Decompositions of Proper Systems

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★★★★★ **Advanced, Innovative and Difficult**, May 29, 2010

By **Pascal Nespeca** (Davis, CA) - See all my reviews
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This review is from: **Linear Systems Theory: A Structural Decomposition Approach (Control Engineering) (Hardcover)**

This book is part of an advanced and innovative presentation of certain topics in system theory and control system design that are very rarely addressed elsewhere (I actually do not know of another source that addresses some of these issues). One might use this book to ask questions like "How do I select actuators/sensors so that transmission zeros are favorable to control design?". Those who have reached a level of knowledge and maturity in systems and control know that this is vital.

This book is not an introductory text into linear system theory. One might read this text alongside other "introductory" linear system theory books by C.T. Chen, Antsaklis and Michel, or Callier and Desoer, etc. This book is mathematically rigorous and there are many special definitions and symbols. The author makes an effort to extend the presentation to engineers with computer codes and numerical examples, which are helpful. Nonetheless, I still feel as though my head is about to explode when I read this book. It is difficult.

☹

Theorem 5.2.1. *Consider the SISO system of (5.2.1). There exist nonsingular state, input and output transformations $\Gamma_s \in \mathbb{R}^{n\times n}$, $\Gamma_i \in \mathbb{R}$ and $\Gamma_o \in \mathbb{R}$, which decompose the state space of Σ into two subspaces, x_a and x_d. These two subspaces correspond to the finite zero and infinite zero structures of Σ, respectively. The new state space, input and output space of the decomposed system are described by the following set of equations:*

$$x = \Gamma_s \tilde{x}, \quad y = \Gamma_o \tilde{y}, \quad u = \Gamma_i \tilde{u}, \tag{5.2.2}$$

$$\tilde{x} = \begin{pmatrix} x_a \\ x_d \end{pmatrix}, \quad x_a \in \mathbb{R}^{n_a}, \quad x_d \in \mathbb{R}^{n_d}, \quad x_d = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n_d} \end{pmatrix}, \tag{5.2.3}$$

and

$$\dot{x}_a = A_{aa} x_a + L_{ad} \tilde{y}, \tag{5.2.4}$$

$$\dot{x}_1 = x_2, \quad \tilde{y} = x_1, \tag{5.2.5}$$

$$\dot{x}_2 = x_3, \tag{5.2.6}$$

$$\vdots$$

$$\dot{x}_{n_d-1} = x_{n_d}, \tag{5.2.7}$$

$$\dot{x}_{n_d} = E_{da} x_a + E_1 x_1 + E_2 x_2 + \cdots + E_{n_d} x_{n_d} + \tilde{u}. \tag{5.2.8}$$

Furthermore, $\lambda(A_{aa})$ contains all the system invariant zeros and n_d is the relative degree of Σ.

$$\Sigma : \dot{x} = A x + B u, \quad y = C x, \tag{5.2.1}$$

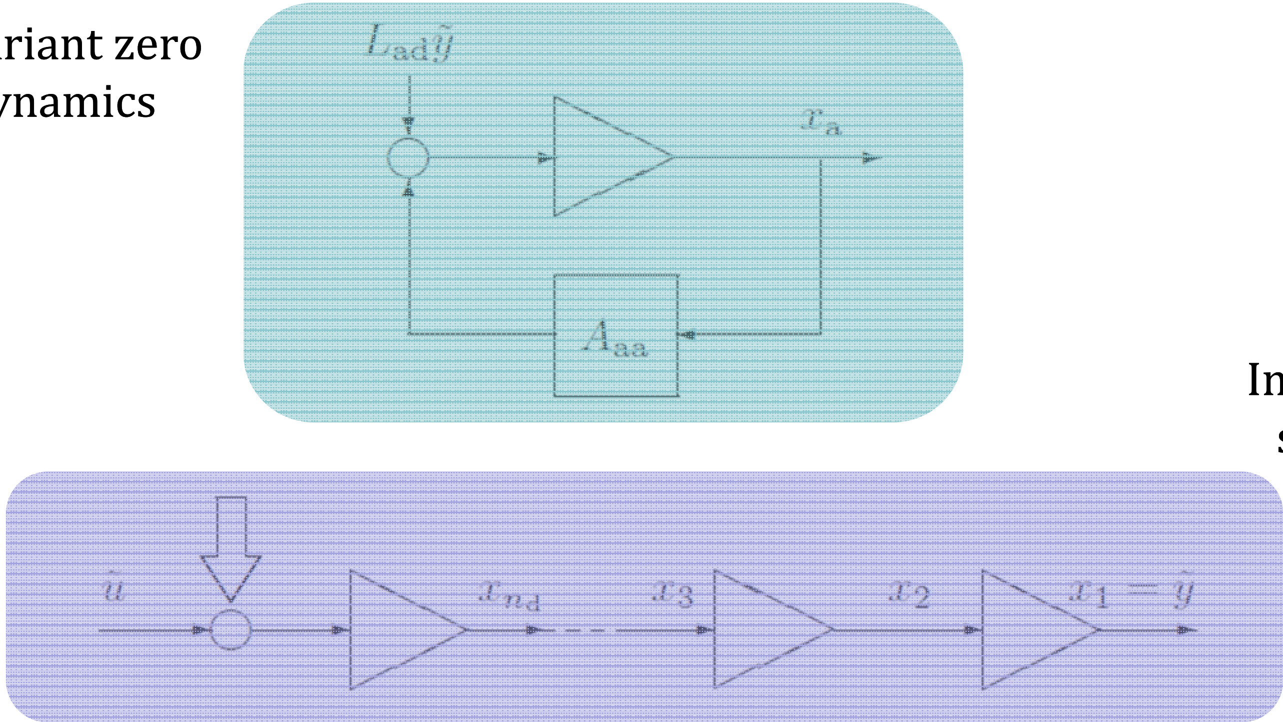
Transformed System:

$$\tilde{A} = \begin{bmatrix} A_{aa} & * & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ * & * & * & * & \cdots & * & * \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$$\tilde{C} = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

$$\tilde{A} + \tilde{B}\tilde{F} = \begin{bmatrix} A_{aa} & * & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

Invariant zero dynamics



Infinite zero structure

Note: the signal given by the double-edged arrow is a linear combination of the states.

Figure 5.2.1: Interpretation of structural decomposition of a SISO system.

Example 5.2.1. Consider a SISO system Σ characterized by (5.2.1) with

$$A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & 3 & 4 & 5 \\ 4 & 5 & 6 & 7 \\ 5 & 6 & 7 & 8 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \tag{5.2.36}$$

and

$$C = [\,0 \quad 3 \quad -2 \quad 0\,]. \tag{5.2.37}$$

The structural decomposition of Σ proceeds as follows:

1. *Differentiating the system output.*

   It involves the following sub-steps.

   (a) First, we have

   $$\dot{y} = C\dot{x} = CAx + CBu = [\,-2 \quad -1 \quad 0 \quad 1\,]\,x + 0 \cdot u.$$

   (b) Since $CB = 0$, we compute

   $$\ddot{y} = CA^2x + CABu = [\,1 \quad -1 \quad -3 \quad 1\,]\,x + 0 \cdot u.$$

   (c) Since $CAB = 0$, we continue on computing

   $$y^{(3)} = CA^3x + CA^2Bu = -[\,8 \quad 10 \quad 12 \quad 17\,]\,x - 6 \cdot u.$$

   We move to the next step as $CA^2B \neq 0$.

2. *Constructing a preliminary state transformation.*

Let Z_0 be a vector such that

$$Z = \begin{bmatrix} Z_0 \\ C \\ CA \\ CA^2 \end{bmatrix}, \tag{5.2.38}$$

is nonsingular. Then, define a new set of state variables $\bar{x}$,

$$\bar{x} = \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} := Zx = \begin{bmatrix} Z_0 \\ C \\ CA \\ CA^2 \end{bmatrix} x = \begin{pmatrix} Z_0 x \\ y \\ \dot{y} \\ \ddot{y} \end{pmatrix}. \tag{5.2.39}$$

It is simple to verify that Z with $Z_0 = [1 \quad 0 \quad 0 \quad 0]$ is a nonsingular matrix. Furthermore,

$$\dot{x}_0 = 8x_0 + x_1 + \frac{8}{3}x_2 - \frac{5}{3}x_3 + u, \tag{5.2.40}$$

$$\dot{x}_1 = x_2, \tag{5.2.41}$$

$$\dot{x}_2 = x_3, \tag{5.2.42}$$

$$\dot{x}_3 = -72x_0 - 9x_1 - 27x_2 + 10x_3 - 6u. \tag{5.2.43}$$

3. *Eliminating u in (5.2.40).*

Equation (5.2.43) implies that

$$u = -12x_0 - \frac{3}{2}x_1 - \frac{9}{2}x_2 + \frac{5}{3}x_3 - \frac{1}{6}\dot{x}_3. \tag{5.2.44}$$

Substituting this into (5.2.40), we obtain

DIY

$$\dot{x}_0 = -4x_0 - \frac{1}{2}x_1 - \frac{11}{6}x_2 - \frac{1}{6}\dot{x}_3. \tag{5.2.45}$$

We have eliminated u in $\dot{x}_0$. Unfortunately, we have also introduced an additional $\dot{x}_3$ in (5.2.45).

4. *Eliminating $\dot{x}_3$ in (5.2.45).*

Define a new variable $\tilde{x}_0$ as

$$\tilde{x}_0 := x_0 + \frac{1}{6}x_3. \tag{5.2.46}$$

DIY

We have

$$\dot{\tilde{x}}_0 = -4\tilde{x}_0 - \frac{1}{2}x_1 - \frac{11}{6}x_2 + \frac{2}{3}x_3, \tag{5.2.47}$$

DIY

and

$$\dot{x}_3 = -72\tilde{x}_0 - 9x_1 - 27x_2 + 22x_3 - 6u. \tag{5.2.48}$$

5. *Eliminating x_2 and x_3 in (5.2.47).*

This step involves two sub-steps.

(a) Letting

$$\tilde{x}_{0,1} := \tilde{x}_0 - \frac{2}{3}x_2, \tag{5.2.49}$$

we have

$$\dot{\tilde{x}}_{0,1} = -4\tilde{x}_{0,1} - \frac{1}{2}x_1 - \frac{9}{2}x_2, \tag{5.2.50}$$

DIY

and

$$\dot{x}_3 = -72\tilde{x}_{0,1} - 9x_1 - 75x_2 + 22x_3 - 6u. \tag{5.2.51}$$

(b) Letting

$$\tilde{x}_{0,2} := \tilde{x}_{0,1} + \frac{9}{2}x_1, \tag{5.2.52}$$

we have

$$\dot{\tilde{x}}_{0,2} = -4\tilde{x}_{0,2} + \frac{35}{2}x_1, \tag{5.2.53}$$

DIY

and

$$\dot{x}_3 = -72\tilde{x}_{0,2} + 315x_1 - 75x_2 + 22x_3 - 6u. \tag{5.2.54}$$

6. *Composing the nonsingular state, output and input transformations.*

Let

$$x_{\mathrm{a}} := \tilde{x}_{0,2} \tag{5.2.55}$$

or equivalently let

$$x = \Gamma_{\mathrm{s}}\tilde{x} = \Gamma_{\mathrm{s}} \begin{pmatrix} x_{\mathrm{a}} \\ x_1 \\ x_2 \\ x_3 \end{pmatrix}, \tag{5.2.56}$$

with

$$\Gamma_{\mathrm{s}} = \left\{ \begin{bmatrix} 1 & 9/2 & -2/3 & 1/6 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & -2 & 0 \\ -2 & -1 & 0 & 1 \\ 1 & -1 & -3 & 1 \end{bmatrix} \right\}^{-1}. \tag{5.2.57}$$

Also, let

$$u = \Gamma_{\mathrm{i}}\tilde{u} = -\frac{1}{6}\tilde{u}, \quad y = \Gamma_{\mathrm{o}}\tilde{y} = 1 \cdot \tilde{y}. \tag{5.2.58}$$

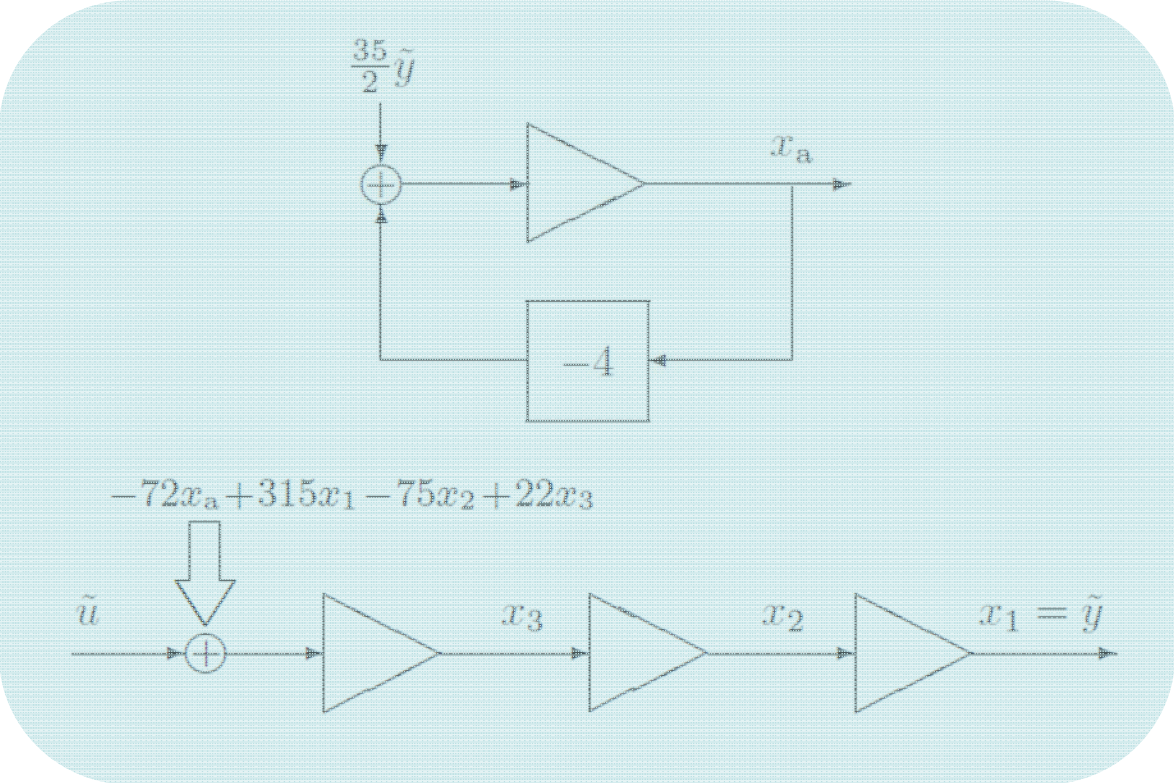
Finally, we obtain the dynamic equations of the transformed system,

$$\dot{x}_a = -4x_a + \frac{35}{2}x_1, \tag{5.2.59}$$

$$\dot{x}_1 = x_2, \quad \tilde{y} = x_1, \tag{5.2.60}$$

$$\dot{x}_2 = x_3, \tag{5.2.61}$$

$$\dot{x}_3 = -72x_a + 315x_1 - 75x_2 + 22x_3 + \tilde{u}, \tag{5.2.62}$$



An invariant zero at – 4

An infinite zero of order 3 = relative degree

5.3 Strictly Proper Systems

Next, we consider a general strictly proper linear system Σ characterized by

$$\begin{cases} \dot{x} = A\ x + B\ u, \\ y = C\ x, \end{cases} \tag{5.3.1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $y \in \mathbb{R}^p$ are the state, input and output. Without loss of generality, we assume that both B and C are of full rank. We have the following structural or special coordinate basis decomposition of Σ.

Theorem 5.3.1. *Consider the strictly proper system Σ characterized by (5.3.1). There exist a nonsingular state transformation, $\Gamma_{\rm s} \in \mathbb{R}^{n \times n}$, a nonsingular output transformation, $\Gamma_{\rm o} \in \mathbb{R}^{p \times p}$, and a nonsingular input transformation, $\Gamma_{\rm i} \in \mathbb{R}^{m \times m}$, that will reveal all the structural properties of Σ. More specifically, we have*

$$x = \Gamma_{\rm s}\tilde{x}, \quad y = \Gamma_{\rm o}\tilde{y}, \quad u = \Gamma_{\rm i}\tilde{u}, \tag{5.3.2}$$

with the new state variables

$$\tilde{x} = \begin{pmatrix} x_{\rm a} \\ x_{\rm b} \\ x_{\rm c} \\ x_{\rm d} \end{pmatrix}, \quad x_{\rm a} \in \mathbb{R}^{n_{\rm a}}, \quad x_{\rm b} \in \mathbb{R}^{n_{\rm b}}, \quad x_{\rm c} \in \mathbb{R}^{n_{\rm c}}, \quad x_{\rm d} \in \mathbb{R}^{n_{\rm d}}, \tag{5.3.3}$$

the new output variables

$$\tilde{y} = \begin{pmatrix} y_{\mathrm{d}} \\ y_{\mathrm{b}} \end{pmatrix}, \quad y_{\mathrm{d}} \in \mathbb{R}^{m_{\mathrm{d}}}, \quad y_{\mathrm{b}} \in \mathbb{R}^{p_{\mathrm{b}}}, \tag{5.3.4}$$

and the new input variables

$$\tilde{u} = \begin{pmatrix} u_{\mathrm{d}} \\ u_{\mathrm{c}} \end{pmatrix}, \quad u_{\mathrm{d}} \in \mathbb{R}^{m_{\mathrm{d}}}, \quad u_{\mathrm{c}} \in \mathbb{R}^{m_{\mathrm{c}}}. \tag{5.3.5}$$

Further, the state variable x_{d} can be decomposed as:

$$x_{\mathrm{d}} = \begin{pmatrix} x_{\mathrm{d},1} \\ x_{\mathrm{d},2} \\ \vdots \\ x_{\mathrm{d},m_{\mathrm{d}}} \end{pmatrix}, \quad y_{\mathrm{d}} = \begin{pmatrix} y_{\mathrm{d},1} \\ y_{\mathrm{d},2} \\ \vdots \\ y_{\mathrm{d},m_{\mathrm{d}}} \end{pmatrix}, \quad u_{\mathrm{d}} = \begin{pmatrix} u_{\mathrm{d},1} \\ u_{\mathrm{d},2} \\ \vdots \\ u_{\mathrm{d},m_{\mathrm{d}}} \end{pmatrix}, \tag{5.3.6}$$

$$x_{\mathrm{d},i} \in \mathbb{R}^{q_i}, \quad x_{\mathrm{d},i} = \begin{pmatrix} x_{\mathrm{d},i,1} \\ x_{\mathrm{d},i,1} \\ \vdots \\ x_{\mathrm{d},i,q_i} \end{pmatrix}, \quad i = 1, 2, \ldots, m_{\mathrm{d}}, \tag{5.3.7}$$

with $q_1 \le q_2 \le \cdots \le q_{m_\mathrm{d}}$. The state variable x_b can be decomposed as

$$x_\mathrm{b} = \begin{pmatrix} x_{\mathrm{b},1} \\ x_{\mathrm{b},2} \\ \vdots \\ x_{\mathrm{b},p_\mathrm{b}} \end{pmatrix}, \quad y_\mathrm{b} = \begin{pmatrix} y_{\mathrm{b},1} \\ y_{\mathrm{b},2} \\ \vdots \\ y_{\mathrm{b},p_\mathrm{b}} \end{pmatrix}, \tag{5.3.8}$$

$$x_{\mathrm{b},i} \in \mathbb{R}^{l_i}, \quad x_{\mathrm{b},i} = \begin{pmatrix} x_{\mathrm{b},i,1} \\ x_{\mathrm{b},i,2} \\ \vdots \\ x_{\mathrm{b},i,l_i} \end{pmatrix}, \quad i = 1, 2, \ldots, p_\mathrm{b}, \tag{5.3.9}$$

with $l_1 \le l_2 \le \cdots \le l_{p_\mathrm{b}}$. And finally, the state variable x_c can be decomposed as

$$x_\mathrm{c} = \begin{pmatrix} x_{\mathrm{c},1} \\ x_{\mathrm{c},2} \\ \vdots \\ x_{\mathrm{c},m_\mathrm{c}} \end{pmatrix}, \quad u_\mathrm{c} = \begin{pmatrix} u_{\mathrm{c},1} \\ u_{\mathrm{c},2} \\ \vdots \\ u_{\mathrm{c},m_\mathrm{c}} \end{pmatrix}, \tag{5.3.10}$$

$$x_{\mathrm{c},i} \in \mathbb{R}^{r_i}, \quad x_{\mathrm{c},i} = \begin{pmatrix} x_{\mathrm{c},i,1} \\ x_{\mathrm{c},i,2} \\ \vdots \\ x_{\mathrm{c},i,r_i} \end{pmatrix}, \quad i = 1, 2, \ldots, m_\mathrm{c}, \tag{5.3.11}$$

with $r_1 \le r_2 \le \cdots \le r_{m_\mathrm{c}}$.

The decomposed system can be expressed in the following dynamical equations:

$$\dot{x}_{\rm a} = A_{\rm aa} x_{\rm a} + L_{\rm ab} y_{\rm b} + L_{\rm ad} y_{\rm d}, \tag{5.3.12}$$

invariant zero

for each subsystem $x_{{\rm b},i}$, $i = 1, 2, \ldots, p_{\rm b}$,

$$\dot{x}_{{\rm b},i,1} = x_{{\rm b},i,2} + L_{{\rm bd},i,1} y_{\rm b} + L_{{\rm b},i,1} y_{\rm d}, \quad y_{{\rm b},i} = x_{{\rm b},i,1}, \tag{5.3.13}$$

$$\dot{x}_{{\rm b},i,2} = x_{{\rm b},i,3} + L_{{\rm bd},i,2} y_{\rm b} + L_{{\rm b},i,2} y_{\rm d}, \tag{5.3.14}$$

$$\vdots$$

$$\dot{x}_{{\rm b},i,l_i} = L_{{\rm bd},i,l_i} y_{\rm b} + L_{{\rm bd},i,l_i} y_{\rm d}, \tag{5.3.15}$$

left invertibility structure

for each subsystem $x_{{\rm c},i}$, $i = 1, 2, \ldots, m_{\rm c}$,

$$\dot{x}_{{\rm c},i,1} = x_{{\rm c},i,2} + L_{{\rm cb},i,1} y_{\rm b} + L_{{\rm cd},i,1} y_{\rm d}, \tag{5.3.16}$$

$$\vdots$$

$$\dot{x}_{{\rm c},i,r_i-1} = x_{{\rm c},i,r_i} + L_{{\rm cb},i,r_i-1} y_{\rm b} + L_{{\rm cd},i,r_i-1} y_{\rm d}, \tag{5.3.17}$$

$$\dot{x}_{{\rm c},i,r_i} = A_{{\rm c},i,{\rm a}} x_{\rm a} + A_{{\rm c},i,{\rm c}} x_{\rm c} + L_{{\rm cb},i,r_i} y_{\rm b} + L_{{\rm cd},i,r_i} y_{\rm d} + u_{{\rm c},i}, \tag{5.3.18}$$

right invertibility structure

and finally, for each subsystem $x_{\mathrm{d},i}$, $i = 1, 2, \ldots, m_{\mathrm{d}}$,

$$\dot{x}_{\mathrm{d},i,1} = x_{\mathrm{d},i,2} + L_{\mathrm{d},i,1} y_{\mathrm{d}}, \quad y_{\mathrm{d},i} = x_{\mathrm{d},i,1}, \tag{5.3.19}$$

$$\dot{x}_{\mathrm{d},i,2} = x_{\mathrm{d},i,3} + L_{\mathrm{d},i,2} y_{\mathrm{d}}, \tag{5.3.20}$$

$$\vdots$$

$$\dot{x}_{\mathrm{d},i,q_i} = A_{\mathrm{d},i,\mathrm{a}} x_{\mathrm{a}} + A_{\mathrm{d},i,\mathrm{c}} x_{\mathrm{c}} + A_{\mathrm{d},i,\mathrm{b}} x_{\mathrm{b}} + A_{\mathrm{d},i,\mathrm{d}} x_{\mathrm{d}} + u_{\mathrm{d},i}, \tag{5.3.21}$$

where $A_{\mathrm{aa}}, L_{\mathrm{ab}}, \ldots, A_{\mathrm{d},i,\mathrm{d}}$ are constant matrices of appropriate dimensions.

infinite zero structure

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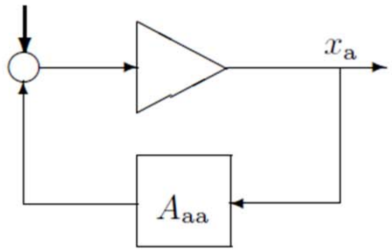
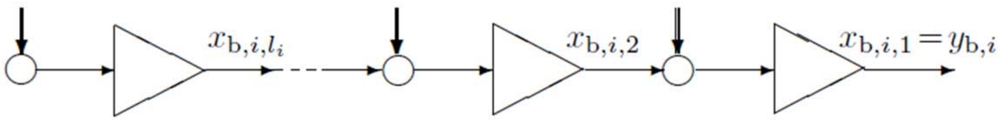
x_{a} – *the subsystem without direct input and output:*



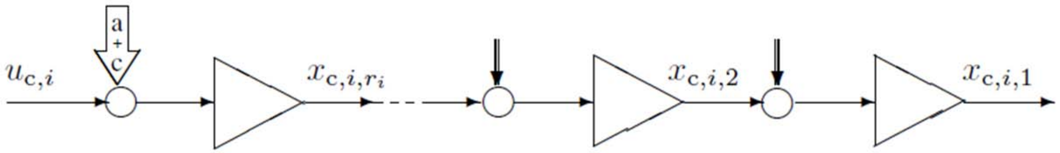
invariant zero

$x_{\mathrm{b},i}$ – *the chain of integrators without a direct input:*



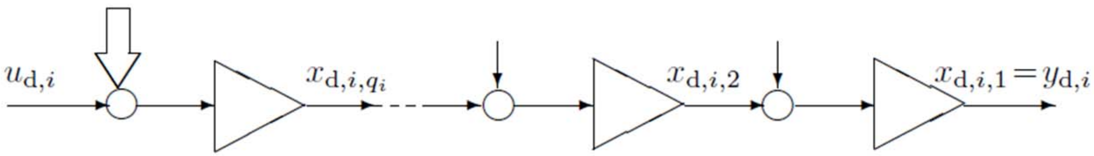
left invertibility structure

$x_{\mathrm{c},i}$ – *the chain of integrators without a direct output:*



right invertibility structure

$x_{\mathrm{d},i}$ – *the chain of integrators with direct input and output:*



infinite zero structure

Example 5.3.1. Consider a strictly proper system Σ characterized by (5.3.1) with

$$A = \begin{bmatrix} 0 & 0 & 2 & -1 & 2 & 0 & -1 & 2 & 0 & -1 & 0 & 2 & 2 \\ 0 & 2 & 4 & -5 & 3 & 2 & -4 & 3 & 2 & -4 & 0 & 5 & 0 \\ 0 & -1 & -1 & 1 & 0 & -1 & 1 & 0 & -1 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 3 & -2 & 0 & 3 & -3 & 0 & 3 & 1 & -1 & 0 \\ 0 & 2 & 2 & 0 & -2 & 2 & 1 & -3 & 2 & 1 & 1 & 1 & -2 \\ 0 & 3 & 3 & -1 & -2 & 3 & 0 & -2 & 3 & 0 & 2 & 2 & -3 \\ 0 & 3 & 3 & -1 & -2 & 3 & -1 & -1 & 3 & 0 & 1 & 3 & -3 \\ 0 & 3 & 3 & -1 & -2 & 3 & -1 & 0 & 3 & 0 & 1 & 4 & -3 \\ 0 & 2 & 2 & 1 & -1 & 2 & 0 & 0 & 2 & 1 & 1 & 3 & -1 \\ 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 2 & 2 & 0 \\ 0 & 0 & -2 & 4 & -2 & 0 & 2 & -2 & 0 & 2 & 1 & -2 & 0 \\ 0 & -1 & -3 & 7 & -3 & -1 & 4 & -3 & -1 & 4 & 2 & -4 & 1 \\ -1 & 0 & 0 & 1 & 1 & 0 & -1 & 2 & 0 & -1 & 0 & 0 & 2 \end{bmatrix},$$

$$B = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 2 & 2 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 & 1 & -1 & 0 & 1 & -1 & 0 & 1 & -1 & -1 & 2 & -1 \\ 0 & 0 & 0 & 1 & -1 & 1 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 & 1 & 0 & -1 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 & 0 & 1 & -1 & 0 & 1 & 1 & -1 & 0 \end{bmatrix}$$

The required state, input and output transformations…

$$\Gamma_s = \begin{bmatrix} -1 & -1 & 1 & -1 & 3 & -3 & 0 & 0 & 0 & 1 & 2 & 1 & 0 \\ -1 & 0 & 5 & 0 & 10 & 0 & 0 & 0 & 0 & 0 & 3 & 2 & 1 \\ 0 & -1 & -2 & 0 & -4 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 2 & 0 & 4 & 0 & 1 & 0 & 1 & 0 & 4 & 1 & 0 \\ 0 & -2 & 2 & 0 & 5 & 0 & 1 & 1 & 1 & 0 & 6 & 2 & 0 \\ 1 & -3 & 7 & 0 & 14 & 1 & 1 & 1 & 2 & -1 & 11 & 4 & 1 \\ 0 & -2 & 4 & 0 & 9 & 0 & 0 & 1 & 1 & 0 & 7 & 3 & 1 \\ 1 & -1 & 5 & 0 & 10 & 0 & 0 & 0 & 1 & 0 & 6 & 2 & 1 \\ 1 & 1 & 0 & 2 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 2 & 1 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 6 & 2 & 0 & 0 & 0 & 0 & 2 & 2 & 1 \end{bmatrix}$$

$$\Gamma_i = \begin{bmatrix} 0 & 1 & -1 & 0 \\ 1 & -1 & 1 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \Gamma_o = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

These transformations are non-unique!

The transformed system $(\tilde{A},\tilde{B},\tilde{C}) = (\Gamma_s^{-1} A \Gamma_s, \Gamma_s^{-1} B \Gamma_i, \Gamma_o^{-1} C \Gamma_s)$

$$\tilde{A} = \left[\begin{array}{cc|cccc|ccc|cccc}
-2 & 0 & 6 & 0 & 11 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
1 & 1 & -2 & 0 & -5 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
\hline
0 & 0 & 1 & 1 & 2 & 0 & 0 & 0 & 0 & 2 & 2 & 0 & 0 \\
0 & 0 & 1 & 0 & 2 & 0 & 0 & 0 & 0 & 2 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 \\
\hline
0 & 0 & -1 & 0 & -2 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 10 & 0 & 16 & 0 & 0 & 0 & 1 & 4 & 8 & 0 & 0 \\
5 & -1 & -1 & 0 & -3 & 0 & 1 & 1 & 2 & 5 & 13 & 0 & 0 \\
\hline
2 & -2 & 12 & 6 & 23 & 8 & 1 & 1 & 2 & 2 & 16 & 6 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
2 & -2 & 8 & 3 & 15 & 4 & 1 & 1 & 2 & 1 & 14 & 5 & 1
\end{array}\right], \quad
\tilde{B} = \left[\begin{array}{cc|cc}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\hline
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
\hline
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{array}\right]$$

$$\tilde{C} = \left[\begin{array}{cc|cccc|ccc|cccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
\hline
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right]$$

F and K

$$\bar{A} = \tilde{A} + \tilde{B}F + K\tilde{C}$$

The essential structures of the system…

$$\bar{A} = \left[\begin{array}{cc|cccc|ccc|cccc}
-2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right], \quad
\tilde{B} = \left[\begin{array}{cc|cc}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\hline
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
\hline
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{array}\right]$$

$$\tilde{C} = \left[\begin{array}{cc|cccc|ccc|cccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
\hline
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right]$$

道

two invariant zeros	left invertibility structure	right invertibility structure	infinite zero structure
$\lambda(A_{\rm aa}) = \{-2, 1\}$	$S_{\rm L}^{\star}(\Sigma) = \{2, 2\}$	$S_{\rm R}^{\star}(\Sigma) = \{1, 2\}$	$S_{\infty}^{\star}(\Sigma) = \{1, 3\}$

5.4 Nonstrictly Proper Systems

We now present in this section the structural decomposition or the special coordinate basis of general nonstrictly proper multivariable systems. We will also present all the structural properties of such a decomposition with rigorous proofs. To be specific, we consider the following nonstrictly proper system Σ characterized by

$$\begin{cases} \dot{x} = A\,x + B\,u, \\ y = C\,x + D\,u, \end{cases} \tag{5.4.1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $y \in \mathbb{R}^p$ are the state, input and output of Σ. Without loss of generality, we assume that both $[\,B' \quad D'\,]$ and $[\,C \quad D\,]$ are of full rank.

The structural decomposition or the special coordinate basis of nonstrictly proper systems follows fairly closely from that of strictly proper systems given in Section 5.3. However, in many applications, it is not necessary to decompose the subsystems $x_{\rm b}$ and $x_{\rm c}$ into chains of integrators. On the other hand, in many situations, it is necessary to further separate $x_{\rm a}$, the subsystem related to the invariant zero dynamics of the given system, into subspaces corresponding to the stable, marginally stable (or marginally unstable) and unstable zero dynamics.

For future use, we rewrite the structural decomposition of Σ in a more compact form:

$$\tilde{A} = \Gamma_s^{-1} A \Gamma_s = A_s + B_0 C_0 = \begin{bmatrix} A_{aa}^- & 0 & 0 & L_{ab}^- C_b & 0 & L_{ad}^- C_d \\ 0 & A_{aa}^0 & 0 & L_{ab}^0 C_b & 0 & L_{ad}^0 C_d \\ 0 & 0 & A_{aa}^+ & L_{ab}^+ C_b & 0 & L_{ad}^+ C_d \\ 0 & 0 & 0 & A_{bb} & 0 & L_{bd} C_d \\ B_c E_{ca}^- & B_c E_{ca}^0 & B_c E_{ca}^+ & L_{cb} C_b & A_{cc} & L_{cd} C_d \\ B_d E_{da}^- & B_d E_{da}^0 & B_d E_{da}^+ & B_d E_{db} & B_d E_{dc} & A_{dd} \end{bmatrix} + B_0 C_0$$

$$\tilde{B} = \Gamma_s^{-1} B \Gamma_i = [\, B_0 \quad B_s \,] = \begin{bmatrix} B_{0a}^- & 0 & 0 \\ B_{0a}^0 & 0 & 0 \\ B_{0a}^+ & 0 & 0 \\ B_{0b} & 0 & 0 \\ B_{0c} & 0 & B_c \\ B_{0d} & B_d & 0 \end{bmatrix}, \quad \tilde{D} = \Gamma_o^{-1} D \Gamma_i = D_s = \begin{bmatrix} I_{m_0} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\tilde{C} = \Gamma_o^{-1} C \Gamma_s = \begin{bmatrix} C_0 \\ C_s \end{bmatrix} = \begin{bmatrix} C_{0a}^- & C_{0a}^0 & C_{0a}^+ & C_{0b} & C_{0c} & C_{0d} \\ 0 & 0 & 0 & 0 & 0 & C_d \\ 0 & 0 & 0 & C_b & 0 & 0 \end{bmatrix}$$

Note: In the compact SCB form…

$$A_{\rm dd} = A_{\rm dd}^* + B_{\rm d} E_{\rm dd} + L_{\rm dd} C_{\rm d},$$

for some constant matrices $L_{\rm dd}$ and $E_{\rm dd}$ of appropriate dimensions, and

$$A_{\rm dd}^* = \text{blkdiag}\left\{ A_{q_1}, A_{q_2}, \ldots, A_{q_{m_{\rm d}}} \right\},$$

$$B_{\rm d} = \text{blkdiag}\left\{ B_{q_1}, B_{q_2}, \ldots, B_{q_{m_{\rm d}}} \right\}, \quad C_{\rm d} = \text{blkdiag}\left\{ C_{q_1}, C_{q_2}, \ldots, C_{q_{m_{\rm d}}} \right\}$$

$$A_{q_i} = \begin{bmatrix} 0 & I_{q_i - 1} \\ 0 & 0 \end{bmatrix}, \quad B_{q_i} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C_{q_i} = [1,\ 0,\ \ldots,\ 0].$$

Moreover, $\lambda(A_{\rm aa}^-) \subset \mathbb{C}^-$, $\lambda(A_{\rm aa}^0) \subset \mathbb{C}^0$

and $\lambda(A_{\rm aa}^+) \subset \mathbb{C}^+$. *Also,* $(A_{\rm cc}, B_{\rm c})$ *is controllable and* $(A_{\rm bb}, C_{\rm b})$ *is observable.*

For a strictly proper system, it has the following form…

$$\tilde{A} = \Gamma_s^{-1} A \Gamma_s = \begin{bmatrix} A_{aa}^- & 0 & 0 & L_{ab}^- C_b & 0 & L_{ad}^- C_d \\ 0 & A_{aa}^0 & 0 & L_{ab}^0 C_b & 0 & L_{ad}^0 C_d \\ 0 & 0 & A_{aa}^+ & L_{ab}^+ C_b & 0 & L_{ad}^+ C_d \\ 0 & 0 & 0 & A_{bb} & 0 & L_{bd} C_d \\ B_c E_{ca}^- & B_c E_{ca}^0 & B_c E_{ca}^+ & L_{cb} C_b & A_{cc} & L_{cd} C_d \\ B_d E_{da}^- & B_d E_{da}^0 & B_d E_{da}^+ & B_d E_{db} & B_d E_{dc} & A_{dd} \end{bmatrix}$$

$$\tilde{B} = \Gamma_s^{-1} B \Gamma_i = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & B_c \\ B_d & 0 \end{bmatrix}$$

$$\tilde{C} = \Gamma_o^{-1} C \Gamma_s = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & C_d \\ 0 & 0 & 0 & C_b & 0 & 0 \end{pmatrix}$$

What can we do with a state feedback gain?…

$$\tilde{A}+\tilde{B}\tilde{F} = \begin{bmatrix} A_{aa}^- & 0 & 0 & L_{ab}^- C_b & 0 & L_{ad}^- C_d \\ 0 & A_{aa}^0 & 0 & L_{ab}^0 C_b & 0 & L_{ad}^0 C_d \\ 0 & 0 & A_{aa}^+ & L_{ab}^+ C_b & 0 & L_{ad}^+ C_d \\ 0 & 0 & 0 & A_{bb} & 0 & L_{bd} C_d \\ 0 & 0 & 0 & 0 & A_{cc} & L_{cd} C_d \\ 0 & 0 & 0 & 0 & 0 & A_{dd} \end{bmatrix} \qquad \tilde{B} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & B_c \\ B_d & 0 \end{bmatrix}$$

$$\tilde{C} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & C_d \\ 0 & 0 & 0 & C_b & 0 & 0 \end{pmatrix}$$

$\mathcal{X}_a^0 \oplus \mathcal{X}_a^+ \oplus \mathcal{X}_b$ is a bad subspace for the state feedback control…

How about observer design?…

…duality…

$\mathcal{X}_a^0 \oplus \mathcal{X}_a^+ \oplus \mathcal{X}_c$ is a bad subspace for observer design… …D.I.Y…

Property 5.4.1. *The given system Σ is observable (detectable) if and only if the pair $(A_{\rm obs}, C_{\rm obs})$ is observable (detectable), where*

$$A_{\rm obs} := \begin{bmatrix} A_{\rm aa} & 0 \\ B_{\rm c}E_{\rm ca} & A_{\rm cc} \end{bmatrix}, \quad C_{\rm obs} := \begin{bmatrix} C_{\rm 0a} & C_{\rm 0c} \\ E_{\rm da} & E_{\rm dc} \end{bmatrix}, \tag{5.4.28}$$

and where

$$A_{\rm aa} := \begin{bmatrix} A_{\rm aa}^- & 0 & 0 \\ 0 & A_{\rm aa}^0 & 0 \\ 0 & 0 & A_{\rm aa}^+ \end{bmatrix}, \quad C_{\rm 0a} := [\, C_{\rm 0a}^- \quad C_{\rm 0a}^0 \quad C_{\rm 0a}^+ \,], \tag{5.4.29}$$

$$E_{\rm da} := [\, E_{\rm da}^- \quad E_{\rm da}^0 \quad E_{\rm da}^+ \,], \quad E_{\rm ca} := [\, E_{\rm ca}^- \quad E_{\rm ca}^0 \quad E_{\rm ca}^+ \,]. \tag{5.4.30}$$

Also, define

$$A_{\rm con} := \begin{bmatrix} A_{\rm aa} & L_{\rm ab}C_{\rm b} \\ 0 & A_{\rm bb} \end{bmatrix}, \quad B_{\rm con} := \begin{bmatrix} B_{\rm 0a} & L_{\rm ad} \\ B_{\rm 0b} & L_{\rm bd} \end{bmatrix}, \tag{5.4.31}$$

$$B_{\rm 0a} := \begin{bmatrix} B_{\rm 0a}^- \\ B_{\rm 0a}^0 \\ B_{\rm 0a}^+ \end{bmatrix}, \quad L_{\rm ab} := \begin{bmatrix} L_{\rm ab}^- \\ L_{\rm ab}^0 \\ L_{\rm ab}^+ \end{bmatrix}, \quad L_{\rm ad} := \begin{bmatrix} L_{\rm ad}^- \\ L_{\rm ad}^0 \\ L_{\rm ad}^+ \end{bmatrix}. \tag{5.4.32}$$

Similarly, Σ is controllable (stabilizable) if and only if the pair $(A_{\rm con}, B_{\rm con})$ is controllable (stabilizable).

Property 5.4.2. *The structural decomposition also shows explicitly the invariant zeros and the normal rank of* Σ. *To be more specific, we have the following properties:*

1. *The normal rank of* $H(s)$ *is equal to* $m_0 + m_{\rm d}$.
2. *Invariant zeros of* Σ *are the eigenvalues of* $A_{\rm aa}$, *which are the unions of the eigenvalues of* $A_{\rm aa}^-$, $A_{\rm aa}^0$ *and* $A_{\rm aa}^+$.

Obviously, Σ *is of minimum phase if and only if* $n_{\rm a}^0 + n_{\rm a}^+ = 0$. *Otherwise, it is of nonminimum phase.*

Property 5.4.4. *Σ has $m_0 = \text{rank}(D)$ infinite zeros of order 0. The infinite zero structure (of order greater than 0) of Σ is given by*

$$S^\star_\infty(\Sigma) = \Big\{ q_1, q_2, \ldots, q_{m_\text{d}} \Big\}. \tag{5.4.36}$$

That is, each q_i corresponds to an infinite zero of order q_i. In particular, for a strictly proper SISO system Σ, we have $S^\star_\infty(\Sigma) = \{q_1\}$, where q_1 is the relative degree of Σ. The given system Σ is said to be of uniform rank if either $m_0 = 0$ and $q_1 = q_2 = \cdots = q_{m_\text{d}}$, or $m_0 \neq 0$ and $S^\star_\infty(\Sigma) = \emptyset$.

Property 5.4.5. *The given system Σ is right invertible if and only if x_b (and hence y_b) are nonexistent, left invertible if and only if x_c (and hence u_c) are nonexistent, and invertible if and only if both x_b and x_c are nonexistent. Moreover, Σ is degenerate if and only if both x_b and x_c are present.*

The structural decomposition decomposes the state space of Σ into several distinct parts. In fact, the state space $\mathcal{X}$ is decomposed as

$$\mathcal{X} = \mathcal{X}_{\rm a}^{-} \oplus \mathcal{X}_{\rm a}^{0} \oplus \mathcal{X}_{\rm a}^{+} \oplus \mathcal{X}_{\rm b} \oplus \mathcal{X}_{\rm c} \oplus \mathcal{X}_{\rm d}. \qquad (5.4.37)$$

Here $\mathcal{X}_{\rm a}^{-}$ is related to the stable invariant zeros, *i.e.*, the eigenvalues of $A_{\rm aa}^{-}$ are the stable invariant zeros of Σ. Similarly, $\mathcal{X}_{\rm a}^{0}$ and $\mathcal{X}_{\rm a}^{+}$ are respectively related to the invariant zeros of Σ located in the marginally stable and unstable regions. On the other hand, $\mathcal{X}_{\rm b}$ is related to the right invertibility, *i.e.*, the system is right invertible if and only if $\mathcal{X}_{\rm b} = \{0\}$, while $\mathcal{X}_{\rm c}$ is related to left invertibility, *i.e.*, the system is left invertible if and only if $\mathcal{X}_{\rm c} = \{0\}$. Finally, $\mathcal{X}_{\rm d}$ is related to zeros of Σ at infinity.

There are interconnections between the subsystems generated by the structural decomposition and various invariant geometric subspaces. The following properties show these interconnections.

Property 5.4.6. *The geometric subspaces defined in Definitions 3.7.2 and 3.7.4 are given by:*

1. $\mathcal{X}_a^- \oplus \mathcal{X}_a^0 \oplus \mathcal{X}_c$ *spans* $\mathcal{V}^-(\Sigma)$.
2. $\mathcal{X}_a^+ \oplus \mathcal{X}_c$ *spans* $\mathcal{V}^+(\Sigma)$.
3. $\mathcal{X}_a^- \oplus \mathcal{X}_a^0 \oplus \mathcal{X}_a^+ \oplus \mathcal{X}_c$ *spans* $\mathcal{V}^*(\Sigma)$.
4. $\mathcal{X}_a^+ \oplus \mathcal{X}_c \oplus \mathcal{X}_d$ *spans* $\mathcal{S}^-(\Sigma)$.
5. $\mathcal{X}_a^- \oplus \mathcal{X}_a^0 \oplus \mathcal{X}_c \oplus \mathcal{X}_d$ *spans* $\mathcal{S}^+(\Sigma)$.
6. $\mathcal{X}_c \oplus \mathcal{X}_d$ *spans* $\mathcal{S}^*(\Sigma)$.
7. $\mathcal{X}_c$ *spans* $\mathcal{R}^*(\Sigma)$.
8. $\mathcal{X}_a^- \oplus \mathcal{X}_a^0 \oplus \mathcal{X}_a^+ \oplus \mathcal{X}_c \oplus \mathcal{X}_d$ *spans* $\mathcal{N}^*(\Sigma)$.

Property 5.4.7. *The geometric subspaces defined in Definition 3.7.5, i.e., $\mathcal{S}_\lambda(\Sigma)$ and $\mathcal{V}_\lambda(\Sigma)$, can be computed as follows:*

$$\mathcal{S}_\lambda(\Sigma) = \text{im}\left\{\Gamma_\text{s}\begin{bmatrix} \lambda I - A_\text{aa} & 0 & 0 & 0 \\ 0 & Y_{\text{b}\lambda} & 0 & 0 \\ 0 & 0 & I_{n_\text{c}} & 0 \\ 0 & 0 & 0 & I_{n_\text{d}} \end{bmatrix}\right\}, \tag{5.4.38}$$

where

$$\text{im}\,\{Y_{\text{b}\lambda}\} = \ker\left[C_\text{b}(A_\text{bb} + K_\text{b}C_\text{b} - \lambda I)^{-1}\right], \tag{5.4.39}$$

and where K_b is any matrix of appropriate dimensions and subject to the constraint that $A_\text{bb} + K_\text{b}C_\text{b}$ has no eigenvalue at λ. We note that such a K_b always exists as $(A_\text{bb}, C_\text{b})$ is observable.

$$\mathcal{V}_\lambda(\Sigma) = \text{im}\left\{\Gamma_\text{s}\begin{bmatrix} X_{\text{a}\lambda} & 0 \\ 0 & 0 \\ 0 & X_{\text{c}\lambda} \\ 0 & 0 \end{bmatrix}\right\}, \tag{5.4.40}$$

where $X_{\text{a}\lambda}$ is a matrix whose columns form a basis for the subspace,

$$\left\{\zeta_\text{a} \in \mathbb{C}^{n_\text{a}} \;\middle|\; (\lambda I - A_\text{aa})\zeta_\text{a} = 0\right\}, \tag{5.4.41}$$

and

$$X_{\text{c}\lambda} := \Big(A_\text{cc} + B_\text{c}F_\text{c} - \lambda I\Big)^{-1} B_\text{c}, \tag{5.4.42}$$

with F_c being any matrix of appropriate dimensions and subject to the constraint that $A_\text{cc} + B_\text{c}F_\text{c}$ has no eigenvalue at λ. Again, we note that the existence of such an F_c is guaranteed by the controllability of $(A_\text{cc}, B_\text{c})$.

Example 5.4.1. Let us reconsider the system Σ of Example 5.3.1, *i.e.*, consider a matrix quadruple (A, B, C, D) with (A, B, C) being the same as those given in Example 5.3.1 and $D = 0$. All the necessary transformations required to transform the given system into the special coordinate basis have already been obtained in Example 5.3.1.

$$\mathcal{V}^-(\Sigma) = \text{im}\left\{\begin{bmatrix} -2 & 0 & 0 & 0 \\ -3 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 4 & 1 & 0 & 1 \\ 2 & 1 & 1 & 1 \\ 6 & 1 & 1 & 2 \\ 2 & 0 & 1 & 1 \\ 4 & 0 & 0 & 1 \\ 2 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\right\}, \quad \mathcal{V}^+(\Sigma) = \text{im}\left\{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 2 & 1 & 1 & 1 \\ 3 & 1 & 1 & 2 \\ 2 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\right\},$$

$$\mathcal{V}^*(\Sigma) = \text{im}\left\{\begin{bmatrix} -2 & 1 & 0 & 0 & 0 \\ -3 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 4 & 1 & 1 & 0 & 1 \\ 2 & 2 & 1 & 1 & 1 \\ 6 & 3 & 1 & 1 & 2 \\ 2 & 2 & 0 & 1 & 1 \\ 4 & 1 & 0 & 0 & 1 \\ 2 & -1 & 0 & 0 & 0 \\ 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}\right\}, \quad \mathcal{R}^*(\Sigma) = \text{im}\left\{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right\},$$

$$\mathcal{S}^-(\Sigma) = \operatorname{im}\left\{\begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 & 2 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 1 & 0 & 4 & 1 & 0 \\ -2 & 1 & 1 & 1 & 0 & 6 & 2 & 0 \\ -3 & 1 & 1 & 2 & -1 & 11 & 4 & 1 \\ -2 & 0 & 1 & 1 & 0 & 7 & 3 & 1 \\ -1 & 0 & 0 & 1 & 0 & 6 & 2 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 2 & 1 \end{bmatrix}\right\}, \quad \mathcal{S}^*(\Sigma) = \operatorname{im}\left\{\begin{bmatrix} 0 & 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 3 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 4 & 1 & 0 \\ 1 & 1 & 1 & 0 & 6 & 2 & 0 \\ 1 & 1 & 2 & -1 & 11 & 4 & 1 \\ 0 & 1 & 1 & 0 & 7 & 3 & 1 \\ 0 & 0 & 1 & 0 & 6 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 2 & 1 \end{bmatrix}\right\},$$

$$\mathcal{S}^+(\Sigma) = \operatorname{im}\left\{\begin{bmatrix} -2 & 0 & 0 & 0 & 1 & 2 & 1 & 0 \\ -3 & 0 & 0 & 0 & 0 & 3 & 2 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & 1 & 0 & 1 & 0 & 4 & 1 & 0 \\ 2 & 1 & 1 & 1 & 0 & 6 & 2 & 0 \\ 6 & 1 & 1 & 2 & -1 & 11 & 4 & 1 \\ 2 & 0 & 1 & 1 & 0 & 7 & 3 & 1 \\ 4 & 0 & 0 & 1 & 0 & 6 & 2 & 1 \\ 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 2 & 1 \end{bmatrix}\right\}, \mathcal{N}^*(\Sigma) = \operatorname{im}\left\{\begin{bmatrix} -2 & -1 & 0 & 0 & 0 & 1 & 2 & 1 & 0 \\ -3 & 0 & 0 & 0 & 0 & 0 & 3 & 2 & 1 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & -1 & 1 & 0 & 1 & 0 & 4 & 1 & 0 \\ 2 & -2 & 1 & 1 & 1 & 0 & 6 & 2 & 0 \\ 6 & -3 & 1 & 1 & 2 & -1 & 11 & 4 & 1 \\ 2 & -2 & 0 & 1 & 1 & 0 & 7 & 3 & 1 \\ 4 & -1 & 0 & 0 & 1 & 0 & 6 & 2 & 1 \\ 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & 1 \end{bmatrix}\right\}$$

$$\mathcal{V}_\lambda(\Sigma) = \text{im}\left\{\begin{bmatrix} -2 & 0 & 0 \\ -3 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 4 & 1 & 1 \\ 2 & 1 & 0 \\ 6 & 1 & 2 \\ 2 & 0 & 1 \\ 4 & 0 & 2 \\ 2 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right\}, \quad \lambda = -2,$$

$$\mathcal{S}_\lambda(\Sigma) = \text{im}\left\{\begin{bmatrix} -2 & 4 & 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 \\ -3 & 0 & -5 & 0 & 0 & 0 & 0 & 3 & 2 & 1 \\ 1 & 2 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & 0 & -2 & 1 & 0 & 1 & 0 & 4 & 1 & 0 \\ 2 & 2 & -1 & 1 & 1 & 1 & 0 & 6 & 2 & 0 \\ 6 & -2 & -8 & 1 & 1 & 2 & -1 & 11 & 4 & 1 \\ 2 & 2 & -3 & 0 & 1 & 1 & 0 & 7 & 3 & 1 \\ 4 & 0 & -5 & 0 & 0 & 1 & 0 & 6 & 2 & 1 \\ 2 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 1 & 0 & 0 & 0 & 0 & 2 & 2 & 1 \end{bmatrix}\right\}, \quad \lambda = 1.$$

We note that $\lambda = -2$ and $\lambda = 1$ correspond respectively to the stable and the unstable invariant zeros of Σ.

Exercise 5.1. Compute a special coordinate basis for the SISO system

$$\dot{x} = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} u, \quad y = \begin{bmatrix} 1 & -1 & 1 & -1 \end{bmatrix} x.$$

Identify the invariant zeros and the relative degree of the given system.

Exercise 5.2. Utilize the properties of the special coordinate basis to construct a fourth order controllable and observable SISO system, Σ, for each of the following five cases:

(a) Σ has no invariant zeros and has a relative degree of 4.

(b) Σ has one invariant zero at $\{1\}$ and has a relative degree of 3.

(c) Σ has two invariant zeros at $\{1, 2\}$, and has a relative degree of 2.

(d) Σ has three invariant zeros at $\{1, 2, 3\}$, and has a relative degree of 1.

(e) Σ has four invariant zeros at $\{\pm j, \pm 1\}$, and has a relative degree of 0.

Exercise 5.5. Utilize the properties of the special coordinate basis to construct a fourth order invertible, controllable and observable MIMO system, Σ, for each of the following cases:

(a) Σ is strictly proper, and has an infinite zero structure $S_\infty^\star = \{1, 3\}$, which implies that Σ is free of invariant zeros.

(b) Σ is strictly proper, and has an infinite zero structure $S_\infty^\star = \{2, 2\}$, which implies that Σ is free of invariant zeros.

(c) Σ is strictly proper, and has one invariant zero at $\{1\}$ and an infinite zero structure $S_\infty^\star = \{1, 2\}$.

(d) Σ is strictly proper, and has two invariant zeros at $\{\pm j\}$ and an infinite zero structure $S_\infty^\star = \{1, 1\}$.

(e) Σ is nonstrictly proper, and has three invariant zeros at $\{1, \pm j\}$ and an infinite zero structure $S_\infty^\star = \{1\}$.

(f) Σ is nonstrictly proper, and has four invariant zeros at $\{\pm 1, \pm j\}$ and no infinite zero of order higher than 0.

Exercise 5.6. Construct a third order strictly proper and right invertible system, Σ, with two inputs and one output, for each of the following cases:

(a) Σ has an infinite zero of order 2, and has no invariant zeros.

(b) Σ has an infinite zero of order 1, and has one invariant zero at $\{-1\}$.

Moreover, the obtained systems must be controllable and unobservable.

Exercise 5.7. Construct a third order strictly proper and left invertible system, Σ, with one input and two outputs, for each of the following cases:

(a) Σ has an infinite zero of order 2, and has no invariant zeros.

(b) Σ has an infinite zero of order 1, and has one invariant zero at $\{-1\}$.

Furthermore, the obtained systems must be uncontrollable and observable.

Exercise 5.8. Construct a second order system, Σ, which has the following properties: (i) Σ is neither left nor right invertible; (ii) Σ is uncontrollable and unobservable; (ii) Σ is free of finite zeros and is free of infinite zeros of order higher than 0; and (iv) Σ is nonstrictly proper with both $[\,C \quad D\,]$ and $[\,B' \quad D'\,]$ being of full rank.

Exercise 5.11. Consider a SISO system, Σ, which is already in the SCB form as given in Theorem 5.2.1, *i.e.*,

$$\begin{aligned} \dot{x}_{\rm a} &= A_{\rm aa}x_{\rm a} + L_{\rm ad}y, \\ \dot{x}_1 &= x_2, \quad y = x_1, \\ \dot{x}_2 &= x_3, \quad \ldots, \quad \dot{x}_{n_{\rm d}-1} = x_{n_{\rm d}}, \\ \dot{x}_{n_{\rm d}} &= E_{\rm da}x_{\rm a} + E_1x_1 + E_2x_2 + \cdots + E_{n_{\rm d}}x_{n_{\rm d}} + u, \end{aligned}$$

or in the matrix form:

$$\dot{x} = Ax + Bu = \begin{bmatrix} A_{\rm aa} & L_{\rm ad} & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ E_{\rm da} & E_1 & E_2 & \cdots & E_{n_{\rm d}} \end{bmatrix} \begin{pmatrix} x_{\rm a} \\ x_1 \\ x_2 \\ \vdots \\ x_{n_{\rm d}} \end{pmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u,$$

and

$$y = Cx = [0 \quad 1 \quad 0 \quad \cdots \quad 0]\, x.$$

Let

$$\tilde{B} := B + \begin{bmatrix} K_{\rm a} \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} K_{\rm a} \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

Construct the special coordinate basis for the new system, $\tilde{\Sigma}$, characterized by $\dot{x} = Ax + \tilde{B}u$, and $y = Cx$. Show that Σ and $\tilde{\Sigma}$ have the same relative degree. Also, show that the invariant zeros of $\tilde{\Sigma}$ are given by the eigenvalues of $\tilde{A}_{\rm aa} := A_{\rm aa} - K_{\rm a}E_{\rm da}$.