Decompositions of Unforced and/or Unsensed Systems

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This lecture is to focus on the structural decomposition of the following systems…

1. An autonomous system characterized by a constant matrix A, *i.e.*,

$$\dot{x} = Ax, \quad x \in \mathbb{R}^n. \tag{4.2.1}$$

2. An unforced system characterized by a matrix pair (C, A), *i.e.*,

$$\dot{x} = Ax, \quad y = Cx, \tag{4.3.1}$$

3. An unsensed system characterized by a matrix pair (A, B), *i.e.*,

$$\dot{x} = Ax + Bu. \tag{4.4.1}$$

Note that the systems in (4.3.1) and (4.4.1) are dual to each other.

Autonomous Systems

In this section, we present two structural decompositions for such an autonomous system, *i.e.*, the stability structural decomposition (SSD) and the real Jordan decomposition (RJD).

Theorem 4.2.1 (SSD). *Consider the autonomous system* Σ *of (4.2.1) characterized by a constant matrix* A*. There exists a nonsingular transformation* $T \in \mathbb{R}^{n\times n}$ *and nonnegative integers* n_-, n_0 *and* n_+ *such that*

$$T^{-1}AT = \tilde{A} = \begin{bmatrix} A_- & 0 & 0 \\ 0 & A_0 & 0 \\ 0 & 0 & A_+ \end{bmatrix}, \tag{4.2.2}$$

where $A_- \in \mathbb{R}^{n_-\times n_-}$ *with* $\lambda(A_-) \subset \mathbb{C}^-$, $A_0 \in \mathbb{R}^{n_0\times n_0}$ *with* $\lambda(A_0) \subset \mathbb{C}^0$, *and* $A_+ \in \mathbb{R}^{n_+\times n_+}$ *with* $\lambda(A_+) \subset \mathbb{C}^+$. *The SSD totally decouples the stable and unstable dynamics as well as those dynamics associated with the imaginary axis eigenvalues.*

Example 4.2.1. Consider an autonomous system Σ of (4.2.1) characterized by

$$A = \begin{bmatrix} -1 & -1 & -3 & -1 & -1 \\ 0 & 2 & 4 & 4 & 4 \\ 0 & -2 & -2 & -3 & -3 \\ 1 & 1 & 1 & 0 & 0 \\ -1 & -1 & -1 & 1 & 1 \end{bmatrix}, \qquad (4.2.18)$$

which has eigenvalues at 0, -1, 1, $-2j$ and $2j$. Following the SSD algorithm of Theorem 4.2.1, which has been implemented with an m-function, `ssd.m`, in [87], we obtain

$$T_1 = \begin{bmatrix} 0.57735 & 0.47385 & 0.66493 & 0 & 0.57735 \\ 0 & -0.81277 & 0.07790 & 0 & 0 \\ 0 & 0.33892 & -0.74283 & 0 & -0.57735 \\ -0.57735 & 0 & 0 & 0.70711 & 0 \\ 0.57735 & 0 & 0 & -0.70711 & 0.57735 \end{bmatrix},$$

which gives the following stability structural decomposition of A,

$$T_1^{-1} A T_1 = \left[\begin{array}{c|ccc|c} -1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0.21932 & 3.44308 & 0 & 0 \\ 0 & -1.17572 & -0.21932 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right].$$

Theorem 4.2.2 (RJD). *Consider the autonomous system Σ of (4.2.1), characterized by $A \in \mathbb{R}^{n \times n}$. There exists a nonsingular transformation $T \in \mathbb{R}^{n \times n}$ and an integer k such that*

$$T^{-1}AT = J = \begin{bmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_k \end{bmatrix}, \tag{4.2.19}$$

where each block J_i, $i = 1, 2, \ldots, k$, has the following form:

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$$J_i = \begin{bmatrix} \lambda_i & 1 & & \\ & \ddots & \ddots & \\ & & \lambda_i & 1 \\ & & & \lambda_i \end{bmatrix}, \tag{4.2.20}$$

if $\lambda_i \in \lambda(A)$ is real, or

$$J_i = \begin{bmatrix} \Lambda_i & I_2 & & \\ & \ddots & \ddots & \\ & & \Lambda_i & I_2 \\ & & & \Lambda_i \end{bmatrix}, \quad \Lambda_i = \begin{bmatrix} \mu_i & \omega_i \\ -\omega_i & \mu_i \end{bmatrix},$$

if $\lambda_i = \mu_i + j\omega_i$, $\bar{\lambda}_i = \mu_i - j\omega_i \in \lambda(A)$ with $\omega_i > 0$.

Camille Jordan
1838–1922
French Mathematician

Example 4.2.2. Consider an autonomous system Σ of (4.2.1) characterized by

$$A = \begin{bmatrix} 3 & 2 & 2 & 2 & 2 & 2 \\ 3 & 1 & 2 & 3 & 2 & 2 \\ 4 & 0 & 2 & 2 & 3 & 2 \\ 4 & 0 & 2 & 1 & 4 & 2 \\ 4 & 0 & 3 & 0 & 4 & 2 \\ -20 & -4 & -12 & -9 & -16 & -11 \end{bmatrix}. \tag{4.2.39}$$

Using the m-function `rjd.m` of [87], we obtain a real Jordan canonical decomposition of A with

$$J = \left[\begin{array}{cc|cc|cc} 0 & 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{array}\right], \tag{4.2.40}$$

and the required state transformation,

$$T = \begin{bmatrix} 0 & 0 & -0.12251 & 0.04504 & -0.07311 & -0.53231 \\ -0.63824 & -0.50617 & 0.26812 & 0.09233 & 0.24639 & -0.13302 \\ -0.13207 & -1.14440 & 0.22084 & 0.48295 & -0.15290 & 0.18649 \\ -0.13207 & -1.14440 & -0.41740 & -0.02322 & -0.15290 & 0.18649 \\ -0.13207 & -1.14440 & 0.08877 & -0.66145 & -0.15290 & 0.18649 \\ 1.03444 & 3.93938 & 0.00090 & -0.01943 & 0.58813 & 0.33547 \end{bmatrix}.$$

Two canonical forms are presented in this section for the unforced system (4.3.1), namely the observability structural decomposition (OSD) and the block diagonal observable structural decomposition (BDOSD). These canonical forms require both state and output transformations. The following theorem characterizes the properties of the OSD.

Theorem 4.3.1 (OSD). *Consider the unforced system of (4.3.1) with C being of full rank. Then, there exist nonsingular state transformation $T_{\rm s} \in \mathbb{R}^{n\times n}$ and nonsingular output transformation $T_{\rm o} \in \mathbb{R}^{p\times p}$ such that, in the transformed state and output,*

$$x = T_{\rm s}\tilde{x}, \quad y = T_{\rm o}\tilde{y}, \tag{4.3.2}$$

where

$$\tilde{x} = \begin{pmatrix} \hat{x}_0 \\ \tilde{x}_1 \\ \vdots \\ \tilde{x}_p \end{pmatrix}, \quad \tilde{x}_i = \begin{pmatrix} \tilde{x}_{i,1} \\ \tilde{x}_{i,2} \\ \vdots \\ \tilde{x}_{i,k_i} \end{pmatrix}, \quad i = 1, 2, \ldots, p, \quad \tilde{y} = \begin{pmatrix} \tilde{y}_1 \\ \tilde{y}_2 \\ \vdots \\ \tilde{y}_p \end{pmatrix}, \tag{4.3.3}$$

we have

$$\dot{\hat{x}}_0 = A_0\hat{x}_0 + L_0\tilde{y}, \tag{4.3.4}$$

and for $i = 1, 2, \ldots, p$,

$$\dot{\tilde{x}}_i = A_i \tilde{x}_i + L_i \tilde{y}, \quad \tilde{y}_i = [\,1 \quad 0\,]\, \tilde{x}_i, \tag{4.3.5}$$

where L_i, $i = 1, 2, \ldots, p$, are some constant matrices of appropriate dimensions and

$$A_i = \begin{bmatrix} 0 & I_{k_i - 1} \\ 0 & 0 \end{bmatrix}. \tag{4.3.6}$$

The matrix A_0 is of dimensions $n_0 \times n_0$, where $n_0 := n - \sum_{i=1}^{p} k_i$, and $\lambda(A_0)$ contains all the unobservable modes of the matrix pair, (C, A). Moreover, the set $\mathcal{O} := \{\, k_1, k_2, \ldots, k_p \,\}$ is the observability index *of (C, A).*

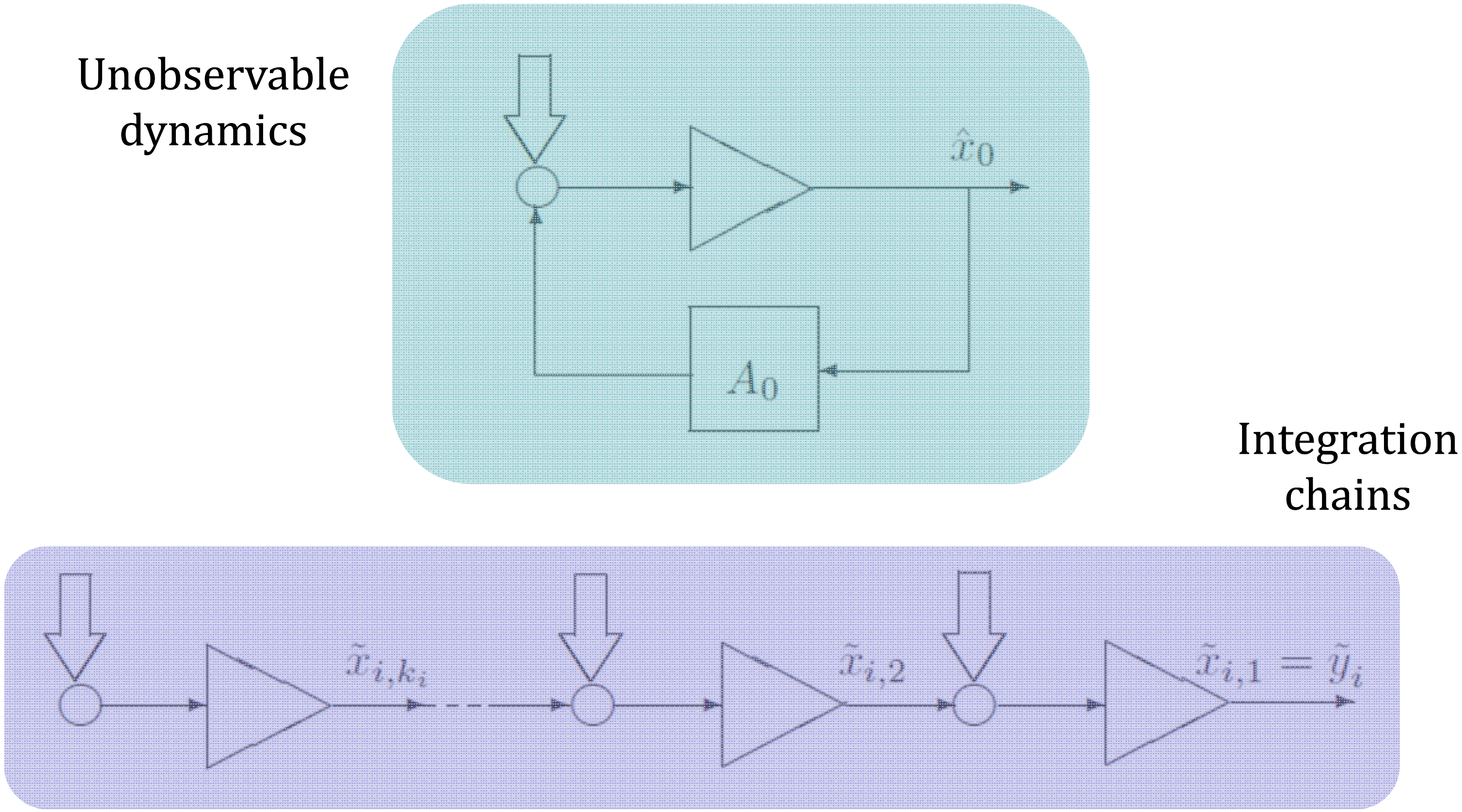
The result of Theorem 4.3.1 can be summarized in a more compact form as follows:

$$T_{\rm s}^{-1} A T_{\rm s} = \begin{bmatrix} A_0 & \star & 0 & \cdots & \star & 0 \\ 0 & \star & I_{k_1 - 1} & \cdots & \star & 0 \\ 0 & \star & 0 & \cdots & \star & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \star & 0 & \cdots & \star & I_{k_p - 1} \\ 0 & \star & 0 & \cdots & \star & 0 \end{bmatrix}, \tag{4.3.7}$$

and

$$T_{\rm o}^{-1} C T_{\rm s} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}, \tag{4.3.8}$$

where $\star$ represents a matrix of less interest.



Note: the signals indicated by double-edged arrows are some linear combinations of $\tilde{y}_i$.

Figure 4.3.1: Interpretation of the observability structural decomposition.

Example 4.3.1. Consider an unforced system (4.3.1) characterized by

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 0 \\ -2 & -1 & 4 & -2 & 3 & 0 \\ -2 & -1 & 3 & -1 & 3 & 0 \\ 1 & 1 & -2 & 3 & -2 & 0 \\ 2 & 1 & -2 & 2 & -3 & 0 \\ 1 & 1 & -1 & 1 & 1 & -1 \end{bmatrix}, \qquad (4.3.46)$$

and

$$C = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ -1 & 0 & 1 & -1 & 1 & 0 \end{bmatrix}. \qquad (4.3.47)$$

The complete required state and output transformations are then given by the following matrices:

$$T_s = (M_2 M_1 W S)^{-1} = \begin{bmatrix} 0 & 2 & 2 & -1 & -0.6667 & -0.5556 \\ 0 & 0 & -2 & 2 & 0.3333 & 0.4444 \\ 0 & -2 & -1 & 3 & 1 & 0.3333 \\ 0 & -7 & -3 & 3 & 2 & 1 \\ 0 & -2 & 0 & 0 & 0.3333 & 0.1111 \\ 1 & -2 & 0 & 0.3333 & 0.6667 & 0 \end{bmatrix},$$

and

$$T_o = W_o^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix},$$

and the resulting transformed system is characterized by

$$T_s^{-1} A T_s = \left[\begin{array}{c|cc|ccc} -1 & 2.3333 & 0 & 4.3333 & 0 & 0 \\ \hline 0 & -2 & 1 & -2 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ \hline 0 & -0 & 0 & 5 & 1 & 0 \\ 0 & -14 & 0 & -14 & 0 & 1 \\ 0 & 6 & 0 & 6 & 0 & 0 \end{array}\right],$$

and

$$T_o^{-1} C T_s = \left[\begin{array}{c|cc|ccc} 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right]$$

after cleaning an appropriate output injection

$$\left[\begin{array}{c|cc|ccc} -1 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The essential structural properties left. All the rest are rubbish!

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Block diagonal observable structural decomposition (BDOSD)…

Theorem 4.3.2 (BDOSD). *Consider the unforced system of (4.3.1) with (C, A) being observable. Then, there exist an integer $k \le p$, a set of k integers κ_1, κ_2, …, κ_k, and nonsingular transformations T_s and T_o such that*

$$T_s^{-1} A T_s = \begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_k \end{bmatrix}, \tag{4.3.48}$$

and

$$T_o^{-1} C T_s = \begin{bmatrix} C_1 & 0 & \cdots & 0 \\ \star & C_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \star & \star & \cdots & C_k \\ \star & \star & \cdots & \star \end{bmatrix}, \tag{4.3.49}$$

where the symbols $\star$ represent some matrices of less interest, and A_i and C_i, $i = 1, 2, \ldots, k$, are in the OSD form

$$A_i = \begin{bmatrix} \star & I_{\kappa_i - 1} \\ \star & 0 \end{bmatrix}, \quad C_i = [\,1 \quad 0 \quad \cdots \quad 0\,]. \tag{4.3.50}$$

Obviously, $\sum_{i=1}^{k} \kappa_i = n$.

Identifying minimal number of output variables to completely observe the system…

Example 4.3.2. Consider an unforced system (4.3.1) characterized by

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & -1 \\ -2 & -1 & 4 & -2 & 3 \\ -2 & -1 & 3 & -1 & 3 \\ 1 & 1 & -2 & 3 & -2 \\ 2 & 1 & -2 & 2 & -3 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 1 & -1 & 1 \end{bmatrix}. \qquad (4.3.51)$$

Using `bdosd.m` of [87], we obtain

$$T_s = \begin{bmatrix} 9.202258 & 9.202258 & 9.202258 & 11.985440 & 20.334987 \\ -18.404516 & 0 & 0 & -15.621333 & -44.080817 \\ -27.606773 & -9.202258 & 0 & 0 & -6.419075 \\ -27.606773 & -18.404516 & -9.202258 & 0 & 9.202258 \\ 0 & 0 & 0 & 2.783182 & 8.349547 \end{bmatrix},$$

$$T_o = \begin{bmatrix} -2.783182 & -0.302446 \\ 0 & 1 \end{bmatrix},$$

$$T_s^{-1} A T_s = \begin{bmatrix} 3 & 1 & 0 & 0 & 0 \\ -3 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

and

$$T_o^{-1} C T_s = \left[\begin{array}{ccccc} 1 & 0 & 0 & 0 & 0 \\ \hline & & & & \end{array} \right]$$

The 2nd output is redundant in observing the system state…

Unsensed Systems

Theorem 4.4.1 (CSD). *Consider the unsensed system of (4.4.1) with B being of full rank. Then, there exist nonsingular state and input transformations $T_s \in \mathbb{R}^{n\times n}$ and $T_i \in \mathbb{R}^{m\times m}$ such that, in the transformed input and state,*

$$x = T_s\tilde{x}, \quad u = T_i\tilde{u}, \tag{4.4.2}$$

where

$$\tilde{x} = \begin{pmatrix} \hat{x}_0 \\ \tilde{x}_1 \\ \vdots \\ \tilde{x}_m \end{pmatrix}, \quad \tilde{x}_i = \begin{pmatrix} \tilde{x}_{i,1} \\ \tilde{x}_{i,2} \\ \vdots \\ \tilde{x}_{i,k_i} \end{pmatrix}, \quad i = 1, 2, \dots, m, \quad \tilde{u} = \begin{pmatrix} \tilde{u}_1 \\ \tilde{u}_2 \\ \vdots \\ \tilde{u}_m \end{pmatrix}, \tag{4.4.3}$$

we have

$$\dot{\hat{x}}_0 = A_0\hat{x}_0, \tag{4.4.4}$$

and for $i = 1, 2, \dots, m$,

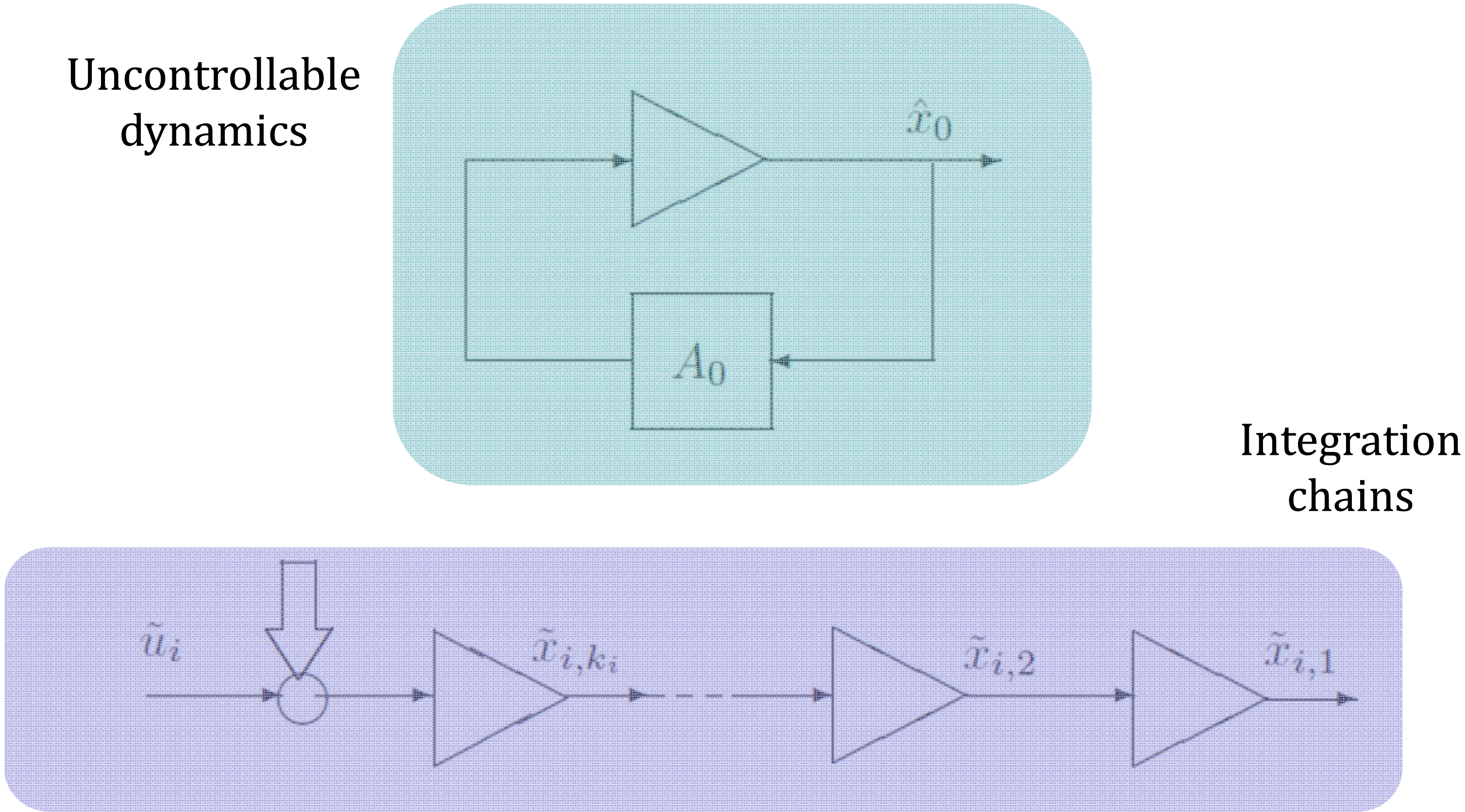
$$\dot{\tilde{x}}_i = A_i\tilde{x}_i + B_i\Big(\tilde{u}_i + E_i\tilde{x}\Big), \tag{4.4.5}$$

where E_i, $i = 1, 2, \dots, m$, are some row vectors of appropriate dimensions, and

$$A_i = \begin{bmatrix} 0 & I_{k_i-1} \\ 0 & 0 \end{bmatrix}, \quad B_i = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{4.4.6}$$

The matrix A_0 is of dimensions $n_0 \times n_0$, where $n_0 = n - \sum_{i=1}^{m} k_i$, and $\lambda(A_0)$ contains all the uncontrollable modes of the matrix pair, (A, B). Moreover, the integer set, $\mathcal{C} := \{k_1, k_2, \dots, k_m\}$, is called the controllability index of (A, B).

Controllability structural decomposition



Note: signals indicated by double-edged arrows are linear combinations of the states.

Figure 4.4.1: Interpretation of the controllability structural decomposition.

Theorem 4.4.1 follows dually from the result of Theorem 4.3.1. The CSD, *i.e.*, the controllability structural decomposition, can be summarized in a matrix form, $(\tilde{A}, \tilde{B}) := (T_s^{-1} A T_s, T_s^{-1} B T_i)$, with

$$\tilde{A} = \begin{bmatrix} A_0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & I_{k_1-1} & \cdots & 0 & 0 \\ \star & \star & \star & \cdots & \star & \star \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & I_{k_m-1} \\ \star & \star & \star & \cdots & \star & \star \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} 0 & \cdots & 0 \\ 0 & \cdots & 0 \\ 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \\ 0 & \cdots & 1 \end{bmatrix}, \qquad (4.4.7)$$

$$\left(\begin{bmatrix} A_0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & I_{k_1-1} & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & I_{k_m-1} \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & \cdots & 0 \\ 0 & \cdots & 0 \\ 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \\ 0 & \cdots & 1 \end{bmatrix} \right)$$

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The essential structural properties left for the unsensed system!

The next theorem deals with the block diagonal controllable structural decomposition (BDCSD).

Theorem 4.4.2 (BDCSD). *Consider the unsensed system of (4.4.1) with (A, B) being controllable. Then, there exist an integer $k \le m$, a set of k integers κ_1, κ_2, ..., κ_k, and nonsingular transformations T_s and T_i such that the transformed system, $(\tilde{A}, \tilde{B}) := (T_s^{-1} A T_s, T_s^{-1} B T_i)$, has the following form:*

$$\tilde{A} = \begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_k \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} B_1 & \star & \cdots & \star & \star \\ 0 & B_2 & \cdots & \star & \star \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & B_k & \star \end{bmatrix}, \tag{4.4.9}$$

where A_i and B_i, $i = 1, 2, \ldots, k$, are in the CSD form

$$A_i = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ \star & \star & \cdots & \star \end{bmatrix}, \quad B_i = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \tag{4.4.10}$$

and $\star$ represents a matrix of less interest. Obviously, $\sum_{i=1}^{k} \kappa_i = n$.

Identifying minimal number of input variables to completely control the system…

Example 4.4.2. Consider the unsensed system (4.4.1) characterized by matrices A and B with

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 6 \\ 2 & 5 \\ 3 & 4 \\ 4 & 3 \\ 5 & 2 \\ 6 & 1 \end{bmatrix}.$$

Using the MATLAB function `bdcsd.m` of [87], we obtain the following necessary transformations and transformed system:

$$T_s = \begin{bmatrix} -2.10371 & 0 & -4.20741 & 0 & -2.10371 & 0.78529 \\ -2.31866 & 0 & -4.63731 & 0 & -2.31866 & -0.71249 \\ -0.21495 & 9.10545 & -3.17845 & -3.17845 & -2.53360 & 0 \\ -5.71205 & 2.53360 & 3.82330 & 2.10371 & -2.74855 & 0 \\ 3.17845 & -0.21495 & 0.21495 & -0.21495 & -2.96350 & 0 \\ -2.96350 & 6.14195 & -6.14195 & 6.14195 & -3.17845 & 0 \end{bmatrix},$$

$$T_i = \begin{bmatrix} -0.48477 & 0 \\ -0.26982 & 0.97828 \end{bmatrix},$$

and

$$T_s^{-1} A T_s = \left[\begin{array}{ccccc|c} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & -1 & 2 & -2 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right], \quad T_s^{-1} B T_i = \left[\begin{array}{c|c} 0 & -0.56147 \\ 0 & -0.29865 \\ 0 & -0.32323 \\ 0 & -0.76184 \\ 1 & -1.20895 \\ \hline 0 & 1 \end{array}\right].$$

Exercise 4.10. Given an unforced system

$$\dot{x} = \begin{bmatrix} \lambda & 1 & & \\ & \ddots & \ddots & \\ & & \lambda & 1 \\ & & & \lambda \end{bmatrix} x, \quad y = [\,\alpha \quad \star \quad \cdots \quad \star\,]\, x,$$

where $\lambda \in \mathbb{R}$ and $\alpha \in \mathbb{R}$, show that the system is observable if and only if $\alpha \neq 0$.

Exercise 4.11. Given an unsensed system

$$\dot{x} = \begin{bmatrix} \Lambda & I & & \\ & \ddots & \ddots & \\ & & \Lambda & I \\ & & & \Lambda \end{bmatrix} x + \begin{bmatrix} \star \\ \vdots \\ \star \\ \beta \end{bmatrix} u,$$

where

$$\Lambda = \begin{bmatrix} \mu & \omega \\ -\omega & \mu \end{bmatrix} \in \mathbb{R}^{2\times 2}, \quad \omega \neq 0, \quad \beta = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} \in \mathbb{R}^2,$$

show that the system is controllable if and only if $\beta \neq 0$.

Exercise 4.12. Given a controllable pair (A, B) with $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$, show that if A has an eigenvalue with a geometric multiplicity of τ, *i.e.*, it has a total number of τ Jordan blocks associated with it, then $m \geq \tau$.