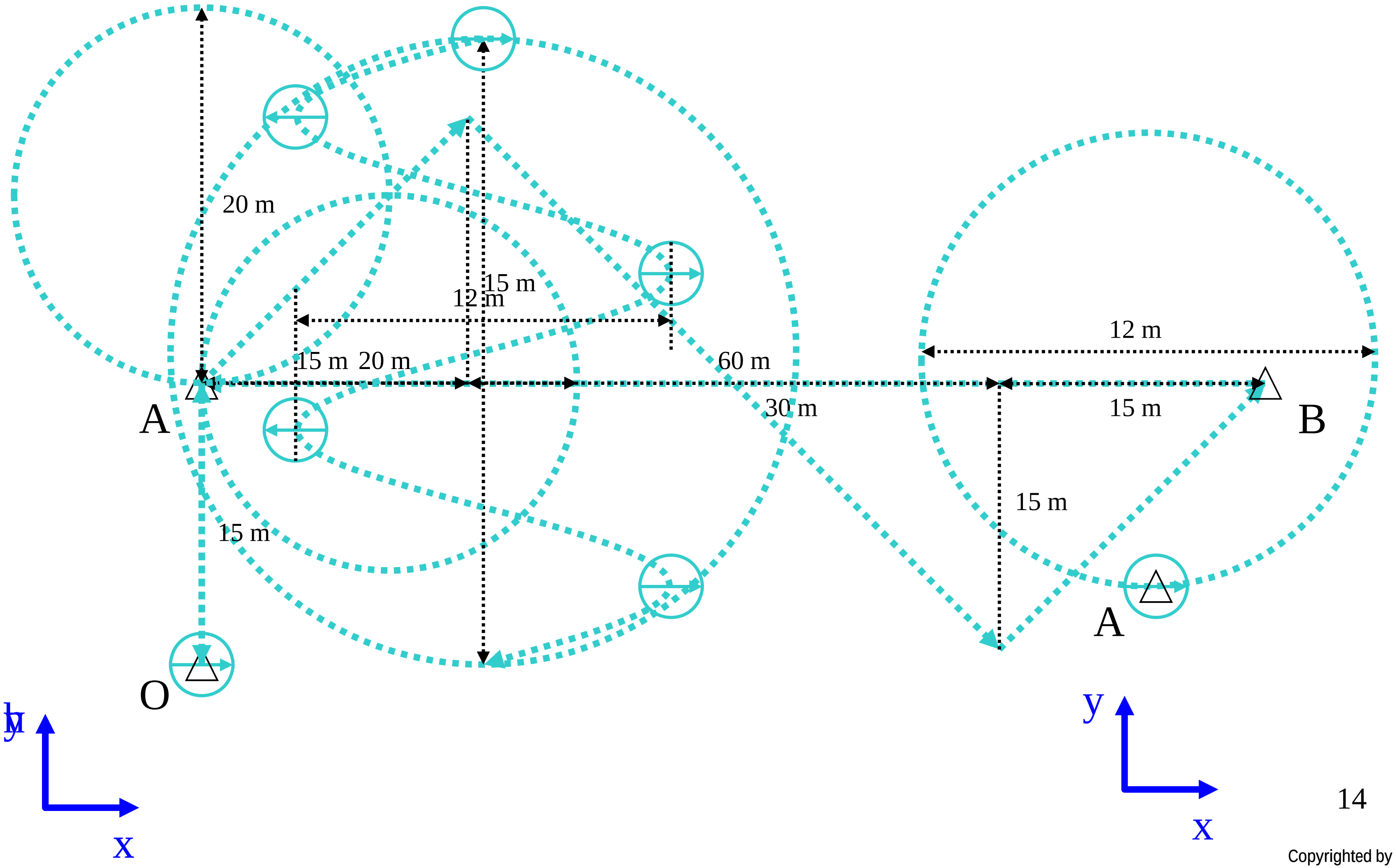
Some Control Systems Examples:



An actual control system demo

Illustration for the video demo of a flight control system…

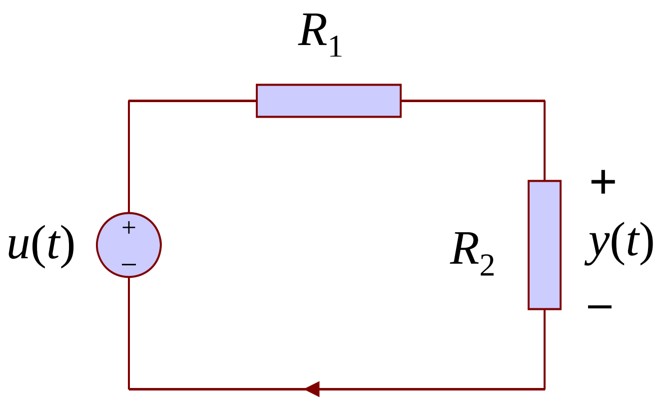
1. standby
2. takeoff
3. hovering
4. slithering
5. head turning
6. pirouetting
7. wheeling
8. backward down spiraling
9. hovering
10. landing
11. standby



Video demo of a fully automatic UAV flight control system

Back to systems – block diagram representation of a system

$u(t)$ is a signal or certain information injected into the system, which is called the system input, whereas $y(t)$ is a signal or certain information produced by the system with respect to the input signal $u(t)$. $y(t)$ is called the system output. For example,



input: voltage source

output: voltage across R_2

$$y(t) = \frac{R_2}{R_1 + R_2} \cdot u(t)$$

Linear systems

Let $y_1(t)$ be the output produced by an input signal $u_1(t)$ and $y_2(t)$ be the output produced by another input signal $u_2(t)$. Then, the system is said to be linear if

a) the input is $\alpha\, u_1(t)$, the output is $\alpha\, y_1(t)$, where α is a scalar; and

b) the input is $u_1(t) + u_2(t)$, the output is $y_1(t) + y_2(t)$.

Or equivalently, the input is $\alpha\, u_1(t) + \beta\, u_2(t)$, the output is $\alpha\, y_1(t) + \beta\, y_2(t)$. Such a property is called ***superposition***. For the circuit example on the previous page,

$$y(t) = \frac{R_2}{R_1 + R_2} \cdot \left[\alpha\, u_1(t) + \beta\, u_2(t)\right] = \alpha \frac{R_2}{R_1 + R_2} u_1(t) + \beta \frac{R_2}{R_1 + R_2} u_2(t) = \alpha\, y_1(t) + \beta\, y_2(t)$$

It is a linear system! We will mainly focus on linear systems in this course.

Example for nonlinear systems

Example: Consider a system characterized by

$$\boxed{y(t) = 100u^2(t)}$$

Step One:

$$y_1(t) = 100u_1^2(t) \quad \& \quad y_2(t) = 100 \cdot u_2^2(t)$$

Step Two: Let $u(t) = u_1(t) + u_2(t)$, we have

$$y(t) = 100u^2(t) = 100[u_1(t) + u_2(t)]^2 = 100[u_1^2(t) + u_2^2(t) + 2u_1(t)u_2(t)]$$

$$= y_1(t) + y_2(t) + 200u_1(t)u_2(t) \neq y_1(t) + y_2(t)$$

The system is nonlinear.

Exercise: Verify that the following system

$$\boxed{y(t) = \cos(u(t))}$$

is a nonlinear system. Give some examples in our daily life, which are nonlinear.

Time invariant systems

A system is said to be time-invariant if for a shift input signal $u(t-t_0)$, the output of the system is $y(t-t_0)$. To see if a system is time-invariant or not, we test

a) Find the output $y_1(t)$ that corresponds to the input $u_1(t)$.

b) Let $u_2(t) = u_1(t-t_0)$ and then find the corresponding output $y_2(t)$.

c) If $y_2(t) = y_1(t-t_0)$, then the system is time-invariant. Otherwise, it is not!

In common words, if a system is time-invariant, then for the same input signal, the output produced by the system today will be ***exactly the same*** as that produced by the system tomorrow or any other time.

Example for time invariant systems

Consider the same circuit, i.e.,

$$\boxed{y(t) = \frac{R_2}{R_1 + R_2} \cdot u(t)}$$

Obviously, whenever you apply a same voltage to the circuit, its output will always be the same. Let us verify this mathematically.

Step One:

$$y_1(t) = \frac{R_2}{R_1 + R_2} \cdot u_1(t) \quad \Rightarrow \quad y_1(t - t_0) = \frac{R_2}{R_1 + R_2} \cdot u_1(t - t_0)$$

Step Two: Let $u_2(t) = u_1(t - t_0)$, we have

$$y_2(t) = \frac{R_2}{R_1 + R_2} \cdot u_2(t) = \frac{R_2}{R_1 + R_2} \cdot u_1(t - t_0) = y_1(t - t_0)$$

By definition, it is time-invariant!

Example for time variant systems

Example 1: Consider a system characterized by

$$\boxed{y(t) = \cos(t) u(t)}$$

Step One:

$$y_1(t) = \cos(t) \cdot u_1(t) \quad \Rightarrow \quad y_1(t - t_0) = \cos(t - t_0) \cdot u_1(t - t_0)$$

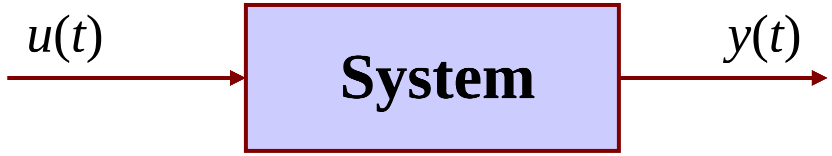
Step Two: Let $u_2(t) = u_1(t - t_0)$, we have

$$y_2(t) = \cos(t) \cdot u_2(t) = \cos(t) \cdot u_1(t - t_0) \neq y_1(t - t_0)$$

The system is not time-invariant. It is time-variant!

Example 2: Consider a financial system such as a stock market. Assume that you invest **$10,000** today in the market and make **$2000**. Is it guaranteed that you will make exactly another **$2000** tomorrow if you invest the same amount of money? Is such a system time-invariant? You know the answer, don't you?

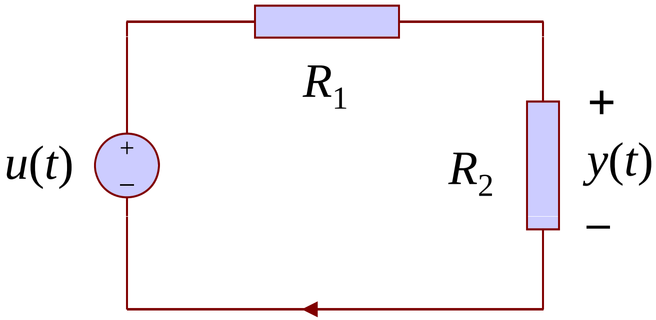
Systems with memory and without memory



A system is said to have memory if the value of $y(t)$ at any particular time t_1 depends on the time from $-\infty$ to t_1. For example,

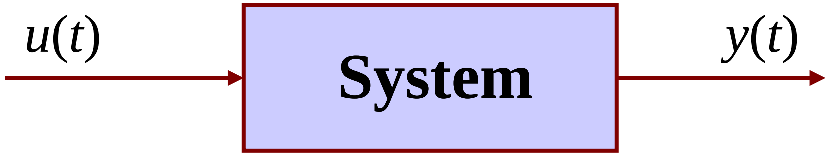
$$u(t) = C\frac{dy(t)}{dt} \quad \Rightarrow \quad y(t) = \frac{1}{C}\int_{-\infty}^{t} u(t)dt$$

On the other hand, a system is said to have no memory if the value of $y(t)$ at any particular time t_1 depends only on the time t_1. For example,



$$y(t) = \frac{R_2}{R_1 + R_2}\cdot u(t)$$

Causal systems



A causal system is a system where the output $y(t)$ at a particular time t_1 depends on the input for $t \le t_1$. For example,

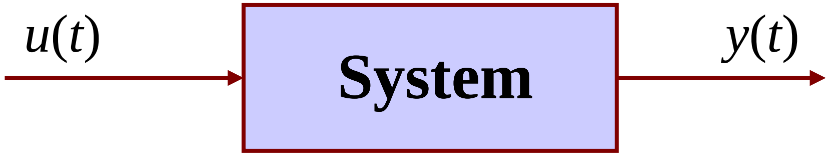
$$u(t) = C\frac{dy(t)}{dt} \quad \Rightarrow \quad y(t) = \frac{1}{C}\int_{-\infty}^{t} u(\tau)d\tau$$

On the other hand, a system is said to be non-causal if the value of $y(t)$ at a particular time t_1 depends on the input $u(t)$ for some $t > t_1$. For example,

$$\boxed{y(t) = u(t+1)}$$

in which the value of $y(t)$ at $t = 0$ depends on the input at $t = 1$.

System stability



The signal $u(t)$ is said to be bounded if $|u(t)| < \beta < \infty$ for all t, where β is real scalar. A system is said to be BIBO (bounded-input bounded-output) stable if its output $y(t)$ produced by any bounded input is bounded.

<u>A BIBO stable system:</u>

$$\boxed{y(t) = e^{u(t)} \quad \Rightarrow \quad |\,y(t)\,| = \left|e^{u(t)}\right| \le \left|e^{\beta}\right| = e^{\beta} < \infty}$$

<u>A BIBO unstable system:</u>

$$\boxed{y(t) = \int_{-\infty}^{t} u(\tau)d\tau}$$

$$\boxed{\text{Let } u(t) = 1, \text{ which is bounded. Then, } y(t) = \int_{-\infty}^{t} u(\tau)d\tau = \int_{-\infty}^{t} d\tau = \infty}$$

Some Preliminary Materials

Operations of complex numbers

Coordinates: Cartesian Coordinate and Polar Coordinate

$$12 + j5 = 13\, e^{j0.39} = \sqrt{12^2 + 5^2}\; e^{j \tan^{-1}\left(\frac{5}{12}\right)}$$

real part imaginary part magnitude argument

Euler's Formula: $\boxed{e^{j\theta} = \cos(\theta) + j\sin(\theta)}$

Additions: It is easy to do additions (subtractions) in Cartesian coordinate.

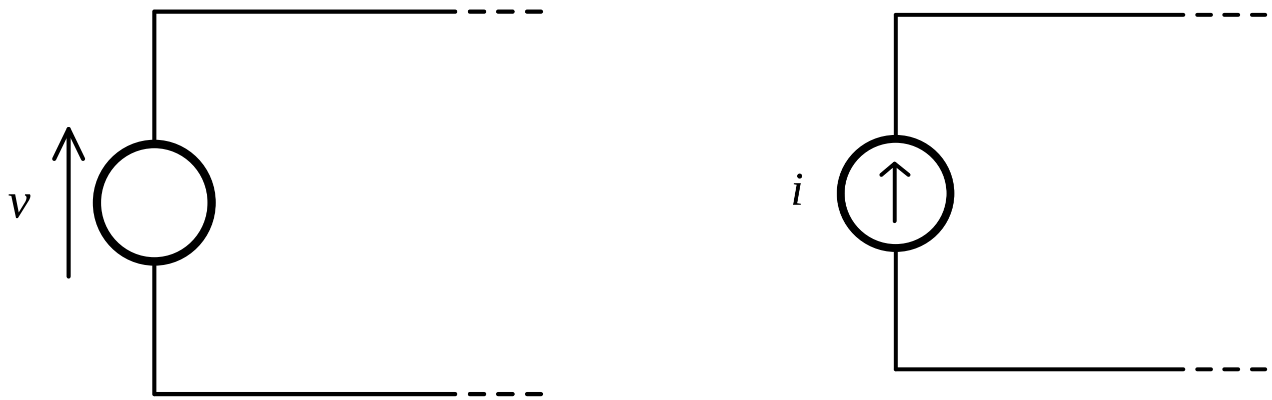
$$\boxed{(a + jb) + (v + jw) = (a + v) + j(b + w)}$$

Multiplication's: It is easy to do multiplication's (divisions) in Polar coordinate.

$$\boxed{re^{j\theta} \cdot ue^{j\omega} = (ru)e^{j(\theta+\omega)}} \qquad \boxed{\frac{re^{j\theta}}{ue^{j\omega}} = \frac{r}{u}\, e^{j(\theta-\omega)}}$$

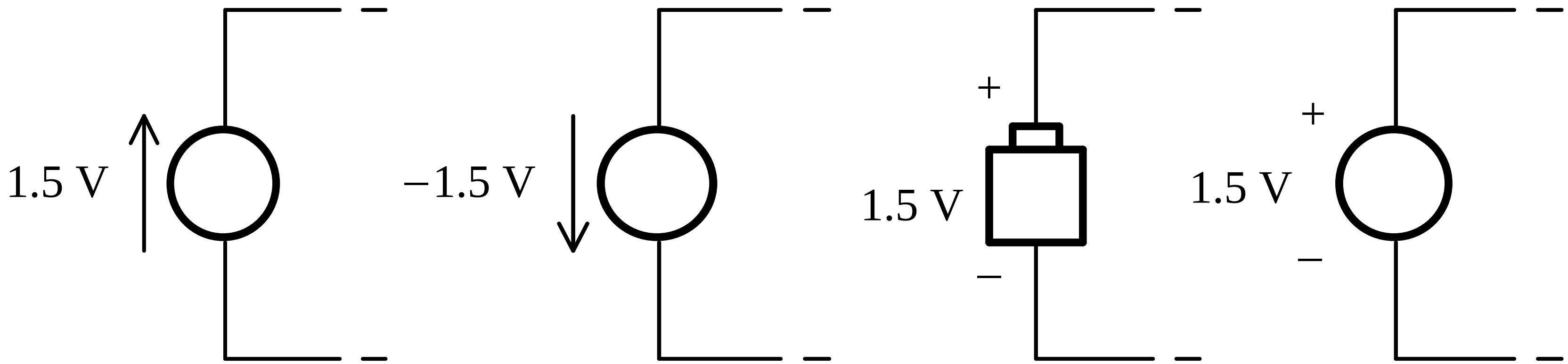
Symbols of voltage and current sources

The circuit symbols of voltage and current sources (either DC or AC) used in this part of the course are:



Basically, the arrow and the value in the voltage source signifies that the top terminal has a potential of v (could be either positive or negative) with respect to the bottom terminal regardless of what has been connected to it. Similarly, the arrow and the value of the current source signifies that there is a current i (could be either positive or negative) flowing upwards.

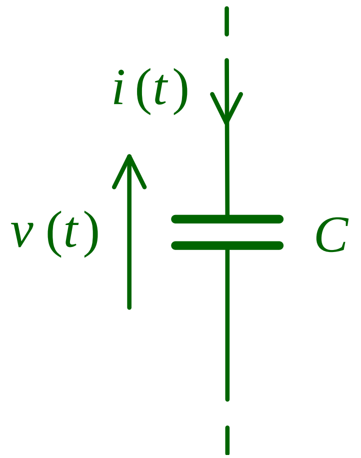
Remark: The following symbols for the voltage source are identical:



Note that on its own, the arrow does not correspond to the positive terminal. Instead, the positive terminal depends on both the arrow and the sign of the voltage which may be negative.

Capacitor

A ***capacitor*** consists of parallel metal plates for storing electric charges. The circuit symbol for an ideal capacitor is:



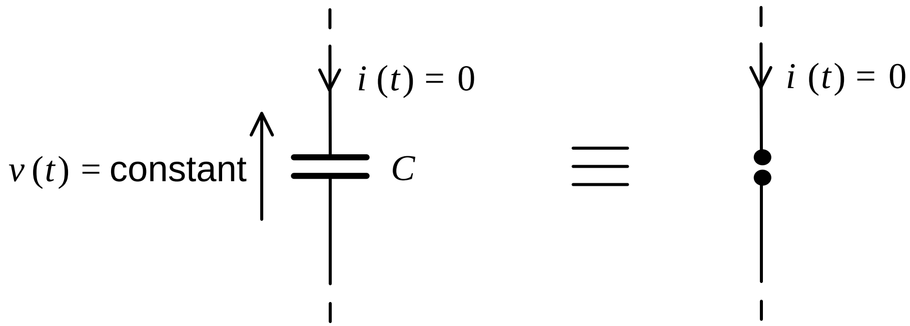
Provided that the voltage and current arrows are in opposite directions, the voltage-current relationship is:

$$i(t) = C\frac{dv(t)}{dt}$$

For dc circuits:

$$v(t) = \text{constant} \Rightarrow \frac{dv(t)}{dt} = 0 \quad \Rightarrow \quad i(t) = 0$$

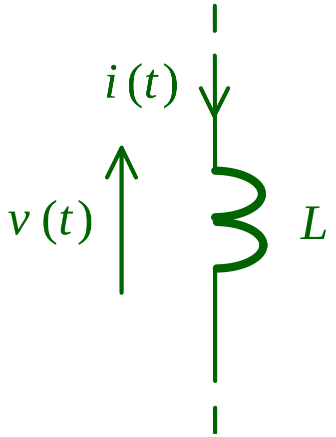
and the capacitor is equivalent to an open circuit:



This is why we don't consider the capacitor in DC circuits.

Inductor

An **inductor** consists of a coil of wires for establishing a magnetic field. The circuit symbol for an ideal inductor is:



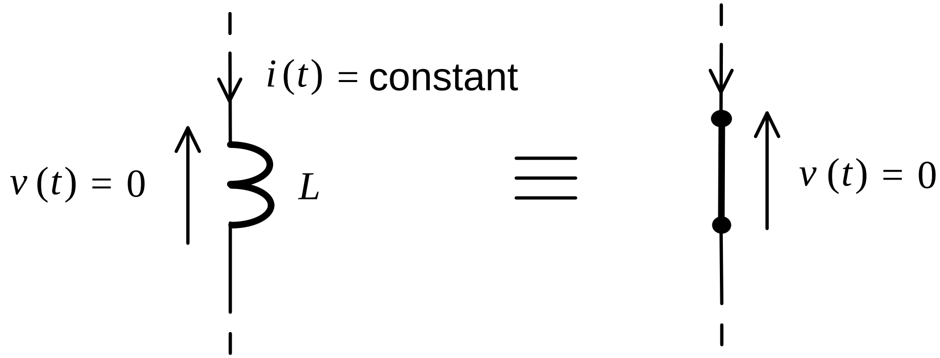
Provided that the voltage and current arrows are in opposite directions, the voltage-current relationship is:

$$\boxed{v(t) = L\frac{di(t)}{dt}}$$

For dc circuits:

$$i(t) = \text{constant} \Rightarrow \frac{di(t)}{dt} = 0 \Rightarrow v(t) = 0$$

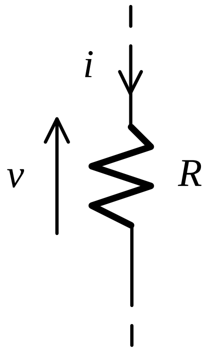
and the inductor is equivalent to a short circuit:



That is why there is nothing interesting about the inductor in DC circuits.

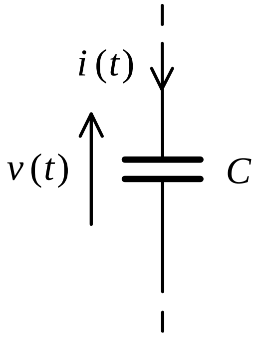
Basic laws for electrical systems

resistor



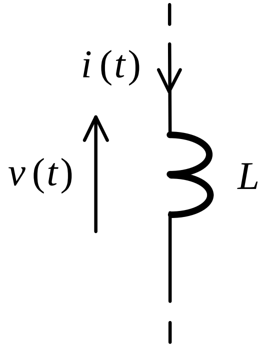
$$\boxed{v = i\,R}$$

capacitor



$$\boxed{i = C\frac{dv}{dt}}$$

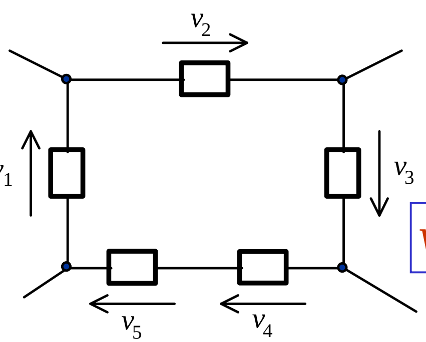
inductor



$$\boxed{v = L\frac{di}{dt}}$$

Kirchhoff's Voltage Law (KVL):

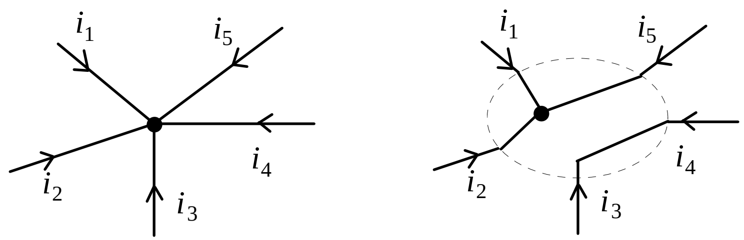
The sum of voltage drops around any close loop in a circuit is 0.



$$\boxed{v_1+v_2+v_3+v_4+v_5 = 0}$$

Kirchhoff's Current Law (KCL):

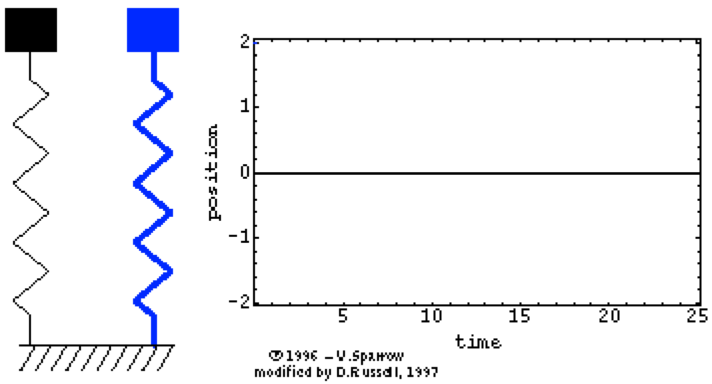
The sum of currents entering/leaving a node/closed surface is 0.



$$\boxed{i_1 + i_2 + i_3 + i_4 + i_5 = 0}$$

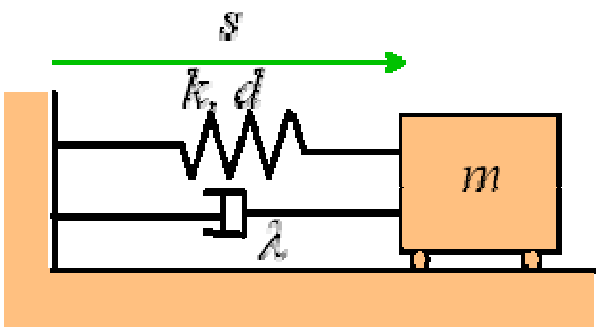
Basic mechanical systems

Spring-mass system

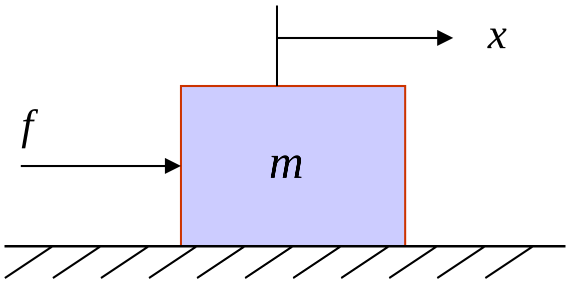


Animation courtesy of Dr. Dan Russell, Kettering University

Mass-spring-damper system



Newton's law of motion



$$f = ma = m\ddot{x}$$

Linear differential equations

General solution:

nth order linear differential equation	$\frac{d^n x(t)}{dt^n} + a_{n-1}\frac{d^{n-1}x(t)}{dt^{n-1}} + \cdots + a_0 x(t) = u(t)$
General solution	$x(t) = x_{ss}(t) + x_{tr}(t)$
Steady state response with no arbitrary constant	$x_{ss}(t) =$ particular integral obtained from assuming solution to have the same form as $u(t)$
Transient response with n arbitrary constants	$x_{tr}(t) =$ general solution of homogeneous equation $\frac{d^n x_{tr}(t)}{dt^n} + a_{n-1}\frac{d^{n-1}x_{tr}(t)}{dt^{n-1}} + \cdots + a_0 x_{tr}(t) = 0$

General solution of homogeneous equation:

nth order linear homogeneous equation	$\frac{d^n x_{tr}(t)}{dt^n} + a_{n-1}\frac{d^{n-1}x_{tr}(t)}{dt^{n-1}} + \cdots + a_0 x_{tr}(t) = 0$
Roots of polynomial from homogeneous equation	roots : $z_1,\cdots,z_n$ given by $(z-z_1)\cdots(z-z_n) = z^n + a_{n-1}z^{n-1} + \cdots + a_0$
General solution (distinct roots)	$x_{tr}(t) = k_1 e^{z_1 t} + \cdots + k_n e^{z_n t}$
General solution (non-distinct roots)	$x_{tr}(t) = (k_1 + k_2 t + k_3 t^2)e^{13t} + (k_4 + k_5 t)e^{22t} + k_6 e^{31t} + k_7 e^{41t}$ if roots are $13, 13, 13, 22, 22, 31, 41$

Particular integral:

$x_{ss}(t)$	Any specific solution (with no arbitrary constant) of $$\frac{d^n x(t)}{dt^n}+a_{n-1}\frac{d^{n-1}x(t)}{dt^{n-1}}+\cdots+a_0 x(t)=u(t)$$
Method to determine $x_{ss}(t)$	Trial and error approach: assume $x_{ss}(t)$ to have the same form as $u(t)$ and substitute into differential equation
Example to find $x_{ss}(t)$ for $\frac{dx(t)}{dt}+2x(t)=e^{3t}$	Try a solution of he^{3t} $$\frac{dx(t)}{dt}+2x(t)=e^{3t}\Rightarrow 3he^{3t}+2he^{3t}=e^{3t}\Rightarrow h=0.2$$ $x_{ss}(t)=0.2e^{3t}$
Standard trial solutions	$u(t)$ — trial solution for $x_{ss}(t)$ $e^{\alpha t}$ — $he^{\alpha t}$ t — ht $te^{\alpha t}$ — $(h_1+h_2 t)e^{\alpha t}$ $a\cos(\omega t)+b\sin(\omega t)$ — $h_1\cos(\omega t)+h_2\sin(\omega t)$

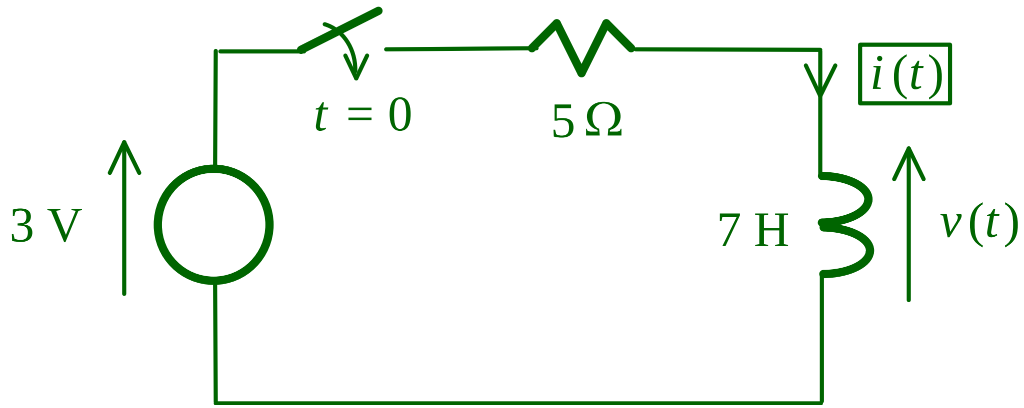
Time-domain System Models

&

Dynamic Responses

RL circuit and governing differential equation

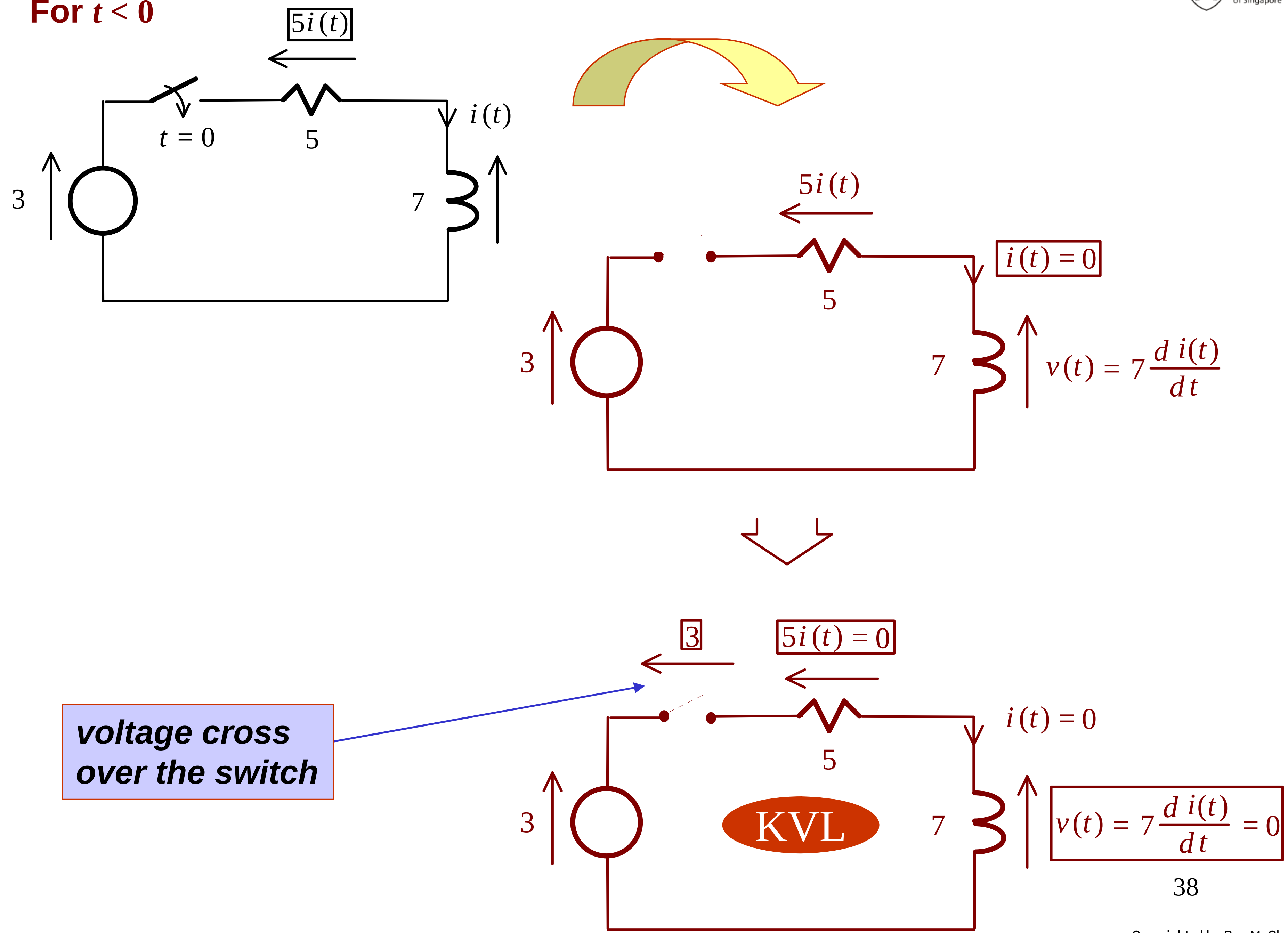
Consider determining $i(t)$ in the following series RL circuit:



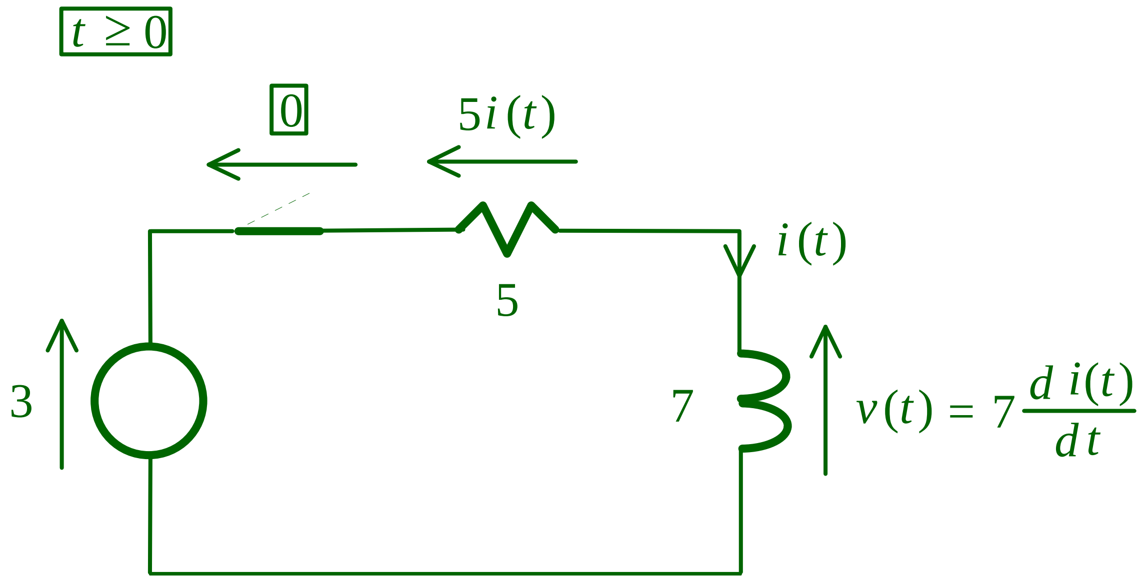
where the switch is open for $t < 0$ and is closed for $t \geq 0$.

Since $i(t)$ and $v(t)$ will not be equal to constants or sinusoids for all time, these cannot be represented as constants or phasors. Instead, the basic general voltage-current relationships for the resistor and inductor have to be used:

For $t < 0$



5i(t)
i(t)
t = 0
5
3
7
5i(t)
i(t) = 0
5
3
7
v(t) = 7 d i(t)/dt
3
5i(t) = 0
i(t) = 0
5
3
KVL
7
v(t) = 7 d i(t)/dt = 0
voltage cross over the switch



Applying KVL:

$$7\frac{di(t)}{dt}+5i(t)=3, \quad t\geq 0$$

and $i(t)$ can be found from determining the ***general solution*** to this first order linear differential equation (d.e.) which governs the behavior of the circuit for $t \geq 0$.

Mathematically, the above d.e. is often written as

$$7\frac{di(t)}{dt}+5i(t)=u(t),\ t\geq 0$$

where the r.h.s. is $u(t)=3,\ t\geq 0$

and corresponds to the dc source or excitation in this example.

Steady state response

Since the r.h.s. of the governing d.e.

$$7\frac{di(t)}{dt}+5i(t)=u(t)=3,\ \ t\geq 0$$

Let us try a steady state solution of

$$i_{ss}(t)=k,\quad t\geq 0$$

which has the same form as $u(t)$, as a possible solution.

$$7\frac{di_{ss}(t)}{dt}+5i_{ss}(t)=3$$

$$\Rightarrow 7(0)+5(k)=3$$

$$\Rightarrow k=\frac{3}{5}$$

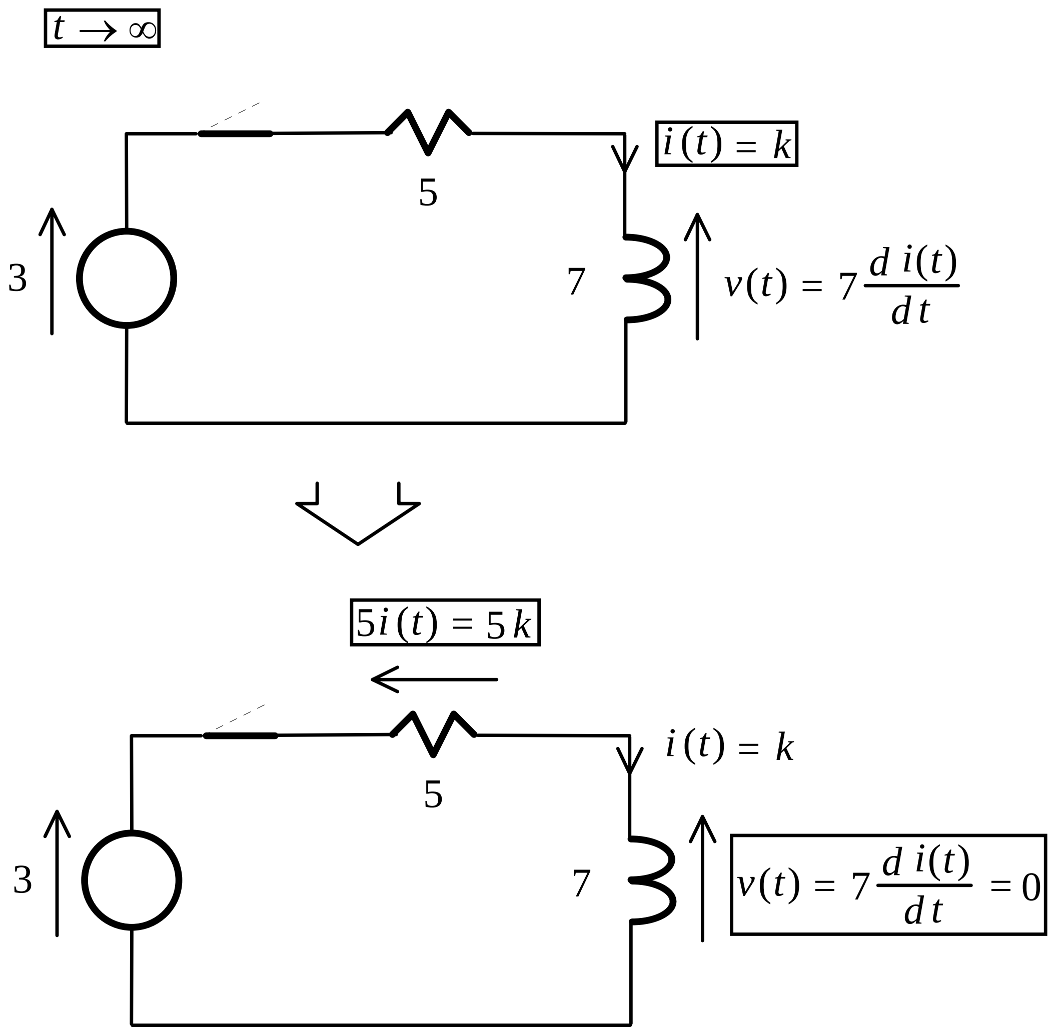
$$i_{ss}(t)=\frac{3}{5},\quad t\geq 0$$

$$7\frac{di_{ss}(t)}{dt}+5i_{ss}(t)=7\frac{d}{dt}\left(\frac{3}{5}\right)+5\left(\frac{3}{5}\right)=3,\quad t\geq 0$$

and is a solution of the governing d.e.

In mathematics, the above solution is called the ***particular integral*** or solution and is found from letting the answer to have the same form as $u(t)$. The word "particular" is used as the solution is only one possible function that satisfy the d.e.

In circuit analysis, the derivation of $i_{ss}(t)$ by letting the answer to have the same form as $u(t)$ can be shown to give the ***steady state response*** of the circuit as $t \to \infty$.



Using KVL, the steady state response is

$$3 = 0 + 5k + 0 = 5k$$

$$\Rightarrow k = \frac{3}{5}$$

$$\Rightarrow i(t) = \frac{3}{5}, \quad t \to \infty$$

This is the same as $i_{ss}(t)$.

Transient response

To determine $i(t)$ for all t, it is necessary to find the complete solution of the governing d.e.

$$7\frac{di(t)}{dt}+5i(t)=u(t)=3,\ \ t\geq 0$$

From mathematics, the complete solution can be obtained from summing a particular solution, say, $i_{ss}(t)$, with $i_{tr}(t)$: $i(t)=i_{ss}(t)+i_{tr}(t),\quad t\geq 0$

where $i_{tr}(t)$ is the general solution of the ***homogeneous*** equation

$$7\frac{di(t)}{dt}+5i(t)=0,\ \ t\geq 0$$

$$7\frac{di_{tr}(t)}{dt}+5i_{tr}(t)\Big|_{\frac{di_{tr}(t)}{dt}\ \text{replaced by}\ z}$$

$$=7z^1+5z^0=7z+5$$

$$\Longrightarrow \qquad z_1=-\frac{5}{7}$$

$$\Longrightarrow \qquad i_{tr}(t)=k_1e^{z_1t}=k_1e^{-\frac{5}{7}t},\ t\geq 0$$

where k_1 is a constant (unknown now).

$$i_{tr}(t)=k_1e^{-\frac{5}{7}t}\rightarrow 0,\ t\rightarrow\infty$$

Thus, it is called ***transient response***.

Complete response

To see that summing $i_{ss}(t)$ and $i_{tr}(t)$ gives the general solution of the governing ODE

$$7\frac{di(t)}{dt}+5i(t)=3,\ \ t\geq 0$$

note that

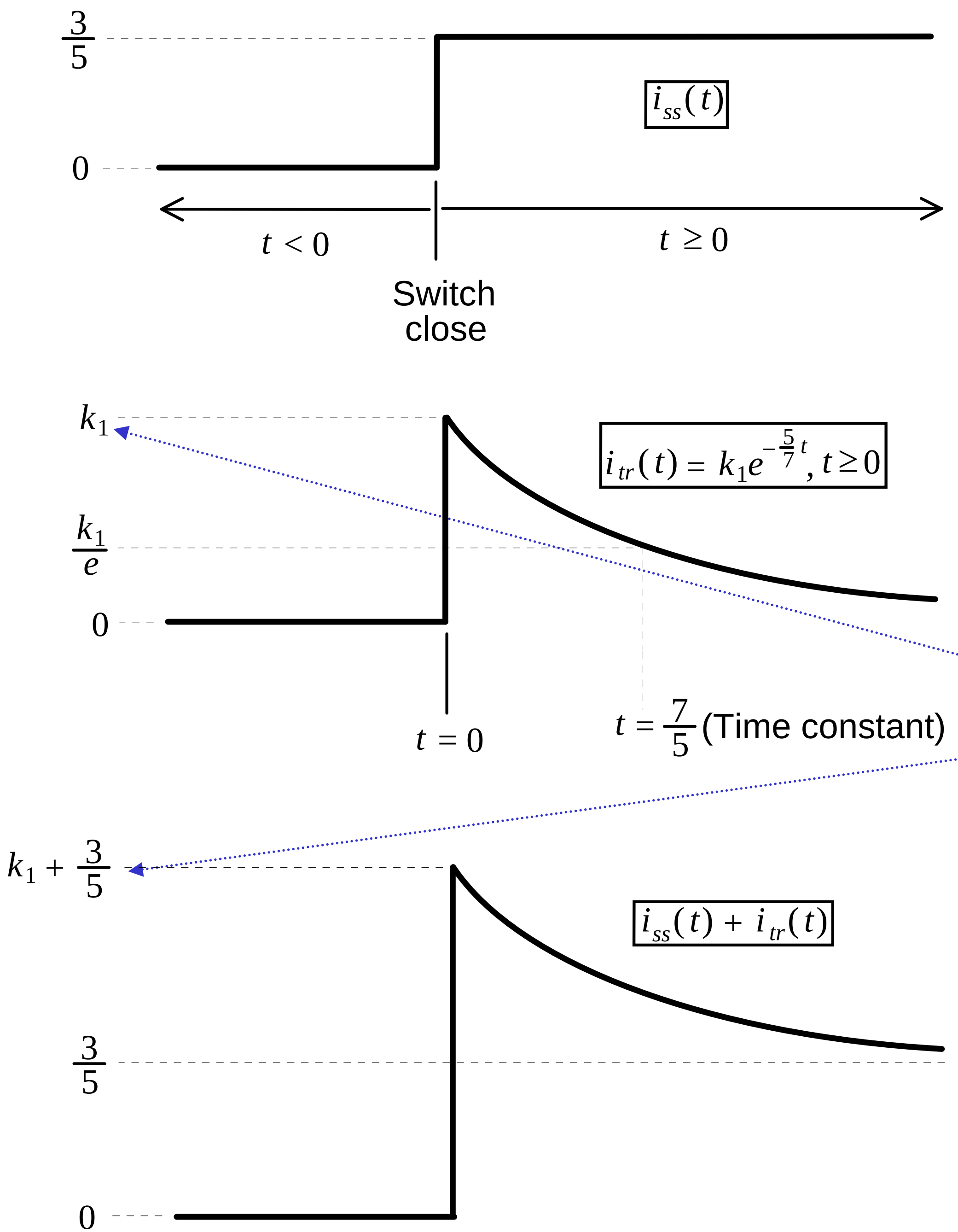
$$i_{ss}(t)=\frac{3}{5}, \quad t\geq 0 \quad \text{satisfies} \quad 7\frac{d}{dt}\left(\frac{3}{5}\right)+5\left(\frac{3}{5}\right)=3, \quad t\geq 0$$

$$i_{tr}(t)=k_1e^{-\frac{5}{7}t}, \quad t\geq 0 \quad \text{satisfies} \quad 7\frac{d}{dt}\left(k_1e^{-\frac{5}{7}t}\right)+5\left(k_1e^{-\frac{5}{7}t}\right)=0, \quad t\geq 0$$

$$\Downarrow$$

$$i_{ss}(t)+i_{tr}(t)=\frac{3}{5}+k_1e^{-\frac{5}{7}t}, \quad t\geq 0 \quad \text{satisfies} \quad 7\frac{d}{dt}\left(\frac{3}{5}+k_1e^{-\frac{5}{7}t}\right)+5\left(\frac{3}{5}+k_1e^{-\frac{5}{7}t}\right)=3$$

$$\Longrightarrow \boxed{i(t)=i_{ss}(t)+i_{tr}(t)=\frac{3}{5}+k_1e^{-\frac{5}{7}t}, \quad t\geq 0}$$ is the general solution of the ODE



steady state response

transient response

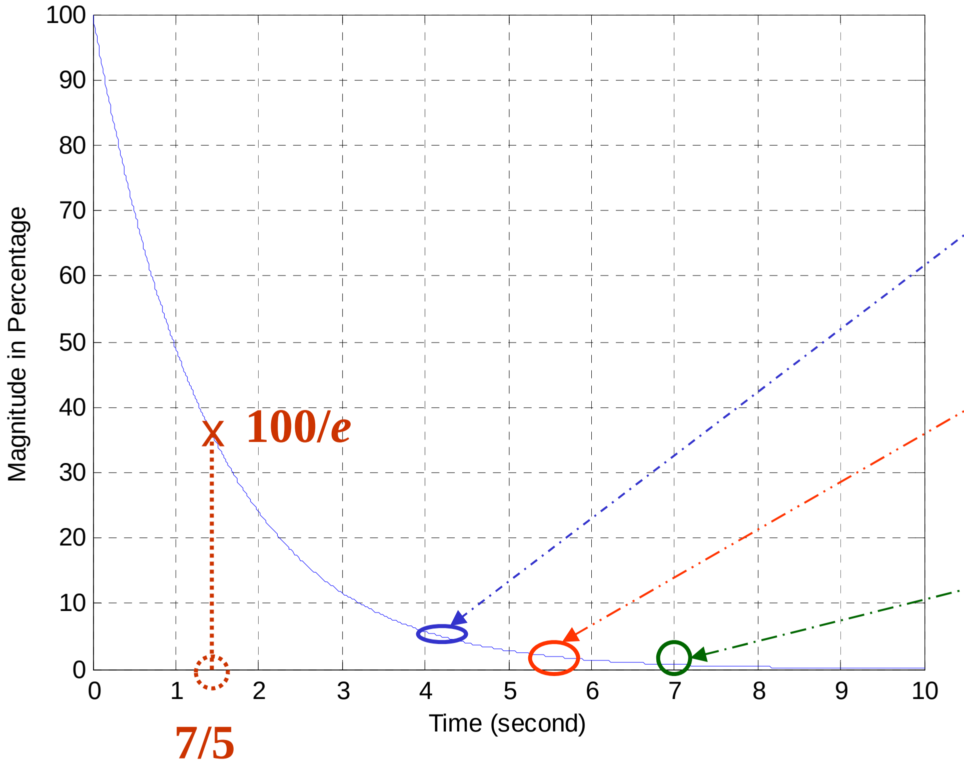
k_1 is to be determined later

complete response

Note that the time it takes for the transient or zero-input response $i_{tr}(t)$ to decay to $1/e$ of its initial value is

$$\text{Time taken for } i_{tr}(t) \text{ to decay to } 1/e \text{ of initial value } = \frac{7}{5}$$

and is called the ***time constant*** of the response or system. We can take the transient response to have died out after a few time constants. For the RC circuit,



At the time equal to **3** time constants, the magnitude is about 5% of the peak.

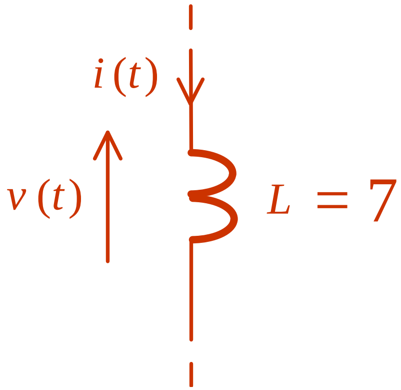
At the time equal to **4** time constants, the magnitude is about 1.83% of the peak.

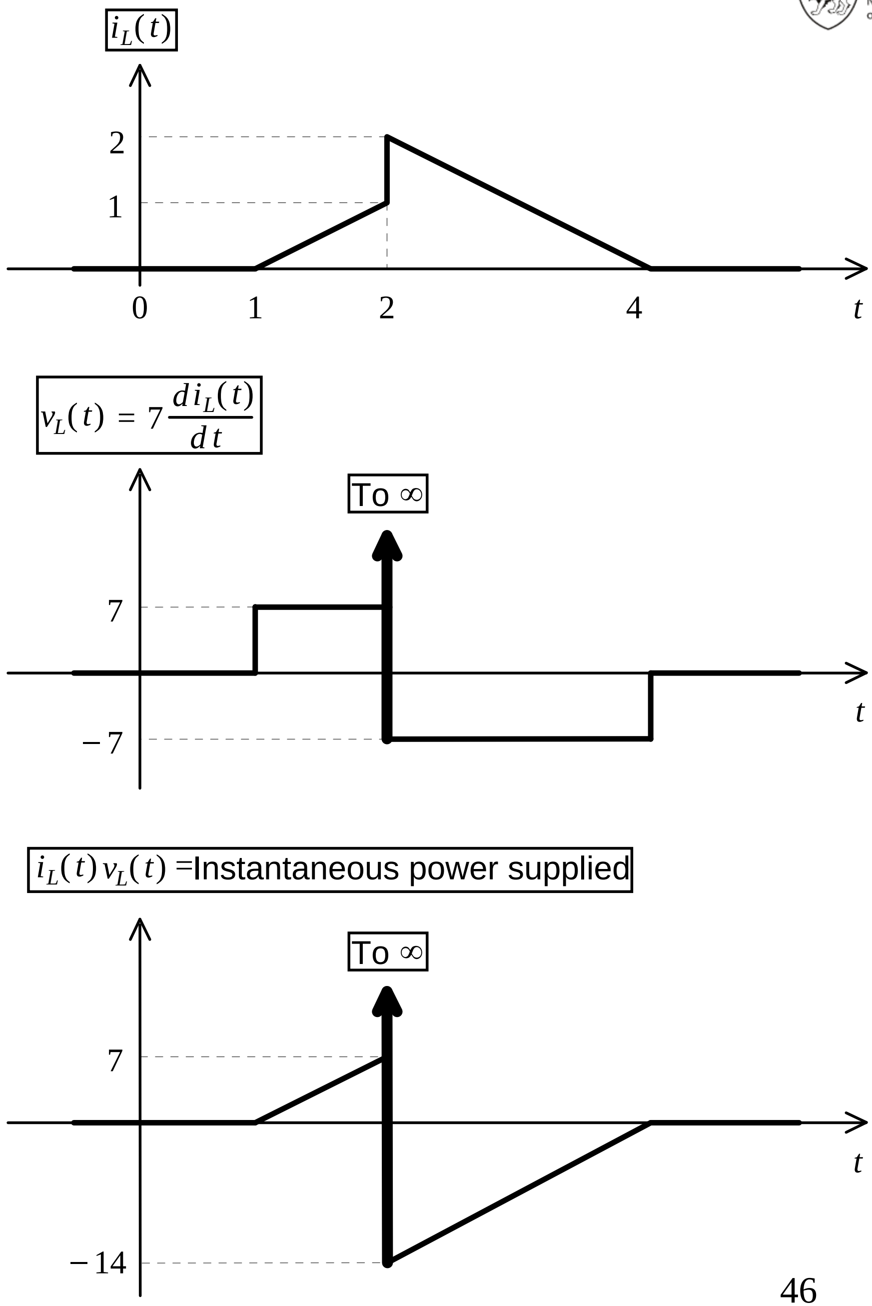
At the time equal to **5** time constants, the magnitude is about 0.68% of the peak

Current continuity for inductor

To determine the constant k_1 in the transient response of the RL circuit, the concept of ***current continuity for an inductor*** has to be used.

Consider the following example for an inductor:





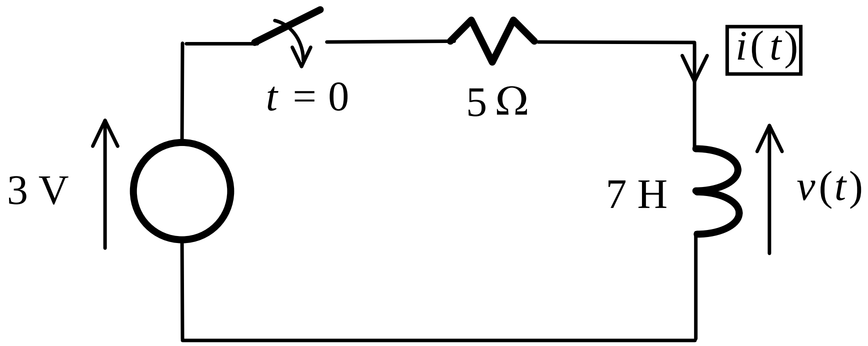
Due to the step change or discontinuity in $i_L(t)$ at $t = 2$, and the power supplied to the inductor at $t = 2$ will go to infinity. Since it is impossible for any system to deliver an infinite amount of power at any time, it is impossible for $i_L(t)$ to change in the manner shown.

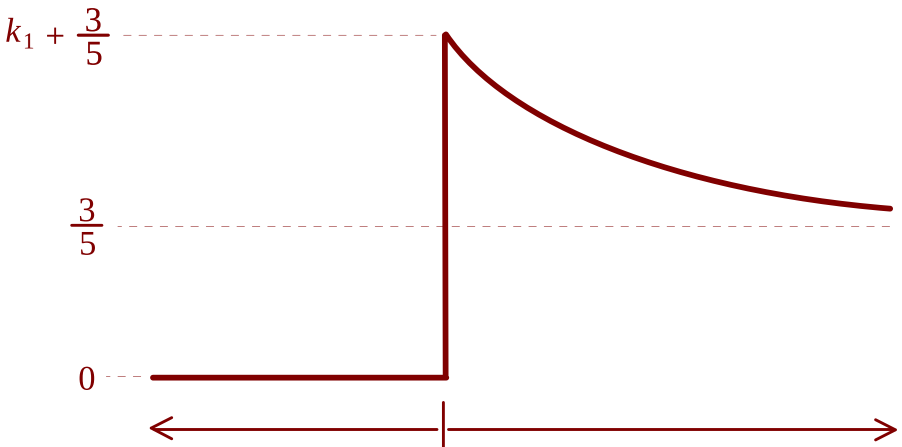
In general, the **current through an inductor must be a continuous function of time** and cannot change in a step manner.

*

Generally speaking, the properties of the current continuity for inductors and the voltage continuity for capacitors (to be covered later) are used to determine, respectively, the initial currents charged to inductors and initial voltages charged to capacitors. These initial voltages and currents are then used to find solutions to transient responses of electric circuits.

Now back to our RL Circuit:





$\boxed{i(t) = 0,\ t < 0}$

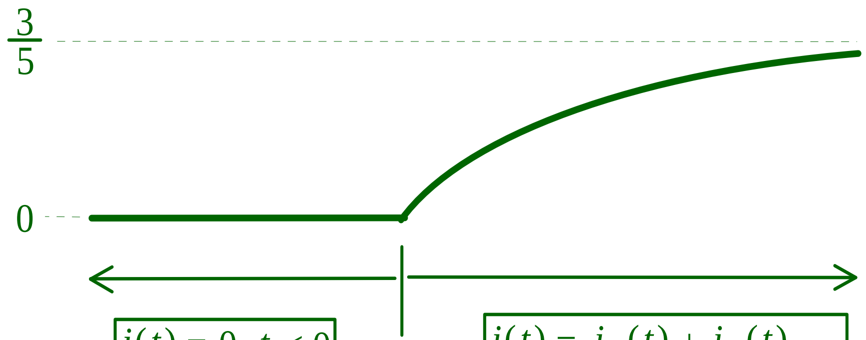
Switch close

$\boxed{\begin{aligned} i(t) &= i_{ss}(t) + i_{tr}(t) \\ &= \frac{3}{5} + k_1 e^{-\frac{5}{7}t},\ t \ge 0 \end{aligned}}$

Using current continuity for an inductor at t = 0:

$$i(t=0) = \frac{3}{5} + k_1 = 0 \quad \Rightarrow \quad k_1 = -\frac{3}{5}$$

$$\Rightarrow \quad i(t) = \begin{cases} 0, & t < 0 \\ \frac{3}{5} - \frac{3}{5} e^{-\frac{5}{7}t}, & t \ge 0 \end{cases}$$



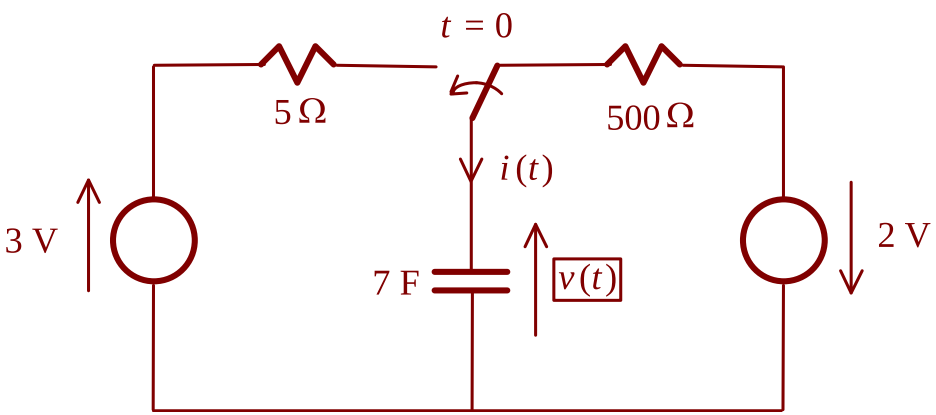
$\boxed{i(t) = 0,\ t < 0}$

Switch close

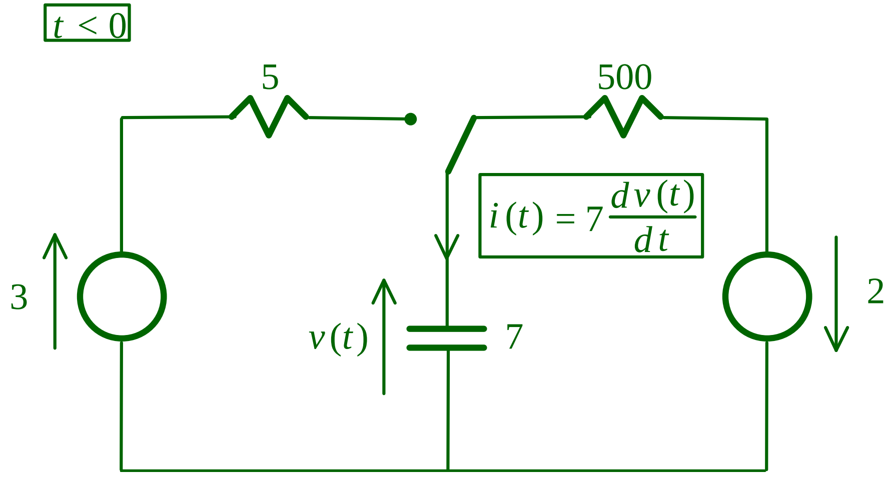
$\boxed{\begin{aligned} i(t) &= i_{ss}(t) + i_{tr}(t) \\ &= \frac{3}{5} - \frac{3}{5} e^{-\frac{5}{7}t},\ t \ge 0 \end{aligned}}$

RC circuit

Consider finding $v(t)$ in the following RC circuit:



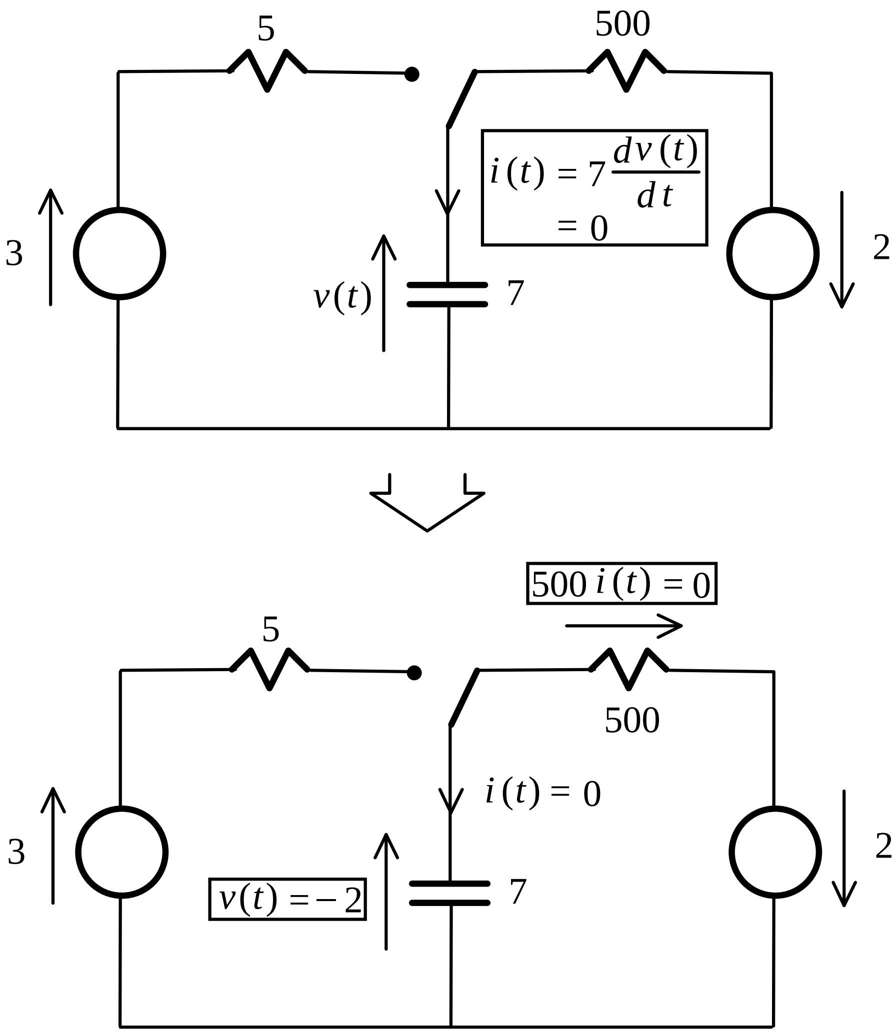
where the switch is in the position shown for $t < 0$ and is in the other position for $t \geq 0$.



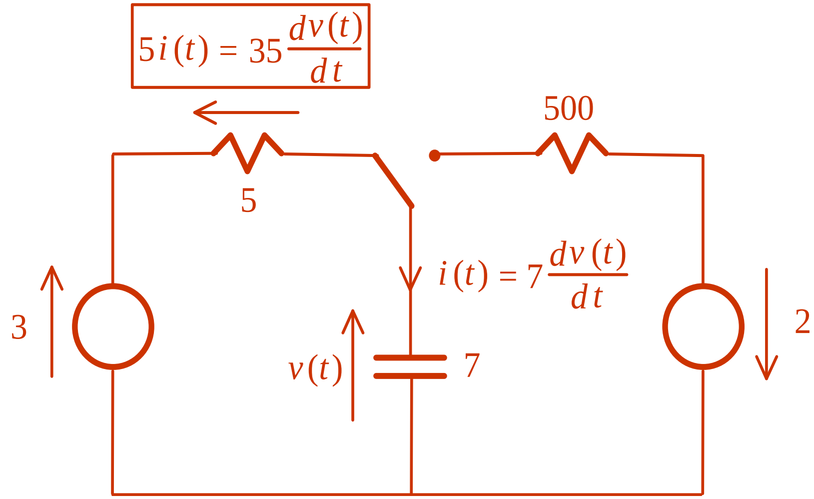
Taking the switch to be in this position starting from $t = -\infty$, the voltages and currents will have settled down to constant values for practically all $t < 0$.

$$i(t) = 7\frac{dv(t)}{dt} = 7\frac{d(\text{constant})}{dt} = 0, \;\; t < 0$$

$t < 0$



$\boxed{t \geq 0}$



Applying KVL:

$$35\frac{dv(t)}{dt} + v(t) = u(t) = 3, \quad t \geq 0$$

which has a solution

$$v(t) = v_{ss}(t) + v_{tr}(t), \quad t \geq 0$$

(1) Steady state response

$$u(t) = 3, \quad t \geq 0$$

$$v_{ss}(t) = k, \quad t \geq 0$$

$$35\frac{dv_{ss}(t)}{dt} + v_{ss}(t) = 3 \Rightarrow 0 + k = 3 \Rightarrow k = 3$$

$$v_{ss}(t) = 3, \quad t \geq 0$$

(2) Transient response

$$35\frac{dv_{tr}(t)}{dt} + v_{tr}(t) = 0, \quad t \geq 0$$

$$35\frac{dv_{tr}(t)}{dt} + v_{tr}(t) \Bigg|_{\frac{dv_{tr}(t)}{dt} \text{ replaced by } z}$$

$$= 35z^1 + z^0 = 35z + 1$$

$$\Longrightarrow \quad z_1 = -\frac{1}{35}$$

$$\Longrightarrow \quad v_{tr}(t) = k_1 e^{z_1 t} = k_1 e^{-\frac{t}{35}}, \quad t \geq 0$$

$$v(t) = \begin{cases} -2, & t < 0 \\ v_{ss}(t) + v_{tr}(t), & t \geq 0 \end{cases} = \begin{cases} -2, & t < 0 \\ 3 + k_1 e^{-\frac{t}{35}}, & t \geq 0 \end{cases}$$

$\Longleftarrow$ **Complete response**

Voltage continuity for capacitor

To determine k_1 in the transient response of the RC circuit, the concept of ***voltage continuity for a capacitor*** has to be used.

Similar to current continuity for an inductor, the voltage $v(t)$ across a capacitor C must be continuous and cannot change in a step manner.

Thus, for the RC circuit we consider, the complete solution was derived as:

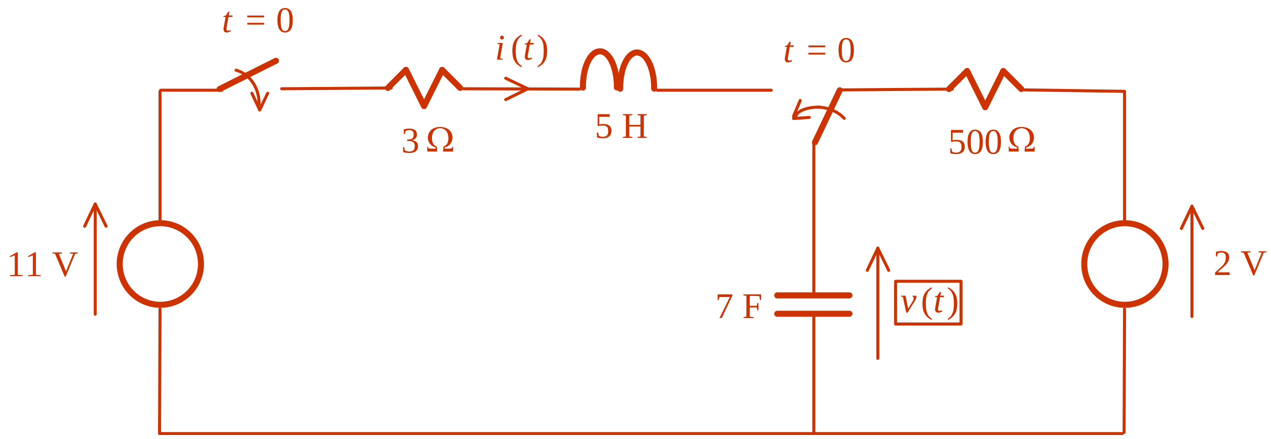
$$v(t) = \begin{cases} -2, & t < 0 \\ v_{ss}(t) + v_{tr}(t), & t \geq 0 \end{cases} = \begin{cases} -2, & t < 0 \\ 3 + k_1 e^{-\frac{t}{35}}, & t \geq 0 \end{cases}$$

At $t = 0$,

$$v(0) = 3 + k_1 = -2 \;\Rightarrow\; k_1 = -5 \quad \Longrightarrow \quad v(t) = \begin{cases} -2, & t < 0 \\ 3 - 5e^{-\frac{t}{35}}, & t \geq 0 \end{cases}$$

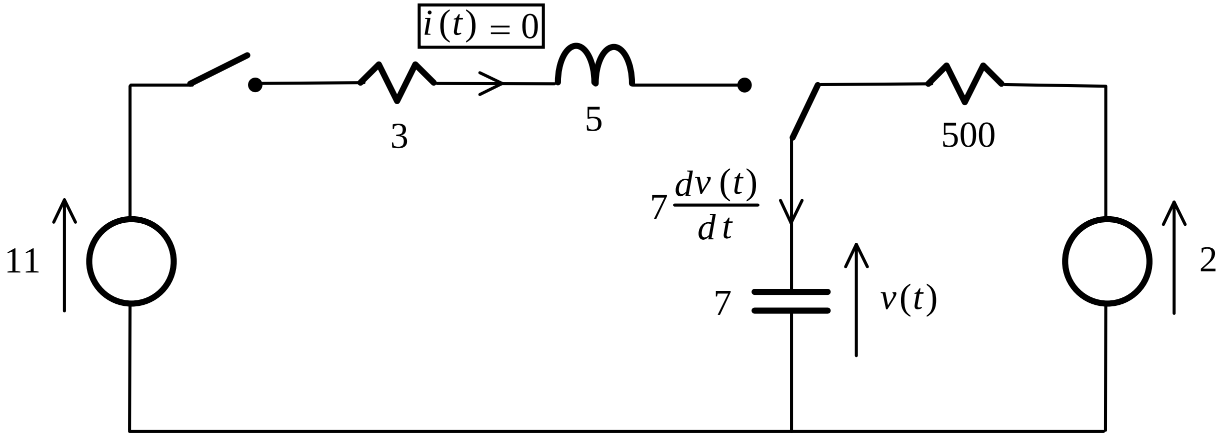
Second order RLC circuit

Consider determining $v(t)$ in the following series RLC circuit:

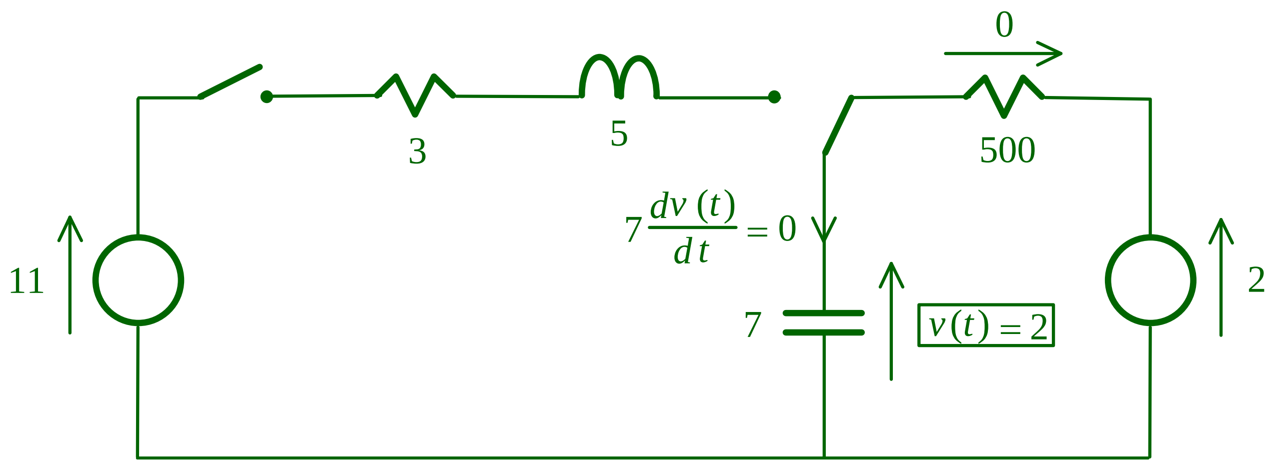


Both switches are in the position shown for $t < 0$ & are in the other positions for $t \geq 0$.

For $t < 0$



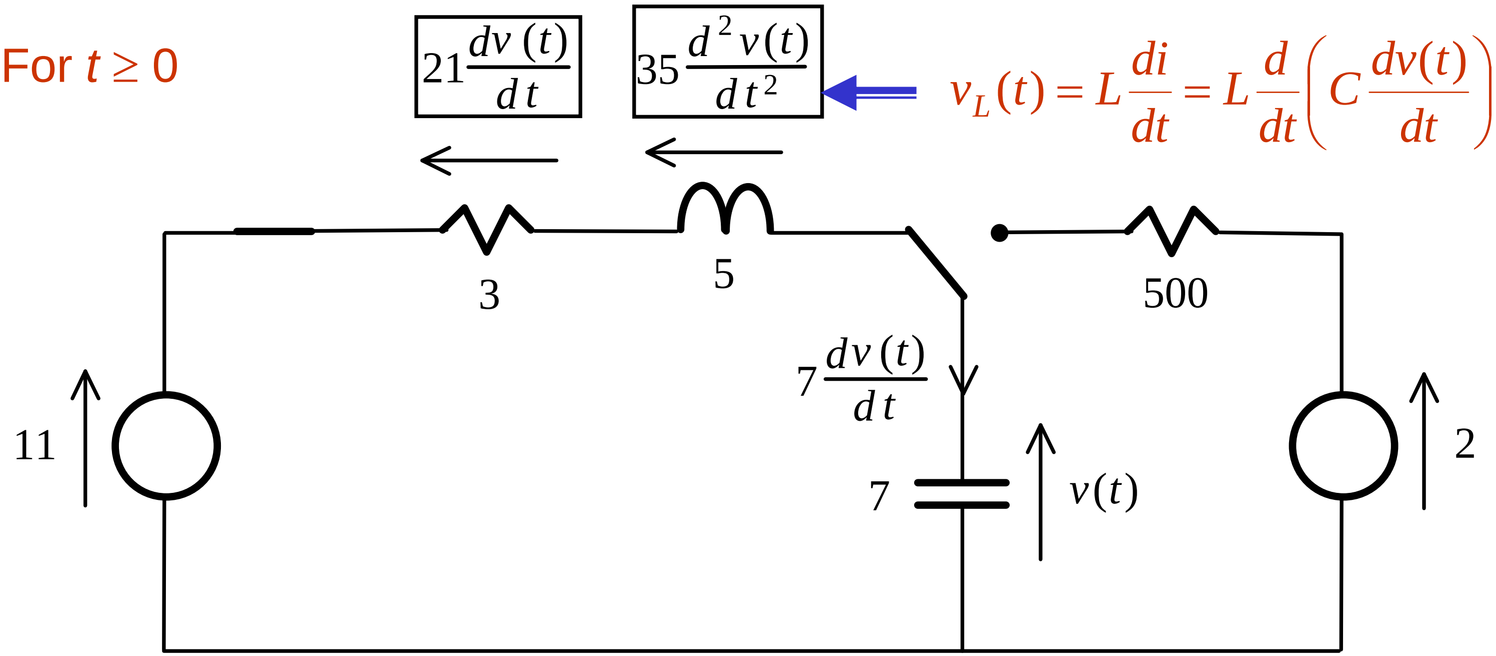
Taking the switches to be in the positions shown starting from $t = -\infty$, the voltages and currents will have settled down to constant values for practically all $t < 0$ and the important voltages and currents are given by:



Mathematically:

$$v(t) = 2, \quad t < 0 \qquad \& \qquad i(t) = 0, \quad t < 0$$

For $t \geq 0$



$$v_L(t) = L\frac{di}{dt} = L\frac{d}{dt}\left(C\frac{dv(t)}{dt}\right) = LC\frac{d^2v(t)}{dt^2}$$

Applying KVL:

$$35\frac{d^2v(t)}{dt^2} + 21\frac{dv(t)}{dt} + v(t) = u(t) = 11, \quad t \geq 0$$

Due to the presence of 2 energy storage elements, the governing d.e. is a second order one and the general solution is

$$v(t) = v_{ss}(t) + v_{tr}(t), \quad t \geq 0$$

(1) Steady state response

$$u(t) = 11, \quad t \ge 0 \quad \Longrightarrow \quad v_{ss}(t) = k, \quad t \ge 0 \quad \Longrightarrow$$

$$35\frac{d^2 v_{ss}(t)}{dt^2} + 21\frac{dv_{ss}(t)}{dt} + v_{ss}(t) = 0 + 0 + k = 11 \quad \Longrightarrow \quad \boxed{v_{ss}(t) = 11, \quad t \ge 0}$$

(2) Transient response

$$35\frac{d^2 v_{tr}(t)}{dt^2} + 21\frac{dv_{tr}(t)}{dt} + v_{tr}(t) = 0, \quad t \ge 0 \quad \Longrightarrow$$

$$35\frac{d^2 v_{tr}(t)}{dt^2} + 21\frac{dv_{tr}(t)}{dt} + v_{tr}(t)\Bigg|_{\frac{dv_{tr}(t)}{dt} \text{ replaced by } z} = 35z^2 + 21z^1 + z^0 = 35z^2 + 21z + 1$$

$$\Longrightarrow \quad z_1, z_2 = \frac{-21 \pm \sqrt{21^2 - 4(35)(1)}}{2(35)} = \frac{-21 \pm 17}{2(35)} = -0.548, -0.052$$

$$\boxed{v_{tr}(t) = k_1 e^{z_1 t} + k_2 e^{z_2 t} = k_1 e^{-0.548t} + k_2 e^{-0.052t}, \quad t \ge 0} \quad \Longleftarrow$$

Complete solution (response) *to be determined*

$$v(t)=\begin{cases}2, & t<0\\ v_{ss}(t)+v_{tr}(t), & t\ge 0\end{cases} = \begin{cases}2, & t<0\\ 11+k_1e^{-0.548t}+k_2e^{-0.052t}, & t\ge 0\end{cases}$$

$$\Rightarrow \quad i(t)=7\frac{dv(t)}{dt}=\begin{cases}0, & t<0\\ 7(-0.548k_1e^{-0.548t}-0.052\,k_2e^{-0.052t}), & t\ge 0\end{cases}$$

To determine k_1 and k_2, voltage continuity for the capacitor and current continuity for the inductor have to be used.

The voltage across the capacitor at $t = 0$:

$$v(0)=11+k_1+k_2=2 \quad \Rightarrow \quad k_1+k_2=-9$$

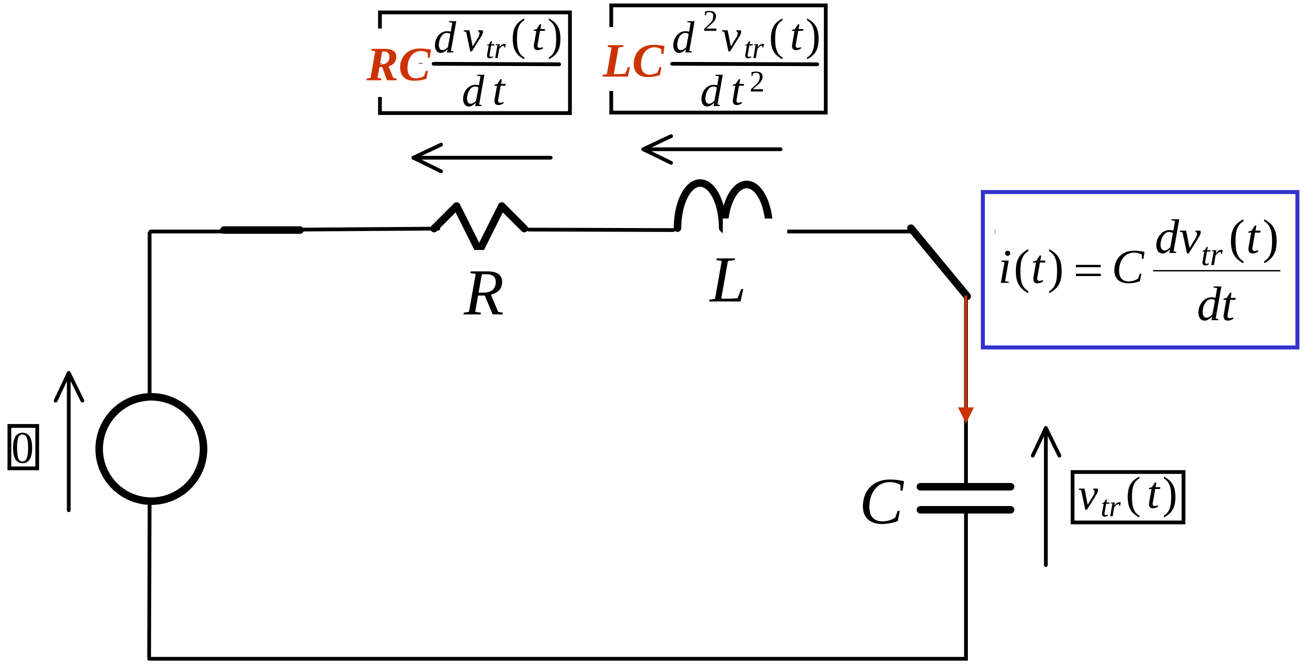
The current passing through the inductor at $t = 0$:

$$i(0)=-0.548k_1-0.052k_2=0 \quad \Rightarrow \quad 0.548k_1+0.052k_2=0$$

$$\boxed{k_1=0.95 \qquad k_2=-9.95}$$

General RLC circuit

For $t \geq 0$



By KVL:

$$LC\frac{d^2v_{tr}(t)}{dt^2} + RC\frac{dv_{tr}(t)}{dt} + v_{tr}(t) = 0, \quad t \geq 0$$

$$\left. LC\frac{d^2v_{tr}(t)}{dt^2} + RC\frac{dv_{tr}(t)}{dt} + v_{tr}(t) \right|_{\frac{dv_{tr}(t)}{dt} \text{ replaced by } z} = LCz^2 + RCz^1 + 1 = 0$$

$$\Rightarrow \quad z_1,\, z_2 = \frac{-RC \pm \sqrt{(RC)^2 - 4LC}}{2LC}$$

Recall that for RLC circuit, the Q factor is defined as

$$Q = \frac{2\pi f_0 L}{R} = \frac{2\pi L}{R} \frac{1}{2\pi\sqrt{LC}} = \frac{L}{R\sqrt{LC}} = \frac{\sqrt{LC}}{RC}$$

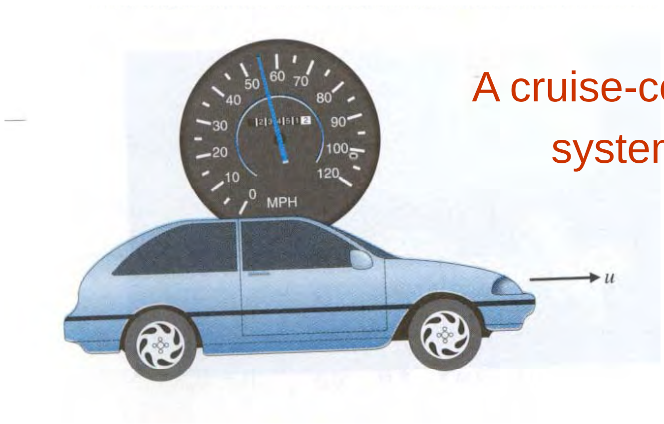
Thus,

$$z_1, z_2 = \frac{-RC \pm \sqrt{(RC)^2 - 4LC}}{2LC} = \frac{-RC \pm RC\sqrt{1 - 4\frac{LC}{(RC)^2}}}{2LC} = \frac{-R \pm R\sqrt{1-4Q^2}}{2L}$$

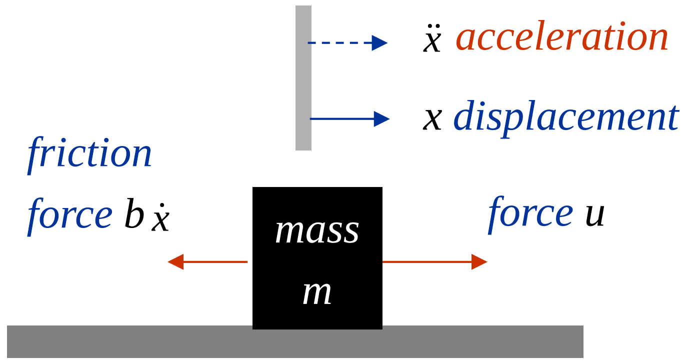
$$= \begin{cases} \text{two real distinct roots if } 1 - 4Q^2 > 0 \text{ or } Q^2 < 1/4 \text{ or } Q < 1/2 \\ \text{two complex conjugate roots if } 1 - 4Q^2 < 0 \text{ or } Q > 1/2 \\ \text{two identical roots if } 1 - 4Q^2 = 0 \text{ or } Q = 1/2 \end{cases}$$

The behaviors of the about cases will be studied in details later…

A cruise control system



A cruise-control system



By the well-known Newton's Law of motion: $f = m\,a$, where f is the total force applied to an object with a mass m and a is the acceleration, we have

$$u - b\dot{x} = m\ddot{x} \qquad \Leftrightarrow \qquad \ddot{x} + \frac{b}{m}\dot{x} = \frac{u}{m}$$

This a 2nd order *Ordinary Differential Equation* with respect to displacement x. It can be written as a 1st order *ODE* with respect to speed v = $\dot{x}$:

$$\boxed{\dot{v} + \frac{b}{m} v = \frac{u}{m}}$$

← model of the cruise control system, u is input force, v is output.

Assume a passenger car weights 1 ton, i.e., $m = 1000$ kg, and the friction coefficient of a certain situation $b = 100$ N·s/m. Assume that the input force generated by the car engine is $u = 1000$ N and the car is initially parked, i.e., $x(0) = 0$ and $v(0) = 0$. Find the solutions for the car velocity $v(t)$ and displacement $x(t)$.

For the velocity model,

$$\dot{v} + \frac{b}{m} v = \frac{u}{m} \quad \Rightarrow \quad \dot{v} + 0.1v = 1$$

The steady state response: It is obvious that $v_{ss} = 10$ m/s $= 36$ km/h

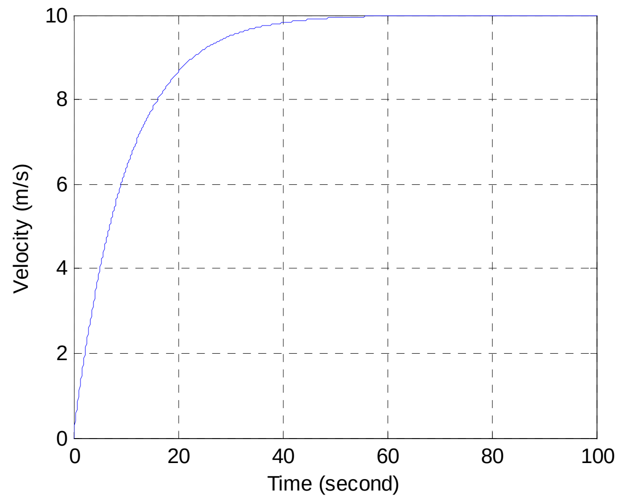
The transient response: Characteristic polynomial $z + 0.1 = 0$, which gives $z_1 = -0.1$.

$$v_{tr}(t) = k_1 e^{-0.1t} \quad \Rightarrow v(t) = v_{ss} + v_{tr}(t) = 10 + k_1 e^{-0.1t}$$

$v(0) = 0$ implies that $k_1 = -10$ and hence

$$\boxed{v(t) = v_{ss} + v_{tr}(t) = 10 - 10e^{-0.1t}}$$

What is the time constant for this system?



For the dynamic model in terms of displacement,

$$u - b\dot{x} = m\ddot{x} \quad \Leftrightarrow \quad \ddot{x} + 0.1\dot{x} = 1$$

The steady state response: From the solution for the velocity, which is a constant, we can conclude that the steady state solution for the displacement is $x_{\rm ss} = v_{\rm ss} t = 10t$.

The transient response: Characteristic polynomial $z^2 + 0.1z = 0$, which gives $z_1 = -0.1$ and $z_2 = 0$. The transient solution is then given by

$$x_{\rm tr}(t) = k_1 e^{-0.1t} + k_2 e^{0t} = k_1 e^{-0.1t} + k_2$$

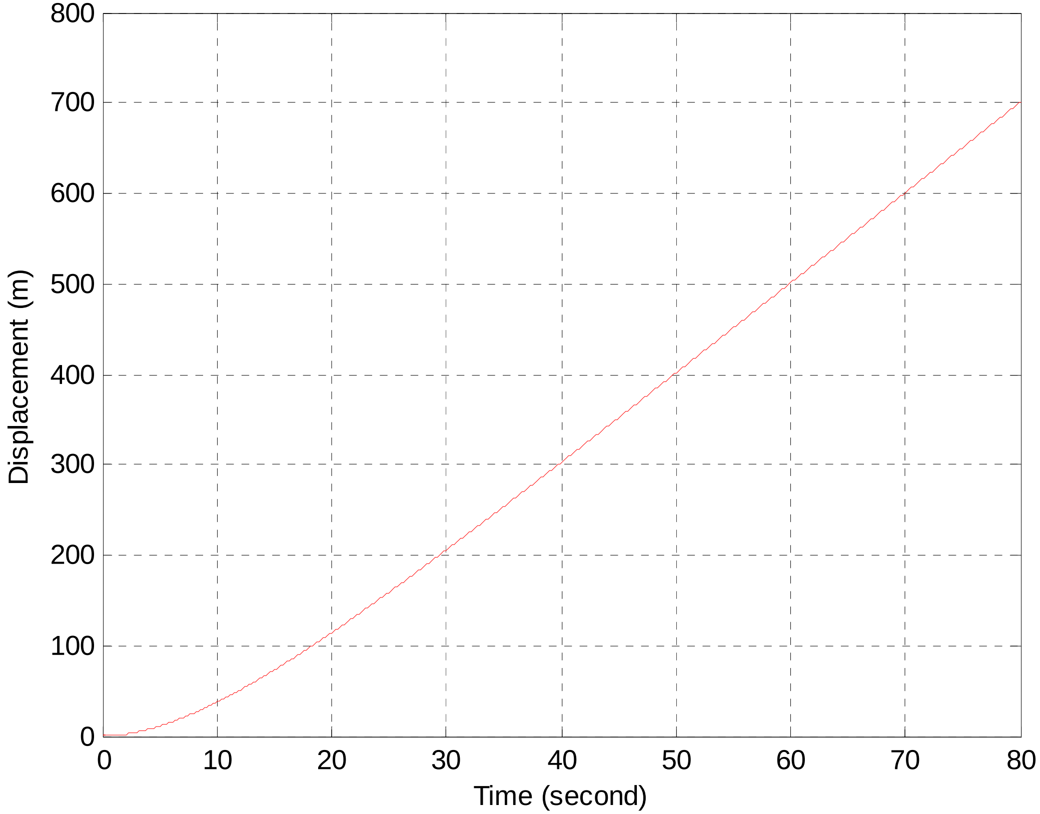
and hence the complete solution

$$x(t) = x_{\rm ss} + x_{\rm tr}(t) = 10t + k_1 e^{-0.1t} + k_2 \quad \Rightarrow \quad v(t) = \dot{x}(t) = 10 - 0.1 k_1 e^{-0.1t}$$

$x(0) = 0$ implies $k_1 + k_2 = 0$ and $v(0) = 0$ implies $10 - 0.1\, k_1 = 0$. Thus, $k_1 = 100$, $k_2 = -100$.

$$\boxed{x(t) = 10t + 100e^{-0.1t} - 100}$$

Complete response for the car displacement



Exercise: Show that the car cruise control system is BIBO stable for its velocity model and it BIBO unstable for its displacement model.

Behaviors of a general 2nd order system

Consider a general 2nd order system (an RLC circuit or a mechanical system or whatever) governed by an ODE

$$\boxed{a\ddot{y}(t) + b\dot{y}(t) + cy(t) = u(t)}$$

Its transient response (or **natural response**) is fully characterized the properties of its homogeneous equation or its characteristic polynomial, i.e.,

$$a\ddot{y}(t) + b\dot{y}(t) + cy(t) = 0 \quad \Rightarrow \quad az^2 + bz + c = 0$$

The latter has two roots at

$$z_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \begin{cases} \text{two real distinct roots if } b^2 - 4ac > 0 \\ \text{two complex conjugate roots if } b^2 - 4ac < 0 \\ \text{two identical roots if } b^2 - 4ac = 0 \end{cases}$$

These different types of roots give different natures of responses.

Overdamped systems

Overdamped response is referred to the situation when the characteristic polynomial has two distinct **negative** real roots, i.e., $ab > 0$ & $b^2 - 4ac > 0$. For example,

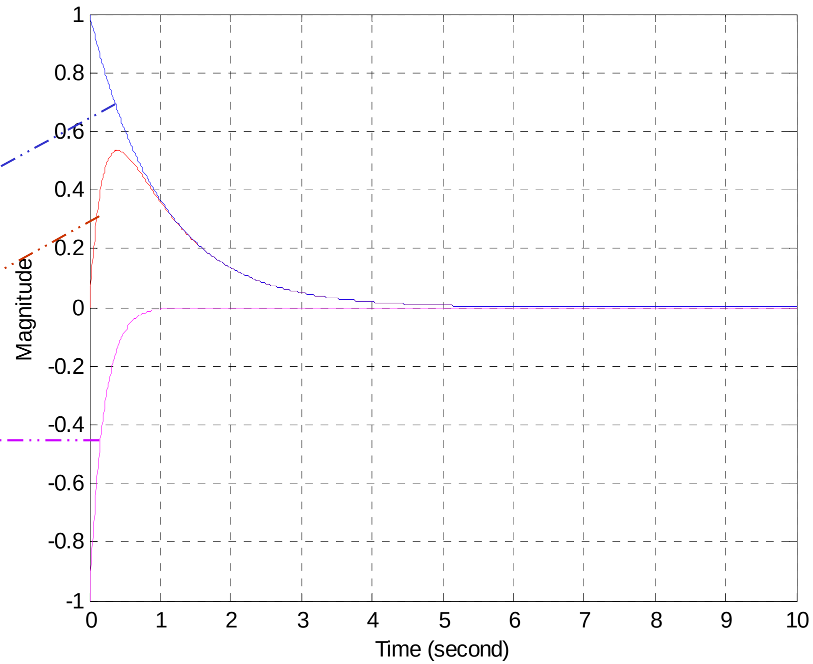
$$\boxed{\ddot{y}(t) + 6\dot{y}(t) + 5y(t) = 0}$$

which has a characteristic polynomial,

$$z^2 + 6z + 5 = 0 \quad \Rightarrow z_{1,2} = -1, -5 \quad \Rightarrow \quad y(t) = k_1 e^{-t} + k_2 e^{-5t}, \quad \dot{y}(t) = -k_1 e^{-t} - 5k_2 e^{-5t}$$

Assume that $y(0) = 0,\ \dot{y}(0) = 4$, which implies

$$\left.\begin{aligned} y(0) &= k_1 + k_2 = 0 \\ \dot{y}(0) &= -k_1 - 5k_2 = 4 \end{aligned}\right\} k_1 = 1,\ k_2 = -1$$



and thus,

$$\boxed{y(t) = e^{-t} - e^{-5t}}$$

What is the dominating time constant?

Underdamped systems

Underdamped response is referred to the situation when the characteristic polynomial has two complex conjugated roots **negative** real part, i.e., $ab > 0$ & $b^2 - 4ac < 0$. For example,

$$\boxed{\ddot{y}(t) + 2\dot{y}(t) + 101y(t) = 0}$$

which has a characteristic polynomial,

$$z^2 + 2z + 101 = 0 \quad \Rightarrow z_{1,2} = -1 \pm j10 \quad \Rightarrow \quad y(t) = k_1 e^{(-1+j10)t} + k_2 e^{(-1-j10)t} = e^{-t}(k_1 e^{j10t} + k_2 e^{-j10t})$$

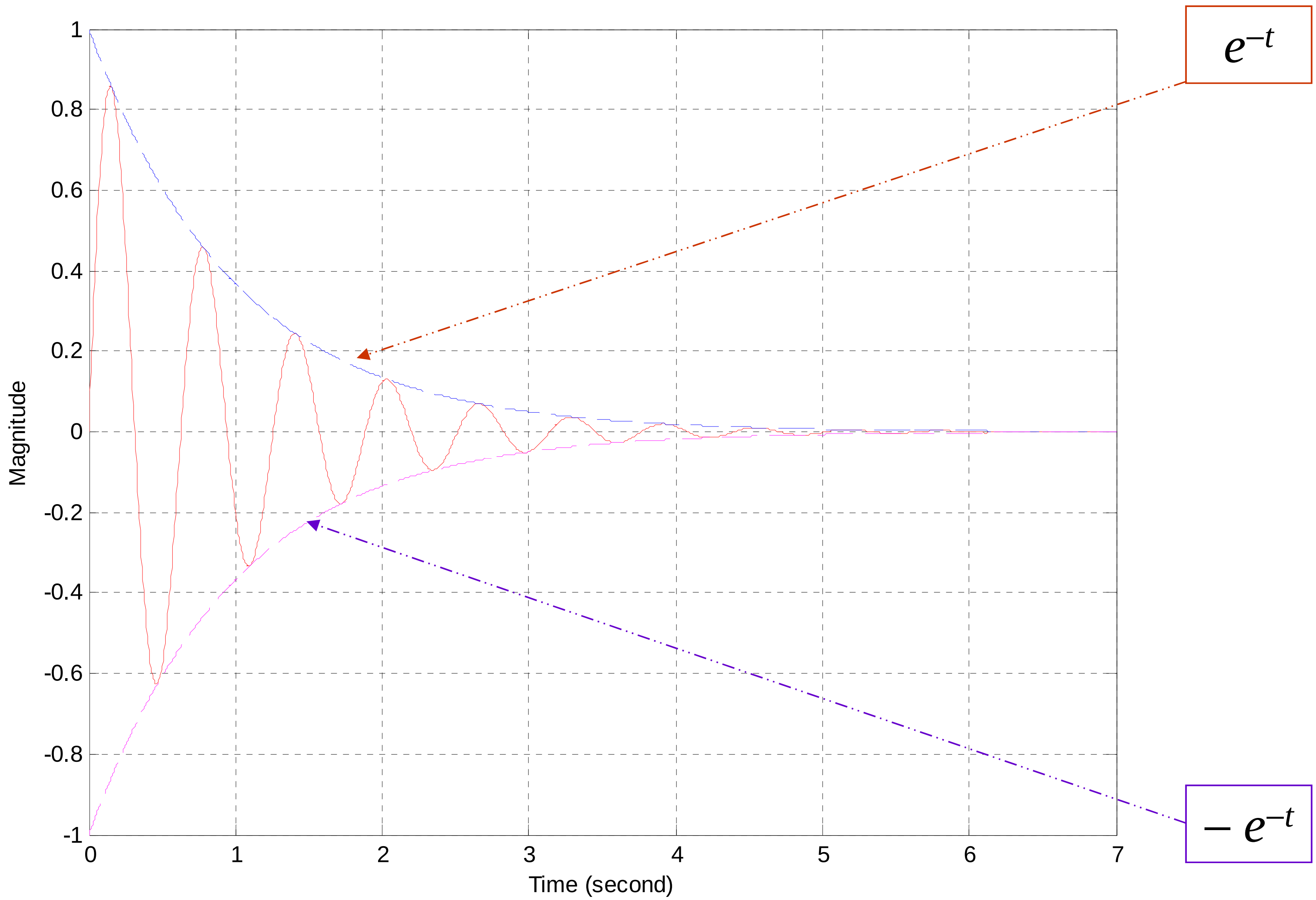
$$y(t) = e^{-t}[k_1(\cos 10t + j\sin 10t) + k_2(\cos 10t - j\sin 10t)] = e^{-t}[(k_1 + k_2)\cos 10t + j(k_1 - k_2)\sin 10t]$$

Assume that $y(0) = 0,\ \dot{y}(0) = 10$, which implies

$$\left.\begin{aligned} y(0) &= k_1 + k_2 = 0 \\ \dot{y}(0) &= -(k_1 + k_2) + j10(k_1 - k_2) = 10 \end{aligned}\right\} \quad j(k_1 - k_2) = 1 \quad \Rightarrow \quad \boxed{y(t) = e^{-t}\sin 10t}$$

The time constant for such a system is determined by the exponential term.

Underdamped response



Critically damped systems

Critically damped response is corresponding to the situation when the characteristic polynomial has two identical **negative** real roots, i.e., $ab > 0$ & $b^2 - 4ac = 0$. For example,

$$\boxed{\ddot{y}(t) + 2\dot{y}(t) + y(t) = 0}$$

which has a characteristic polynomial,

$$z^2 + 2z + 1 = 0 \quad \Rightarrow z_{1,2} = -1 \quad \Rightarrow \quad y(t) = k_1 e^{-t} + k_2 t e^{-t} = e^{-t}(k_1 + k_2 t)$$

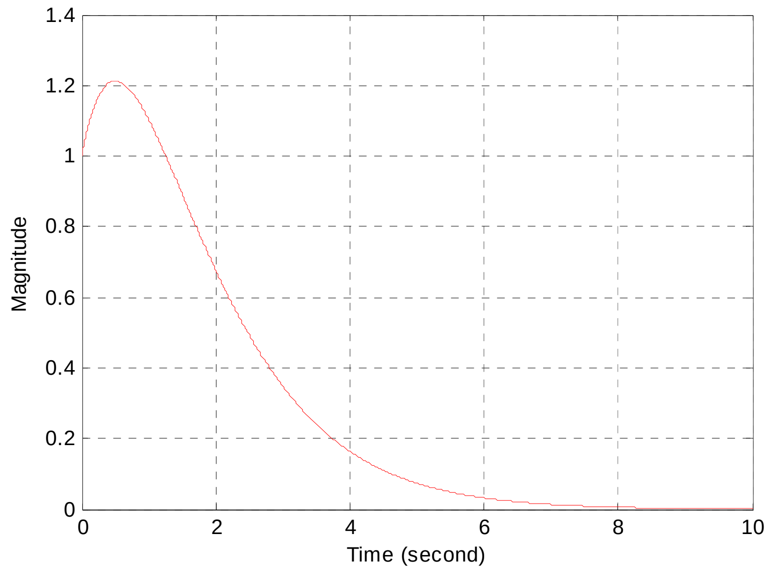
$$\dot{y}(t) = e^{-t}(k_2 - k_1 - k_2 t)$$

Assume that $y(0) = 1,\ \dot{y}(0) = 1$, which implies

$$\left.\begin{aligned} y(0) &= k_1 = 1 \\ \dot{y}(0) &= k_2 - k_1 = 1 \end{aligned}\right\} \quad k_1 = 1, k_2 = 2$$

and thus

$$\boxed{y(t) = e^{-t}(1 + 2t)}$$



Never damped (unstable) systems

Never damped response is corresponding to the situation when the characteristic polynomial has at least one root with a **nonnegative** real part. For example,

$$\boxed{\ddot{y}(t) - y(t) = 0}$$

which has a characteristic polynomial,

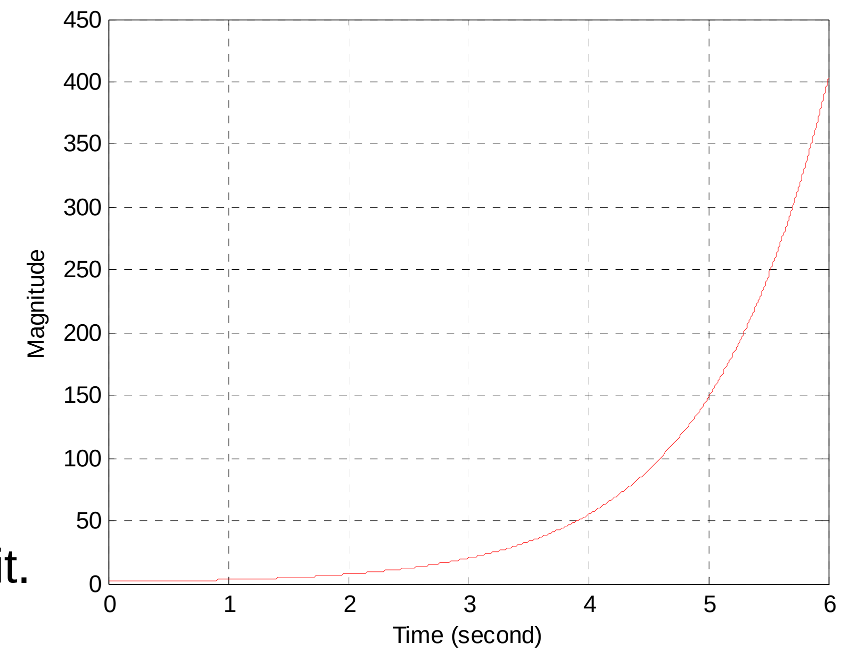
$$z^2 - 1 = 0 \quad \Rightarrow z_{1,2} = \pm 1 \quad \Rightarrow \quad y(t) = k_1 e^{-t} + k_2 e^t \quad \Rightarrow \quad \dot{y}(t) = -k_1 e^{-t} + k_2 e^t$$

Assume that $y(0) = 2,\ \dot{y}(0) = 0$, which implies

$$\left.\begin{aligned} y(0) &= k_1 + k_2 = 2 \\ \dot{y}(0) &= k_2 - k_1 = 0 \end{aligned}\right\} \quad k_1 = k_2 = 1$$

and thus

$$\boxed{y(t) = e^{-t} + e^t}$$

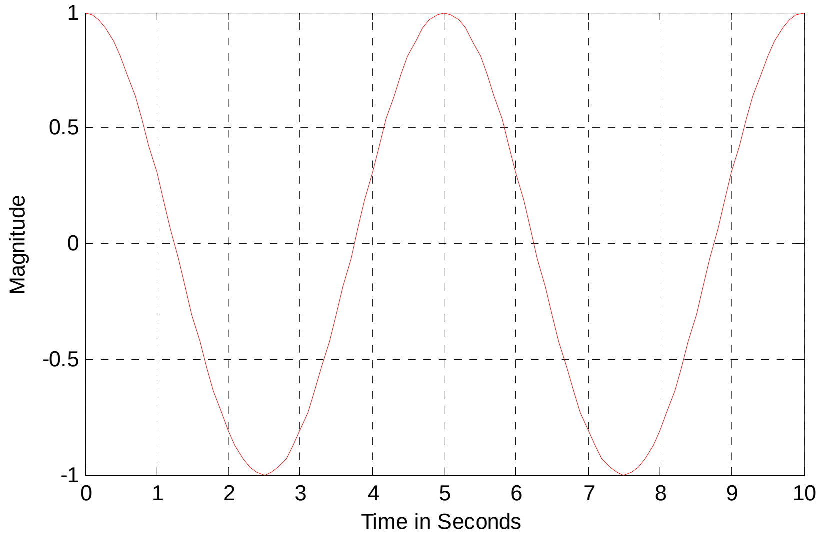


It is an unstable system. We'll study more on it.

Review of Laplace Transforms

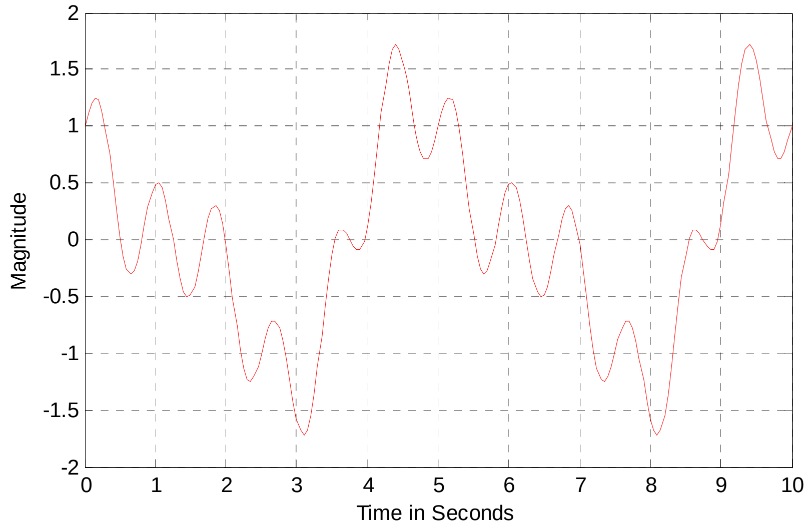
Introduction

Let us first examine the following time-domain functions:



A cosine function with a frequency $f = 0.2$ Hz.

Note that it has a period $T = 5$ seconds.



$$x(t) = \cos(0.4\pi t) + \sin(0.8\pi t)\cos(1.6\pi t)$$

What are frequencies of this function?

Laplace transform is a tool to convert time-domain functions into a frequency-domain ones in which information about frequencies of the function can be captured. It is often much easier to solve problems in frequency-domain with the help of Laplace transform.

Laplace Transform

Given a time-domain function $f(t)$, the one-sided Laplace transform is defined as follows:

$$\boxed{F(s) = L\{f(t)\} = \int_{0^-}^{\infty} f(t) e^{-st} dt, \quad s = \sigma + j\omega}$$

where the lower limit of integration is set to 0^- to include the origin ($t = 0$) and to capture any discontinuities of the function at $t = 0$.

The integration for the Laplace transform might not convert to a finite solution for arbitrary time-domain function. In order for the integration to convert to a final value, which implies that the Laplace transform for the given function is existent, we need

$$|F(s)| = \left| \int_{0^-}^{\infty} f(t) e^{-(\sigma + j\omega)t} dt \right| \le \int_{0^-}^{\infty} |f(t)| \cdot e^{-\sigma t} \cdot \left| e^{-j\omega t} \right| dt = \int_{0^-}^{\infty} |f(t)| \cdot e^{-\sigma t} dt < \infty$$

for some real scalar $\sigma = \sigma_c$. Clearly, the integration exists for all $\sigma \ge \sigma_c$, which is called the **region of convergence** (ROC). Laplace transform is undefined outside of ROC.

Example 1: Find the Laplace transform of a **unit step** function $f(t) = 1(t) = 1,\ t \ge 0.$

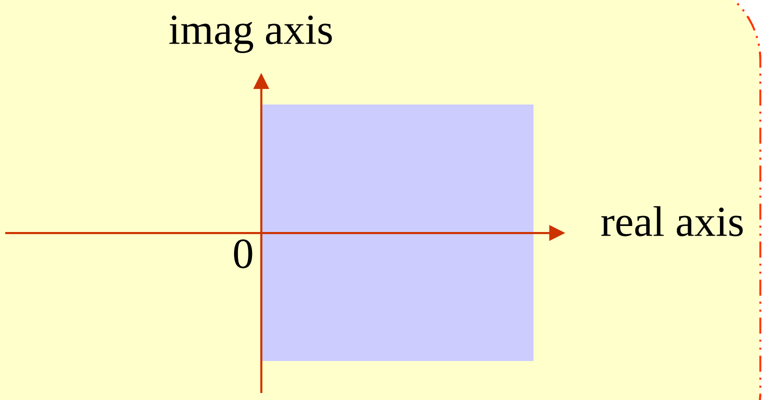
$$F(s) = \int_0^\infty 1(t) e^{-st} dt = \int_0^\infty e^{-st} dt = -\frac{1}{s} e^{-st} \Big|_0^\infty = -\frac{1}{s} e^{-\infty} - \left(-\frac{1}{s} e^0 \right) = -\frac{1}{s} \cdot 0 - \left(-\frac{1}{s} \cdot 1 \right) = \frac{1}{s}$$

Clearly, the about result is valid for $s = \sigma + j\omega$ with $\sigma > 0$.

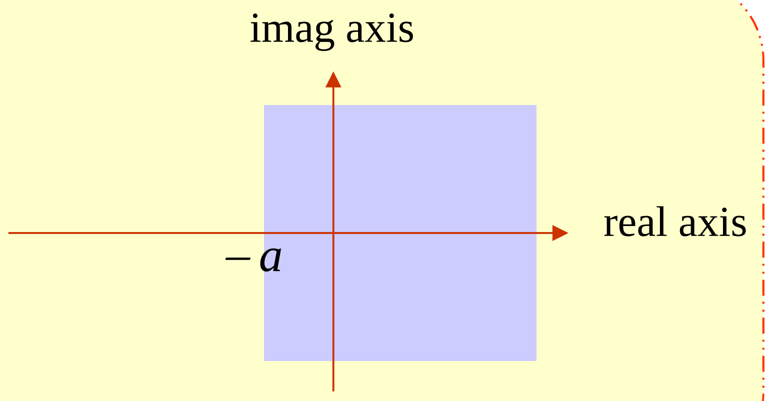
Example 2: Find the Laplace transform of an exponential function $f(t) = e^{-at},\ t \ge 0.$

$$F(s) = \int_0^\infty f(t) e^{-st} dt = \int_0^\infty e^{-at} e^{-st} dt = \int_0^\infty e^{-(s+a)t} dt = -\frac{1}{s+a} e^{-(s+a)t} \Big|_0^\infty = \frac{1}{s+a}$$

Again, the result is only valid for all $s = \sigma + j\omega$ with $\sigma > -a$.



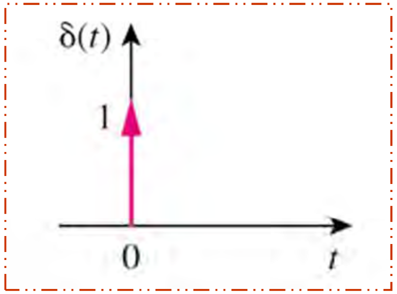
ROC for Example 1



ROC for Example 2

Example 3: Find the Laplace transform of a **unit impulse** function $\delta(t)$, which has the following properties:

1. $\delta(t) = 0$ for $t \neq 0$
2. $\delta(0) \to \infty$
3. $\int_{-\infty}^{\infty} \delta(t)dt = 1, \; \int_{-\infty}^{\infty} f(t)\delta(t)dt = f(0)$ for any $f(t)$
4. $\delta(t)$ is an even function, i.e., $\delta(t) = \delta(-t)$



Its Laplace transform is significant to many system and control problems. By definition,

$$L\{\delta(t)\} = \int_{0^-}^{\infty} \delta(t)e^{-st}dt = e^0 = 1$$

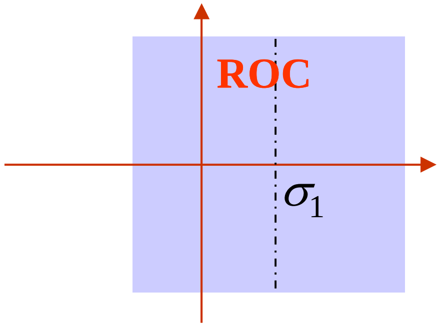
Its ROC is the whole complex plane.

Obviously, impulse functions are non-existent in real life. We will learn from a tutorial question on how to approximate such a function.

Inverse Laplace Transform

Given a frequency-domain function $F(s)$, the inverse Laplace transform is to convert it back to its original time-domain function $f(t)$.

$$f(t) = L^{-1}\left[F(s)\right] = \frac{1}{2\pi j}\int_{\sigma_1 - j\infty}^{\sigma_1 + j\infty} F(s)e^{st}ds$$



This process is quite complex because it requires knowledge about complex analysis.

We use look-up table rather than evaluating these complex integrals.

The functions $f(t)$ and $F(s)$ are one-to-one pairing each other and are called Laplace transform pair. Symbolically,

$$f(t) \Leftrightarrow F(s)$$

Properties of Laplace transform:

1. Superposition or linearity:

$$L\{a_1 f_1(t) + a_2 f_2(t)\} = a_1 L\{f_1(t)\} + a_2 L\{f_2(t)\} = a_1 F_1(s) + a_2 F_2(s)$$

Example: Find the Laplace transform of $\cos(\omega t)$. By Euler's formula, we have

$$\cos(\omega t) = \frac{1}{2}\left[e^{j\omega t} + e^{-j\omega t}\right]$$

By the superposition property, we have

$$L[\cos(\omega t)] = L\left[\frac{1}{2}\left[e^{j\omega t} + e^{-j\omega t}\right]\right] = \frac{1}{2} L\left[e^{j\omega t}\right] + \frac{1}{2} L\left[e^{-j\omega t}\right]$$

$$= \frac{1}{2}\left[\left(\frac{1}{s - j\omega}\right) + \left(\frac{1}{s + j\omega}\right)\right] = \frac{1}{2} \cdot \frac{s + j\omega + s - j\omega}{(s - j\omega)(s + j\omega)} = \frac{s}{s^2 + \omega^2}$$

2. Scaling:

If $F(s)$ is the Laplace transform of $f(t)$, then

$$\boxed{L[f(at)] = \frac{1}{a} F\left(\frac{s}{a}\right)}$$

Example: It was shown in the previous example that

$$L[\cos(\omega t)] = \frac{s}{s^2 + \omega^2}$$

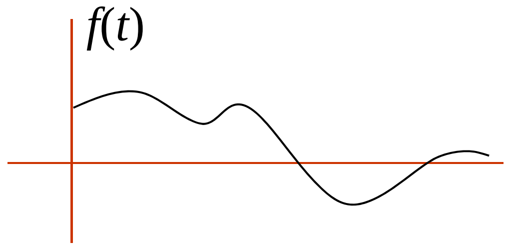
By the scaling property, we have

$$L[\cos(2\omega t)] = \frac{1}{2} \frac{\frac{s}{2}}{\frac{s^2}{4} + \omega^2} = \frac{1}{2}\left[\frac{\frac{s}{2}}{\frac{s^2 + 4\omega^2}{4}}\right] = \frac{s}{s^2 + 4\omega^2}$$

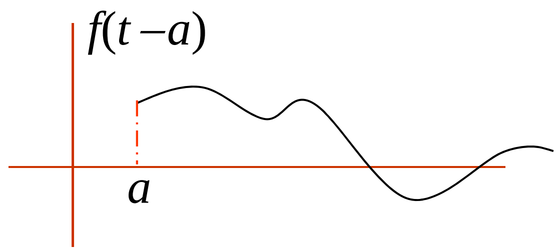
which may also be obtained by replacing ω by 2ω.

3. Time shift (delay):

If $F(s)$ is the Laplace transform of $f(t)$, then



$$\boxed{L\left[f(t-a)\cdot 1(t-a)\right]=e^{-as}F(s)}$$



For a function delayed by 'a' in time-domain, the equivalence in s-domain is multiplying its original Laplace transform of the function by e^{-as}.

Example:

$$L[\cos\omega t]=\frac{s}{s^2+\omega^2} \quad\Rightarrow\quad \boxed{L\{\cos(\omega(t-a))\cdot 1(t-a)\}=e^{-as}\frac{s}{s^2+\omega^2}}$$

4. Frequency shift:

If $F(s)$ is the Laplace transform of $f(t)$, then

$$\boxed{L[e^{-at} f(t)] = F(s+a)}$$

Example: Given

$$\cos(\omega t) \Leftrightarrow \frac{s}{s^2+\omega^2} \qquad \text{and} \qquad \sin(\omega t) \Leftrightarrow \frac{\omega}{s^2+\omega^2}$$

Using the shift property, we obtain the Laplace transforms of the damped sine and cosine functions as

$$\boxed{L\left\{e^{-at}\cos(\omega t)\right\} = \frac{s+a}{(s+a)^2+\omega^2}} \qquad \text{and} \qquad \boxed{L\left\{e^{-at}\sin(\omega t)\right\} = \frac{\omega}{(s+a)^2+\omega^2}}$$

5. Differentiation:

$$L\left\{\frac{df(t)}{dt}\right\} = L\left\{\dot{f}(t)\right\} = sL\left\{f(t)\right\} - f(0^-) = sF(s) - f(0^-)$$

$$L\left\{\frac{d^2 f(t)}{dt^2}\right\} = L\left\{\ddot{f}(t)\right\} = s^2 L\left\{f(t)\right\} - sf(0^-) - f'(0^-) = s^2 F(s) - sf(0^-) - f'(0^-)$$

6. Integration:

$$L\left\{\int_0^t f(\zeta)d\zeta\right\} = \frac{1}{s}L\left\{f(t)\right\} = \frac{1}{s}F(s)$$

Example: The derivative of a unit step function $1(t)$ is a unit impulse function $\delta(t)$.

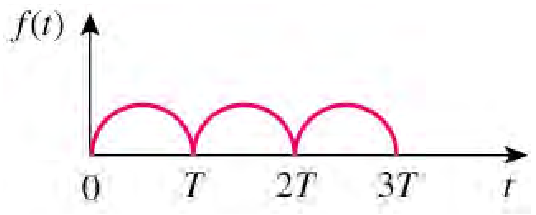
$$\boxed{\frac{d}{dt}1(t) = \delta(t)}$$

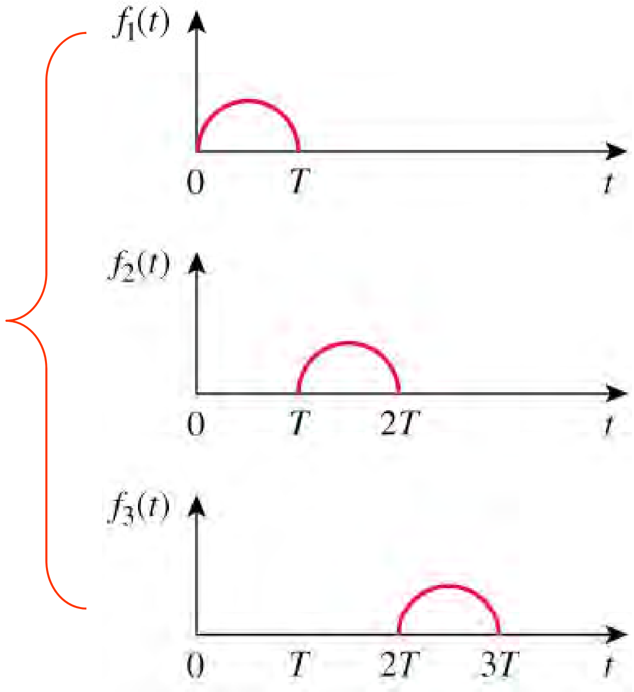
$$\boxed{L\left\{\delta(t)\right\} = L\left\{\frac{d}{dt}1(t)\right\} = sL\left\{1(t)\right\} - 1(0^-) = s\frac{1}{s} - 0 = 1}$$

$$\boxed{L\left\{1(t)\right\} = L\left\{\int_{0^-}^t \delta(\zeta)d\zeta\right\} = \frac{1}{s}L\left\{\delta(t)\right\} = \frac{1}{s}}$$

7. Periodic functions:

If $f(t)$ is a periodic function, then it can be represented as the sum of time-shifted functions as follows:





$$f(t) = f_1(t) + f_2(t) + f_3(t) + \cdots$$
$$= f_1(t) + f_1(t-T) + f_1(t-2T) + \cdots$$

Applying time-shift property, we obtain

$$F(s) = F_1(s) + F_1(s)e^{-sT} + F_1(s)e^{-s2T} \cdots$$
$$= F_1(s)(1 + e^{-sT} + e^{-s2T} \cdots) = \frac{F_1(s)}{1-e^{-sT}} = \frac{\int_{0^-}^{T} f(t)e^{-st}dt}{1-e^{-sT}}$$

8. Initial value theorem:

Examining the differentiation property of the Laplace transform, i.e.,

$$L\left\{\frac{df(t)}{dt}\right\} = L\left\{\dot{f}(t)\right\} = sL\left\{f(t)\right\} - f(0^-) = sF(s) - f(0^-)$$

we have

$$sF(s) - f(0^-) = L\left\{\frac{df(t)}{dt}\right\} = \int_{0^-}^{\infty} \frac{df(t)}{dt} e^{-st} dt = \int_{0^-}^{\infty} e^{-st} df(t) \to 0, \text{ as } s \to \infty$$

Thus,

$$f(0^-) \to sF(s), \text{ as } s \to \infty \quad \text{or} \quad \boxed{f(0^-) = \lim_{s \to \infty} [sF(s)]}$$

This is called the **initial value theorem** of Laplace transform. For example, recall that

$$f(t) = e^{-at}\cos(\omega t) \quad \Leftrightarrow \quad F(s) = \frac{s+a}{(s+a)^2 + \omega^2}$$

$$f(0) = e^0 \cos(0) = 1 \quad \Leftrightarrow \quad \lim_{s \to \infty}[sF(s)] = \lim_{s \to \infty} \frac{s(s+a)}{(s+a)^2 + \omega^2} = \lim_{s \to \infty} \frac{s^2 + as}{s^2 + 2as + (a^2 + \omega^2)} = 1$$

9. Final value theorem:

Examining again the differentiation property of the Laplace transform, i.e.,

$$L\left\{\frac{df(t)}{dt}\right\} = L\left\{\dot{f}(t)\right\} = sL\left\{f(t)\right\} - f(0^-) = sF(s) - f(0^-)$$

we have

$$\lim_{s\to 0}[sF(s) - f(0^-)] = \lim_{s\to 0} L\left\{\frac{df(t)}{dt}\right\} = \lim_{s\to 0}\int_{0^-}^{\infty}\frac{df(t)}{dt}e^{-st}dt = \int_{0^-}^{\infty}\frac{df(t)}{dt}e^{-0t}dt = f(t)\Big|_{0^-}^{\infty} = f(\infty) - f(0^-)$$

Thus,

$$\boxed{f(\infty) = \lim_{s\to 0}[sF(s)]}$$

The result is only valid for a function whose $F(s)$ has all its poles in the open left-half plane (a simple pole at $s = 0$ permitted)!

This is called the **final value theorem** of Laplace transform. For example,

$$f(t) = e^{-at}\cos(\omega t) \quad \Leftrightarrow \quad F(s) = \frac{s+a}{(s+a)^2 + \omega^2}$$

$$f(\infty) = e^{-\infty}\cos(\infty) = 0 \quad \Leftrightarrow \quad \lim_{s\to 0}[sF(s)] = \lim_{s\to 0}\frac{s(s+a)}{(s+a)^2+\omega^2} = \lim_{s\to 0}\frac{s^2+as}{s^2+2as+(a^2+\omega^2)} = 0$$

Summary of Laplace transform properties

Property	$f(t)$	$F(s)$
Linearity	$a_1 f_1(t) + a_2 f_2(t)$	$a_1 F_1(s) + a_2 F_2(s)$
Scaling	$f(at)$	$\frac{1}{a} F(\frac{s}{a})$
Time shift	$f(t-a)u(t-a)$	$e^{-as} F(s)$
Frequency shift	$e^{-at} f(t)$	$F(s+a)$
Time derivative	$\frac{d^n f(t)}{dt^n}$	$s^n F(s) - s^{n-1} f(0^-) - s^{n-2} f'(0^-) - \ldots - s^0 f^{(n-1)}(0^-)$
Time integration	$\int_0^t f(\zeta) d\zeta$	$\frac{1}{s} F(s)$
Time periodicity	$f(t) = f(t+nT)$	$\frac{F_1(s)}{1-e^{-sT}}$
Initial value	$f(0^-)$	$\lim_{s \to \infty} [sF(s)]$
Final value	$f(\infty)$	$\lim_{s \to 0} [sF(s)]$
Convolution	$f_1(t) \otimes f_2(t)$	$F_1(s) F_2(s)$

Courtesy of Dr Melissa Tao

Some commonly used Laplace transform pairs

$f(t)$	$\Leftrightarrow$	$F(s)$
$\delta(t)$	$\Leftrightarrow$	1
$1(t)$	$\Leftrightarrow$	$\dfrac{1}{s}$
t	$\Leftrightarrow$	$\dfrac{1}{s^2}$
t^n	$\Leftrightarrow$	$\dfrac{n!}{s^{n+1}}$
e^{-at}	$\Leftrightarrow$	$\dfrac{1}{s+a}$
te^{-at}	$\Leftrightarrow$	$\dfrac{1}{(s+a)^2}$

$f(t)$	$\Leftrightarrow$	$F(s)$
$\sin \omega t$	$\Leftrightarrow$	$\dfrac{\omega}{s^2+\omega^2}$
$\cos \omega t$	$\Leftrightarrow$	$\dfrac{s}{s^2+\omega^2}$
$\sin(\omega t+\theta)$	$\Leftrightarrow$	$\dfrac{s\sin\theta+\omega\cos\theta}{s^2+\omega^2}$
$\cos(\omega t+\theta)$	$\Leftrightarrow$	$\dfrac{s\cos\theta-\omega\sin\theta}{s^2+\omega^2}$
$e^{-at}\sin \omega t$	$\Leftrightarrow$	$\dfrac{\omega}{(s+a)^2+\omega^2}$
$e^{-at}\cos \omega t$	$\Leftrightarrow$	$\dfrac{s+a}{(s+a)^2+\omega^2}$

Why Laplace transform?

Allows us to work with algebraic equations rather than differential equations.

Applicable to a wider variety of inputs than phasor analysis

Laplace transform is significant for many reasons

Provides an easy way to solve system problems involving initial conditions.

Capable of giving us the total response (natural and forced) of the circuit in one single operation.

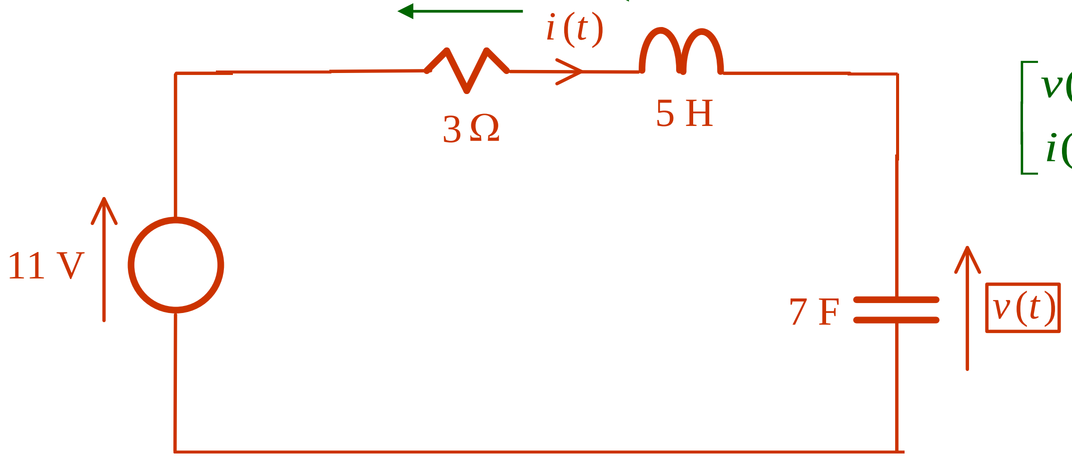
Courtesy of Dr Melissa Tao

Frequency-domain Descriptions of Linear Systems

Reconsider the series RLC circuit

$$3i = 21\frac{dv}{dt}$$

$$v_L = L\frac{di}{dt} = 5\frac{d}{dt}\left(C\frac{dv}{dt}\right) = 35\frac{d^2v}{dt^2}$$



$$\begin{bmatrix} v(0) \\ i(0) \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$$i(0) = C\left.\frac{dv}{dt}\right|_{t=0}$$

$$\Downarrow$$

$$v'(0) = \frac{i(0)}{C} = 0$$

Applying KVL:

$$35\frac{d^2v(t)}{dt^2} + 21\frac{dv(t)}{dt} + v(t) = u(t) = 11, \quad t \ge 0$$

It was solved earlier by finding solutions directly on the time domain. The problem can be solved in the frequency domain. From this point onwards, we will make use of the Laplace transform to solve circuit and system problems.

Taking Laplace transform on the both sides of the ODE, i.e.,

$$L\left\{35\frac{d^2v(t)}{dt^2}+21\frac{dv(t)}{dt}+v(t)\right\}=L\{u(t)\}=L\{11\}=U(s)$$

Recall that

$$L\left\{\frac{df(t)}{dt}\right\}=sF(s)-f(0^-),\qquad L\left\{\frac{d^2f(t)}{dt^2}\right\}=s^2F(s)-sf(0^-)-f'(0^-)$$

we have

$$35\cdot[s^2V(s)-sv(0)-v'(0)]+21\cdot[sV(s)-v(0)]+V(s)=U(s)=\frac{11}{s}$$

$$\Longrightarrow\qquad 35\cdot[s^2V(s)-2s]+21\cdot[sV(s)-2]+V(s)=U(s)$$

$$\Longrightarrow\quad (35s^2+21s+1)V(s)=U(s)+(70s+42)\quad\Longrightarrow\quad \boxed{V(s)=\frac{U(s)}{35s^2+21s+1}+\frac{70s+42}{35s^2+21s+1}}$$

frequency domain model of the circuit

due to external force

i.e., ***force response***

due to initial conditions

i.e., ***natural response***

Examining

$$V(s)=\frac{U(s)}{35s^2+21s+1}+\frac{70s+42}{35s^2+21s+1}=\frac{11}{s(35s^2+21s+1)}+\frac{70s+42}{35s^2+21s+1}$$

$$=\frac{{}^{11}\!/_{35}}{s(s+0.052)(s+0.548)}+\frac{2s+{}^{42}\!/_{35}}{(s+0.052)(s+0.548)}$$

$$=\left[\frac{A}{s}+\frac{B}{s+0.052}+\frac{C}{s+0.548}\right]+\left[\frac{D}{s+0.052}+\frac{E}{s+0.548}\right]$$

Partial Fraction

The next step is to obtain these coefficients. Take the first term first…

$$V_1(s)=\frac{{}^{11}\!/_{35}}{s(s+0.052)(s+0.548)}=\frac{A}{s}+\frac{B}{s+0.052}+\frac{C}{s+0.548}$$

Multiplying both sides by s and let $s = 0$, we obtain

$$\frac{{}^{11}\!/_{35}}{(0+0.052)(0+0.548)}\leftarrow\frac{{}^{11}\!/_{35}\,s}{s(s+0.052)(s+0.548)}=\frac{As}{s}+\frac{Bs}{s+0.052}+\frac{Cs}{s+0.548}\rightarrow A+\frac{B\cdot 0}{0+0.052}+\frac{C\cdot 0}{0+0.548}$$

and

$$\boxed{A=sV_1(s)\big|_{s=0}=11}$$

Similarly, multiplying both sides by s + 0.052 and let s = – 0.052, we obtain

$$\frac{\frac{11}{35}(s+0.052)}{s(s+0.052)(s+0.548)}=\frac{A(s+0.052)}{s}+\frac{B(s+0.052)}{s+0.052}+\frac{C(s+0.052)}{s+0.548}$$

$$\frac{\frac{11}{35}}{s(s+0.548)}=\frac{A(s+0.052)}{s}+B+\frac{C(s+0.052)}{s+0.548}$$

$$-12.185 \leftarrow \frac{\frac{11}{35}}{-0.052(-0.052+0.548)}=\frac{A(-0.052+0.052)}{-0.052}+B+\frac{C(-0.052+0.052)}{-0.052+0.548} \rightarrow B$$

$$B=(s+0.052)V_1(s)\big|_{s=-0.052}$$
$$=-12.185$$

Lastly, multiplying both sides by s + 0.548 and let s = – 0.548, we obtain

$$\frac{\frac{11}{35}(s+0.548)}{s(s+0.052)(s+0.548)}=\frac{A(s+0.548)}{s}+\frac{B(s+0.548)}{s+0.052}+\frac{C(s+0.548)}{s+0.548}$$

$$\frac{\frac{11}{35}}{s(s+0.052)}=\frac{A(s+0.548)}{s}+\frac{B(s+0.548)}{s+0.052}+C$$

$$1.156=\frac{\frac{11}{35}}{-0.548(-0.548+0.052)}=\frac{A(-0.548+0.548)}{-0.548}+\frac{B(-0.548+0.548)}{s+0.052}+C \rightarrow C$$

$$C=(s+0.548)V_1(s)\big|_{s=-0.548}$$
$$=1.156$$

Thus, we obtain

$$\frac{11}{s(s+0.052)(s+0.548)} = \frac{A}{s} + \frac{B}{s+0.052} + \frac{C}{s+0.548} = \frac{11}{s} - \frac{12.185}{s+0.052} + \frac{1.156}{s+0.548}$$

The above method is called the ***residual method***. Let us derive D and E for the 2nd term of $V(s)$ using a so-called ***coefficient matching method***…

$$\frac{2s + {}^{42}\!/_{35}}{(s+0.052)(s+0.548)} = \frac{D}{s+0.052} + \frac{E}{s+0.548} =$$

$$\frac{D(s+0.548)+E(s+0.052)}{(s+0.052)(s+0.548)} = \frac{(D+E)s+(0.548D+0.052E)}{(s+0.052)(s+0.548)}$$

which implies

$$\left.\begin{aligned} D+E&=2 \\ 0.548D+0.052E&=1.2 \end{aligned}\right\} \Rightarrow \begin{cases} D=2.21 \\ E=-0.21 \end{cases}$$

$$\frac{2s + {}^{42}\!/_{35}}{(s+0.052)(s+0.548)} = \frac{2.21}{s+0.052} - \frac{0.21}{s+0.548}$$

Finally, we get

$$V(s) = \frac{11}{s(s+0.052)(s+0.548)} + \frac{70s+42}{(s+0.052)(s+0.548)} = \frac{11}{s} - \frac{12.185}{s+0.052} + \frac{1.156}{s+0.548} + \frac{2.21}{s+0.052} - \frac{0.21}{s+0.548}$$

and

$$v(t) = L^{-1}\{V(s)\} = L^{-1}\overbrace{\left\{\frac{11}{s} - \frac{12.185}{s+0.052} + \frac{1.156}{s+0.548}\right\}}^{\textit{force response}} + L^{-1}\overbrace{\left\{\frac{2.21}{s+0.052} - \frac{0.21}{s+0.548}\right\}}^{\textit{natural response}}$$

$$= \underbrace{11 - 12.185e^{-0.052t} + 1.156e^{-0.548t}}_{\textit{force response}} + \underbrace{2.21e^{-0.052t} - 0.21e^{-0.548t}}_{\textit{natural response}} = 11 - \underbrace{9.975e^{-0.052t} + 0.946e^{-0.548t}}_{\textit{transient response}}$$

which is 'almost' the same as that obtained directly from ODE, i.e.,

$$v(t) = \begin{cases} 2, & t < 0 \\ v_{ss}(t) + v_{tr}(t), & t \ge 0 \end{cases} = \begin{cases} 2, & t < 0 \\ 11 - 9.95e^{-0.548t} + 0.95e^{-0.052t}, & t \ge 0 \end{cases}$$

The difference is some computational errors.

What have we learnt? The same circuit problem can be solved using a totally different approach!

Summary of partial fraction technique

simple real poles	repeated real poles	complex poles pairs
$F(s)=\dfrac{N(s)}{(s+p_1)(s+p_2)}$	$F(s)=\dfrac{N(s)}{(s+p)^2}$	$F(s)=\dfrac{A_1s+A_2}{s^2+as+b}$
$F(s)=\dfrac{k_1}{(s+p_1)}+\dfrac{k_2}{(s+p_2)}$	$F(s)=\dfrac{k_1}{(s+p)}+\dfrac{k_2}{(s+p)^2}$	$F(s)=\dfrac{A_1(s+\alpha)}{(s+\alpha)^2+\beta^2}+\dfrac{B_1\beta}{(s+\alpha)^2+\beta^2}$

residual method or coefficient matching

$f(t)=k_1e^{-p_1t}+k_2e^{-p_2t}$	$f(t)=k_1e^{-pt}+k_2te^{-pt}$	$f(t)=A_1e^{-\alpha t}\cos\beta t+B_1e^{-\alpha t}\sin\beta t$

Courtesy of Dr Melissa Tao

Reconsider the cruise control dynamic model for displacement

$$u - b\dot{x} = m\ddot{x} \qquad \Leftrightarrow \qquad \ddot{x} + 0.1\dot{x} = 1$$

with $x(0) = 0$ and $x'(0) = 0$. Taking Laplace transform on both sides, we have

$$s^2 X(s) + 0.1sX(s) \leftarrow L\{\ddot{x} + 0.1\dot{x}\} = L\{1\} \rightarrow \frac{1}{s}$$

$$X(s) = \frac{1}{(s^2 + 0.1s)s} = \frac{1}{s^2(s+0.1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+0.1}$$

Using the coefficient matching method, we obtain

$$X(s) = \frac{1}{s^2(s+0.1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+0.1}$$

$$= \frac{As(s+0.1) + B(s+0.1) + Cs^2}{s^2(s+0.1)}$$

$$= \frac{(A+C)s^2 + (0.1A+B)s + 0.1B}{s^2(s+0.1)}$$

$$\left.\begin{aligned} A + C &= 0 \\ 0.1A + B &= 0 \\ 0.1B &= 1 \end{aligned}\right\} \quad \begin{aligned} A &= -100 \\ B &= 10 \\ C &= 100 \end{aligned}$$

$$X(s) = \frac{-100}{s} + \frac{10}{s^2} + \frac{100}{s+0.1}$$

$$\boxed{x(t) = 10t + 100e^{-0.1t} - 100}$$

Frequency domain model of linear systems – transfer functions

A linear system expressed in terms of an ODE is called a model in the time domain. In what follows, we will learn that the same system can be expressed in terms of a rational function of s, the Laplace transform variable. Recall the cruise control system

$$u - b\dot{x} = m\ddot{x} \qquad \Leftrightarrow \qquad m\ddot{x} + b\dot{x} = u$$

Assume that the initial conditions are zero. Taking Laplace transform on its both sides,

$$(ms^2 + bs)X(s) = ms^2 X(s) + bsX(s) \leftarrow L\{m\ddot{x} + b\dot{x}\} = L\{u\} \rightarrow U(s)$$

we obtain a rational function of s, i.e.,

$$\boxed{H(s) = \frac{X(s)}{U(s)} = \frac{1}{ms^2 + bs}} \quad \Rightarrow \quad X(s) = H(s)U(s) = \frac{1}{ms^2 + bs}U(s)$$

$H(s)$ is the ratio of the system output (displacement) and input (force) in frequency domain. Such a function is called the ***transfer function*** of the system, which fully characterizes the system properties.

More example: the series RLC circuit

The ODE for the general series RLC circuit was derived earlier as the following:

$$LC\frac{d^2 v_C(t)}{dt^2} + RC\frac{dv_C(t)}{dt} + v_C(t) = v(t)$$

Assume that all initial conditions are zero (note that for deriving transfer functions, we always assume initial conditions are zero!).

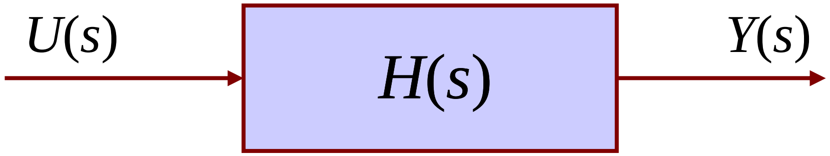
$$(LCs^2 + RCs + 1)V_C(s) \leftarrow L\left\{LC\frac{d^2 v_C(t)}{dt^2} + RC\frac{dv_C(t)}{dt} + v_C(t)\right\} = L\{v(t)\} \rightarrow V(s)$$

$$H(s) = \frac{V_C(s)}{V(s)} = \frac{1}{LCs^2 + RCs + 1} \quad \Longleftrightarrow \quad V_C(s) = H(s)V(s) = \frac{1}{LCs^2 + RCs + 1}V(s)$$

$H(s)$ is the ratio of the system output (capacitor voltage) and input (voltage source) in frequency domain. The circuit (or the system) is fully characterized by the ***transfer function***. If the $H(s)$ and $V(s)$ are known, we can compute the system output. As such, it is important to study the properties of $H(s)$!

System poles and zeros

As we have seen from the previous examples, a general linear time-invariant system can be expressed in a frequency-domain model or transfer function:



with

$$H(s) = \frac{N(s)}{D(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0}, \qquad m \le n$$

n is called the ***order*** of the system. The roots of the numerator of $H(s)$, i.e., $N(s)$, are called the ***system zeros*** (because the transfer function is equal to 0 at these points), and the roots of the denominator of $H(s)$, i.e., $D(s)$, are called the ***system poles*** (the transfer function is singular at these points). It turns out that the system properties are fully captured by the locations of these poles and zeros…

Examples:

The cruise control system has a transfer function $H(s)=\dfrac{1}{ms^2+bs}=\dfrac{1/m}{s(s+b/m)}$

It has no zero at all and two poles at $s=0,\ s=-b/m$

The RLC circuit has a transfer function $H(s)=\dfrac{1}{LCs^2+RCs+1}$

It has no zero and two poles at

$$s_1=\frac{-RC-\sqrt{(RC)^2-4LC}}{2LC},\quad s_2=\frac{-RC+\sqrt{(RC)^2-4LC}}{2LC}$$

which are precisely the same as the roots of the characteristic polynomial of its ODE.

The system $H(s)=\dfrac{5s^2-10s+5}{s^3+6s^2+11s+6}=\dfrac{5(s-1)^2}{(s+1)(s+2)(s+3)}$ has two zeros (repeated)

at $s=1$ and three poles at $s=-1,\ s=-2,\ s=-3$, respectively.

Response to sinusoidal inputs

Let us consider the series RL circuit whose current governed by the following ODE

$$5\frac{di(t)}{dt}+5i(t)=v(t) \quad\Rightarrow\quad (5s+5)I(s)=V(s) \quad\Rightarrow\quad H(s)=\frac{I(s)}{V(s)}=\frac{0.2}{s+1}$$

Let the voltage source be a sinusoidal $v(t) = \cos(2t)$, which has a Laplace transform

$$V(s)=\frac{s}{s^2+4} \quad\Rightarrow\quad I(s)=\frac{0.2}{s+1}V(s)=\frac{0.2}{s+1}\cdot\frac{s}{s^2+4}=\frac{A}{s+1}+\frac{Bs+C}{s^2+4}$$

Using the coefficient matching method, we have to match

$$(A+B)s^2+(B+C)s+(4A+C)=0.2s$$

$$\Rightarrow \begin{cases} A+B=0 \\ B+C=0.2 \\ 4A+C=0 \end{cases} \Rightarrow \begin{cases} A=-0.04 \\ B=0.04 \\ C=0.16 \end{cases}$$

$$I(s)=\frac{0.04s+0.16}{s^2+4}-\frac{0.04}{s+1}=\frac{0.04s}{s^2+2^2}+\frac{0.08\times 2}{s^2+2^2}-\frac{0.04}{s+1}$$

$$i(t)=0.04[\cos(2t)+2\sin(2t)-e^{-t}]$$

The steady state response is the given by

$$i_{ss}(t)=0.04[\cos(2t)+2\sin(2t)]=0.0894\cos(2t-63.4349^\circ).$$

Let us solve the problem using another approach. Noting that $v(t) = \cos(2t)$, which has an angular frequency of $\omega = 2$ rad/sec, and noting that the transfer function

$$H(s) = \frac{I(s)}{V(s)} = \frac{0.2}{s+1}$$

is nothing more than the ratio or gain between the system input and output in the frequency domain. Let us calculate the 'gain' of $H(s)$ at the particular frequency coinciding with the input signal, i.e., at $s = j\omega$ with $\omega = 2$.

$$H(j2) = H(j\omega)\Big|_{\omega=2} = \left.\frac{0.2}{j\omega+1}\right|_{\omega=2} = \frac{0.2}{j2+1} = 0.0894e^{-j1.1071} = 0.0894\angle -63.4349^\circ$$

It simply means that at $\omega = 2$ rad/sec, the input signal is amplified by 0.0894 and its phase is shifted by – 63.4349 degrees. It is obvious then the system output, i.e., the current in the circuit at the steady state is given by

$$i_{ss}(t) = 0.0894\cos(2t - 63.4349^\circ).$$

which is identical to what we have obtained earlier. Actually, by letting $s = j\omega$, the above approach is the same as the phasor technique for AC circuits.

Frequency response – amplitude and phase responses

It can be seen from the previous example that for an AC circuits or for a system with a sinusoidal input, the system steady state output can be easily evaluated once the amplitude and phase of its transfer function are known. Thus, it is useful to 'compute' the amplitude and phase of the transfer function, i.e.,

$$H(j\omega) = H(s)\big|_{s=j\omega} = \left|H(j\omega)\right| \cdot \angle H(j\omega)$$

It is obvious that both magnitude and phase of $H(j\omega)$ are functions of ω. The plot of $|H(j\omega)|$ is called the magnitude (amplitude) response of $H(j\omega)$ and the plot of $\angle H(j\omega)$ is called the phase response of $H(j\omega)$. Together they are called the frequency response of the transfer function. In particularly, $|H(0)|$ is called the ***DC gain*** of the system (DC is equivalent to $k\cos(\omega t)$ with $\omega = 0$).

The frequency response of a circuit or a system is an important concept in system theory. It can be used to characterize the properties of the circuit or system, and used to evaluated the system output response.

Example:

Let us reconsider the series RL circuit with the following transfer function:

$$H(s)=\frac{I(s)}{V(s)}=\frac{0.2}{s+1} \quad \Rightarrow \quad H(j\omega)=\frac{0.2}{j\omega+1}=|H(j\omega)|\cdot\angle H(j\omega)=\frac{0.2}{\sqrt{1+\omega^2}}\angle-\tan^{-1}\omega$$

Thus, it is straightforward, but very tedious, to compute its amplitude and phase.

$$\omega=0.01 \quad\Rightarrow\quad |H(j\omega)|=0.2,\ \angle H(j\omega)=-0.57^\circ$$

$$\omega=0.1 \quad\Rightarrow\quad |H(j\omega)|=0.199,\ \angle H(j\omega)=-5.71^\circ$$

$$\omega=0.2 \quad\Rightarrow\quad |H(j\omega)|=0.196,\ \angle H(j\omega)=-11.31^\circ$$

$$\omega=0.5 \quad\Rightarrow\quad |H(j\omega)|=0.179,\ \angle H(j\omega)=-26.57^\circ$$

$$\omega=1 \quad\Rightarrow\quad |H(j\omega)|=0.141,\ \angle H(j\omega)=-45^\circ$$

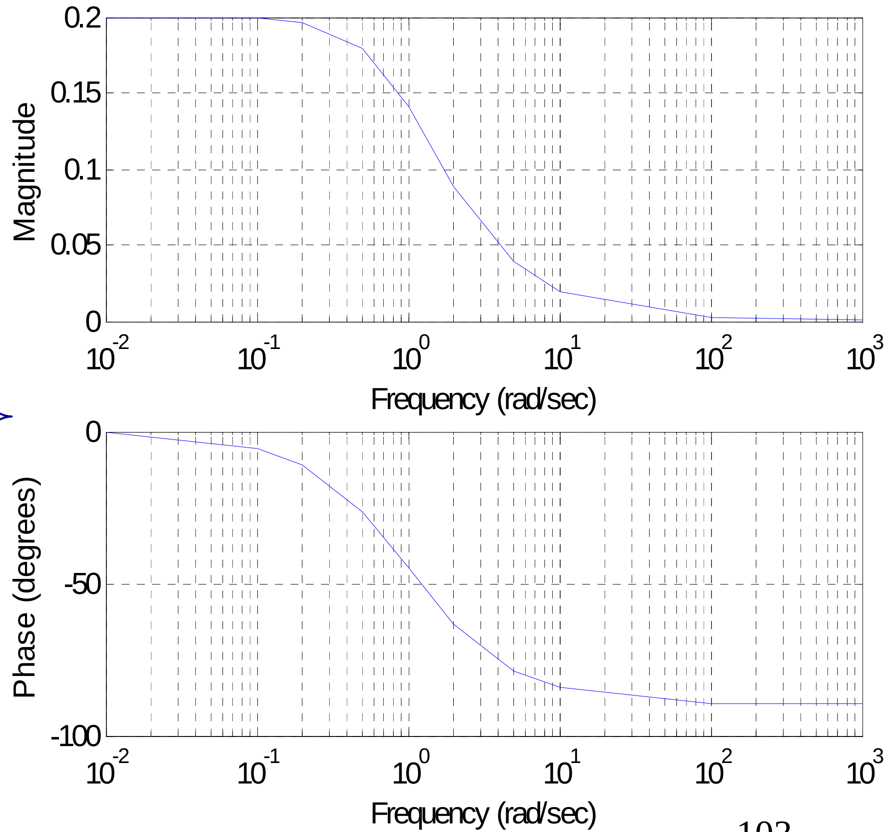
$$\omega=2 \quad\Rightarrow\quad |H(j\omega)|=0.089,\ \angle H(j\omega)=-63.43^\circ$$

$$\omega=5 \quad\Rightarrow\quad |H(j\omega)|=0.039,\ \angle H(j\omega)=-78.69^\circ$$

$$\omega=10 \quad\Rightarrow\quad |H(j\omega)|=0.020,\ \angle H(j\omega)=-84.29^\circ$$

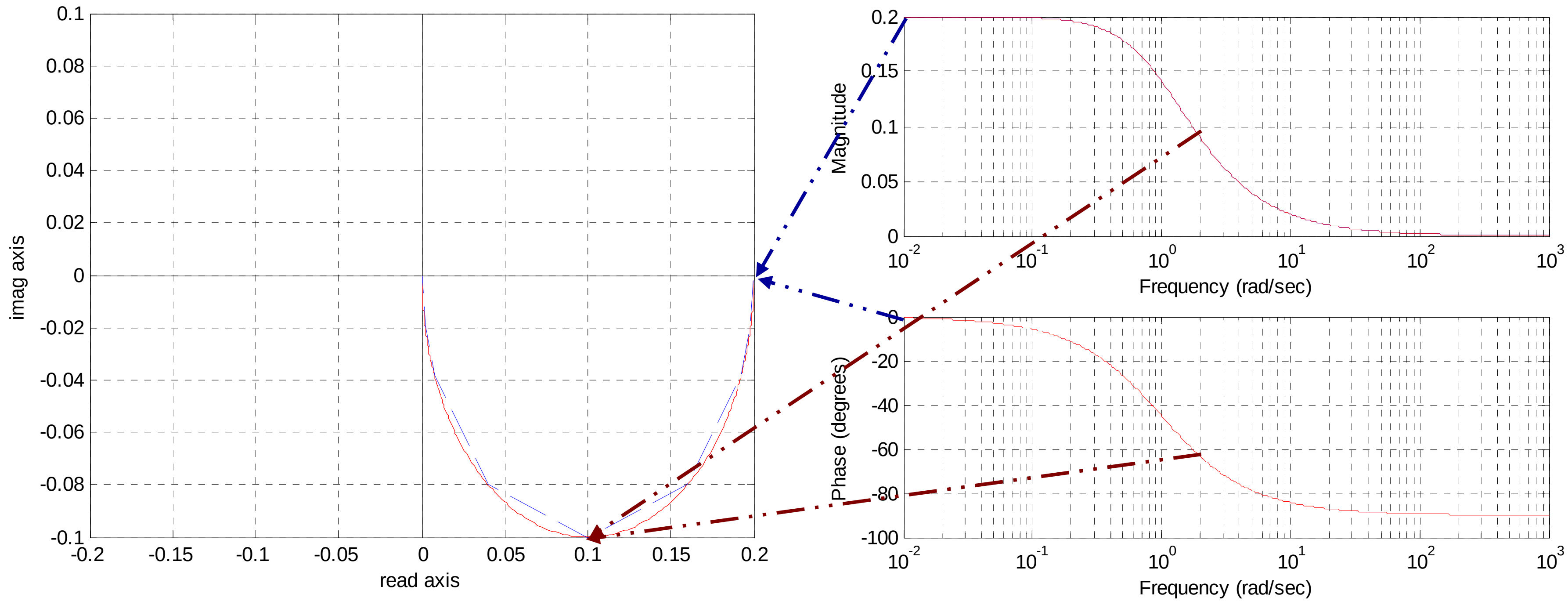
$$\omega=100 \quad\Rightarrow\quad |H(j\omega)|=0.002,\ \angle H(j\omega)=-89.43^\circ$$

$$\omega=1000 \quad\Rightarrow\quad |H(j\omega)|=0.0002,\ \angle H(j\omega)=-89.94^\circ$$



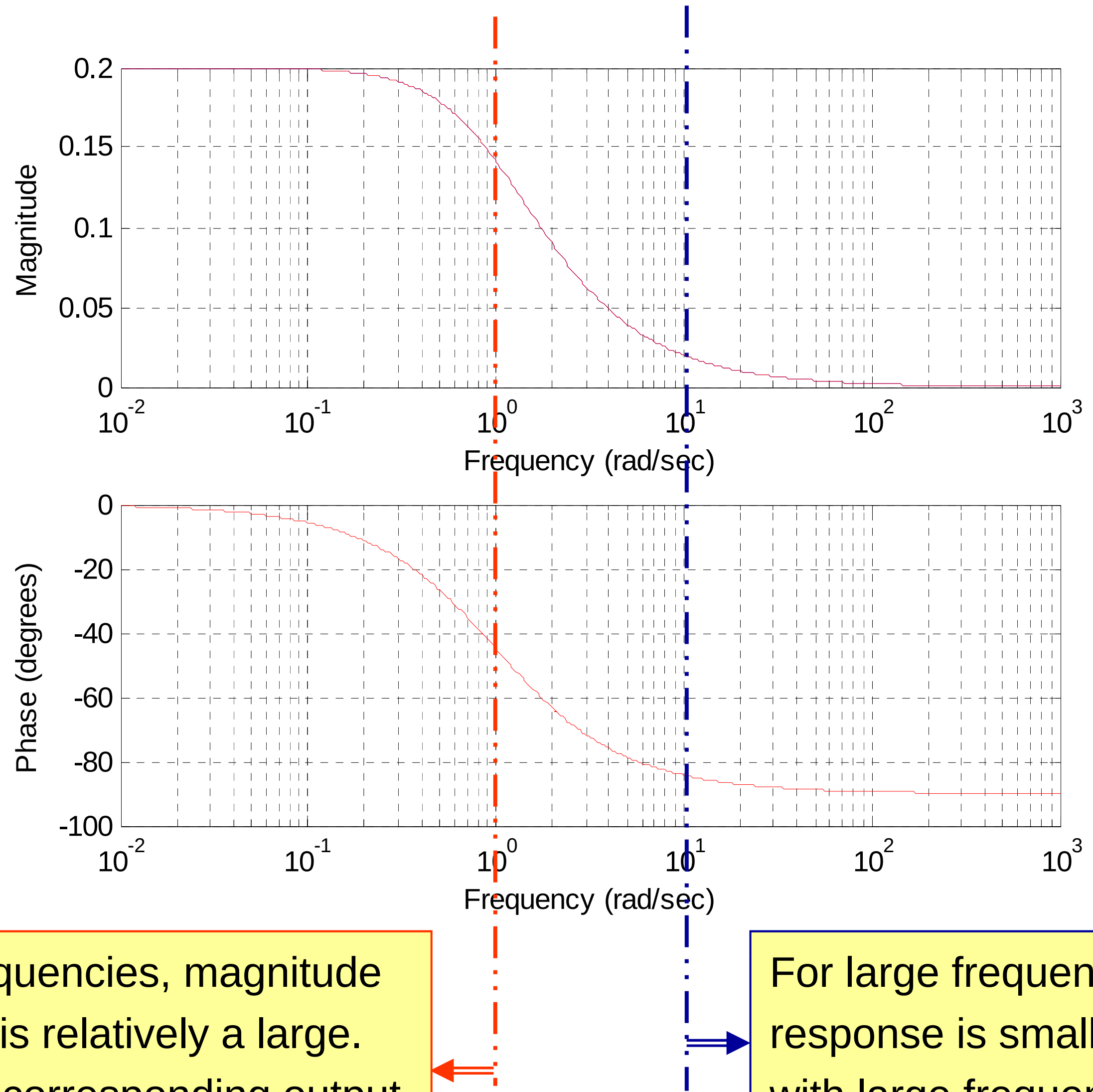
Note that in the plots of the magnitude and phase responses, we use a log scale for the frequency axis. If we draw in a normal scale, the responses will look awful.

There is another way to draw the frequency response. i.e., directly draw both magnitude and phase on a complex plane, which is called the polar plot. For the example considered, its polar plot is given as follows:



The red-line curves are more accurate plots using MATLAB.

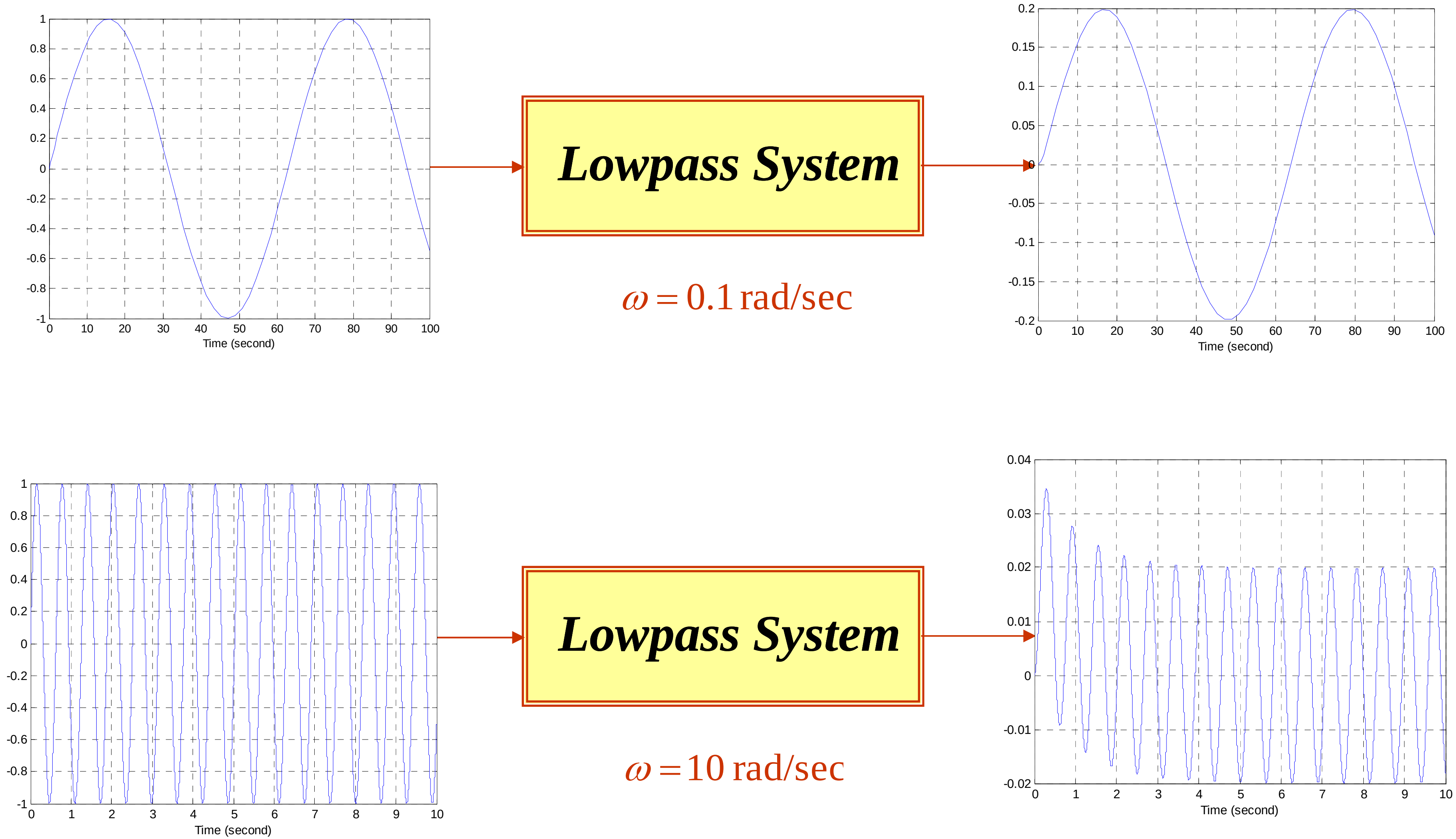
What can we observe from the frequency response?



At low frequencies, magnitude response is relatively a large. Thus, the corresponding output will be large.

For large frequencies, the magnitude response is small and thus signals with large frequencies is attenuated or blocked.

Lowpass systems



Bode plot

The plots of the magnitude and phase responses of a transfer function are called the Bode plot. The easiest way to draw Bode plot is to use MATLAB (i.e., **bode** function). However, there are some tricks that can help us to sketch Bode plots (approximation) without computing detailed values. To do this, we need to introduce a scale called dB (decibel). Given a positive scalar a, its decibel is defined as $20 \cdot \log_{10}(a)$. For example,

$$a = 1 \quad \Rightarrow \quad 20 \cdot \log_{10}(a) = 0 \quad \Rightarrow \quad a = 0\,\text{dB}$$

$$a = 10 \quad \Rightarrow \quad 20 \cdot \log_{10}(a) = 20 \quad \Rightarrow \quad a = 20\,\text{dB}$$

$$a = 100 \quad \Rightarrow \quad 20 \cdot \log_{10}(a) = 40 \quad \Rightarrow \quad a = 40\,\text{dB}$$

$$a = \alpha \cdot \beta \quad \Rightarrow \quad 20 \cdot \log_{10}(\alpha \cdot \beta) = 20 \cdot \log_{10}(\alpha) + 20 \cdot \log_{10}(\beta) \quad \Rightarrow \quad a = \alpha \text{ in dB} + \beta \text{ in dB}$$

$$a = \alpha/\beta \quad \Rightarrow \quad 20 \cdot \log_{10}(\alpha/\beta) = 20 \cdot \log_{10}(\alpha) - 20 \cdot \log_{10}(\beta) \quad \Rightarrow \quad a = \alpha \text{ in dB} - \beta \text{ in dB}$$

In the dB scale, the product of two scalars becomes an addition and the division of two scalars becomes a subtraction.

Bode plot – an integrator

We start with finding the Bode plot asymptotes for a simple system characterized by

$$H(s)=\frac{1}{s} \quad\Rightarrow\quad H(j\omega)=\frac{1}{j\omega}=|H(j\omega)|\angle H(j\omega)=\frac{1}{|\omega|}\angle -90^\circ$$

Examining the amplitude in dB scale, i.e.,

$$20\cdot\log_{10}|H(j\omega)|=20\cdot\log_{10}\frac{1}{|\omega|}=-20\cdot\log_{10}|\omega|\ \text{dB}$$

it is simple to see that

$$\omega=1 \quad\Rightarrow\quad 20\cdot\log_{10}|H(j1)|=-20\cdot\log_{10}1=0\ \text{dB}$$

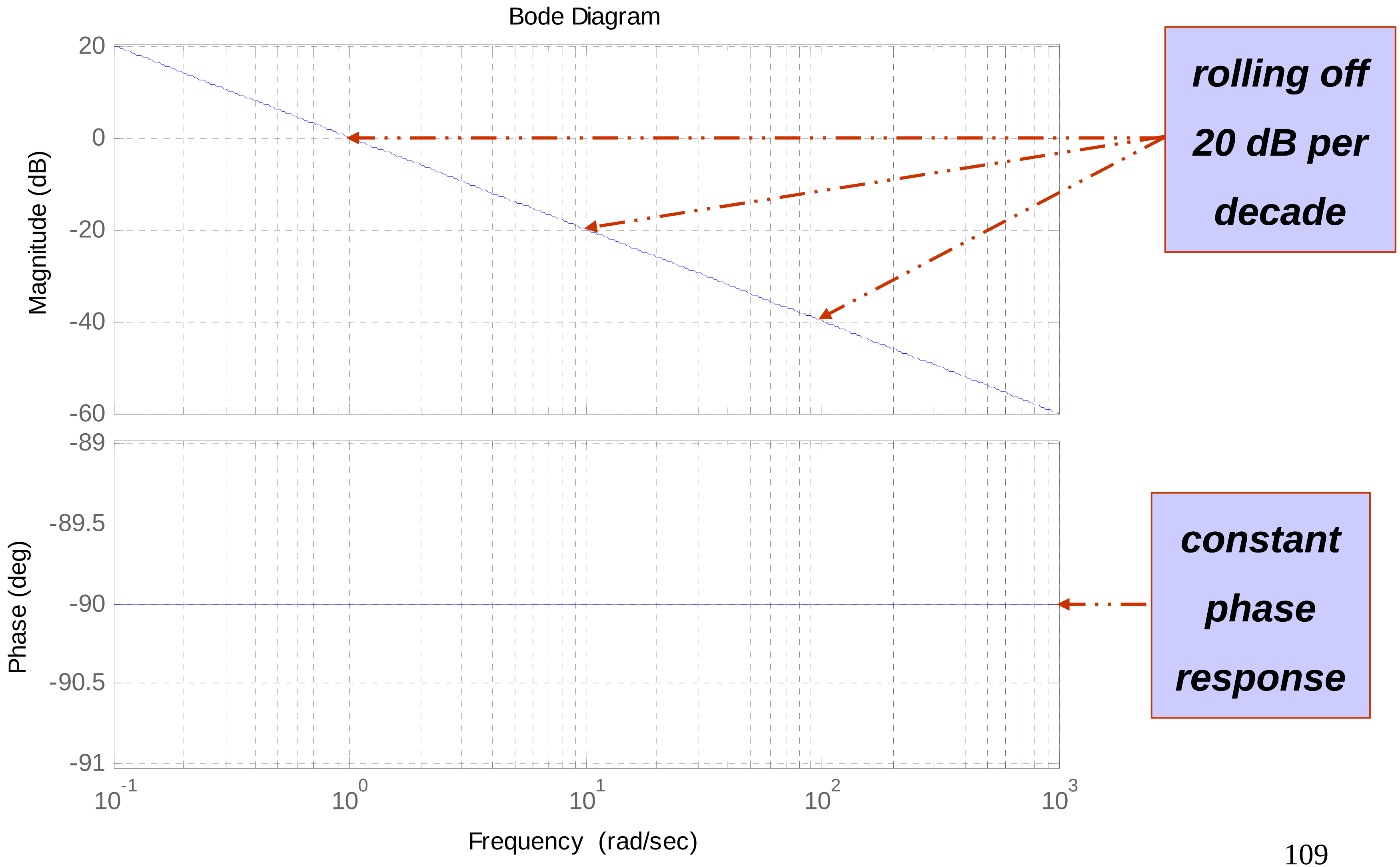
$$\omega=10 \quad\Rightarrow\quad 20\cdot\log_{10}|H(j10)|=-20\cdot\log_{10}10=-20\ \text{dB}$$

$$\omega=\omega_2=10\omega_1 \quad\Rightarrow\quad 20\cdot\log_{10}|H(j\omega_2)|=\underline{-20\cdot\log_{10}\omega_2}=-20\cdot\log_{10}10\omega_1$$

$$=-20\cdot\log_{10}10-20\cdot\log_{10}\omega_1=\underline{-20-20\cdot\log_{10}\omega_1\ \text{dB}}$$

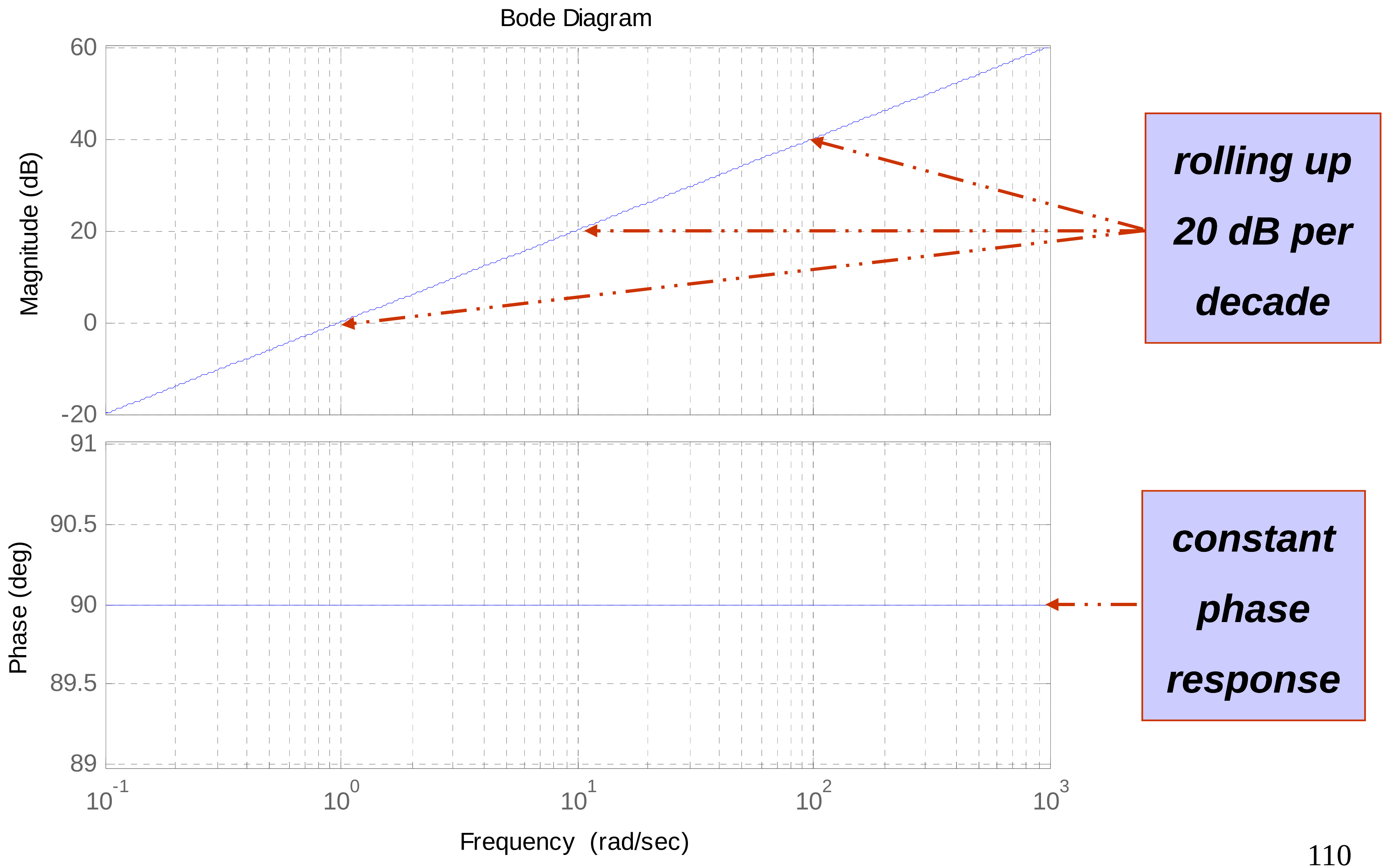
Thus, the above expressions clearly indicate that the magnitude is reduced by **– 20 dB** when the frequency is increased by **10 times**. It is equivalent to say that the magnitude is rolling off **20 dB** per decade.

The phase response of an integrator is – 90 degrees, a constant. The Bode plot of an integrator is given by



Bode plot – an differentiator

The Bode plot of $H(s) = s$ can be done similarly…



Bode plot – a general first order system

The Bode plot of a first order system characterized by a simple pole, i.e.,

$$H(s)=\frac{\omega_1}{s+\omega_1}=\frac{1}{1+{s}/{\omega_1}} \quad\Rightarrow\quad H(j\omega)=\frac{1}{1+j\left({\omega}/{\omega_1}\right)}=\frac{1}{\sqrt{1+\left({\omega}/{\omega_1}\right)^2}}\angle-\tan^{-1}\left({\omega}/{\omega_1}\right)$$

Let us examine the following situations.

$$\omega=\omega_1 \quad\Rightarrow\quad H(j\omega_1)=\frac{1}{\sqrt{1+\left({\omega_1}/{\omega_1}\right)^2}}\angle-\tan^{-1}\left({\omega_1}/{\omega_1}\right)=0.707\angle-45^\circ \quad (0.707=-3\,\text{dB})$$

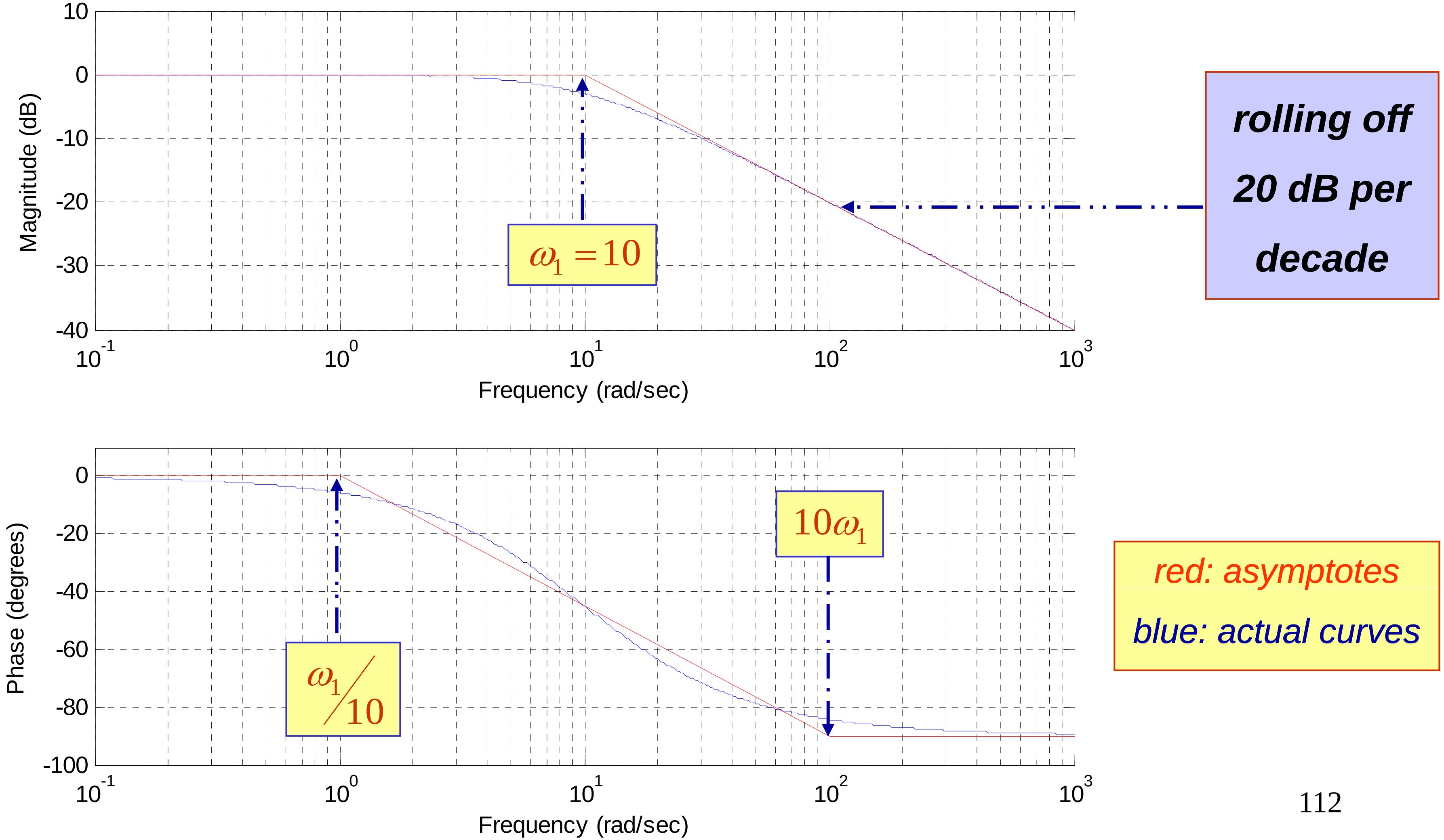
$$\omega<<\omega_1 \quad\Rightarrow\quad H(j\omega)=\frac{1}{\sqrt{1+\left({\omega}/{\omega_1}\right)^2}}\angle-\tan^{-1}\left({\omega}/{\omega_1}\right)\approx 1\angle 0^\circ$$

$$\omega>>\omega_1 \quad\Rightarrow\quad H(j\omega)=\frac{1}{\sqrt{1+\left({\omega}/{\omega_1}\right)^2}}\angle-\tan^{-1}\left({\omega}/{\omega_1}\right)\approx \frac{\omega_1}{\omega}\angle-90^\circ$$

These give us the approximation (asymptotes) of the Bode curves…

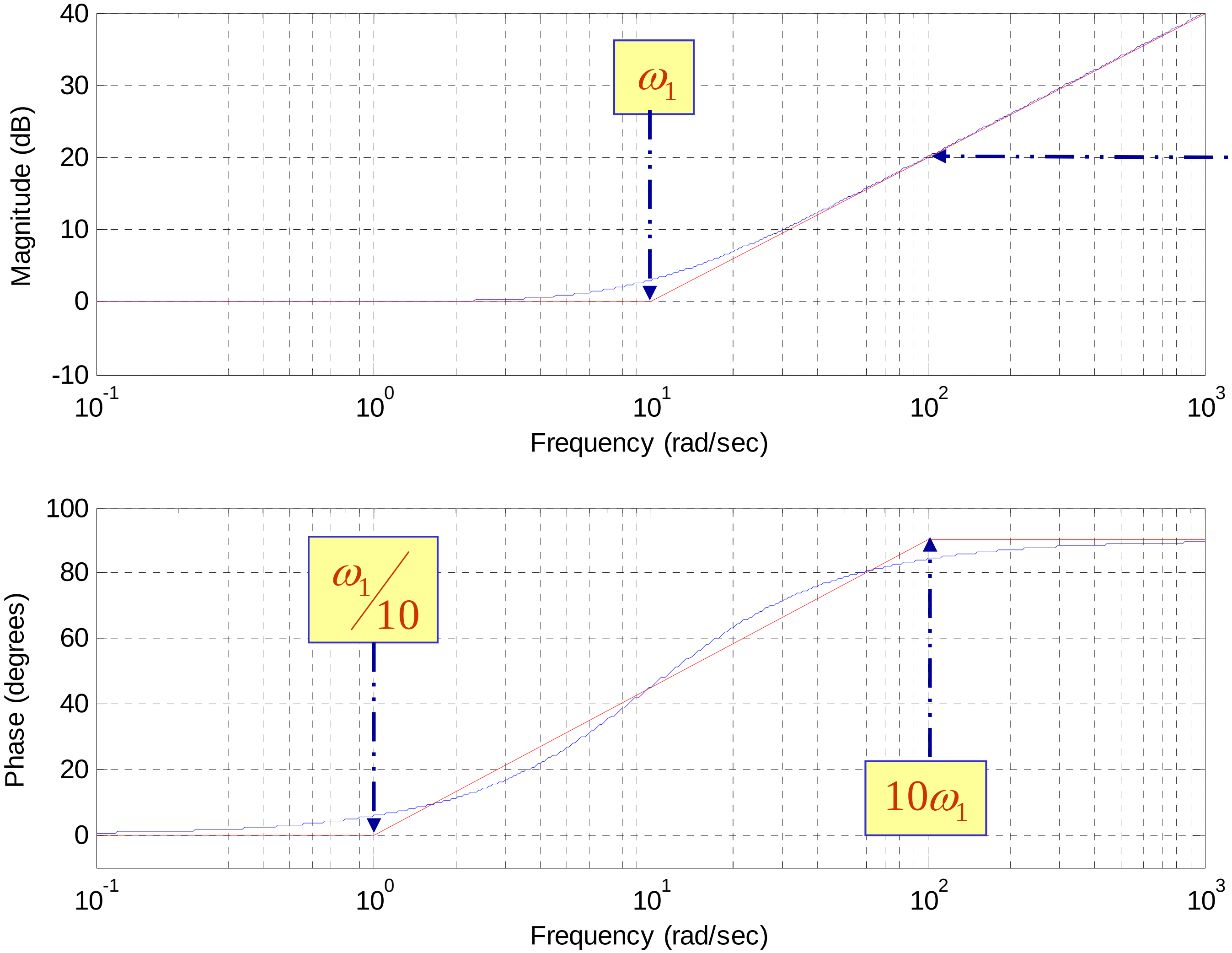
Example

Consider a 1st order system $H(s) = \dfrac{1}{1 + s/10}$ where $\omega_1 = 10$ rad/sec is called the corner frequency.



Bode plot – a simple zero factor

The Bode plot of $H(s) = 1 + {s}/{\omega_1}$ can be done similarly…



rolling up 20 dB per decade

red: asymptotes

blue: actual curves

Bode plot – putting all together

Assume a given system has only simple poles and zeros, i.e.,

$$H(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0} = \frac{b_m (s+z_1)\cdots(s+z_m)}{(s+p_1)\cdots(s+p_m)}$$

or

$$\boxed{H(s) = \frac{b_m (s+z_1)\cdots(s+z_m)}{(s+p_1)\cdots(s+p_m)} = \frac{k(1+ {}^{s}\!/_{z_1})\cdots(1+ {}^{s}\!/_{p_{m_1}})}{s^q (1+ {}^{s}\!/_{p_1})\cdots(1+ {}^{s}\!/_{p_{n_1}})}}$$

In the dB scale, we have

$$\boxed{|H(j\omega)|\ \mathrm{dB} = |k| \text{ in dB} + \left|1+ {}^{j\omega}\!/_{z_1}\right| \text{ in dB} + \cdots + \left|1+ {}^{j\omega}\!/_{z_{m_1}}\right| \text{ in dB} - |\omega| \text{ in dB} \times q - \left|1+ {}^{j\omega}\!/_{p_1}\right| \text{ in dB} - \cdots - \left|1+ {}^{j\omega}\!/_{p_{n_1}}\right| \text{ in dB}}$$

Similarly,

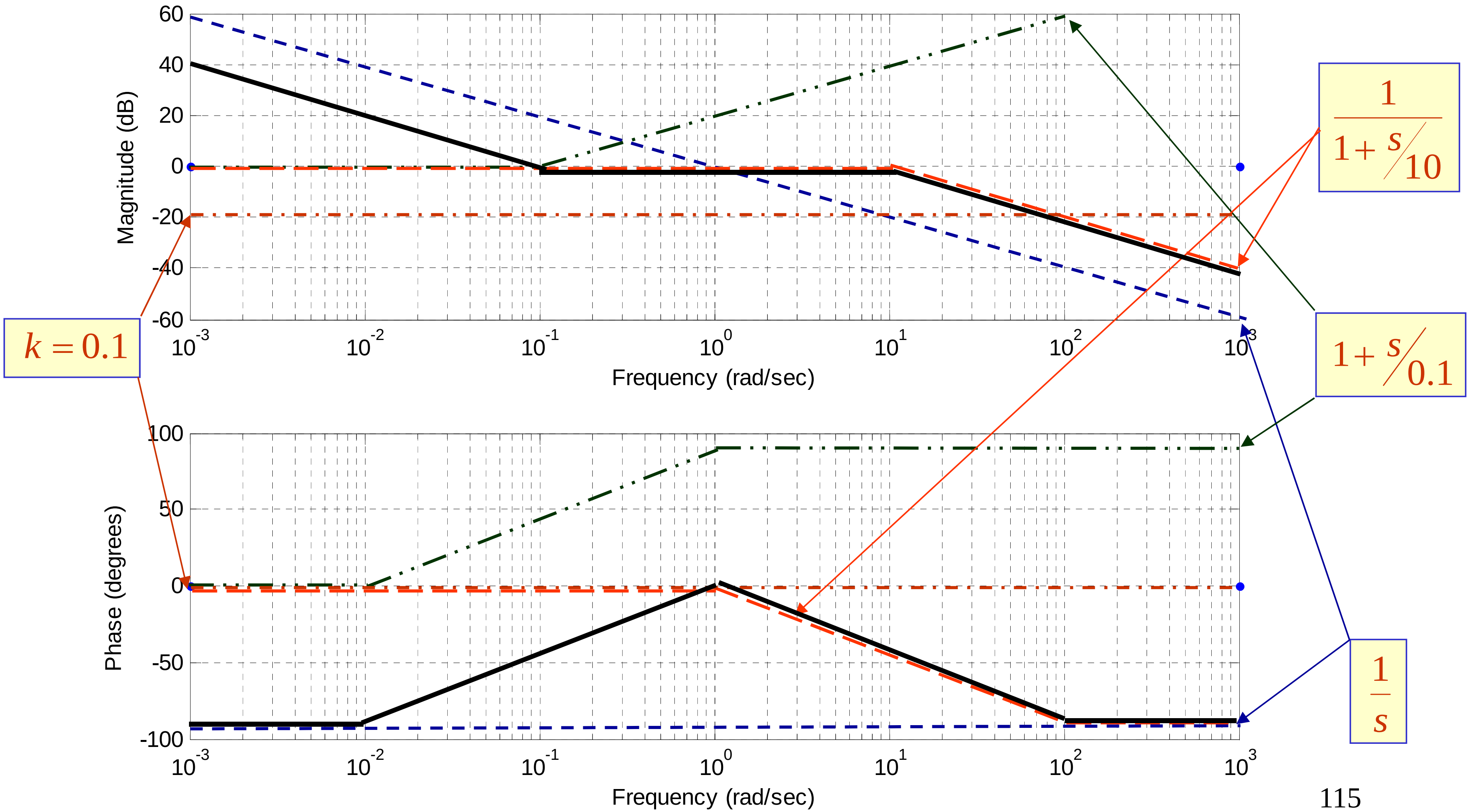
$$\boxed{\angle H(j\omega) = \angle k + \angle\left(1+ {}^{j\omega}\!/_{z_1}\right) + \cdots + \angle\left(1+ {}^{j\omega}\!/_{z_{m_1}}\right) - 90^\circ \times q - \angle\left(1+ {}^{j\omega}\!/_{p_1}\right) - \cdots - \angle\left(1+ {}^{j\omega}\!/_{p_{n_1}}\right)}$$

Thus, the Bode plot of a complex system can be broken down to the additions and subtractions of some simple systems…

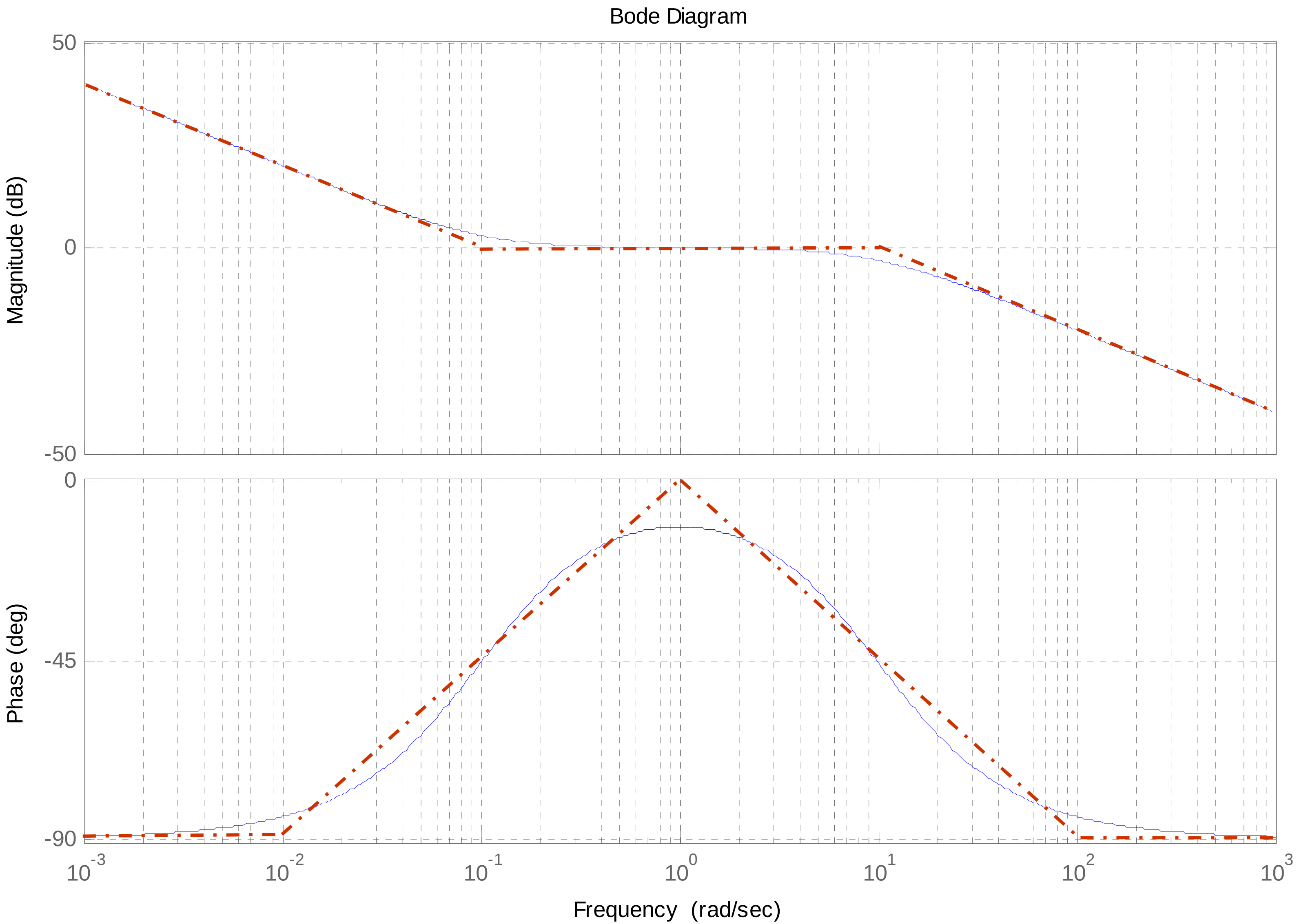
Example

We consider a system characterized by

$$H(s)=\frac{10(s+0.1)}{s(s+10)}=\frac{0.1(1+s/0.1)}{s(1+s/10)}$$



The actual Bode plot

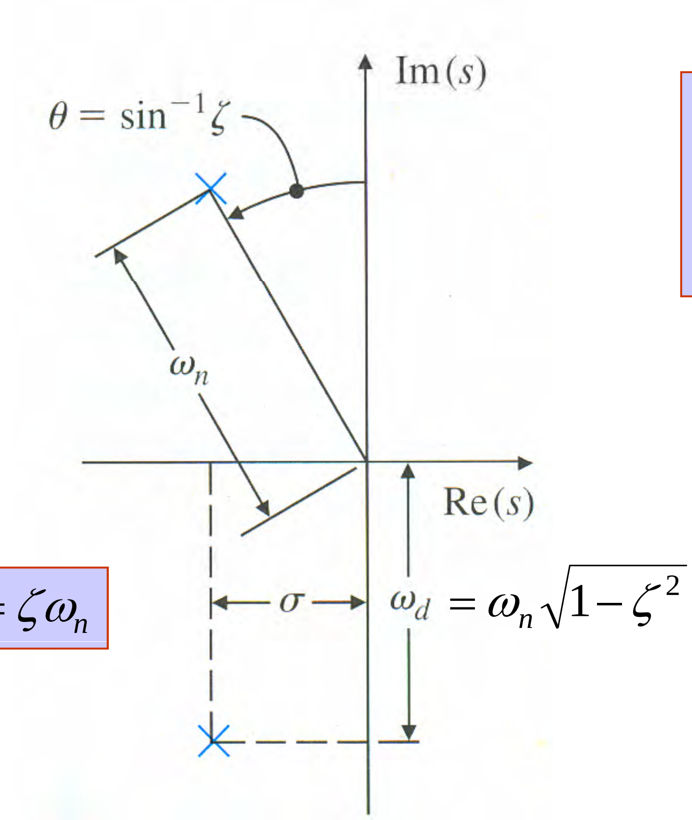


Bode plot of a typical 2nd order system with complex poles

So far, we haven't touched the case when the system has complex poles. Consider

$$H(s)=\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}=\frac{\omega_n^2}{(s+\zeta\omega_n)^2+\omega_n^2(1-\zeta^2)}=\frac{1}{1+2\zeta\left(s/\omega_n\right)+\left(s/\omega_n\right)^2}$$

When $\zeta<1$, it has two complex conjugated poles at



$\sigma=\zeta\omega_n$

ζ is called the damping ratio of the system

ω_n is called the natural frequency

This 2nd order prototype is the most important system for classical control, which is to be covered in Part 2.

Examining

$$H(j\omega)=\frac{1}{1+2\zeta\left(\frac{j\omega}{\omega_n}\right)+\left(\frac{j\omega}{\omega_n}\right)^2}=\frac{1}{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]+j2\zeta\left(\frac{\omega}{\omega_n}\right)}$$

we have

$$\omega \ll \omega_n \quad\Rightarrow\quad H(j\omega)=\frac{1}{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]+j2\zeta\left(\frac{\omega}{\omega_n}\right)}\to 1=1\angle 0^\circ$$

0 dB in magnitude and 0 degree phase at low frequencies

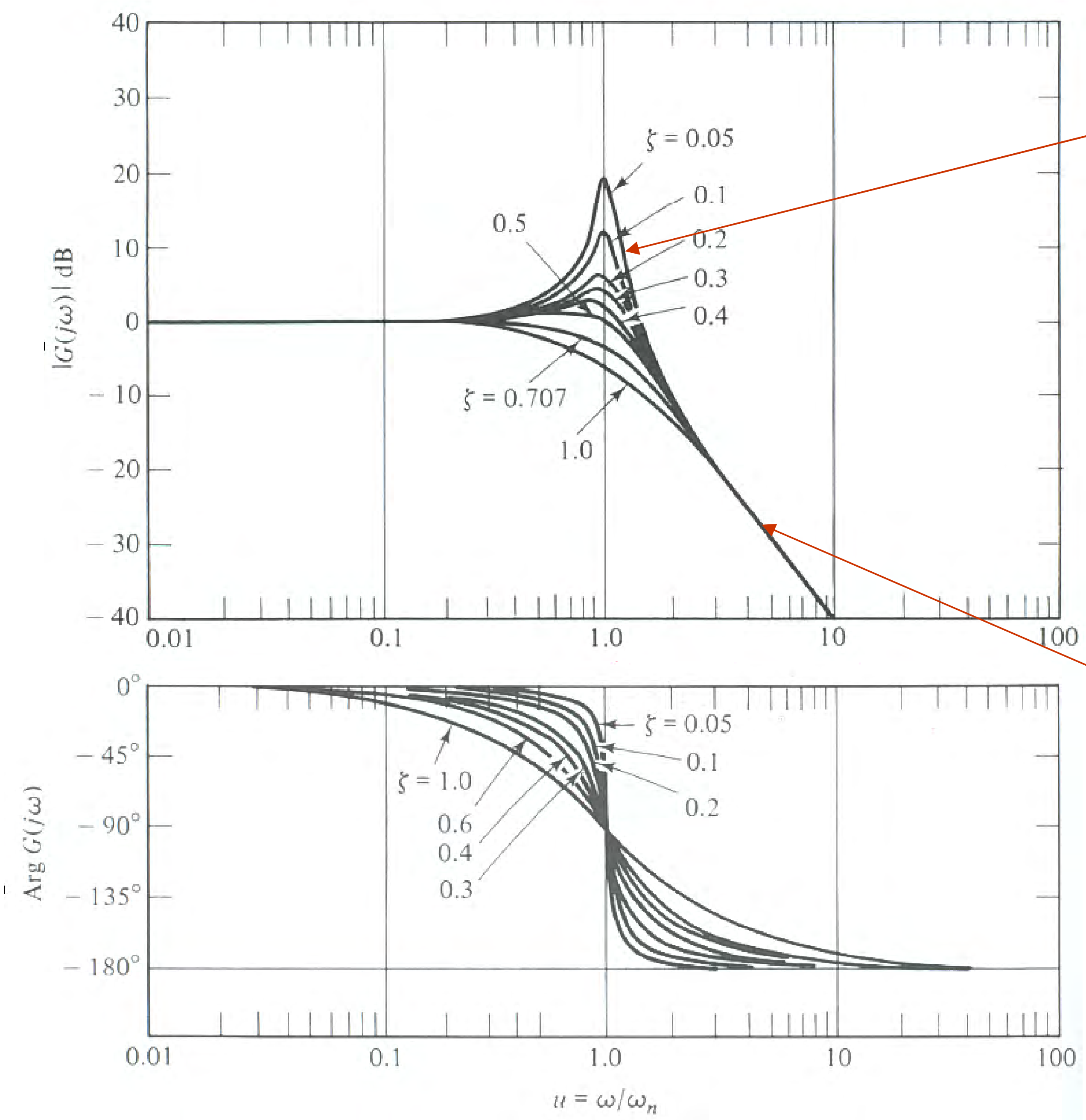
$$\omega \gg \omega_n \quad\Rightarrow\quad H(j\omega)=\frac{1}{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]+j2\zeta\left(\frac{\omega}{\omega_n}\right)}\to -\left(\frac{\omega_n}{\omega}\right)^2=\left(\frac{\omega_n}{\omega}\right)^2\angle -180^\circ$$

roll off 40 dB per decade at high frequencies

$$\omega=\omega_n \quad\Rightarrow\quad \begin{cases} H(j\omega)=\dfrac{1}{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]+j2\zeta\left(\frac{\omega}{\omega_n}\right)}=\dfrac{1}{j2\zeta}=\dfrac{1}{2\zeta}\angle -90^\circ \\ \left|H(j\omega)\right| \text{ in dB}=20\log_{10}\left(\dfrac{1}{2\zeta}\right)=20\log_{10}\left(\dfrac{1}{2}\right)+20\log_{10}\left(\dfrac{1}{\zeta}\right)=-6\text{ dB}-20\log_{10}\zeta \end{cases}$$

Bode plot of the 2nd order prototype

$$|H(j\omega_n)| \text{ peak in dB} = -6 \text{ dB} - 20\log_{10}\zeta$$



peak magnitudes:

$\zeta = 0.05$, peak = 20 dB

$\zeta = 0.1$, peak = 14 dB

$\zeta = 0.2$, peak = 8 dB

$\zeta = 0.3$, peak = 4.5 dB

$\zeta = 0.4$, peak = 2 dB

$\zeta = 0.5$, peak = 0 dB

rolling off 40 dB per decade

Polar plot

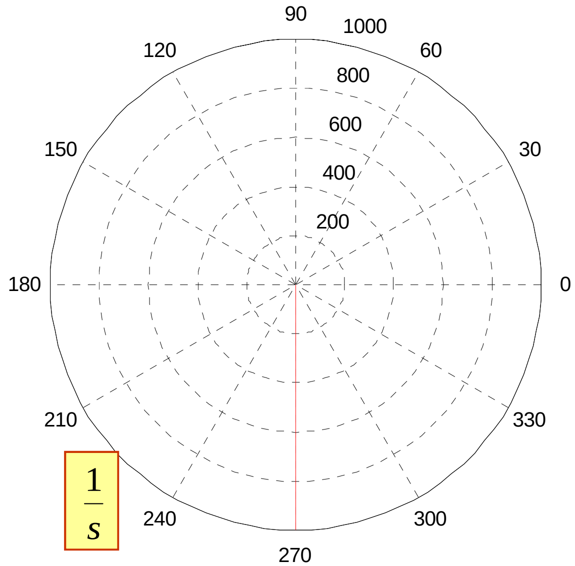
As mentioned earlier, the polar plot is another form to draw the frequency response of a system transfer function, which is general a complex function of ω. The Bode plot is to draw the magnitude and phase responses in a separate form. If we draw the magnitude and phase as a whole directly on a complex plane, it is called ***polar plot***.

The easiest way to draw a polar plot for a system, especially for a complicated system, is to use a computer. This is true for the Bode plots as well.

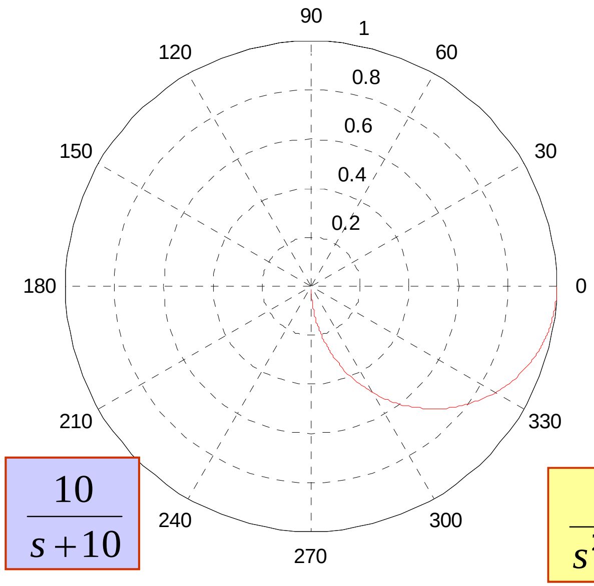
By the way, polar plots will be again in the second part in the topic related to Nyquist plots and Nyquist stability criterion of control systems. Bode plots will be heavily used in the second part as well.

In what follows, we will show the polar plots of some systems studied earlier. Once again, they are done using MATLAB…

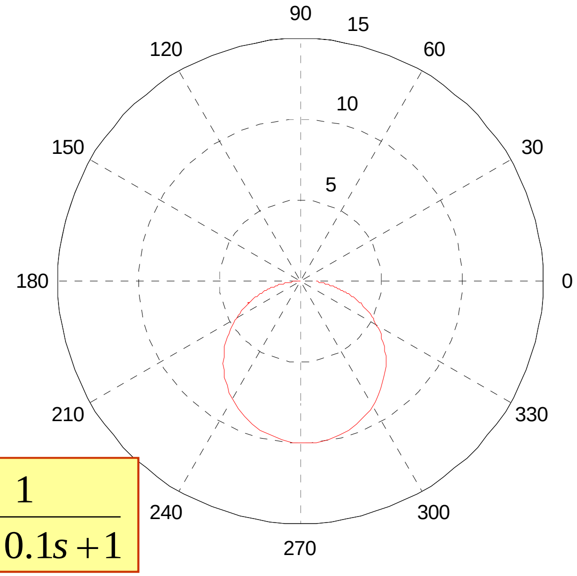
Polar plots of some examples



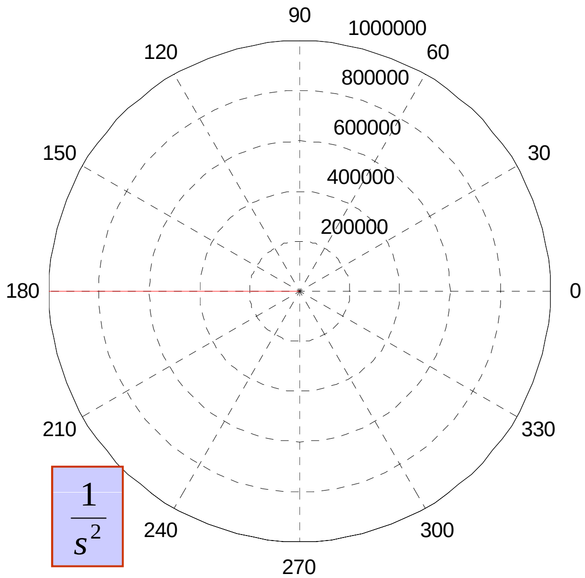
$$\frac{1}{s}$$



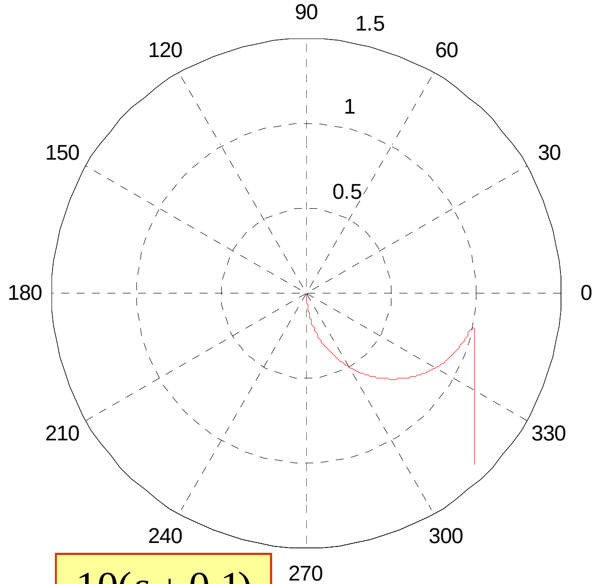
$$\frac{10}{s+10}$$



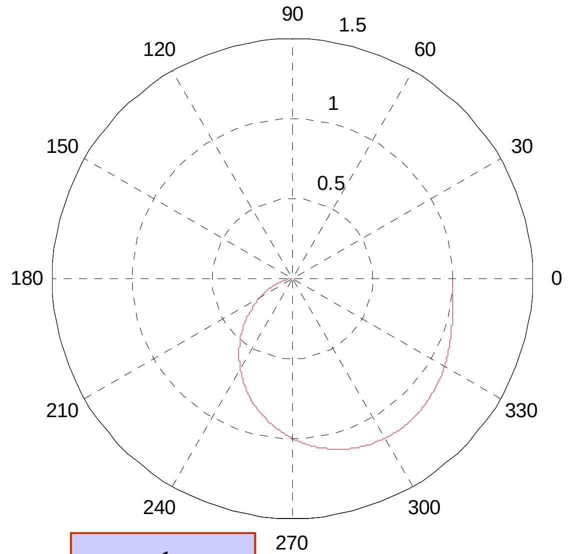
$$\frac{1}{s^2+0.1s+1}$$



$$\frac{1}{s^2}$$



$$\frac{10(s+0.1)}{s(s+10)}$$

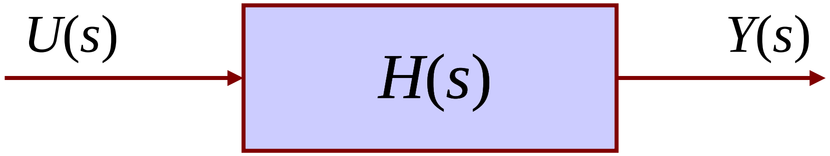


$$\frac{1}{s^2+s+1}$$

Properties of Linear Time Invariant Systems

Impulse response

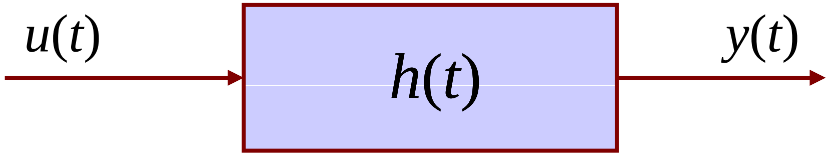
Given a linear time invariant system



its unit impulse response is defined as the resulting output response corresponding to a unit impulse input, i.e., $u(t) = \delta(t)$. Recall that the Laplace transform of $\delta(t)$ is 1. Thus, the unit impulse response is given by

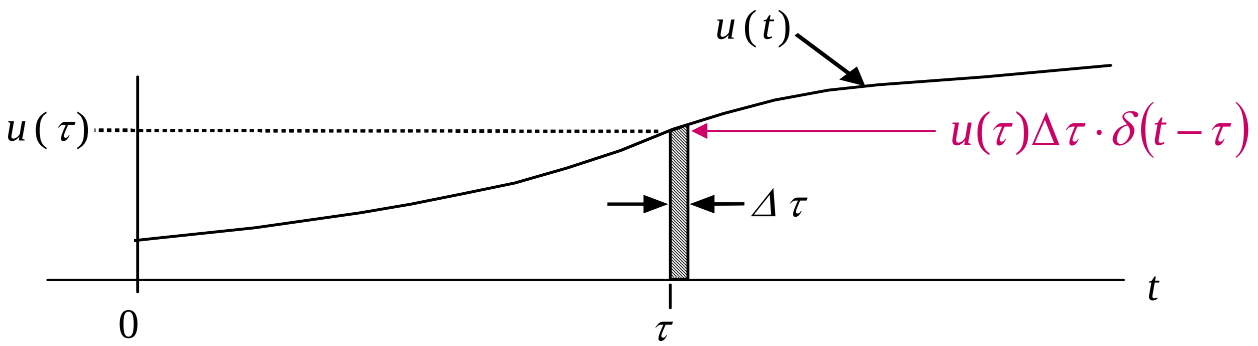
$$\boxed{h(t) = y(t) = L^{-1}\left[Y(s)\right] = L^{-1}\left[H(s)U(s)\right] = L^{-1}\left[H(s)\right]}$$

which implies the unit impulse response characterizes the properties of the system as well. In particular, we can characterize the given system in the time domain as



What is the relationship among u, h and y?

Let us take a small piece of the input signal $u(t)$ at $t = \tau$, i.e.,



When $\Delta\tau$ is sufficiently small, we can regard the shaded portion in the above figure as an impulse function. Since $\delta(t) \to h(t)$, by the properties of Laplace transform, we have

$$\boxed{u(\tau)\Delta\tau\,\delta(t-\tau) \quad \to \quad \Delta y(t) = u(\tau)\Delta\tau\, h(t-\tau)}$$

which is the output corresponding to the shaded portion of the input. The output response corresponding to the whole input signal $u(t)$ is then given by

$$y(t) = \int_0^\infty dy(t) = \int_0^\infty u(\tau)h(t-\tau)d\tau = u(t)\otimes h(t) = h(t)\otimes u(t)$$

convolution

Example

Find the output response of the following system using the convolution integral

$$u(t) = 1(t) \longrightarrow \boxed{H(s) = \frac{1}{s+1}} \longrightarrow y(t)$$

The unit impulse response is

$$h(t) = L^{-1}[H(s)] = L^{-1}\left[\frac{1}{s+1}\right] = e^{-t}$$

The output response

$$y(t) = h(t) \otimes u(t) = \int_0^\infty h(\tau)u(t-\tau)d\tau = \int_0^t h(\tau)u(t-\tau)d\tau = \int_0^t e^{-\tau}d\tau = -e^{-\tau}\Big|_0^t = 1 - e^{-t}$$

If we compute it using Laplace and inverse Laplace transforms, i.e.,

$$y(t) = L^{-1}[Y(s)] = L^{-1}[H(s)U(s)] = L^{-1}\left[\frac{1}{s+1}\cdot\frac{1}{s}\right] = L^{-1}\left[\frac{1}{s} - \frac{1}{s+1}\right] = 1 - e^{-t}$$

which yields the same solution! However, it is much easier to do it using by the latter.

Example

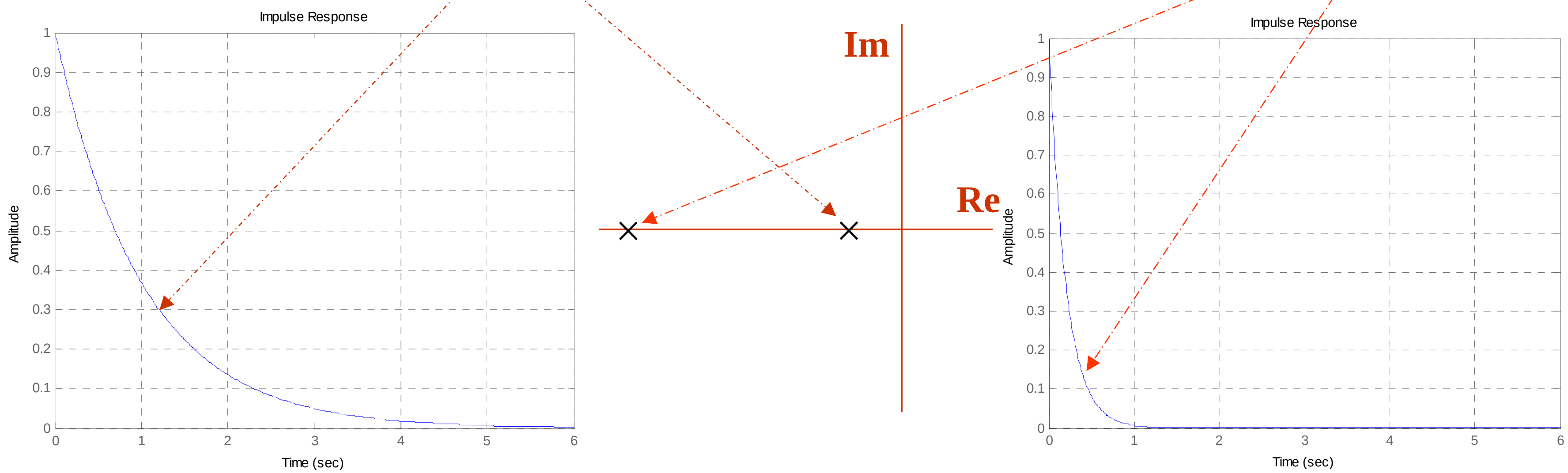
Consider the unit impulse response of the following two systems

$$H_1(s) = \frac{1}{s+1}$$

$$h_1(t) = L^{-1}[H_1(s)] = L^{-1}\left[\frac{1}{s+1}\right] = e^{-t}$$

$$H_2(s) = \frac{1}{s+5}$$

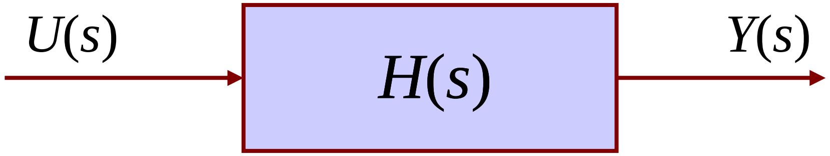
$$h_2(t) = L^{-1}[H_2(s)] = L^{-1}\left[\frac{1}{s+5}\right] = e^{-5t}$$



Observation: The farther the system pole is located to the left in the left-half plane (LHP), the faster the output response is decaying to zero.

Step response

Given a linear time invariant system



its unit step response is defined as the resulting output response corresponding to a unit step input, i.e., $u(t) = 1(t)$. Recall that the Laplace transform of $1(t)$ is $1/s$. Thus, the unit step response is given by

$$\boxed{y(t) = L^{-1}\left[Y(s)\right] = L^{-1}\left[\frac{1}{s}H(s)\right]}$$

It can also be evaluated by using the convolution integral, i.e.,

$$y(t) = 1(t) \otimes h(t) = \int_0^t u(\tau)h(t-\tau)d\tau = \int_0^t h(\tau)u(t-\tau)d\tau = \int_0^t h(\tau)d\tau$$

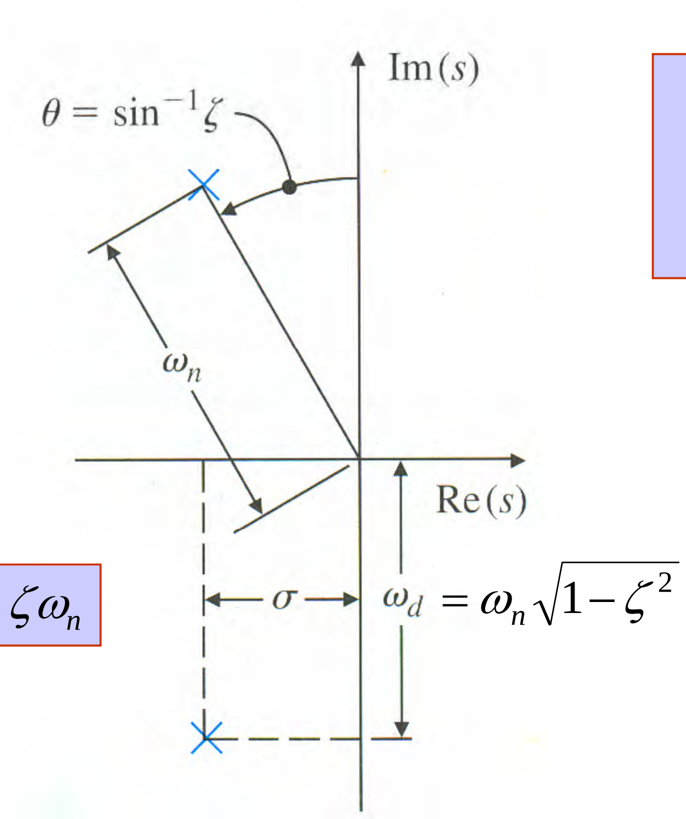
Unit step response for the 2nd order prototype

This is very important for the 2nd part of this course in designing a meaningful control system. We consider the 2nd order prototype

$$H(s)=\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}=\frac{\omega_n^2}{(s+\zeta\omega_n)^2+\omega_n^2(1-\zeta^2)}$$

ζ is the damping ratio of the system

ω_n is the natural frequency



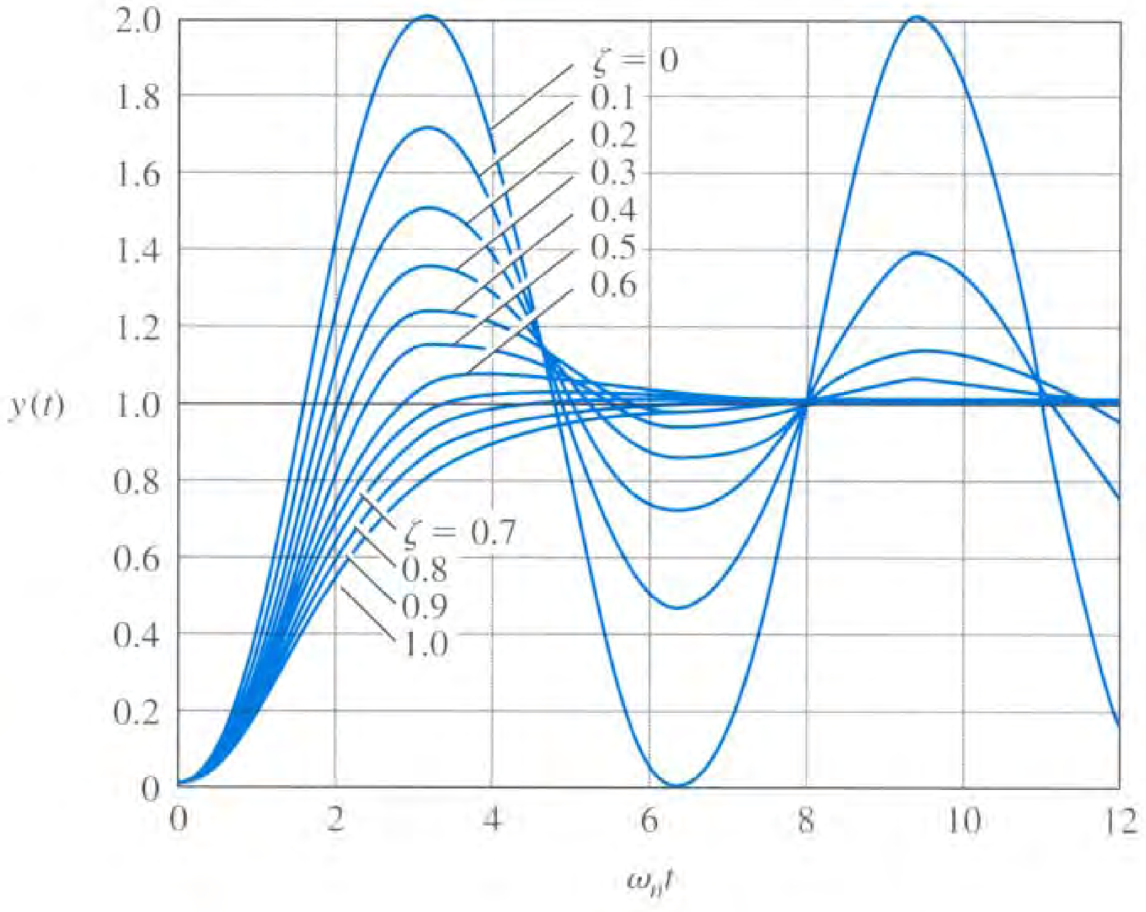
$\sigma = \zeta\omega_n$

It can be shown that its unit step response is given as

$$y(t)=1-e^{-\sigma t}\left(\cos\omega_d t+\frac{\sigma}{\omega_d}\sin\omega_d t\right)$$

Unit step response for the 2nd order prototype (cont.)

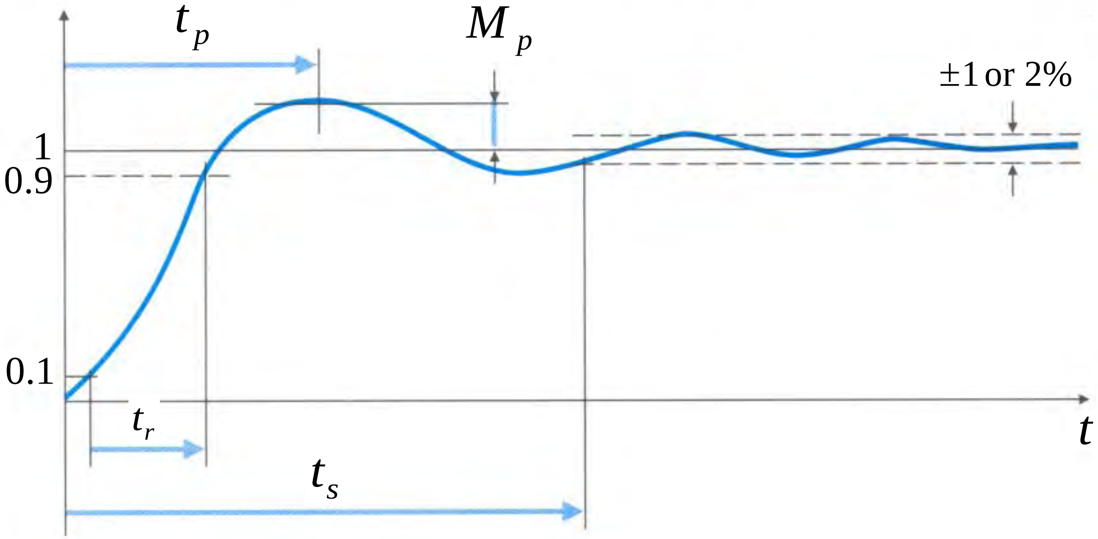
Graphically,

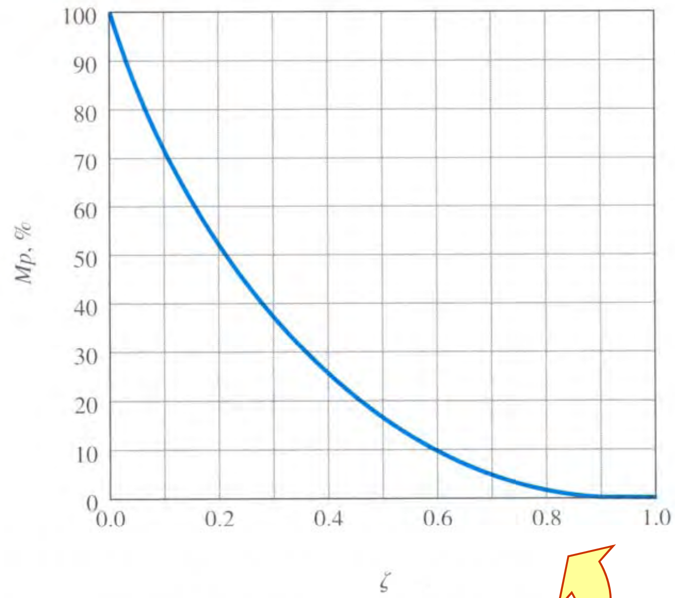


It can be observed that the smaller damping ratio yields larger overshoot.

Unit step response for the 2nd order prototype (cont.)

The typical step response can be depicted as





$$y(t) = 1 - e^{-\sigma t}\left(\cos\omega_d t + \frac{\sigma}{\omega_d}\sin\omega_d t\right)$$

overshoot $M_p = e^{-\pi\zeta/\sqrt{1-\zeta^2}}$

rise time $t_r \cong \dfrac{1.8}{\omega_n}$

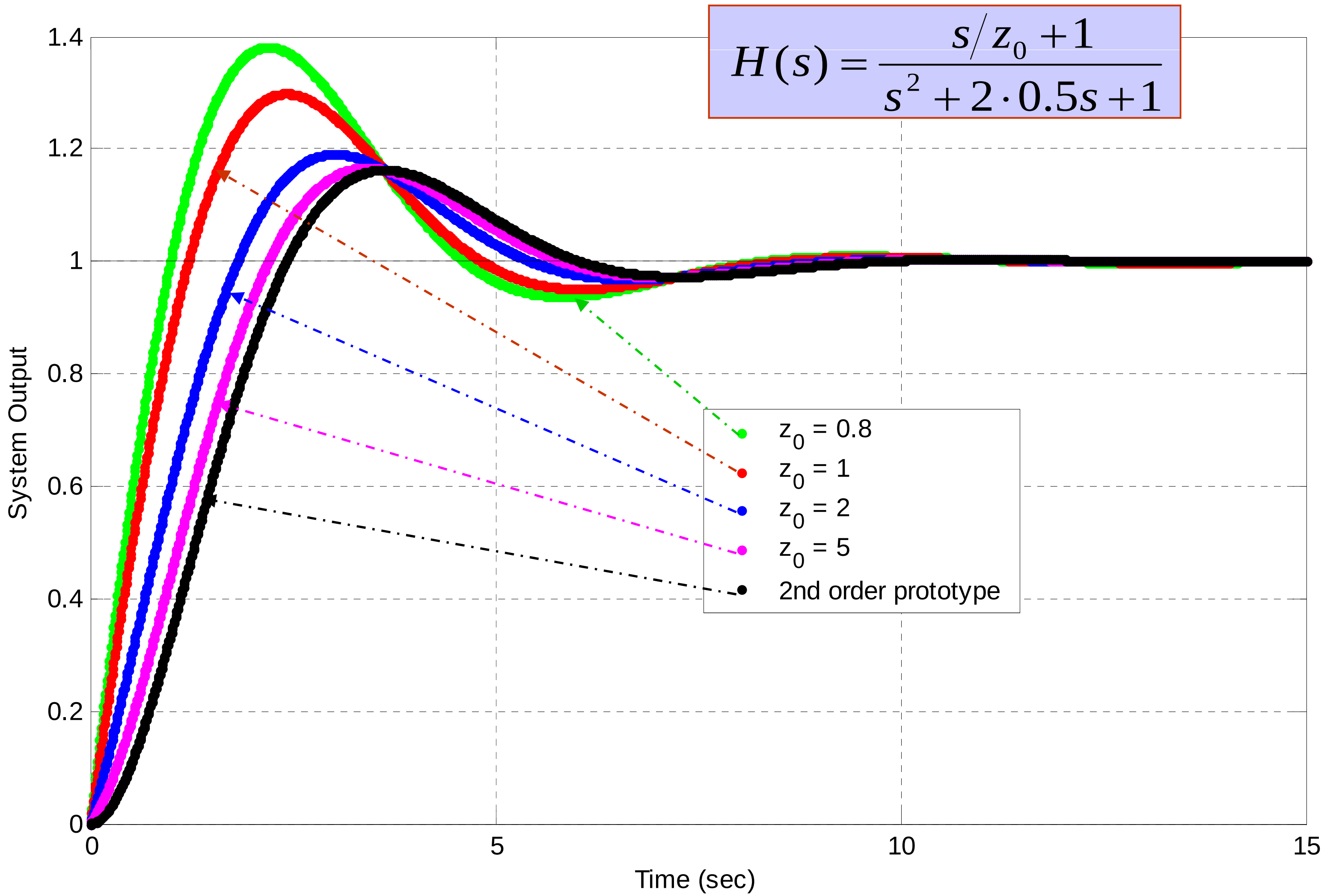
1% settling time $t_s \cong \dfrac{4.6}{\zeta\omega_n}$

2% settling time $t_s \cong \dfrac{4}{\zeta\omega_n}$

peak time $t_p \cong \dfrac{\pi}{\omega_n\sqrt{1-\zeta^2}}$

all in sec

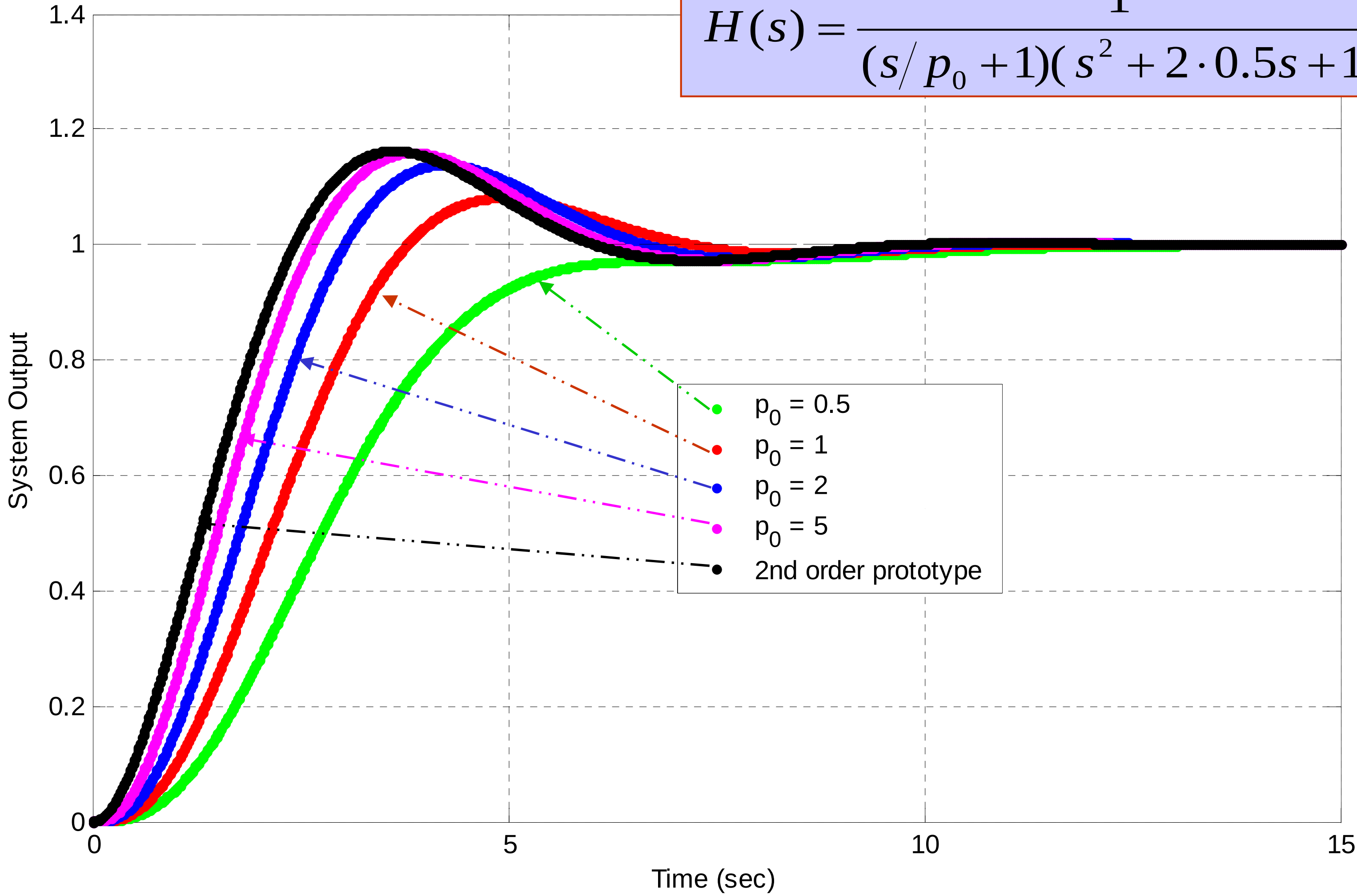
Effect of an additional system zero



Clearly, the farther the system zero is located to the left on LHP, the lesser the system output response is affected. However, settling time is about the same for all cases.

Effect of an additional system pole

$$H(s) = \frac{1}{(s/p_0 + 1)(s^2 + 2 \cdot 0.5 s + 1)}$$



Again, the farther the additional system pole is located to the left on LHP, the lesser the system output response is affected. For small p_0, rise time & settling time are longer.

System stability

Example 1: Consider a system with a transfer function,

$$U(s) = 1 \longrightarrow \boxed{H(s) = \frac{1}{s^2 - 1}} \longrightarrow Y(s)$$

$$e^{-at} \Leftrightarrow \frac{1}{s+a}$$

$$\downarrow$$

$$0.5e^{-t} \Leftrightarrow \frac{0.5}{s+1}$$

&

$$0.5e^{t} \Leftrightarrow \frac{0.5}{s-1}$$

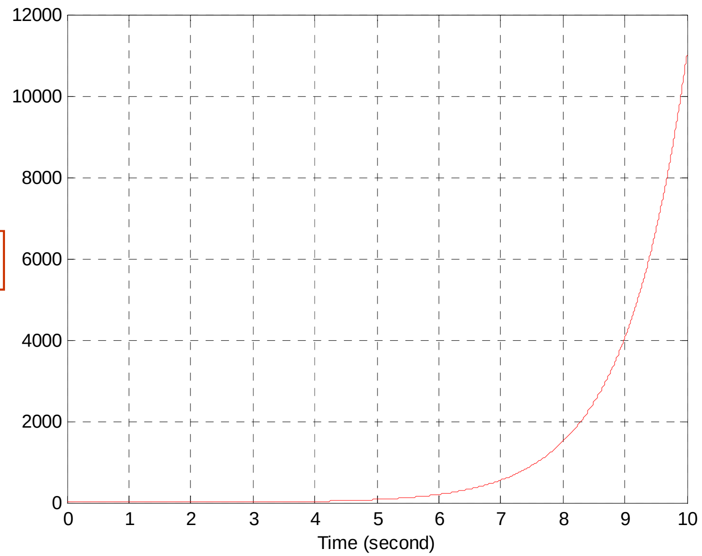
We have

$$Y(s) = H(s)U(s) = \frac{1}{s^2 - 1} = \frac{1}{(s+1)(s-1)} = \frac{0.5}{s-1} - \frac{0.5}{s+1}$$

From the Laplace transform table, we obtain

$$y(t) = 0.5(e^{t} - e^{-t})$$

$y(t)$



This system is said to be *unstable* because the output response $y(t)$ goes to infinity as time t is getting larger and large. This happens because the denominator of $H(s)$ has one positive root at $s = 1$.

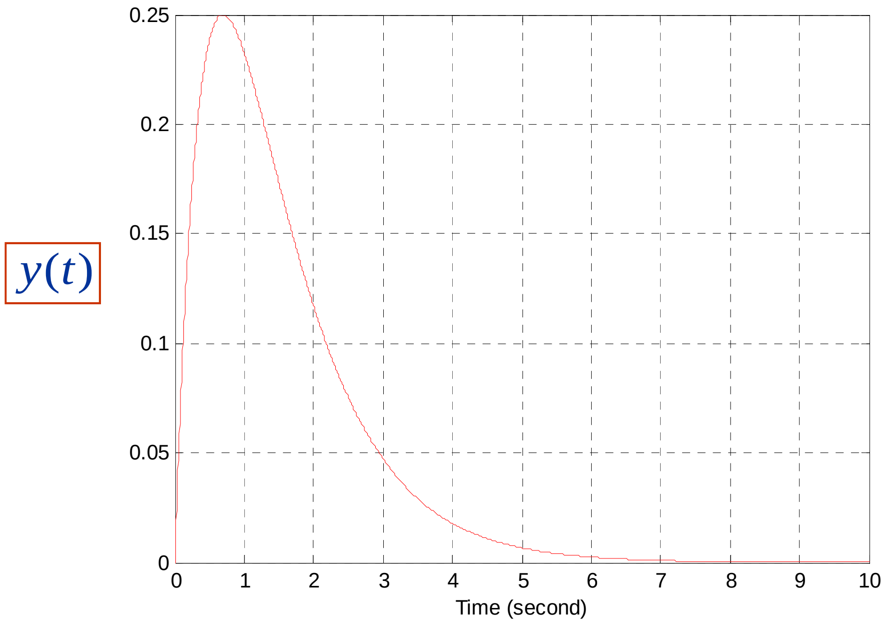
Example 2: Consider a closed-loop system with,

$$U(s) = 1 \longrightarrow \boxed{H(s) = \frac{1}{s^2 + 3s + 2}} \longrightarrow Y(s)$$

We have

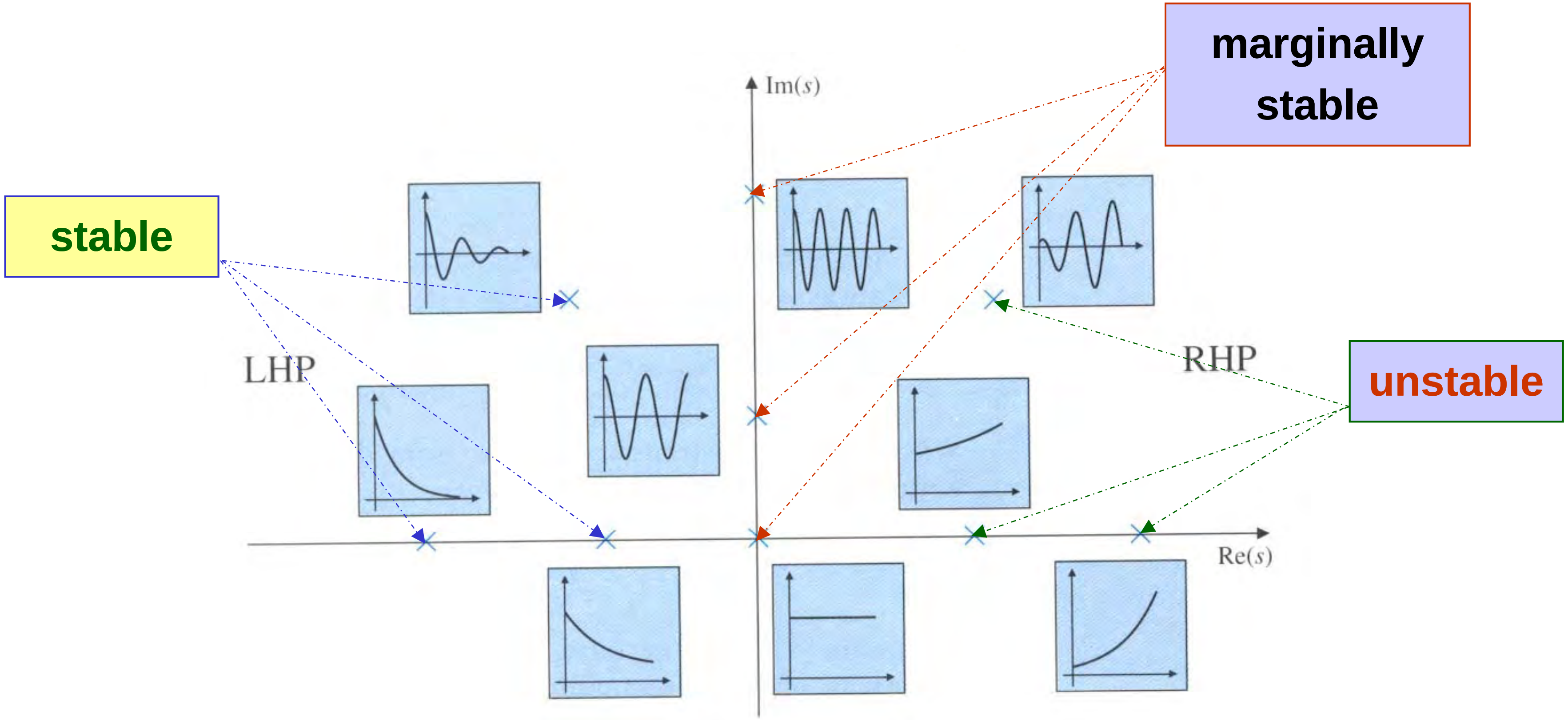
$$Y(s) = H(s)U(s) = \frac{1}{s^2 + 3s + 2} = \frac{1}{(s+1)(s+2)} = \frac{1}{s+1} - \frac{1}{s+2}$$

From the Laplace transform table, we obtain $y(t) = e^{-t} - e^{-2t}$



This system is said to be ***stable*** because the output response $y(t)$ goes to 0 as time t is getting larger and large. This happens because the denominator of $H(s)$ has no positive roots.

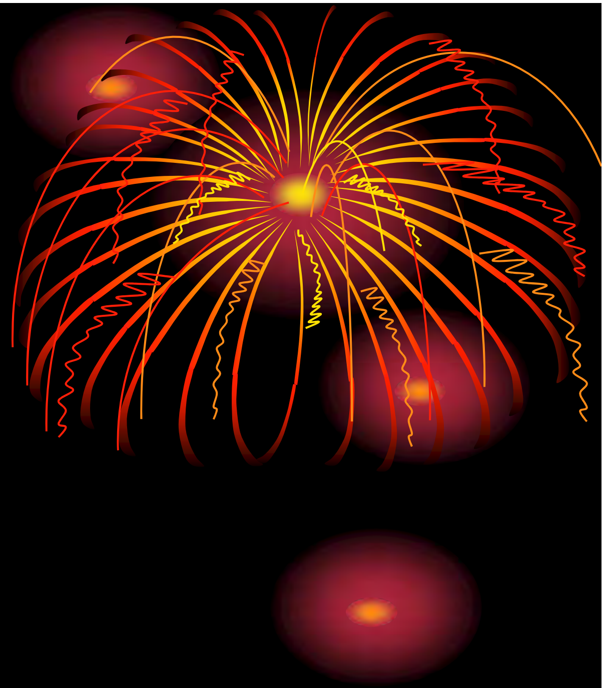
A given system is ***stable*** if the system does not have poles on the right-half plane (RHP). It is ***unstable*** if it has at least one pole on the RHP. In particular,



The above diagram also shows the relationship the locations of poles and natural responses.

Annex: some useful MATLAB commands

- **LAPLACE** is to generate the Laplace transform of the scalar time-domain function.
- **ILAPLACE** is to generate the inverse Laplace transform of the scalar s-domain function.
- **IMPULSE** plots the impulse response of a linear system.
- **STEP** plots the step response of a linear system.
- **BODE** gives the Bode plot of a linear system.
- **ROOTS** computes the roots of a polynomial (can be used to compute system poles & zeros).
- **PLOT** is to generate a linear plot (there are many options available for plotting curves).
- **SEMILOGX** is to generate a semi-log scale plot (see those Bode plots).
- **POLAR** is to generate a polar plot of a complex-valued function.
- **LOG10** computes common (base 10) logarithm. **LOG** is for the natural logarithm.
- **ANGLE** is to compute the angular of a complex number.
- **ABS** is to compute the magnitude of a complex number.
- **EXP** is the exponential function.
- **SIN**, **COS**, **TAN**, **ASIN**, **ACOS**, **ATAN**. To learn more about MATLAB functions, use **HELP**…



That’s all, Folks!

Thank You…