Time-Scale and Eigenstructure Assignment

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10.2 Continuous-time Systems

In this section, we describe the technique of the asymptotic time-scale and eigenstructure assignment (ATEA) design for continuous-time systems together with its applications in solving H_2 and H_∞ control as well as disturbance decoupling problems. Consider a continuous-time linear system Σ characterized by

$$\begin{cases} \dot{x} = A\,x + B\,u, \\ y = C\,x + D\,u, \end{cases} \tag{10.2.1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $y \in \mathbb{R}^p$ are the state, input and output of Σ. Without loss of generality, we assume that (A, B) is stabilizable, and both B and C are of full rank. As indicated earlier, we assume that Σ does not have any invariant zeros on the imaginary axis.

The key idea behind the ATEA design is to decompose the given system into various subsystems using the special coordinate basis (SCB) technique and then tackle the subsystems one by one in accordance with their structural properties through the selection of appropriate control gains…

STEP ATEA-C.1.

Transform Σ into the structural decomposition or the special coordinate basis form as given by Theorem 5.4.1, that is, compute nonsingular state, input and output transformations Γ_s, Γ_i and Γ_o that transform the given system Σ into the special coordinate basis form of Theorem 5.4.1, which can also be put in the following compact form:

$$\tilde{A} = \Gamma_s^{-1} A \Gamma_s = \begin{bmatrix} A_{aa}^- & 0 & L_{ab}^- C_b & 0 & L_{ad}^- C_d \\ 0 & A_{aa}^+ & L_{ab}^+ C_b & 0 & L_{ad}^+ C_d \\ 0 & 0 & A_{bb} & 0 & L_{bd} C_d \\ B_c E_{ca}^- & B_c E_{ca}^+ & L_{cb} C_b & A_{cc} & L_{cd} C_d \\ B_d E_{da}^- & B_d E_{da}^+ & B_d E_{db} & B_d E_{dc} & A_{dd} \end{bmatrix} + \begin{bmatrix} B_{0a}^- \\ B_{0a}^+ \\ B_{0b} \\ B_{0c} \\ B_{0d} \end{bmatrix} \begin{bmatrix} C_{0a}^- & C_{0a}^+ & C_{0b} & C_{0c} & C_{0d} \end{bmatrix}, \qquad (10.2.2)$$

$$\tilde{B} = \Gamma_s^{-1} B \Gamma_i = \begin{bmatrix} B_{0a}^- & 0 & 0 \\ B_{0a}^+ & 0 & 0 \\ B_{0b} & 0 & 0 \\ B_{0c} & 0 & B_c \\ B_{0d} & B_d & 0 \end{bmatrix}, \qquad (10.2.3)$$

$$\tilde{C} = \Gamma_o^{-1} C \Gamma_s = \begin{bmatrix} C_{0a}^- & C_{0a}^+ & C_{0b} & C_{0c} & C_{0d} \\ 0 & 0 & 0 & 0 & C_d \\ 0 & 0 & C_b & 0 & 0 \end{bmatrix}, \qquad (10.2.4)$$

$$\tilde{D} = \Gamma_o^{-1} D \Gamma_i = \begin{bmatrix} I_{m_0} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad (10.2.5)$$

$$A_{ss} = \begin{bmatrix} A_{aa}^+ & L_{ab}^+ C_b \\ 0 & A_{bb} \end{bmatrix}, \quad B_{0s} = \begin{bmatrix} B_{0a}^+ \\ B_{0b} \end{bmatrix}$$

$$L_{sd} = \begin{bmatrix} L_{ad}^+ \\ L_{bd} \end{bmatrix}, \quad B_s = \begin{bmatrix} B_{0s} & L_{sd} \end{bmatrix}$$

bad subsystems

good subsystems

fast subsystems

nonexistent for strictly proper systems

bad subsystems

STEP ATEA-C.2.

Let F_s be chosen such that

$$\lambda(A^c_{ss}) = \lambda(A_{ss} - B_s F_s) \subset \mathbb{C}^-, \tag{10.2.9}$$

and partition F_s in conformity with (10.2.7) and (10.2.8) as

$$F_s = \begin{bmatrix} F_{s0} \\ F_{s1} \end{bmatrix} = \begin{bmatrix} F^+_{a0} & F_{b0} \\ F^+_{a1} & F_{b1} \end{bmatrix}. \tag{10.2.10}$$

It follows from the property of the special coordinate basis that the pair (A_{ss}, B_s) is controllable provided that the pair (A, B) is stabilizable. Then, we further partition $F_{s1} = [\, F^+_{a1} \quad F_{b1} \,]$ as

$$F_{s1} = [\, F^+_{a1} \quad F_{b1} \,] = \begin{bmatrix} F^+_{a11} & F_{b11} \\ F^+_{a12} & F_{b12} \\ \vdots & \vdots \\ F^+_{a1m_d} & F_{b1m_d} \end{bmatrix}, \tag{10.2.11}$$

where F^+_{a1i} and F_{b1i} are of dimensions $1 \times n^+_a$ and $1 \times n_b$, respectively.

good subsystems

STEP ATEA-C.3.

Let F_c be any arbitrary $m_c \times n_c$ matrix subject to the constraint that

$$A_{cc}^{c} = A_{cc} - B_c F_c \tag{10.2.12}$$

is a stable matrix. Note that the existence of such an F_c is guaranteed by the property that (A_{cc}, B_c) is controllable.

fast subsystems

STEP ATEA-C.4.

This step makes use of the fast subsystems, $i = 1, 2, \ldots, m_{\rm d}$, represented by (5.4.11). Let

$$\Lambda_i = \{\lambda_{i1}, \lambda_{i2}, \ldots, \lambda_{iq_i}\}, \quad i = 1, 2, \ldots, m_{\rm d},$$

be the sets of q_i elements, all in $\mathbb{C}^-$, which are closed under complex conjugation, where q_i and $m_{\rm d}$ are as defined in Theorem 5.4.1. Then, we let $\Lambda_{\rm d} := \Lambda_1 \cup \Lambda_2 \cup \cdots \cup \Lambda_{m_{\rm d}}$. For $i = 1, 2, \ldots, m_{\rm d}$, we define

$$p_i(s) := \prod_{j=1}^{q_i} (s - \lambda_{ij}) = s^{q_i} + F_{i1} s^{q_i - 1} + \cdots + F_{iq_i - 1} s + F_{iq_i}, \quad (10.2.13)$$

and a sub-gain matrix parameterized by tuning parameter, ε,

$$\tilde{F}_i(\varepsilon) := \frac{1}{\varepsilon^{q_i}} \Big[F_{iq_i}, \ \varepsilon F_{iq_i - 1}, \ \ldots, \ \varepsilon^{q_i - 1} F_{i1} \Big]. \quad (10.2.14)$$

high gains

STEP ATEA-C.5.

In this step, various gains calculated in STEPS ATEA-C.2 to ATEA-C.4 are put together to form a composite state feedback gain for the given system Σ. Let

$$\tilde{F}^+_{\rm a1}(\varepsilon) := \begin{bmatrix} F^+_{\rm a11} F_{1q_1} / \varepsilon^{q_1} \\ F^+_{\rm a12} F_{2q_2} / \varepsilon^{q_2} \\ \vdots \\ F^+_{{\rm a1}m_{\rm d}} F_{m_{\rm d} q_{m_{\rm d}}} / \varepsilon^{q_{m_{\rm d}}} \end{bmatrix}, \tag{10.2.15}$$

$$\tilde{F}_{\rm b1}(\varepsilon) := \begin{bmatrix} F_{\rm b11} F_{1q_1} / \varepsilon^{q_1} \\ F_{\rm b12} F_{2q_2} / \varepsilon^{q_2} \\ \vdots \\ F_{{\rm b1}m_{\rm d}} F_{m_{\rm d} q_{m_{\rm d}}} / \varepsilon^{q_{m_{\rm d}}} \end{bmatrix}, \tag{10.2.16}$$

and

$$\tilde{F}_{\rm s1}(\varepsilon) = \begin{bmatrix} \tilde{F}^+_{\rm a1}(\varepsilon) & \tilde{F}_{\rm b1}(\varepsilon) \end{bmatrix}. \tag{10.2.17}$$

one of the prices paid for controlling bad subsystems is high gain

Then, the ATEA state feedback gain is given by

$$F(\varepsilon) = -\Gamma_{\rm i} \Big(\tilde{F}(\varepsilon) + \tilde{F}_0 \Big) \Gamma_{\rm s}^{-1}, \tag{10.2.18}$$

where

$$\tilde{F}(\varepsilon) = \begin{bmatrix} 0 & F^+_{\rm a0} & F_{\rm b0} & 0 & 0 \\ 0 & \tilde{F}^+_{\rm a1}(\varepsilon) & \tilde{F}_{\rm b1}(\varepsilon) & 0 & \tilde{F}_{\rm d}(\varepsilon) \\ 0 & 0 & 0 & F_{\rm c} & 0 \end{bmatrix}, \tag{10.2.19}$$

$$\tilde{F}_0 = \begin{bmatrix} C^-_{\rm 0a} & C^+_{\rm 0a} & C_{\rm 0b} & C_{\rm 0c} & C_{\rm 0d} \\ E^-_{\rm da} & E^+_{\rm da} & E_{\rm db} & E_{\rm dc} & E_{\rm dd} \\ E^-_{\rm ca} & E^+_{\rm ca} & 0 & 0 & 0 \end{bmatrix}, \tag{10.2.20}$$

and where

$$\tilde{F}_{\rm d}(\varepsilon) = {\rm diag}\Big[\tilde{F}_1(\varepsilon),\ \tilde{F}_2(\varepsilon),\ \ldots,\ \tilde{F}_{m_{\rm d}}(\varepsilon) \Big]. \tag{10.2.21}$$

The Overall State Feedback Gain

Theorem 10.2.1. *Consider the given system Σ of (10.2.1). Then, the ATEA state feedback law $u = F(\varepsilon)x$ with $F(\varepsilon)$ being given as in (10.2.18) has the following properties:*

1. *There exists a scalar $\varepsilon^* > 0$ such that for every $\varepsilon \in (0, \varepsilon^*]$, the closed-loop system comprising the given system Σ and the ATEA state feedback law is asymptotically stable. Moreover, as $\varepsilon \to 0$, the closed-loop eigenvalues are given by*

$$\lambda(A_{\mathrm{aa}}^-), \quad \lambda(A_{\mathrm{cc}}^{\mathrm{c}}), \quad \lambda(A_{\mathrm{ss}}^{\mathrm{c}}) + 0(\varepsilon), \quad \frac{\Lambda_{\mathrm{d}}}{\varepsilon} + 0(1). \qquad (10.2.22)$$

stability

There are a total number of n_{d} closed-loop eigenvalues, which have infinite negative real parts as $\varepsilon \to 0$.

2. *Let*

$$C_{\mathrm{s}} = \Gamma_{\mathrm{o}} \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & C_{\mathrm{b}} \end{bmatrix}, \quad D_{\mathrm{s}} = \Gamma_{\mathrm{o}} \begin{bmatrix} I_{m_0} & 0 \\ 0 & I_{m_{\mathrm{d}}} \\ 0 & 0 \end{bmatrix}. \qquad (10.2.23)$$

Then, we have

$$H(s,\varepsilon) := [C + DF(\varepsilon)][sI - A - BF(\varepsilon)]^{-1} \to [0 \quad H_{\mathrm{s}}(s) \quad 0 \quad 0]\,\Gamma_{\mathrm{s}}^{-1}, \qquad (10.2.24)$$

pointwise in s as $\varepsilon \to 0$, where

$$H_{\mathrm{s}}(s) = (C_{\mathrm{s}} - D_{\mathrm{s}} F_{\mathrm{s}})(sI - A_{\mathrm{ss}} + B_{\mathrm{s}} F_{\mathrm{s}})^{-1}. \qquad (10.2.25)$$

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H_2 Control, H_∞ Control, Disturbance Decoupling Problems

10.2.2 H_2 Control, H_∞ Control and Disturbance Decoupling

In a typical control system design, the given specifications are usually transformed into a performance index, and then control laws which would minimize a certain norm, say H_2 or H_∞ norm, of the performance index are sought. In what follows, we will demonstrate that by properly choosing the sub-feedback gain matrix F_s in STEP ATEA-C.2, the ATEA design can be trivially adopted to solve the well-known H_2 and H_∞ control as well as disturbance decoupling problems.

To be specific, we consider a generalized continuous-time system Σ with a state-space description:

$$\Sigma : \begin{cases} \dot{x} = A\,x + B\,u + E\,w, \\ y = \quad x, \\ h = C\,x + D\,u, \end{cases} \tag{10.2.61}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the control input, $w \in \mathbb{R}^q$ is the external disturbance input, $y = x$ is the measurement output, and $h \in \mathbb{R}^p$ is the controlled output of Σ. We assume that (A, B) is stabilizable and (A, B, C, D) has no invariant zeros on the imaginary axis. Then, the standard optimization problem is to find a control law

$$u = Fx, \tag{10.2.62}$$

such that when it is applied to the given system (10.2.61), the resulting closed-loop system is internally stable, *i.e.*, $\lambda(A + BF) \subset \mathbb{C}^-$, and a certain norm of the resulting closed-loop transfer function from the disturbance input w to the controlled output h, *i.e.*,

$$H_{hw}(s) = (C + DF)(sI - A - BF)^{-1}E, \tag{10.2.63}$$

is minimized.

We will consider in this section the problems of H_2 optimal control and H_∞ control. In particular, H_2 optimal control is to minimize the H_2-norm of $H_{hw}(s)$ over all the possible internally stabilizing state feedback control laws (see Definition 2.4.5 of Chapter 2 for the definition of the H_2-norm of continuous-time systems). For future use, we define

$$\gamma_2^* := \inf \left\{ \|H_{hw}\|_2 \;\middle|\; u = Fx \text{ internally stabilizes } \Sigma \right\}. \tag{10.2.64}$$

Similarly, the standard H_∞ control is to minimize the H_∞-norm of $H_{hw}(s)$ over all the possible internally stabilizing state feedback control laws (see Definition 2.4.5 of Chapter 2 for the definition of the H_∞-norm of continuous-time systems). For future use, we define

$$\gamma_\infty^* := \inf \left\{ \|H_{hw}\|_\infty \;\middle|\; u = Fx \text{ internally stabilizes } \Sigma \right\}. \tag{10.2.65}$$

We note that the determination of this γ_∞^* is rather tedious. For a fairly large class of systems, γ_∞^* can be exactly computed using some numerically stable algorithms. In general, an iterative scheme is required to determine γ_∞^*. We refer interested readers to the work of Chen [22] for a detailed treatment of this particular issue. For simplicity, we assume throughout this section that γ_∞^* has been determined and hence it is known.

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We summarize in the following the solutions to the H_2 and H_∞ control as well as disturbance decoupling problems. We assume that Γ_s, Γ_i and Γ_o are the nonsingular state, input and output transformations that transform the matrix quadruple (A, B, C, D) into the special coordinate basis as in (10.2.2)–(10.2.5). Let

$$\tilde{E} := \Gamma_s^{-1} E = \begin{bmatrix} E_a^- \\ E_a^+ \\ E_b \\ E_c \\ E_d \end{bmatrix} \quad \text{and} \quad E_s := \begin{bmatrix} E_a^+ \\ E_b \end{bmatrix}. \tag{10.2.66}$$

The disturbance w can be attenuated from the controlled output if and only if

$$E_s = 0 \quad \Rightarrow \quad \mathrm{Im}(E) \subset S^+(A,B,C,D) = \text{span of } X_a^- \oplus X_c \oplus X_d$$

This condition is automatically satisfied if the given system (A, B, C, D) is of minimum phase and right invertible.

H_2 Control

Theorem 10.2.2. *Consider the generalized continuous-time system Σ characterized by (10.2.61). The ATEA design can be easily adapted to solve the H_2 and H_∞ control as well as disturbance decoupling problems for Σ. More specifically, we have*

1. *If the sub-feedback gain matrix F_s in* STEP ATEA-C.2 *is chosen to be*

$$F_s = (D_s' D_s)^{-1}(B_s' P_s + D_s' C_s), \tag{10.2.67}$$

where $P_s > 0$ is a solution of the algebraic Riccati equation

$$P_s A_{ss} + A_{ss}' P_s + C_s' C_s - (P_s B_s + C_s' D_s)(D_s' D_s)^{-1}(B_s' P_s + D_s' C_s) = 0, \tag{10.2.68}$$

then the resulting closed-loop transfer function from w to h under the corresponding ATEA state feedback law has the following property:

$$\|H_{hw}\|_2 = \left\|[C + DF(\varepsilon)][sI - A - BF(\varepsilon)]^{-1}E\right\|_2 \to \gamma_2^*, \tag{10.2.69}$$

as $\varepsilon \to 0$, i.e., the corresponding ATEA state feedback law solves the H_2 suboptimal control problem for Σ. Furthermore,

$$\gamma_2^* = \sqrt{\text{trace}\,(E_s' P_s E_s)}. \tag{10.2.70}$$

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H_∞ Control

2. *Given a scalar $\gamma > \gamma^*_\infty \geq 0$, if F_s in* STEP ATEA-C.2 *is chosen to be*

$$F_s = (D_s' D_s)^{-1}(B_s' P_s + D_s' C_s), \tag{10.2.71}$$

where $P_s > 0$ is a solution of the algebraic Riccati equation

$$\begin{aligned} &P_s A_{ss} + A_{ss}' P_s + C_s' C_s + P_s E_s E_s' P_s/\gamma^2 \\ &\qquad -(P_s B_s + C_s' D_s)(D_s' D_s)^{-1}(B_s' P_s + D_s' C_s) = 0, \end{aligned} \tag{10.2.72}$$

then the resulting closed-loop transfer function from w to h under the corresponding ATEA state feedback law has the following property:

$$\|H_{hw}\|_\infty = \left\| [C + DF(\varepsilon)][sI - A - BF(\varepsilon)]^{-1} E \right\|_\infty < \gamma, \tag{10.2.73}$$

for sufficiently small ε, i.e., the corresponding ATEA state feedback law solves the H_∞ γ-suboptimal control problem for Σ.

disturbance rejection

3. *If $E_s = 0$, which has been shown in [22] to be the necessary and sufficient condition for the solvability of the disturbance decoupling problem for Σ, then the ATEA state feedback law with any arbitrarily chosen F_s (subject to the constraint on the stability of A_{ss}^c) has a resulting closed-loop transfer function $H_{hw}(s, \varepsilon)$ with*

$$H_{hw}(s, \varepsilon) \to 0, \quad \text{pointwise in } s \text{ as } \varepsilon \to 0, \tag{10.2.74}$$

i.e., any ATEA state feedback control law solves the disturbance decoupling problem for Σ.

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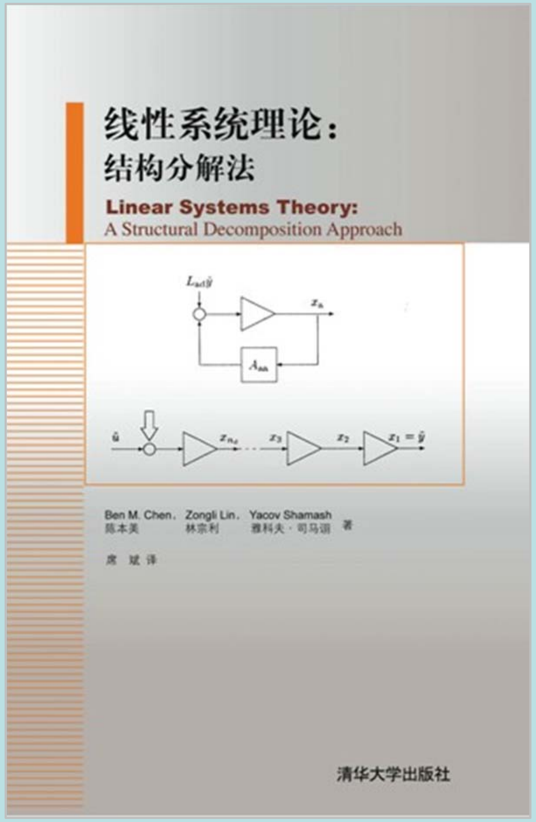

本书的内容充分表明这种系统结构分解方法在适用领域上的广泛性和系统分析与综合上的方便性，如在结构分解的框架下，本书中定理 10.2.2 就简单明了地揭示了H_2控制和H_∞控制以及干扰解耦控制之间的内在关系。。。

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Example 10.2.1. Consider a given system (10.2.61) with

$$A = \left[\begin{array}{cc|ccc} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ \hline 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{array}\right], \quad B = \left[\begin{array}{cc} 0 & 0 \\ 0 & 0 \\ \hline 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{array}\right], \quad E = \left[\begin{array}{cc} 1 & 1 \\ 0 & 1 \\ \hline 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{array}\right], \quad (10.2.78)$$

and

$$C = \left[\begin{array}{cc|ccc} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right], \quad D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \quad (10.2.79)$$

The quadruple (A, B, C, D) is already in the form of the special coordinate basis presented in Chapter 5. It is invertible and hence its associated $\mathcal{X}_b$ and $\mathcal{X}_c$ are nonexistent. It has two unstable invariant zeros both at $s = 1$ and two infinite zeros of orders 1 and 2, respectively. Moreover, we have

$$A_{ss} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad B_s = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad E_s = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix},$$

and

$$C_s = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad D_s = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Since $E_s \neq 0$, the disturbance decoupling problem for the given system is not solvable. We will thus focus on solving the H_2 and H_∞ suboptimal control problems for the system. Following the construction procedures of the ATEA algorithm in the previous section, we obtain a state feedback

$$F(\varepsilon) = -\begin{bmatrix} F_{s11}/\varepsilon + 1 & F_{s12}/\varepsilon + 1 & 1/\varepsilon + 1 & 1 & 1 \\ 2F_{s21}/\varepsilon^2 + 1 & 2F_{s22}/\varepsilon^2 + 1 & 1 & 2/\varepsilon^2 + 1 & 2/\varepsilon + 1 \end{bmatrix}, \tag{10.2.80}$$

where

$$F_s = \begin{bmatrix} F_{s11} & F_{s12} \\ F_{s21} & F_{s22} \end{bmatrix} \tag{10.2.81}$$

is to be selected to solve either the H_2 or H_∞ control problem. The closed-loop eigenvalues of $A + BF$ are asymptotically placed at $\lambda(A_{ss} - B_s F_s)$, $-1/\varepsilon$ and $-1/\varepsilon \pm j/\varepsilon$, respectively.

1. H_2 *Control*. Solving the H_2 algebraic Riccati equation of (10.2.68), we get

$$P_s = \begin{bmatrix} 4 & -2 \\ -2 & 2 \end{bmatrix},$$

which gives a sub-feedback gain,

$$F_s = \begin{bmatrix} 2 & 0 \\ -2 & 2 \end{bmatrix},$$

and $\gamma_2^* = \sqrt{\text{trace}\,(E_s' P_s E_s)} = \sqrt{6}$. Thus, it follows from (10.2.80) and (10.2.81) that the H_2 suboptimal control law is given by $u = F(\varepsilon)x$, with

$$F(\varepsilon) = -\begin{bmatrix} 2/\varepsilon + 1 & 1 & 1/\varepsilon + 1 & 1 & 1 \\ -4/\varepsilon^2 + 1 & 4/\varepsilon^2 + 1 & 1 & 2/\varepsilon^2 + 1 & 2/\varepsilon + 1 \end{bmatrix}.$$

Figure 10.2.1 shows the values of the H_2-norm of the resulting closed-loop system versus $1/\varepsilon$. Clearly, it shows that the H_2-norm of the resulting closed-loop system tends to $\gamma_2^* = \sqrt{6} = 2.4495$ as $1/\varepsilon \to \infty$.

H_2 Control

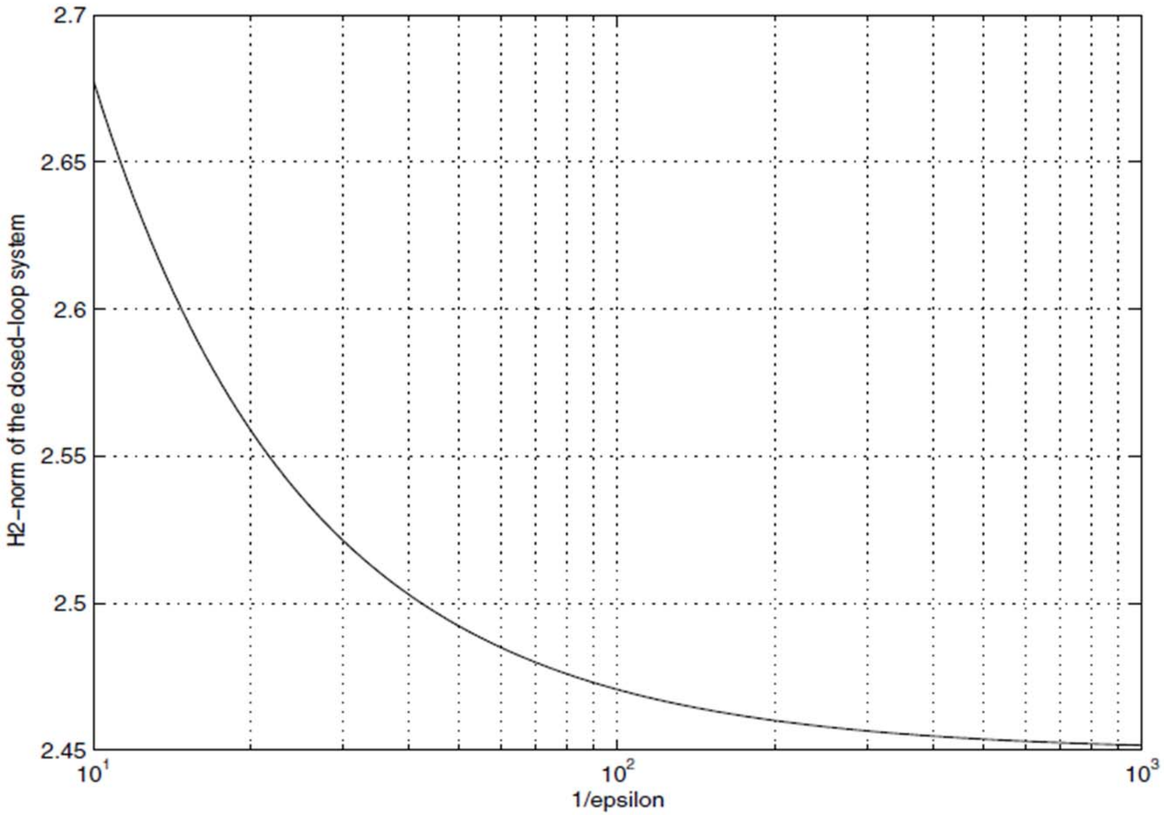


Figure 10.2.1: The H_2-norm of the closed-loop system *vs* $1/\varepsilon$.

H_2 Control

2. H_∞ *Control.* For the case when the quadruple, (A, B, C, D), is right invertible, it was shown in Chen [22] that the H_∞ algebraic Riccati equation of (10.2.72) can be explicitly obtained by solving the two Lyapunov equations

$$A_{ss}S_s + S_sA'_{ss} = B_sB'_s \quad \text{and} \quad A_{ss}T_s + T_sA'_{ss} = E_sE'_s.$$

Solving the above Lyapunov equations, we obtain

$$S_s = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \quad \text{and} \quad T_s = \frac{1}{2}\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}.$$

It then follows from Chen [22] that

$$\gamma_\infty^* = \sqrt{\lambda_{\max}(T_sS_s^{-1})} = 1.618034,$$

and for any $\gamma > \gamma_\infty^*$, the solution to (10.2.72) can be expressed as

$$P_s = \left(S_s - T_s/\gamma^2\right)^{-1} = \frac{2\gamma^2}{\gamma^4 - 3\gamma^2 + 1}\begin{bmatrix} 2\gamma^2 - 1 & 1 - \gamma^2 \\ 1 - \gamma^2 & \gamma^2 - 2 \end{bmatrix},$$

and the sub-feedback gain F_s is given by

$$F_s = \frac{2\gamma^2}{\gamma^4 - 3\gamma^2 + 1}\begin{bmatrix} \gamma^2 & -1 \\ 1 - \gamma^2 & \gamma^2 - 2 \end{bmatrix}.$$

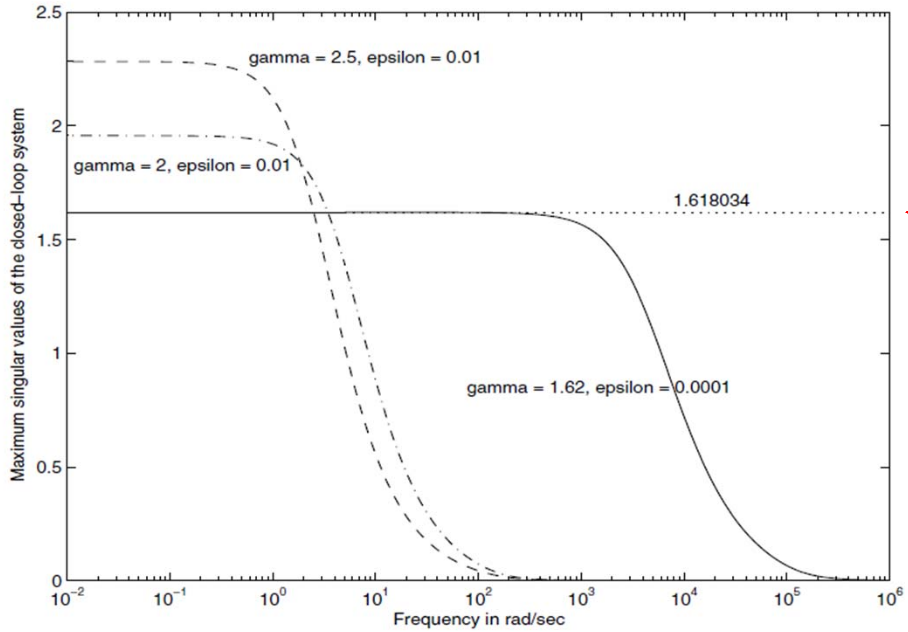
H_∞ Control

Hence, given a $\gamma > \gamma_\infty^*$, it follows from (10.2.80) and (10.2.81) that the control law $u = F(\gamma, \varepsilon)x$, with

$$F(\gamma,\varepsilon) = -\begin{bmatrix} \frac{2\gamma^4}{\varepsilon(\gamma^4-3\gamma^2+1)}+1 & \frac{4\gamma^2(1-\gamma^2)}{\varepsilon^2(\gamma^4-3\gamma^2+1)}+1 \\ \frac{-2\gamma^2}{\varepsilon(\gamma^4-3\gamma^2+1)}+1 & \frac{4\gamma^2(\gamma^2-2)}{\varepsilon^2(\gamma^4-3\gamma^2+1)}+1 \\ \frac{1}{\varepsilon}+1 & 1 \\ 1 & \frac{2}{\varepsilon^2}+1 \\ 1 & \frac{2}{\varepsilon}+1 \end{bmatrix}',$$

An explicitly parameterized gain matrix in terms of γ and ε

is an H_∞ γ-suboptimal controller for sufficiently small ε.



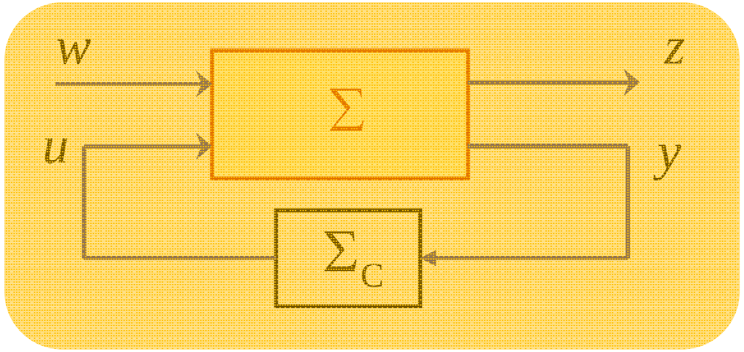
optimal value γ^*

H_∞ Control

Figure 10.2.2: The maximum singular values of the closed-loop system.

Measurement feedback cases…

Consider a stabilizable and detectable linear time-invariant system Σ with a proper controller Σ_C



$$\Sigma : \begin{cases} \dot{x} = A\,x + B\,u + E\,w \\ y = C_1\,x + 0\,u + D_1\,w \\ z = C_2\,x + D_2\,u + D_{22}\,w \end{cases}$$

$$\Sigma_C : \begin{cases} \dot{v} = A_C\,v + B_C\,y \\ u = C_C\,v + D_C\,y \end{cases}$$

We will only focus on the case when D_{22} either is 0 or can be made to 0.

$$\begin{cases} x \in R^n \Leftrightarrow \text{state variable} \\ y \in R^p \Leftrightarrow \text{measurement} \\ z \in R^q \Leftrightarrow \text{controlled output} \end{cases} \quad \& \quad \begin{matrix} u \in R^m \Leftrightarrow \text{control input} \\ w \in R^l \Leftrightarrow \text{disturbance} \\ v \in R^k \Leftrightarrow \text{controller state} \end{matrix}$$

H_2 control with measurement feedback…

$D_{22} = 0$

1. Find a state feedback gain matrix F for the quadruple (A, B, C_2, D_2) using the ATEA method for H_2 control (or other appropriate methods);

2. Find an observer gain matrix K for the quadruple (A', C'_1, E', D'_1) using the ATEA method for H_2 control (or other appropriate methods);

3. The H_2 control output feedback law is then given by

$$\Sigma_C : \begin{cases} \dot{v} = \left(A + BF + KC_1\right) v - K\ y \\ u = \qquad\quad F \quad v \end{cases}$$

H_2 Control

H_∞ Control

7.3. Full Order Output Feedback

This section deals with H_∞ suboptimal and optimal design using full order measurement output feedback laws, i.e., the dynamical order of these control laws will be exactly the same as that of the given system. To be more specific, we consider the following measurement feedback system

$$\Sigma : \begin{cases} \dot{x} = A\ x + B\ u + E\ w, \\ y = C_1\ x + D_1\ w, \\ h = C_2\ x + D_2\ u + D_{22}\ w, \end{cases} \tag{7.3.1}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the control input, $w \in \mathbb{R}^q$ is the external disturbance input, $y \in \mathbb{R}^p$ is the measurement output, and $h \in \mathbb{R}^\ell$ is the controlled output of Σ. Again, we let Σ_P be the subsystem characterized by the matrix quadruple (A, B, C_2, D_2) and Σ_Q be the subsystem characterized by the matrix quadruple (A, E, C_1, D_1). The following assumptions are made first:

Assumption 7.M.1: (A, B) is stabilizable;

Assumption 7.M.2: Σ_P has no invariant zero on the imaginary axis;

Assumption 7.M.3: $\mathrm{Im}\,(E) \subset \mathcal{V}^-(\Sigma_P) + \mathcal{S}^-(\Sigma_P)$;

Assumption 7.M.4: (A, C_1) is detectable;

Assumption 7.M.5: Σ_Q has no invariant zero on the imaginary axis;

Assumption 7.M.6: $\mathrm{Ker}\,(C_2) \supset \mathcal{V}^-(\Sigma_Q) \cap \mathcal{S}^-(\Sigma_Q)$; and

Assumption 7.M.7: $D_{22} = 0$.

fairly common assumptions

The procedure for obtaining the closed-form of the H_∞ suboptimal output feedback laws for any $\gamma > \gamma^*$ proceeds as follows.

Step 7.M.1: Define an auxiliary full state feedback system

$$\begin{cases} \dot{x} = A\ x + B\ u + E\ w, \\ y = x \\ h = C_2\ x + D_2\ u + D_{22}\ w, \end{cases}$$

and proceed to perform Steps 7.F.1 to 7.F.5 of Section 7.2 to obtain the gain matrix $F(\gamma, \varepsilon, \Lambda_{d\text{P}}, \Delta_{c\text{P}})$. Also, define

$$P(\gamma) := (\Gamma_{s\text{P}}^{-1})' \begin{bmatrix} (S_{x\text{P}} - \gamma^{-2} T_{x\text{P}})^{-1} & 0 \\ 0 & 0 \end{bmatrix} \Gamma_{s\text{P}}^{-1}. \tag{7.3.2}$$

Step 7.M.2: Define another auxiliary full state feedback system as follows,

$$\begin{cases} \dot{x} = A'\ x + C_1'\ u + C_2'\ w, \\ y = x \\ h = E'\ x + D_1'\ u + D_{22}'\ w, \end{cases} \tag{7.3.3}$$

and proceed to perform Steps 7.F.1 to 7.F.5 of Section 7.2 but for this auxiliary system to obtain a gain matrix $F(\gamma, \varepsilon, \Lambda_{d\text{Q}}, \Delta_{c\text{Q}})$. Define $K(\gamma, \varepsilon, \Lambda_{d\text{Q}}, \Delta_{c\text{Q}}) := F(\gamma, \varepsilon, \Lambda_{d\text{Q}}, \Delta_{c\text{Q}})'$. Also, define

$$Q(\gamma) := (\Gamma_{s\text{Q}}^{-1})' \begin{bmatrix} (S_{x\text{Q}} - \gamma^{-2} T_{x\text{Q}})^{-1} & 0 \\ 0 & 0 \end{bmatrix} \Gamma_{s\text{Q}}^{-1}. \tag{7.3.4}$$

H_∞ Control

Step 7.M.3: Construct the following full order observer based controller,

$$\Sigma_{\rm cmp} : \begin{cases} \dot{v} = A_{\rm cmp}\, v + B_{\rm cmp}\, y, \\ u = C_{\rm cmp}\, v + \;\; 0 \;\; y, \end{cases} \tag{7.3.5}$$

where

$$\begin{aligned} A_{\rm cmp} = A &+ \gamma^{-2}EE'P(\gamma) + BF(\gamma,\varepsilon,\Lambda_{d\rm P},\Delta_{c\rm P}) \\ &+ \left[I-\gamma^{-2}Q(\gamma)P(\gamma)\right]^{-1}\Big\{K(\gamma,\varepsilon,\Lambda_{d\rm Q},\Delta_{c\rm Q})\left[C_1+\gamma^{-2}D_1E'P(\gamma)\right] \\ &+ \gamma^{-2}Q(\gamma)\left[A'P(\gamma)+P(\gamma)A+C_2'C_2+\gamma^{-2}P(\gamma)EE'P(\gamma)\right] \\ &+ \gamma^{-2}Q(\gamma)\left[P(\gamma)B+C_2'D_2\right]F(\gamma,\varepsilon,\Lambda_{d\rm P},\Delta_{c\rm P})\Big\}, \end{aligned} \tag{7.3.6}$$

$$B_{\rm cmp} = -\left[I-\gamma^{-2}Q(\gamma)P(\gamma)\right]^{-1}K(\gamma,\varepsilon,\Lambda_{d\rm Q},\Delta_{c\rm Q}), \tag{7.3.7}$$

$$C_{\rm cmp} = F(\gamma,\varepsilon,\Lambda_{d\rm P},\Delta_{c\rm P}). \tag{7.3.8}$$

It is to be shown that $\Sigma_{\rm cmp}$ is indeed a γ-suboptimal controller. Clearly, it has a dynamical order of n, i.e., it is a full order output feedback controller. [A]

H_∞ Control

If (A, E, C_1, D_1) is left invertible and of minimum phase, then it can be proved that $Q = 0$ and H_∞ control law can be much simplified…

Step 7.M.3: Construct the following full order observer based controller,

$$\Sigma_{\rm cmp}: \begin{cases} \dot{v} = A_{\rm cmp}\, v + B_{\rm cmp}\, y, \\ u = C_{\rm cmp}\, v + \;\; 0 \;\; y, \end{cases} \qquad (7.3.5)$$

where

$$A_{\rm cmp} = A + \quad BF$$

$$+ \quad K \quad C_1$$

$$B_{\rm cmp} = - \quad K \qquad (7.3.7)$$

$$C_{\rm cmp} = F \qquad (7.3.8)$$

It is to be shown that $\Sigma_{\rm cmp}$ is indeed a γ-suboptimal controller. Clearly, it has a dynamical order of n, i.e., it is a full order output feedback controller.

H_∞ Control

H_∞ disturbance decoupling with measurement feedback…

More specifically, we consider the general H_∞-ADDPMS and the general H_∞-ADDPS, for the following general continuous-time linear system,

$$\Sigma : \begin{cases} \dot{x} = A\ x + B\ u + E\ w, \\ y = C_1\ x + D_1\ w, \\ h = C_2\ x + D_2\ u + D_{22}\ w, \end{cases} \tag{8.1.1}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the control input, $y \in \mathbb{R}^\ell$ is the measurement, $w \in \mathbb{R}^q$ is the disturbance and $h \in \mathbb{R}^p$ is the output to be controlled. As usual, for convenient reference in future development, throughout this chapter, we define Σ_P to be the subsystem characterized by the matrix quadruple (A, B, C_2, D_2) and Σ_Q to be the subsystem characterized by the matrix quadruple (A, E, C_1, D_1). The following dynamic feedback control laws are investigated:

$$\Sigma_\mathrm{cmp} : \begin{cases} \dot{v} = A_\mathrm{cmp}\ v + B_\mathrm{cmp}\ y, \\ u = C_\mathrm{cmp}\ v + D_\mathrm{cmp}\ y. \end{cases} \tag{8.1.2}$$

The controller Σ_cmp of (8.1.2) is said to be internally stabilizing when applied to the system Σ, if the following matrix is asymptotically stable:

$$A_\mathrm{cl} := \begin{bmatrix} A + BD_\mathrm{cmp}C_1 & BC_\mathrm{cmp} \\ B_\mathrm{cmp}C_1 & A_\mathrm{cmp} \end{bmatrix} \tag{8.1.3}$$

disturbance rejection

H_∞ disturbance decoupling problem…

Definition 8.1.1. The H_∞ almost disturbance decoupling problem with measurement feedback and with internal stability (the H_∞-ADDPMS) for the continuous time system (8.1.1) is said to be solvable if, for any given positive scalar $\gamma > 0$, there exists at least one controller of the form (8.1.2) such that,

1. in the absence of disturbance, the closed-loop system comprising the system (8.1.1) and the controller (8.1.2) is asymptotically stable, i.e., the matrix $A_{\rm cl}$ as given by (8.1.3) is asymptotically stable; and
2. the closed-loop system has an L_2-gain, from the disturbance w to the controlled output h, that is less than or equal to γ, i.e.,

$$\|h\|_2 \le \gamma \|w\|_2, \quad \forall w \in L_2 \text{ and for } (x(0), v(0)) = (0, 0). \tag{8.1.4}$$

Equivalently, the H_∞-norm of the closed-loop transfer matrix from w to h, T_{hw}, is less than or equal to γ, i.e., $\|T_{hw}\|_\infty \le \gamma$.

In the case that $C_1 = I$ and $D_1 = 0$, the general H_∞-ADDPMS as defined above becomes the general H_∞-ADDPS, where only a static state feedback, instead the dynamic output feedback (8.1.2) is necessary.

disturbance rejection

H_∞ disturbance decoupling solvability conditions…

Theorem 8.2.1. Consider the general measurement feedback system (8.1.1). Then the general H_∞ almost disturbance decoupling problem for (8.1.1) with internal stability (H_∞-ADDPMS) is solvable, if and only if the following conditions are satisfied:

1. (A, B) is stabilizable;
2. (A, C_1) is detectable;
3. $D_{22} + D_2 S D_1 = 0$, where $S = -(D_2' D_2)^\dagger D_2' D_{22} D_1' (D_1 D_1')^\dagger$;
4. $\operatorname{Im}(E + BSD_1) \subset \mathcal{S}^+(\Sigma_P) \cap \left\{ \cap_{\lambda \in \mathbb{C}^0} \mathcal{S}_\lambda(\Sigma_P) \right\}$;
5. $\operatorname{Ker}(C_2 + D_2 S C_1) \supset \mathcal{V}^+(\Sigma_Q) \cup \left\{ \cup_{\lambda \in \mathbb{C}^0} \mathcal{V}_\lambda(\Sigma_Q) \right\}$; and
6. $\mathcal{V}^+(\Sigma_Q) \subset \mathcal{S}^+(\Sigma_P)$. coupling term

common assumptions

it is entire space if no jω axis zero

it is empty space if no jω axis zero

W. M. Wonham

Jan C. Willems

Carsten Scherer

……

Bugs Bunny

A special but very useful case…

Remark 8.2.1. Note that if Σ_P is right invertible and of minimum phase, and Σ_Q is left invertible and of minimum phase, then Conditions 4 to 6 of Theorem 8.2.1 are automatically satisfied. Hence, the solvability conditions of the H_∞-ADDPMS for such a case reduce to:

1. (A, B) is stabilizable;
2. (A, C_1) is detectable; and
3. $D_{22} + D_2 S D_1 = 0$, where $S = -(D_2' D_2)^\dagger D_2' D_{22} D_1' (D_1 D_1')^\dagger$. ®

If $D_{22} = 0$, then condition 3 is automatically satisfied…

disturbance rejection

Exercise 10.1. Consider the H_∞ control for the system given in (10.2.61). Assume that (A, B, C, D) is right invertible. It follows from (10.2.23) that $C_s \equiv 0$ and thus, the corresponding H_∞-ARE (10.2.72) can be rewritten as

$$P_s A_{ss} + A'_{ss} P_s + P_s E_s E'_s P_s / \gamma^2 - P_s B_s (D'_s D_s)^{-1} B'_s P_s = 0.$$

Show that the above ARE has a positive definite solution if and only if

$$\gamma^2 > (\gamma^*_\infty)^2 = \lambda_{\max}(T_s S_s^{-1}),$$

where $S_s > 0$ and $T_s \geq 0$ are respectively the solutions of the Lyapunov equations

$$A_{ss} S_s + S_s A'_{ss} = B_s (D'_s D_s)^{-1} B'_s \quad \text{and} \quad A_{ss} T_s + T_s A'_{ss} = E_s E'_s.$$

Also, show that, for $\gamma > \gamma^*_\infty$, the positive definite solution to the H_∞ ARE is given by

$$P_s = (S_s - T_s / \gamma^2)^{-1}.$$

In fact, γ^*_∞ is the infimum for the given H_∞ control problem.

Exercise 10.2. Consider a continuous-time system characterized by (10.2.61) with

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad E = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix},$$

and

$$C = [0 \quad 0 \quad 1 \quad 0], \quad D = 0.$$

It is simple to see that (A, B, C, D) is already in the form of the special coordinate basis with two invariant zeros at 1 and 2, and a relative degree of 2.

(a) Solve the corresponding H_2-ARE (10.2.68) for $P_s > 0$, and compute the infimum γ_2^* and an H_2 suboptimal state feedback gain matrix $F(\varepsilon)$, explicitly parameterized in ε.

(b) Determine the infimum γ_∞^*. Given a $\gamma > \gamma_\infty^*$, solve the corresponding H_∞-ARE (10.2.72) for $P_s > 0$. Also, calculate an H_∞ suboptimal state feedback gain matrix $F(\gamma, \varepsilon)$, explicitly parameterized in γ and ε.

Exercise 10.3. Consider a general singular H_2 or H_∞ control problem for

$$\Sigma : \begin{cases} \dot{x} = A\ x + B\ u + E\ w, \\ y = \quad x, \\ h = C\ x + D\ u. \end{cases}$$

The problem can also be solved by a so-called perturbation approach (see, e.g., [160]), in which we define a new auxiliary controlled output,

$$h_{\mathbf{aux},\varepsilon} = \begin{pmatrix} h \\ \varepsilon x \\ \varepsilon u \end{pmatrix} = C_\varepsilon x + D_\varepsilon u = \begin{bmatrix} C \\ \varepsilon I \\ 0 \end{bmatrix} x + \begin{bmatrix} D \\ 0 \\ \varepsilon I \end{bmatrix} u.$$

Then, the H_2 suboptimal control law for the system can be computed by solving the following ε-perturbed H_2-ARE,

$$A'P_\varepsilon + P_\varepsilon A + C_\varepsilon' C_\varepsilon - (P_\varepsilon B + C_\varepsilon' D_\varepsilon)(D_\varepsilon' D_\varepsilon)^{-1}(D_\varepsilon' C_\varepsilon + B' P_\varepsilon) = 0,$$

for $P_\varepsilon > 0$. The H_2 suboptimal state feedback gain matrix is given by

$$F(\varepsilon) = -(D_\varepsilon' D_\varepsilon)^{-1}(D_\varepsilon' C_\varepsilon + B' P_\varepsilon).$$

Similarly, given a $\gamma > \gamma_\infty^*$, the H_∞ suboptimal control law for the system can be computed by solving the following ε-perturbed H_∞-ARE,

$$A'P_\varepsilon + P_\varepsilon A + C_\varepsilon' C_\varepsilon + P_\varepsilon E E' P_\varepsilon / \gamma^2 \\ - (P_\varepsilon B + C_\varepsilon' D_\varepsilon)(D_\varepsilon' D_\varepsilon)^{-1}(D_\varepsilon' C_\varepsilon + B' P_\varepsilon) = 0,$$

for $P_\varepsilon > 0$. The H_∞ suboptimal state feedback gain matrix is given by

$$F(\gamma, \varepsilon) = -(D_\varepsilon' D_\varepsilon)^{-1}(D_\varepsilon' C_\varepsilon + B' P_\varepsilon).$$

Let us now consider the system given in Exercise 10.2.

(a) Verify that the solution to the ε-perturbed H_2-ARE satisfies

$$P_\varepsilon \to \begin{bmatrix} P_\mathrm{s} & 0 \\ 0 & 0 \end{bmatrix}, \quad \text{as} \quad \varepsilon \to 0,$$

where P_s is the solution obtained in Part (a) of Exercise 10.2.

(b) Given a $\gamma > \gamma_\infty^*$, verify that the solution to the ε-perturbed H_∞-ARE satisfies

$$P_\varepsilon \to \begin{bmatrix} P_\mathrm{s} & 0 \\ 0 & 0 \end{bmatrix}, \quad \text{as} \quad \varepsilon \to 0,$$

where P_s is the solution obtained in Part (b) of Exercise 10.2.

Robust and Perfect Tracking Control

Consider

$$\Sigma : \begin{cases} \dot{x} = A\ x + B\ u + E\ w, & x(0) = x_0, \\ y = C_1\ x + D_1\ w, \\ h = C_2\ x + D_2\ u + D_{22}\ w, \end{cases} \tag{9.1.1}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the control input, $w \in \mathbb{R}^q$ is the external disturbance, $y \in \mathbb{R}^p$ is the measurement output, and $h \in \mathbb{R}^\ell$ is the output to be controlled. We also assume that the pair (A, B) is stabilizable and (A, C_1) is detectable. For future references, we define Σ_P and Σ_Q to be the subsystems characterized by the matrix quadruples (A, B, C_2, D_2) and (A, E, C_1, D_1), respectively. Given the external disturbance $w \in L_p$, $p \in [1, \infty)$, and any reference signal vector, $r \in \mathbb{R}^\ell$ with $r, \dot{r}, \cdots, r^{(\kappa-1)}$, $\kappa \geq 1$, being available, and $r^{(\kappa)}$ being either a vector of delta functions or in L_p, the robust and perfect tracking (RPT) problem for the system (9.1.1) is to find a parameterized dynamic measurement control law of the following form

$$\begin{cases} \dot{v} = A_{\rm cmp}(\varepsilon)v + B_{\rm cmp}(\varepsilon)y + G_0(\varepsilon)r + \cdots + G_{\kappa-1}(\varepsilon)r^{(\kappa-1)}, \\ u = C_{\rm cmp}(\varepsilon)v + D_{\rm cmp}(\varepsilon)y + H_0(\varepsilon)r + \cdots + H_{\kappa-1}(\varepsilon)r^{(\kappa-1)}, \end{cases} \tag{9.1.2}$$

such that when (9.1.2) is applied (9.1.1), we have

1. There exists an $\varepsilon^* > 0$ such that the resulting closed-loop system with $r = 0$ and $w = 0$ is asymptotically stable for all $\varepsilon \in (0, \varepsilon^*]$; and

2. Let $h(t, \varepsilon)$ be the closed-loop controlled output response and let $e(t, \varepsilon)$ be the resulting tracking error, i.e., $e(t, \varepsilon) := h(t, \varepsilon) - r(t)$. Then, for any initial condition of the state, $x_0 \in \mathbb{R}^n$,

$$J_p(x_0, w, r, \varepsilon) := \|e\|_p \to 0 \quad \text{as} \quad \varepsilon \to 0. \tag{9.1.3}$$

the system

the assumptions

the controller

the problem

The solvability conditions for RPT control problem…

Theorem 9.2.1. Consider the given system (9.1.1) with its external disturbance $w \in L_p$, $p \in [1, \infty)$, and its initial condition $x(0) = x_0$. Then, for any reference signal $r(t)$, which has all its i-th order derivatives, $i = 0, 1, \cdots, \kappa - 1$, $\kappa \geq 1$, being available and $r^{(\kappa)}(t)$ being either a vector of delta functions or in L_p, the proposed robust and perfect tracking (RPT) problem is solvable by the control law of (9.1.2) if and only if the following conditions are satisfied:

1. (A, B) is stabilizable and (A, C_1) is detectable;
2. $D_{22} + D_2 S D_1 = 0$, where $S = -(D_2' D_2)^\dagger D_2' D_{22} D_1' (D_1 D_1')^\dagger$;
3. Σ_P, i.e., (A, B, C_2, D_2), is right invertible and of minimum phase;
4. $\mathrm{Ker}\,(C_2 + D_2 S C_1) \supset C_1^{-1}\{\mathrm{Im}\,(D_1)\}$. ⊤

Condition 1 is necessary for all control problem. Condition 2 is automatically satisfied when $D_{22} = 0$.

We note that for the case when $D_1 = 0$, then the direct feedthrough term D_{22} must be a zero matrix as well, and the last condition, i.e., Item 4, of Theorem 9.2.1 reduces to $\mathrm{Ker}\,(C_2) \supset \mathrm{Ker}\,(C_1)$.

Corollary 9.2.1. Consider the given system (9.1.1) with its external disturbance $w \in L_p$, $p \in [1, \infty)$, its initial condition $x(0) = x_0$. Assume that all its states are measured for feedback, i.e., $C_1 = I$ and $D_1 = 0$. Then, for any reference signal $r(t)$, which has all its i-th order derivatives, $i = 1, 2, \cdots, \kappa - 1$, $\kappa \geq 1$, being available and $r^{(\kappa)}(t)$ being either a vector of delta functions or in L_p, the proposed robust and perfect tracking (RPT) problem is solvable by the control law of (9.1.2) if and only if the following conditions are satisfied:

1. (A, B) is stabilizable;
2. $D_{22} = 0$;
3. Σ_P, i.e., (A, B, C_2, D_2), is right invertible and of minimum phase. ©

Rewrite the given reference signal in a state space form as

$$\frac{d}{dt}\begin{pmatrix} r \\ \vdots \\ r^{(\kappa-2)} \\ r^{(\kappa-1)} \end{pmatrix} = \begin{bmatrix} 0 & I_\ell & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I_\ell \\ 0 & 0 & \cdots & 0 \end{bmatrix} \begin{pmatrix} r \\ \vdots \\ r^{(\kappa-2)} \\ r^{(\kappa-1)} \end{pmatrix} + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ I_\ell \end{bmatrix} r^{(\kappa)}. \tag{9.2.27}$$

Combining (9.2.27) with the given system, we obtain the following augmented system,

$$\Sigma_{\text{AUG}} : \begin{cases} \dot{\boldsymbol{x}} = \boldsymbol{A}\ \boldsymbol{x} + \boldsymbol{B}\ u + \boldsymbol{E}\ \boldsymbol{w} \\ \boldsymbol{y} = \quad \boldsymbol{x} \\ e = \boldsymbol{C}_2\ \boldsymbol{x} + \boldsymbol{D}_2\ u \end{cases} \tag{9.2.28}$$

an artificial system

where

$$\boldsymbol{w} := \begin{pmatrix} w \\ r^{(\kappa)} \end{pmatrix}, \quad \boldsymbol{x} := \begin{pmatrix} r \\ \vdots \\ r^{(\kappa-2)} \\ r^{(\kappa-1)} \\ x \end{pmatrix}, \tag{9.2.29}$$

$$\boldsymbol{A} = \begin{bmatrix} 0 & I_\ell & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & I_\ell & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & A \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ B \end{bmatrix}, \quad \boldsymbol{E} = \begin{bmatrix} 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ 0 & I_\ell \\ E & 0 \end{bmatrix}, \tag{9.2.30}$$

and

$$\boldsymbol{C}_2 = [\, -I_\ell \quad 0 \quad 0 \quad \cdots \quad 0 \quad C_2 \,], \quad \boldsymbol{D}_2 = D_2. \tag{9.2.31}$$

Design an appropriate state feedback gain using the ATEA technique and then

Step 9.S.5. Finally, we partition

$$\boldsymbol{F}(\varepsilon) = [\, H_0(\varepsilon) \quad \cdots \quad H_{\kappa-1}(\varepsilon) \quad F(\varepsilon) \,], \tag{9.2.42}$$

where $H_i(\varepsilon) \in \mathbb{R}^{m \times \ell}$ and $F(\varepsilon) \in \mathbb{R}^{m \times n}$. This ends the algorithm. $\boxed{\text{A}}$

The RPT control law is given by

$$u = F(\varepsilon)x + H_0(\varepsilon)r + \cdots + H_{\kappa-1}(\varepsilon)r^{(\kappa-1)}$$

Remark 9.2.1. Note that the required gain matrices for the state feedback RPT problem might be computed by solving the following Riccati equation,

$$\boldsymbol{P}\tilde{\boldsymbol{A}}+\tilde{\boldsymbol{A}}'\boldsymbol{P}+\tilde{\boldsymbol{C}}_2'\tilde{\boldsymbol{C}}_2-\left(\boldsymbol{P}\boldsymbol{B}+\tilde{\boldsymbol{C}}_2'\tilde{\boldsymbol{D}}_2\right)\left(\tilde{\boldsymbol{D}}_2'\tilde{\boldsymbol{D}}_2\right)^{-1}\left(\boldsymbol{P}\boldsymbol{B}+\tilde{\boldsymbol{C}}_2'\tilde{\boldsymbol{D}}_2\right)'=0, \quad (9.2.43)$$

for a positive definite solution $\boldsymbol{P} > 0$, where

$$\tilde{\boldsymbol{C}}_2 = \begin{bmatrix} \boldsymbol{C}_2 \\ \varepsilon I_{\kappa\ell+n} \\ 0 \end{bmatrix}, \quad \tilde{\boldsymbol{D}}_2 = \begin{bmatrix} \boldsymbol{D}_2 \\ 0 \\ \varepsilon I_m \end{bmatrix}, \quad (9.2.44)$$

$$\tilde{\boldsymbol{A}} = \begin{bmatrix} \tilde{A}_0 & 0 \\ 0 & A \end{bmatrix}, \quad \tilde{A}_0 = -\varepsilon I_{\kappa\ell} + \begin{bmatrix} 0 & I_\ell & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I_\ell \\ 0 & 0 & \cdots & 0 \end{bmatrix}, \quad (9.2.45)$$

and where $\boldsymbol{B}$, $\boldsymbol{C}_2$ and $\boldsymbol{D}_2$ are as defined in (9.2.30) and (9.2.31). The required gain matrix is then given by

$$\tilde{\boldsymbol{F}}(\varepsilon) = -\left(\tilde{\boldsymbol{D}}_2'\tilde{\boldsymbol{D}}_2\right)^{-1}\left(\boldsymbol{P}\boldsymbol{B}+\tilde{\boldsymbol{C}}_2'\tilde{\boldsymbol{D}}_2\right)' = \left[\, H_0(\varepsilon) \quad \cdots \quad H_{\kappa-1}(\varepsilon) \quad F(\varepsilon) \,\right], \quad (9.2.46)$$

where $H_i(\varepsilon) \in \mathbb{R}^{m\times\ell}$ and $F(\varepsilon) \in \mathbb{R}^{m\times n}$. Finally, we note that solutions to the Riccati equation (9.2.43) might have severe numerical problems for small ε. Ⓡ

Example 9.3.1. Consider a linear system given in the form of (9.1.1) with

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad E = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad x_0 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \tag{9.3.12}$$

and

$$C_2 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad D_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad D_{22} = 0. \tag{9.3.13}$$

For easy verification, we assume that the external disturbance w is given by

$$w = \begin{bmatrix} 1 \\ \sin(\pi t) \end{bmatrix} \cdot 1(t) \in L_\infty. \tag{9.3.14}$$

Let the reference input be given as,

$$r = \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} = \begin{bmatrix} t \\ \cos(2t) \end{bmatrix} \cdot 1(t). \tag{9.3.15}$$

A. State Feedback Case. We first consider the case when all the state variables of the given system are measurable, i.e., $C_1 = I$ and $D_1 = 0$. It is simple to verify that the subsystem Σ_P is invertible and of minimum phase with one invariant zero at $s = -1$ and two infinite zeros of orders 0 and 2, respectively. Hence, the general robust and perfect problem for the system with the given reference is solvable. Following the constructive algorithm for the state feedback case, we obtain a parameterized control law,

$$u = \begin{bmatrix} -1 & -1 & 0 \\ 1-\frac{2}{\varepsilon} & -1-\frac{2}{\varepsilon^2} & -\frac{2}{\varepsilon} \end{bmatrix} x + \begin{bmatrix} 1 & 0 \\ -1 & \frac{2}{\varepsilon^2} \end{bmatrix} r. \qquad (9.3.16)$$

The poles of the closed-loop system comprising the given plant and the above control law are located at -1, $-1/\varepsilon \pm j/\varepsilon$. Hence, the closed-loop system is stable for any positive ε. Figure 9.3.1 shows the responses of the error signal $e_2(t) = h_2(t) - r_2(t)$, corresponding to $\varepsilon = 0.1$, 0.05 and 0.01, respectively. Note that $e_1(t) = h_1(t) - r_1(t) \equiv 0$ for all $t \geq 0$. The results clearly show that the general robust and perfect tracking is achieved.

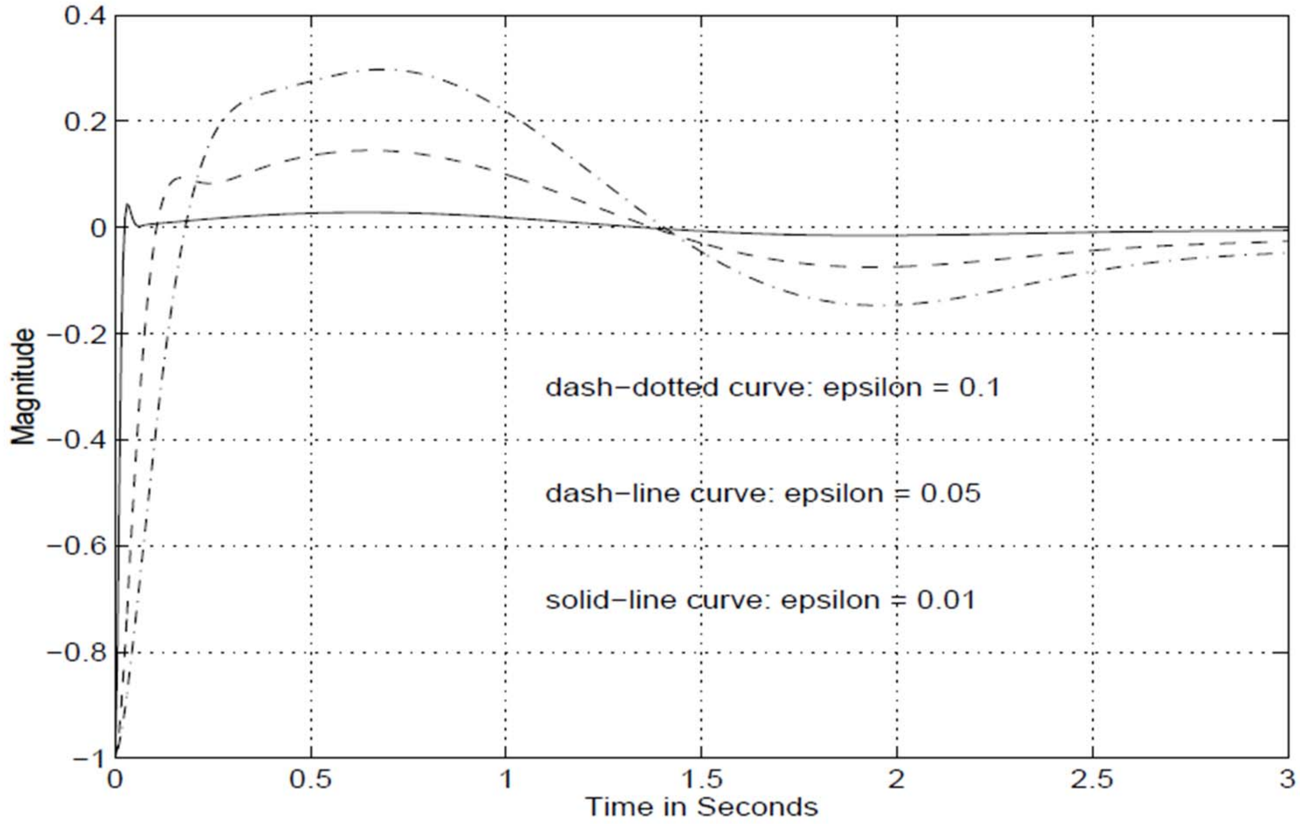


Figure 9.3.1: Tracking error e_2 under state feedback.

9.2.2. Solutions to Measurement Feedback Case

Step 9.F.1. For the given reference $r(t)$ and the given system (9.1.1), we first assume that all the state variables of (9.1.1) are measurable and follow the procedures of the previous subsection to define an auxiliary system,

$$\begin{cases} \dot{x} = A\,x + B\,u + E\,w \\ y = x \\ e = C_2\,x + D_2\,u \end{cases} \tag{9.2.48}$$

Then, we follow Steps 9.S.1 to 9.S.5 of the algorithm of the previous subsection to construct a state feedback gain matrix

$$\boldsymbol{F}(\varepsilon) = [\,H_0(\varepsilon) \quad \cdots \quad H_{\kappa-1}(\varepsilon) \quad F(\varepsilon)\,]\,. \tag{9.2.49}$$

Step 9.F.2. Let $\Sigma_{\mathrm{Q}a}$ be characterized by a matrix quadruple

$$(A_{\mathrm{Q}a}, E_{\mathrm{Q}a}, C_{\mathrm{Q}a}, D_{\mathrm{Q}a}) := (A, [\,E \quad I_n\,], C_1, [\,D_1 \quad 0\,])\,. \tag{9.2.50}$$

This step is to transform this $\Sigma_{\mathrm{Q}a}$ into the special coordinate basis of Theorem 2.4.1. Because of the special structure of the matrix $E_{\mathrm{Q}a}$, it is simple to show that $\Sigma_{\mathrm{Q}a}$ is always right invertible and is free of invariant zeros. Utilize the results of Theorem 2.4.1 to find nonsingular state, input and output transformation $\Gamma_{s\mathrm{Q}}$, $\Gamma_{i\mathrm{Q}}$ and $\Gamma_{o\mathrm{Q}}$ such that

$$\Gamma_{s\mathrm{Q}}^{-1} A \Gamma_{s\mathrm{Q}} = \begin{bmatrix} A_{c c\mathrm{Q}} & L_{cd\mathrm{Q}} \\ E_{dc\mathrm{Q}} & A_{dd\mathrm{Q}} \end{bmatrix} + \begin{bmatrix} B_{0c\mathrm{Q}} \\ B_{0d\mathrm{Q}} \end{bmatrix} [\,C_{0c\mathrm{Q}} \quad 0\,], \tag{9.2.51}$$

$$\Gamma_{s\mathrm{Q}}^{-1} E_{\mathrm{Q}a} \Gamma_{i\mathrm{Q}} = \begin{bmatrix} B_{0c\mathrm{Q}} & 0 & I_{n-k} & 0 \\ B_{0d\mathrm{Q}} & I_k & 0 & 0 \end{bmatrix}, \tag{9.2.52}$$

and

$$\Gamma_{o\mathrm{Q}}^{-1} C_1 \Gamma_{s\mathrm{Q}} = \begin{bmatrix} C_{0c\mathrm{Q}} & 0 \\ 0 & I_k \end{bmatrix}, \quad \Gamma_{o\mathrm{Q}}^{-1} [\,D_1 \quad 0\,] \Gamma_{i\mathrm{Q}} = \begin{bmatrix} I_{\mathrm{p}-k} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{9.2.53}$$

where $k = \mathrm{p} - \mathrm{rank}(D_1)$. It can be verified that the pair (A, C_1) is detectable if and only if the pair

$$\left(A_{cc\mathrm{Q}}, \begin{bmatrix} C_{0c\mathrm{Q}} \\ E_{dc\mathrm{Q}} \end{bmatrix} \right) \tag{9.2.54}$$

is detectable.

Step 9.F.3. Let $K_{c\mathrm{Q}}$ be an appropriate dimensional constant matrix such that the eigenvalues of the matrix

$$A^c_{cc\mathrm{Q}} = A_{cc\mathrm{Q}} - K_{c\mathrm{Q}} \begin{bmatrix} C_{0c\mathrm{Q}} \\ E_{dc\mathrm{Q}} \end{bmatrix} = A_{cc\mathrm{Q}} - \begin{bmatrix} K_{c0\mathrm{Q}} & K_{cd\mathrm{Q}} \end{bmatrix} \begin{bmatrix} C_{0c\mathrm{Q}} \\ E_{dc\mathrm{Q}} \end{bmatrix} \tag{9.2.55}$$

are all in $\mathbb{C}^-$. Next, we define a parameterized observer gain matrix,

$$K(\varepsilon) = -\Gamma_{s\mathrm{Q}} \begin{bmatrix} B_{0c\mathrm{Q}} + K_{c0\mathrm{Q}} & L_{cd\mathrm{Q}} + K_{cd\mathrm{Q}}/\varepsilon \\ B_{0d\mathrm{Q}} & A_{dd\mathrm{Q}} + I_k/\varepsilon \end{bmatrix} \Gamma_{o\mathrm{Q}}^{-1}. \tag{9.2.56}$$

Step 9.F.4. Finally, we obtain the following full order measurement feedback control law,

$$\begin{cases} \dot{v} = A_{\mathrm{cmp}}(\varepsilon)\, v - K(\varepsilon)\, y + BH_0(\varepsilon)\, r + \cdots + BH_{\kappa-1}(\varepsilon)\, r^{(\kappa-1)}, \\ u = F(\varepsilon)\, v + H_0(\varepsilon)\, r + \cdots + H_{\kappa-1}(\varepsilon)\, r^{(\kappa-1)}, \end{cases} \tag{9.2.57}$$

where $A_{\mathrm{cmp}}(\varepsilon) = A + BF(\varepsilon) + K(\varepsilon)C_1$. This completes the construction of the full order measurement feedback controller. A

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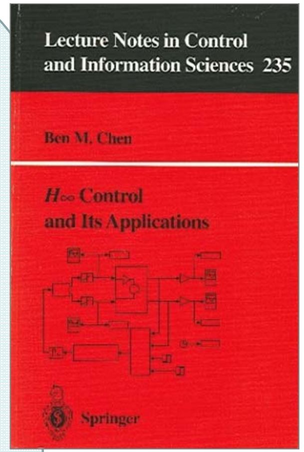
Book review

H_∞ control and its applications
Ben M. Chen; Springer, London, 1998,
ISBN 1-85233-026-0

We summarize that this book is an excellent research reference for insights and proofs pertaining to the geometric approach in H_∞-control. The careful and detailed exposition illustrates the far-reaching potentials of these techniques for a variety of other control problems beyond H_∞-theory.

We therefore believe that this book closes a gap in literature by providing a thorough basis for entering the rich field of applying geometric techniques in H_∞-control.



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