Decompositions of Descriptor Systems

We consider a continuous-time system Σ characterized by

$$\Sigma : \begin{cases} E\,\dot{x} = A\,x + B\,u, \\ \quad y = C\,x + D\,u, \end{cases} \tag{6.1.1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $y \in \mathbb{R}^p$ are respectively the state, input and output of the system, and E, A, B, C and D are constant matrices of appropriate dimensions. The system Σ is said to be singular if $\text{rank}(E) < n$. As usual, in order to avoid any ambiguity in the solutions to the system, we assume throughout this chapter that the given descriptor system Σ is *regular*, *i.e.*, $\det(sE - A) \not\equiv 0$, for all $s \in \mathbb{C}$.

We present the result of the structural decomposition of descriptor systems for the single-input-and-single-output (SISO) case. Results for general multi-input-and-multi-output (MIMO) case can be found in the reference…

BEN M. CHEN, NUS ECE

Theorem 6.2.1. *Consider the descriptor system* Σ *of (6.1.1) with* $p = m = 1$ *satisfying the usual regularity assumption, i.e.,* $\det(sE - A) \not\equiv 0$ *for* $s \in \mathbb{C}$. *There exist nonsingular state, input and output transformations* $\Gamma_{\rm s} \in \mathbb{R}^{n \times n}$, $\Gamma_{\rm i} \in \mathbb{R}$ *and* $\Gamma_{\rm o} \in \mathbb{R}$, *and an* $n \times n$ *nonsingular matrix* $\Gamma_{\rm e}(s)$, *whose elements are polynomials of* s, *which together give a structural decomposition of* Σ *described by the set of equations*

$$x = \Gamma_{\rm s}\tilde{x}, \quad \tilde{x} = \begin{pmatrix} x_{\rm z} \\ x_{\rm e} \\ x_{\rm a} \\ x_{\rm d} \end{pmatrix}, \quad x_{\rm d} \in \mathbb{R}^{n_{\rm d}}, \quad x_{\rm d} = \begin{pmatrix} x_{\rm d1} \\ x_{\rm d2} \\ \vdots \\ x_{{\rm d}n_{\rm d}} \end{pmatrix}, \tag{6.2.1}$$

$$x_{\rm z} \in \mathbb{R}^{n_{\rm z}}, \quad x_{\rm e} \in \mathbb{R}^{n_{\rm e}}, \quad x_{\rm a} \in \mathbb{R}^{n_{\rm a}}, \quad y = \Gamma_{\rm o}\tilde{y}, \quad u = \Gamma_{\rm i}\tilde{u}, \tag{6.2.2}$$

and

$$\left.\begin{aligned} x_{\rm z} &= 0, \\ x_{\rm e} &= \alpha_{\rm e}\tilde{u}^{(v)}, \\ \dot{x}_{\rm a} &= A_{\rm aa}x_{\rm a} + L_{\rm ad}y_{\rm d}, \\ \dot{x}_{\rm d1} &= x_{\rm d2}, \\ \dot{x}_{\rm d2} &= x_{\rm d3}, \\ &\ \vdots \\ \dot{x}_{{\rm d}n_{\rm d}} &= M_{\rm da}x_{\rm a} + L_{\rm dd}y_{\rm d} + \tilde{u}^{(v)}, \quad \tilde{y} = y_{\rm d} = x_{\rm d1}. \end{aligned}\right\} \tag{6.2.4}$$

static

input derivative

proper system

Here, v *is a nonnegative integer,* $A_{\rm aa}$, $B_{\rm 0a}$, $\bar{C}$, $\bar{D}$, $L_{\rm ad}$, $M_{\rm da}$ *and* $L_{\rm dd}$, *if existent, are constant matrices of appropriate dimensions, and* $\alpha_{\rm e}$ *is a nonzero scalar.*

Exercise 6.1. It is well known that the commonly used PID control law,

$$u(t) = K_\mathrm{p} e(t) + K_\mathrm{i} \int_0^t e(\tau) d\tau + K_\mathrm{d} \dot{e}(t),$$

cannot be expressed in a strictly proper or proper state-space form. But, it can be represented by a descriptor system. Show that the following descriptor system is a realization of the above PID control law:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} e(t),$$

and

$$u(t) = [\, K_\mathrm{i} \quad K_\mathrm{p} \quad K_\mathrm{d} \,]\, x,$$

where the state variable is given by

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad x_1 = \int_0^t e(\tau) d\tau, \ x_2 = e(t), \ x_3 = \dot{e}(t).$$

Exercise 6.2. Verify that the following descriptor system is another realization of the PID control law given in Exercise 6.1:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \dot{x} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} e(t),$$

and

$$u(t) = [\, K_\mathrm{i} \quad K_\mathrm{p} \quad K_\mathrm{d} \,]\, x.$$

Structural Mappings of Bilinear Transformations

BEN M. CHEN, NUS ECE

In this chapter, we present a comprehensive study of how the structures, *i.e.*, the finite and infinite zero structures, invertibility structures, as well as the geometric subspaces of a general continuous-time (discrete-time) linear time-invariant system are mapped to those of its discrete-time (continuous-time) counterpart under the well-known bilinear (inverse bilinear) transformation

$$s = a\left(\frac{z-1}{z+1}\right) \quad \Leftrightarrow \quad z = \frac{a+s}{a-s}$$

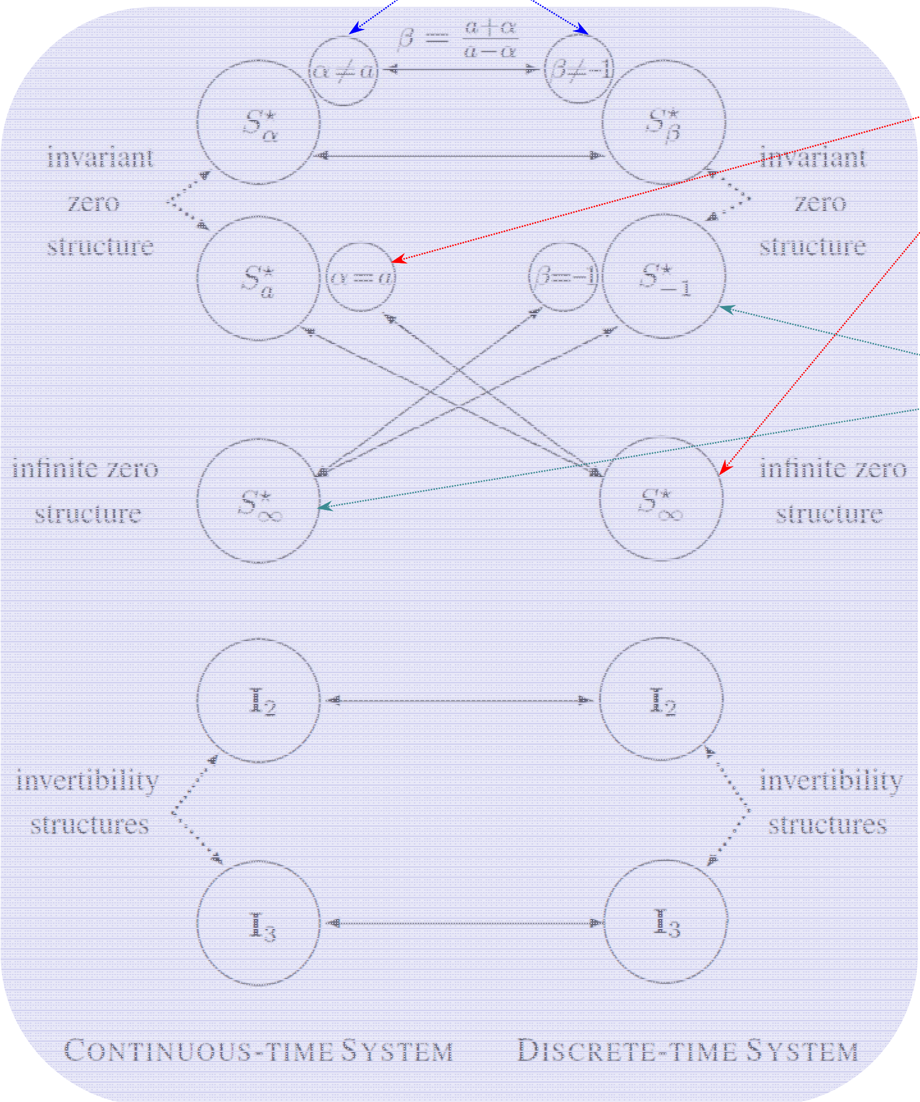
$$\Sigma_{\rm c} \ : \ \begin{cases} \dot{x} = A\ x + B\ u \\ y = C\ x + D\ u \end{cases} \quad \Leftrightarrow \quad \Sigma_{\rm d} \ : \ \begin{cases} x(k+1) = \tilde{A}\ x(k) + \tilde{B}\ u(k) \\ \quad y(k) \quad = \tilde{C}\ x(k) + \tilde{D}\ u(k) \end{cases}$$

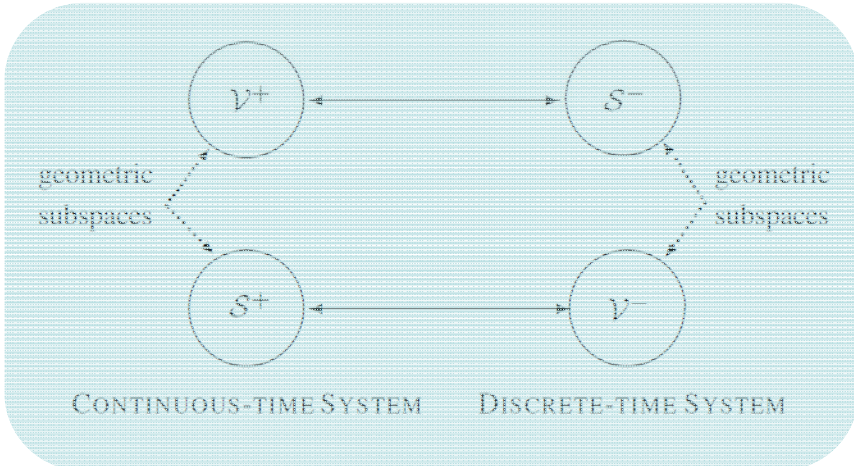
How are the structural properties of the continuous-time system and the discrete-time counterpart mapped under the bilinear transformation?…

$$S_\alpha^\star(\Sigma_c) = \{n_{\alpha,1}, n_{\alpha,2}, \dots, n_{\alpha,\tau_\alpha}\}$$
$$S_\beta^\star(\Sigma_d) = \{n_{\alpha,1}, n_{\alpha,2}, \dots, n_{\alpha,\tau_\alpha}\}$$

$$S_a^\star(\Sigma_c) = \{n_{a,1}, n_{a,2}, \dots, n_{a,\tau_a}\}$$
$$S_\infty^\star(\Sigma_d) = \{n_{a,1}, n_{a,2}, \dots, n_{a,\tau_a}\}$$

$$S_{-1}^\star(\Sigma_d) = \{q_1, q_2, \dots, q_{m_d}\}$$
$$S_\infty^\star(\Sigma_c) = \{q_1, q_2, \dots, q_{m_d}\}$$



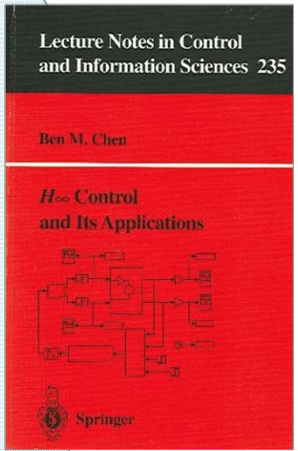


Automatica 36 (2000) 1247–1248

Book review

H_∞ control and its applications
Ben M. Chen; Springer, London, 1998,
ISBN 1-85233-026-0

…… Chapter 4 comprises a thorough investigation of the transformation properties of geometric subspaces and Riccati equations if transforming the underlying system from a discrete-time into a continuous-time description. Although it is conceptually a bit unclear why the inherent symmetry is broken and the corresponding Caley transformation and its inverse are treated separately, this chapter provides a very complete and fully proved reference list of relations that are useful for a variety of problems that involve the translation of continuous- to discrete-time results. As an impressive demonstration, it is revealed in Chapters 8–10 how these preparations render the proofs of H_∞-results for discrete-time systems almost into a routine exercise.



Carsten W. Scherer
Mechanical Engineering Systems and Control Group, Delft University of Technology, Mekelweg 2, 2628 CD Delft, Netherlands
E-mail address: c.w.scherer@wbmt.tudelft.nl

Structural Assignment via Sensor/ Actuator Selection

The problem: Given a linear unsensed system characterized by

$$\dot{x} = Ax + Bu, \tag{9.1.1}$$

where $x \in \mathbb{R}^n$ is the system state and $u \in \mathbb{R}^m$ is the control input. The problem of structural assignments or sensor selection is to find a constant matrix, C, or equivalently, a measurement output,

$$y = Cx, \tag{9.1.2}$$

such that the resulting system characterized by the matrix triple (A, B, C) would have the pre-specified desired structural properties, including finite and infinite zero structures and invertibility structures. We note that this technique can be applied to solve the dual problem of actuator selection, *i.e.*, to find a matrix B provided that matrices A and C are given such that the resulting system characterized by the triple (A, B, C) would have the pre-specified desired structural properties. Throughout this chapter, a set of complex scalars, say $\mathcal{W}$, is said to be self-conjugated if for any $w \in \mathcal{W}$, its complex conjugate $w^* \in \mathcal{W}$.

9.2 Simultaneous Finite and Infinite Zero Placement

9.2.1 SISO Systems

We consider in this subsection the finite and infinite zero assignment problem for system (9.1.1) with $m = 1$. We first have the following theorem, the proof of which is constructive and gives an explicit expression of a set of output matrices, $\mathbf{\Omega}$, such that for any element in $\mathbf{\Omega}$, the corresponding system has the prescribed finite zero and infinite zero structures.

Theorem 9.2.1. *Consider the unsensed system (9.1.1) characterized by* (A, B) *with* $A \in \mathbb{R}^{n\times n}$ *and* $B \in \mathbb{R}^{n\times 1}$. *Let* $\mathcal{C} := \{k_1\}$ *be the controllability index of* (A, B) *and let the number of uncontrollable modes be* n_o. *Also, let* $\{\nu_1, \nu_2, \dots, \nu_{n_\mathrm{o}}\}$ *be the uncontrollable modes of* (A, B). *Then for any given integer* q_1, $0 < q_1 \le k_1$, *and a set of self-conjugated scalars,* $\{z_1, z_2, \dots, z_{k_1-q_1}\}$, *there exists a nonempty set of output matrices* $\mathbf{\Omega} \subset \mathbb{R}^{1\times n}$ *such that for any* $C \in \mathbf{\Omega}$ *the resulting system* (A, B, C) *has* $n_\mathrm{o} + k_1 - q_1$ *invariant zeros at* $\{\nu_1, \nu_2, \dots, \nu_{n_\mathrm{o}}, z_1, z_2, \dots, z_{k_1-q_1}\}$ *and has an infinite zero structure* $\mathcal{S}_\infty^\star = \{q_1\}$, *i.e., the relative degree of* (A, B, C) *is equal to* q_1.

Proof. It follows from Theorem 4.4.1 that there exist nonsingular state and input transformations T_s and T_i such that (A, B) can be transformed into the controllability structural decomposition form of (4.4.7). Next, we rewrite (4.4.7) as follows,

uncontrollable modes

$$\tilde{A} = \begin{bmatrix} A_o & 0 & 0 & 0 & 0 \\ 0 & 0 & I_{k_1-q_1-1} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & I_{q_1-1} \\ \star & \star & \star & \star & \star \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \tag{9.2.1}$$

CSD

where $\star$ represents a matrix of less interest.

Let

$$a(s) := s^{k_1-q_1} + a_1 s^{k_1-q_1-1} + \cdots + a_{k_1-q_1} \tag{9.2.2}$$

be a polynomial having roots at $z_1, z_2, \ldots, z_{k_1-q_1}$. Also, let us define

$$\underline{a} := \begin{bmatrix} a_{k_1-q_1-1} & \cdots & a_2 & a_1 \end{bmatrix}.$$

new invariant zeros

Then the desired set of output matrices $\mathbf{\Omega}$ is given by

$$\mathbf{\Omega} := \left\{ C \in \mathbb{R}^{1\times n} \,\middle|\, C = \alpha [\quad a_{k_1-q_1} \quad \underline{a} \quad 1 \quad 0] T_s^{-1}, 0 \neq \alpha \in \mathbb{R}, \underline{d} \in \mathbb{R}^{1\times n_o} \right\}.$$

Example 9.2.1. Consider a system characterized by

$$\dot{x} = Ax + Bu = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u. \tag{9.2.12}$$

It is simple to see that the pair (A, B) is already in the form of the controllability structure decomposition with a controllability index $\mathcal{C} = \{3\}$. Then it follows from Theorem 9.2.1 that one has freedom to choose output matrices such that the resulting systems have: 1) infinite zero structure $\mathcal{S}_\infty^\star = \{3\}$ with no invariant zero, 2) $\mathcal{S}_\infty^\star = \{2\}$ with one invariant zero, and 3) $\mathcal{S}_\infty^\star = \{1\}$ with two invariant zeros. The systems with the following output matrices respectively have such properties:

$$C_1 = \alpha \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \qquad H_1(s) := C_1(sI_3 - A)^{-1}B = \frac{\alpha}{s(s^2-1)},$$

$$C_2 = \alpha \begin{bmatrix} a_1 & 1 & 0 \end{bmatrix}, \qquad H_2(s) := C_2(sI_3 - A)^{-1}B = \frac{\alpha(s+a_1)}{s(s^2-1)},$$

$$C_3 = \alpha \begin{bmatrix} a_2 & a_1 & 1 \end{bmatrix} \qquad H_3(s) := C_3(sI_3 - A)^{-1}B = \frac{\alpha(s^2+a_1 s+a_2)}{s(s^2-1)}.$$

9.2.2 MIMO Systems

Theorem 9.2.2. *Consider the unsensed system (9.1.1) characterized by* (A, B) *with* $A \in \mathbb{R}^{n\times n}$ *and* $B \in \mathbb{R}^{n\times m}$*. Assume that* B *is of full rank. Let the controllability index of* (A, B) *be given by* $\mathcal{C} := \{k_1, k_2, \dots, k_m\}$ *and let the pair* (A, B) *have* n_o *uncontrollable modes. Also, let* $\{\nu_1, \nu_2, \dots, \nu_{n_o}\}$ *be the uncontrollable modes of* (A, B)*. Then for any given set of integers,* $\mathcal{S}_\infty^\star := \{q_1, q_2, \dots, q_m\}$ *with* $0 < q_i \le k_i$, $i = 1, 2, \dots, m$*, and a set of self-conjugated scalars,* $\{z_1, z_2, \dots, z_\ell\}$ *where* $\ell := \sum_{i=1}^m (k_i - q_i)$*, there exists a nonempty set* $\mathbf{\Omega} \subset \mathbb{R}^{m\times n}$ *such that for any* $C \in \mathbf{\Omega}$*, the corresponding system characterized by* (A, B, C) *has the following properties:*

1. (A, B, C) *is square and invertible;*
2. (A, B, C) *has* $n_o + \ell$ *invariant zeros at* $\{\nu_1, \dots, \nu_{n_o}, z_1, \dots, z_\ell\}$*; and*
3. (A, B, C) *has an infinite zero structure* $\mathcal{S}_\infty^\star = \{q_1, \dots, q_m\}$.

Key Idea… CSD

$$\tilde{A} = \begin{bmatrix} A_o & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & I_{k_1-q_1-1} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_{q_1} & \cdots & 0 & 0 & 0 & 0 \\ \star & \star & \star & \star & \star & \cdots & \star & \star & \star & \star \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & I_{k_m-q_m-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & I_{q_m-1} \\ \star & \star & \star & \star & \star & \cdots & \star & \star & \star & \star \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} 0 & \cdots & 0 \\ 0 & \cdots & 0 \\ 0 & \cdots & 0 \\ 0 & \cdots & 0 \\ 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \\ 0 & \cdots & 0 \\ 0 & \cdots & 0 \\ 0 & \cdots & 1 \end{bmatrix}$$

1 1

$$\tilde{A}_{\rm aa} := \begin{bmatrix} 0 & I_{k_1-q_1-1} & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & I_{k_m-q_m-1} \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}, \quad \tilde{L}_{\rm ad} := \begin{bmatrix} 0 & \cdots & 0 \\ 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \\ 0 & \cdots & 1 \end{bmatrix}.$$

Note that $(\tilde{A}_{\rm aa}, \tilde{L}_{\rm ad})$ is controllable and, in fact, is in the controllability structural decomposition form.

Let us define

$$\boldsymbol{F}_{\mathrm{a}} := \left\{ \tilde{F}_{\mathrm{a}} \in \mathbb{R}^{m\times \ell} \;\middle|\; \lambda\left(\tilde{A}_{\mathrm{aa}} - \tilde{L}_{\mathrm{ad}}\tilde{F}_{\mathrm{a}}\right) = \{\, z_1,\, z_2,\, \ldots,\, z_\ell \,\} \right\}. \tag{9.2.17}$$

For any $\tilde{F}_{\mathrm{a}} \in \boldsymbol{F}_{\mathrm{a}}$, we partition it in conformity with (9.2.16) as

$$\tilde{F}_{\mathrm{a}} = \begin{bmatrix} F_{11}^{0} & F_{11}^{1} & \cdots & F_{1m}^{0} & F_{1m}^{1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ F_{m1}^{0} & F_{m1}^{1} & \cdots & F_{mm}^{0} & F_{mm}^{1} \end{bmatrix}, \tag{9.2.18}$$

and define a corresponding $m \times n$ matrix, in conformity with (9.2.13) and (9.2.14),

$$\tilde{C} := \begin{bmatrix} \boxed{} & F_{11}^{0} & F_{11}^{1} & 1 & 0 & \cdots & F_{1m}^{0} & F_{1m}^{1} & 0 & 0 \\ & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ & F_{m1}^{0} & F_{m1}^{1} & 0 & 0 & \cdots & F_{mm}^{0} & F_{mm}^{1} & 1 & 0 \end{bmatrix}, \tag{9.2.19}$$

arbitrary

The desired set of output matrices is give by

$$\boldsymbol{\Omega} := \left\{ C \in \mathbb{R}^{n\times m} \;\middle|\; C = \Gamma \tilde{C} T_{\mathrm{s}}^{-1} \text{ with } \boxed{},\ \tilde{F}_{\mathrm{a}} \in \boldsymbol{F}_{\mathrm{a}},\ \Gamma \in \mathbb{R}^{m\times m} \text{ and } \det(\Gamma) \neq 0 \right\}.$$

Example 9.2.2. Consider a two-input system characterized by

$$\dot{x} = Ax + Bu = \begin{bmatrix} -2 & -5 & -4 & -4 \\ 2 & 3 & 3 & 3 \\ -2 & 0 & 0 & -1 \\ 1 & 0 & 0 & 1 \end{bmatrix} x + \begin{bmatrix} -2 & 0 \\ 1 & 1 \\ 0 & -2 \\ 0 & 1 \end{bmatrix} u. \qquad (9.2.22)$$

Using the software package of [87], we obtain that

$$T_\mathrm{s} = \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad T_\mathrm{i} = I_2,$$

and the controllability structural decomposition form of (A, B) is given by

$$\tilde{A} = T_\mathrm{s}^{-1} A T_\mathrm{s} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & -2 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & -2 & 1 & 1 \end{bmatrix}, \quad \tilde{B} = T_\mathrm{s}^{-1} B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix},$$

with a controllability index $\mathcal{C} = \{2, 2\}$.

We have

$$\mathbf{\Omega}_1 = \left\{ \Gamma \begin{bmatrix} a_1 & 1 & a_2 & 0 \\ a_3 & 0 & a_4 & 1 \end{bmatrix} T_s^{-1} \,\middle|\, a_1 + a_4 = a_1 a_4 - a_2 a_3 = 2, \quad \Gamma \in \mathbb{R}^{2\times 2} \text{ with } \det(\Gamma) \neq 0 \right\}$$

such that for any $C \in \mathbf{\Omega}_1$ the resulting system (A, B, C) has an infinite zero structure $S_\infty^\star = \{1, 1\}$ and two invariant zeros at $-1 \pm j1$. The following is another set of output matrices that we obtain,

$$\mathbf{\Omega}_2 = \left\{ \Gamma \begin{bmatrix} 1 & 1 & 0 & 0 \\ a & 0 & 1 & 0 \end{bmatrix} T_s^{-1} \,\middle|\, a \in \mathbb{R},\ \Gamma \in \mathbb{R}^{2\times 2} \text{ with } \det(\Gamma) \neq 0 \right\}.$$

It is easy to verify that for any $C \in \mathbf{\Omega}_2$ the corresponding system (A, B, C) has an infinite zero structure $S_\infty^\star = \{1, 2\}$ and one invariant zero at -1.

We can work out as many different combinations as we want!

9.3 Complete Structural Assignment

Having studied in the previous chapters all the structural properties of linear systems, *i.e.*, the finite zero and infinite zero structures as well as the invertibility structures, we are now ready to present in the following theorem the result of the general system structural assignment.

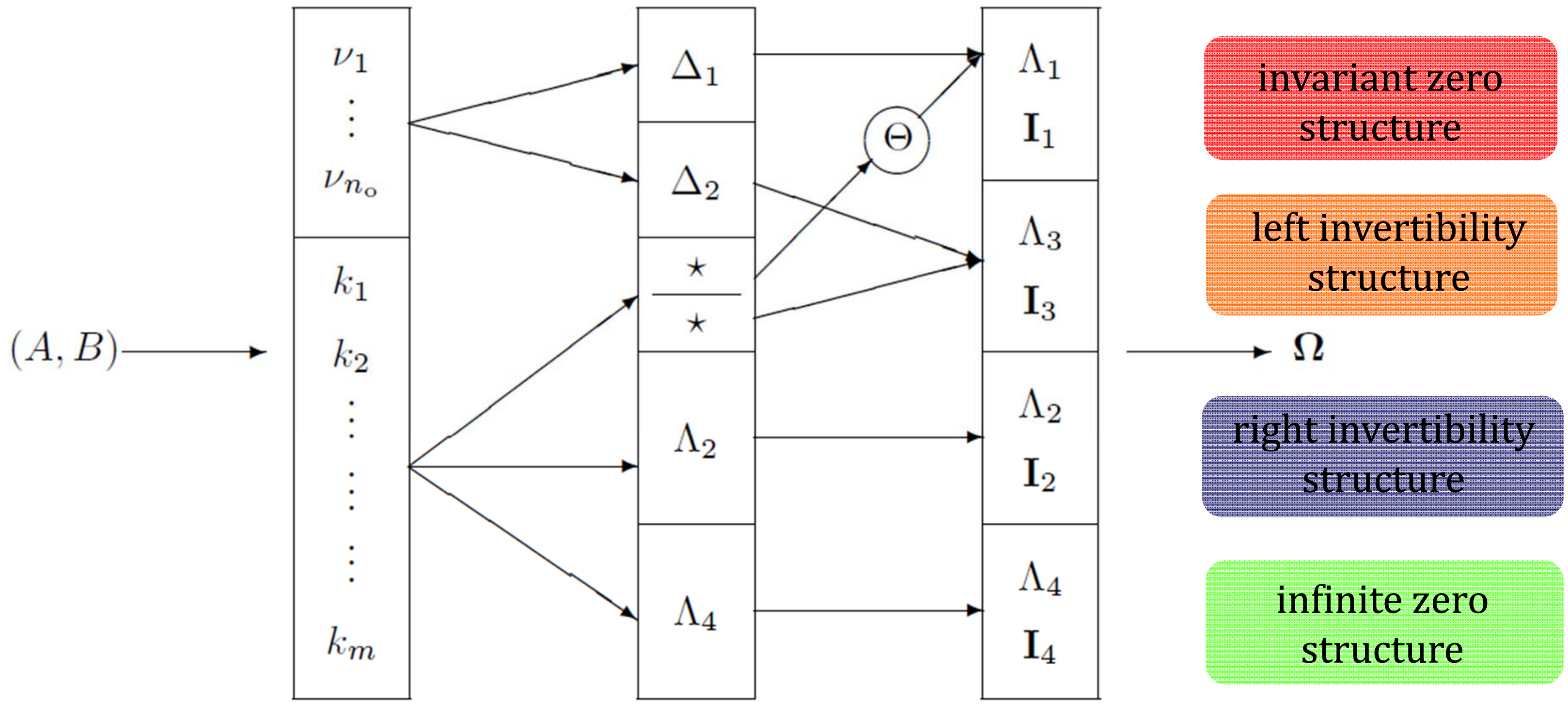


Figure 9.3.1: Graphical summary of the general structural assignment.

Example 9.3.2. We consider a benchmark problem for robust control of a flexible mechanical system proposed by Wie and Bernstein [149]. Although simple in nature, this problem will however provide an interesting example how sensor selection can affect the design performance. The problem is to control the displacement of the second mass by applying a force to the first mass as shown in Figure 9.3.2. The dynamic model of the system is given by

$$m_1\ddot{x}_1 = k(x_2 - x_1) + u + w_1, \tag{9.3.20}$$

$$m_2\ddot{x}_2 = k(x_1 - x_2) + w_2, \tag{9.3.21}$$

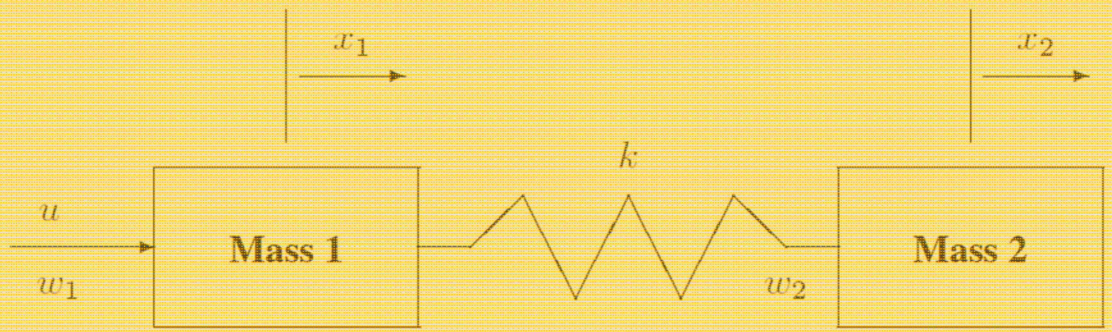


Figure 9.3.2: A two-mass-spring flexible mechanical system.

or in the state space representation,

$$\begin{pmatrix} \dot{x}_1 \\ \ddot{x}_1 \\ \dot{x}_2 \\ \ddot{x}_2 \end{pmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k}{m_1} & 0 & \frac{k}{m_1} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k}{m_2} & 0 & -\frac{k}{m_2} & 0 \end{bmatrix} \begin{pmatrix} x_1 \\ \dot{x}_1 \\ x_2 \\ \dot{x}_2 \end{pmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m_1} \\ 0 \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 & 0 \\ \frac{1}{m_1} & 0 \\ 0 & 0 \\ 0 & \frac{1}{m_2} \end{bmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix},$$

where x_1 and x_2 are respectively the positions of Mass 1 (with a mass of m_1) and Mass 2 (with a mass of m_2), k is the spring constant, u is the input force, and w_1 and w_2 are the frictions (disturbances). For simplicity, we choose $m_1 = m_2 = 1$ and $k = 1$. It is natural to define an output to be controlled as $h = x_2$, *i.e.*, the position of the second mass. Thus, the plant model used for robust control synthesis is given by

$$\dot{x} = Ax+Bu+Ew = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{bmatrix} \begin{pmatrix} x_1 \\ \dot{x}_1 \\ x_2 \\ \dot{x}_2 \end{pmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}, \tag{9.3.22}$$

and

$$h = C_2 x = [\,0 \quad 0 \quad 1 \quad 0\,]\, x. \tag{9.3.23}$$

It is simple to verify that the subsystem (A, B, C_2) is of minimum-phase and invertible. Hence, the disturbance w can be totally decoupled from the output to be controlled, *i.e.*, h, under the full state feedback. Our objective next is to identify sets of measurement output or the locations of sensors that would yield the same performance as that of the state feedback case. It follows from the results of [22,147] that this can be made possible by choosing a measurement output,

$$y = C_1 x, \tag{9.3.24}$$

such that the resulting subsystem (A, E, C_1) is left invertible and of minimum-phase. Following the procedure given in the previous section, we first transform the pair (A, E) into the controllability structural decomposition (CSD) form of Theorem 4.4.1. This can be done by the state and input transformation

$$T_0 = \begin{bmatrix} 0.316228 & 0 & 0.707107 & 0 \\ 0 & 0.316228 & 0 & 0.707107 \\ -0.316228 & 0 & 0.707107 & 0 \\ 0 & -0.316228 & 0 & 0.707107 \end{bmatrix},$$

and

$$T_{\mathrm{i}} = \begin{bmatrix} 0.316228 & 0.707107 \\ -0.316228 & 0.707107 \end{bmatrix}.$$

The controllability structural decomposition form of the pair (A, E) is given by

$$A_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad E_1 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \tag{9.3.25}$$

with a controllability index of (A, E) being $\{2, 2\}$. Following the proof of Theorem 9.3.1, we obtain the following set of measurement matrices,

$$\boldsymbol{\Omega}_1 = \left\{ \Gamma_{\rm o} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \,\middle|\, \Gamma_{\rm o} \in \mathbb{R}^{2\times 2}, \ \det(\Gamma_{\rm o}) \neq 0 \right\}, \tag{9.3.26}$$

such that for any $C_1 \in \boldsymbol{\Omega}_1$, the resulting subsystem (A, E, C_1) is square invertible with two infinite zeros of order 2 and with no invariant zeros. Hence, it is of minimum-phase. It is well-known that higher orders of infinite zeros would yield higher controller gains, which is in general not desirable in practical situations. In what follows, we will identify a set of measurement matrices, $\boldsymbol{\Omega}_2$, such that for any $C_1 \in \boldsymbol{\Omega}_2$, the resulting subsystem (A, E, C_1) is of minimum-phase and square invertible with two infinite zeros of order 1 and two invariant zeros at -1.

The following $\boldsymbol{\Omega}_2$ is such a set obtained again using the procedure given in the proof of Theorem 9.3.1:

$$\boldsymbol{\Omega}_2 = \left\{ \Gamma_\mathrm{o} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \;\middle|\; \Gamma_\mathrm{o} \in \mathbb{R}^{2\times 2},\ \det(\Gamma_\mathrm{o}) \neq 0 \right\}. \tag{9.3.27}$$

Thus, it is straightforward to verify that the H_∞ almost disturbance decoupling is achievable for the flexible mechanical system of (9.3.22)–(9.3.23) together with a measurement output $y = C_1 x$, where $C_1 \in \boldsymbol{\Omega}_1$ or $C_1 \in \boldsymbol{\Omega}_2$. In fact, we can show that the H_∞ almost disturbance decoupling for the system cannot be achieved if there is only one sensor allowed to be placed in the system, *i.e.*, one would have to place two or more sensors in the system in order to decouple the disturbance (the frictions) from the position of the second mass.

Exercise 9.1. Consider a linear system characterized by

$$\dot{x} = Ax + Bu = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u,$$

and

$$y = Cx = [\,0 \quad 1 \quad 0 \quad 0\,]\, x,$$

which has an unstable invariant zero at 1 and a relative degree of 3.

(a) Determine a new measurement matrix, $\tilde{C}_1$, such that the resulting new system characterized by $(A, B, \tilde{C}_1)$ has an invariant zero at -1 and has the same relative degree as the original system characterized by (A, B, C).

(b) Determine a new measurement matrix, $\tilde{C}_2$, such that the resulting new system characterized by $(A, B, \tilde{C}_2)$ has two invariant zeros at -1 and -2, and has a relative degree of 2.

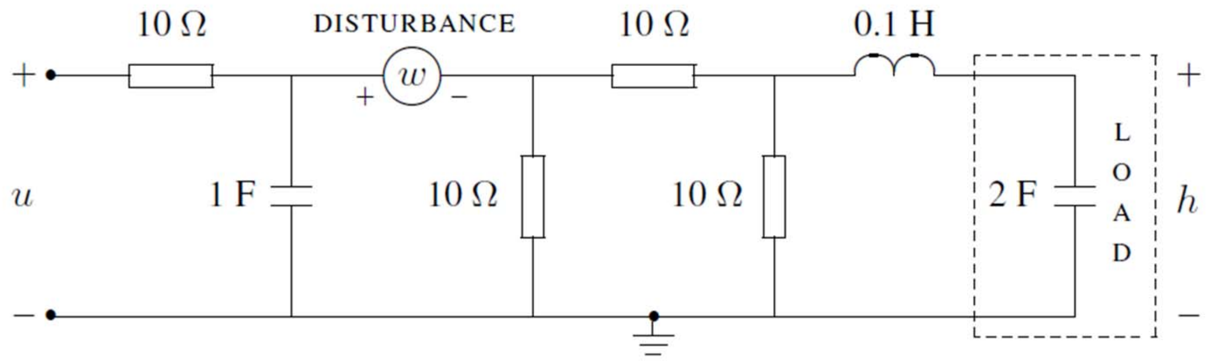
(c) Determine a new measurement matrix, $\tilde{C}_3$, such that the resulting new system characterized by $(A, B, \tilde{C}_3)$ has three invariant zeros at -1, -2 and -3, and has a relative degree of 1.

(d) Determine a new control matrix, $\tilde{B}_1$, such that the resulting new system characterized by $(A, \tilde{B}_1, C)$ has an invariant zero at -1 and has the same relative degree as the original system characterized by (A, B, C).

(e) Determine a new control matrix, $\tilde{B}_2$, such that the resulting new system characterized by $(A, \tilde{B}_2, C)$ has two invariant zeros at -1 and -2, and has a relative degree of 2.

(f) Determine a new control matrix, $\tilde{B}_3$, such that the resulting new system characterized by $(A, \tilde{B}_3, C)$ has three invariant zero at -1, -2 and -3, and has a relative degree of 1.

Exercise 9.2. Consider an electric system given in the circuit below, in which the voltage of the circuit load, *i.e.*, the controlled output, h, cannot be measured, and the disturbance input, w, is to be rejected.



Circuit for Exercise 9.2

(a) Verify that the state-space realization of the system from the control input, u, to the controlled output, h, can be expressed as follows:

$$\begin{cases} \dot{x} = A\ x + B\ u + E\ w, \\ h = C_2\ x + D_2\ u, \end{cases}$$

with

$$A = \begin{bmatrix} -0.25 & -0.5 & 0 \\ 5 & -50 & -10 \\ 0 & 0.5 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0.1 \\ 0 \\ 0 \end{bmatrix}, \quad E = \begin{bmatrix} 0.15 \\ -5 \\ 0 \end{bmatrix},$$

and

$$C_2 = [0 \quad 0 \quad 1], \quad D_2 = 0, \quad x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix},$$

where x_1 is the voltage across the 1 F capacitor, x_2 is the current through the 0.1 H inductor, and finally x_3 is the voltage across the 2 F capacitor.

(b) Show that if the inductor current is the only measurement available, *i.e.*,

$$y = C_1 x + D_1 w = [0 \quad 1 \quad 0]\,x + 0 \cdot w,$$

the resulting subsystem from the disturbance, w, to the measurement output, y, is of nonminimum phase. In this case, it is not possible to find a proper and stabilizing controller for the system that can achieve H_∞ almost disturbance decoupling from w to h.

(c) Show that if the 1 F capacitor voltage can be measured, *i.e.*,

$$y = C_1 x + D_1 w = [1 \quad 0 \quad 0]\, x + 0 \cdot w,$$

then the resulting subsystem from w to y is of minimum phase, and thus, there exists a proper and stabilizing controller for the circuit such that the disturbance, w, can be almost decoupled from the controlled output, h.