Review of Basic Linear Systems Theory

Dynamical Responses

Given a linear time-invariant system

$$\Sigma \ : \ \begin{cases} \dot{x}(t) = A\ x(t) + B\ u(t), & x(t) \in \mathbb{R}^n\ , u(t) \in \mathbb{R}^m \\ y(t) = C\ x(t) + D\ u(t), & y(t) \in \mathbb{R}^p \end{cases} \tag{3.1.1}$$

The solution of the state variable or the state response, $x(t)$, of Σ with an initial condition $x_0 = x(0)$ can be uniquely expressed as

$$x(t) = \mathrm{e}^{At} x_0 + \int_0^t \mathrm{e}^{A(t-\tau)} B u(\tau) d\tau, \quad t \geq 0, \tag{3.2.1}$$

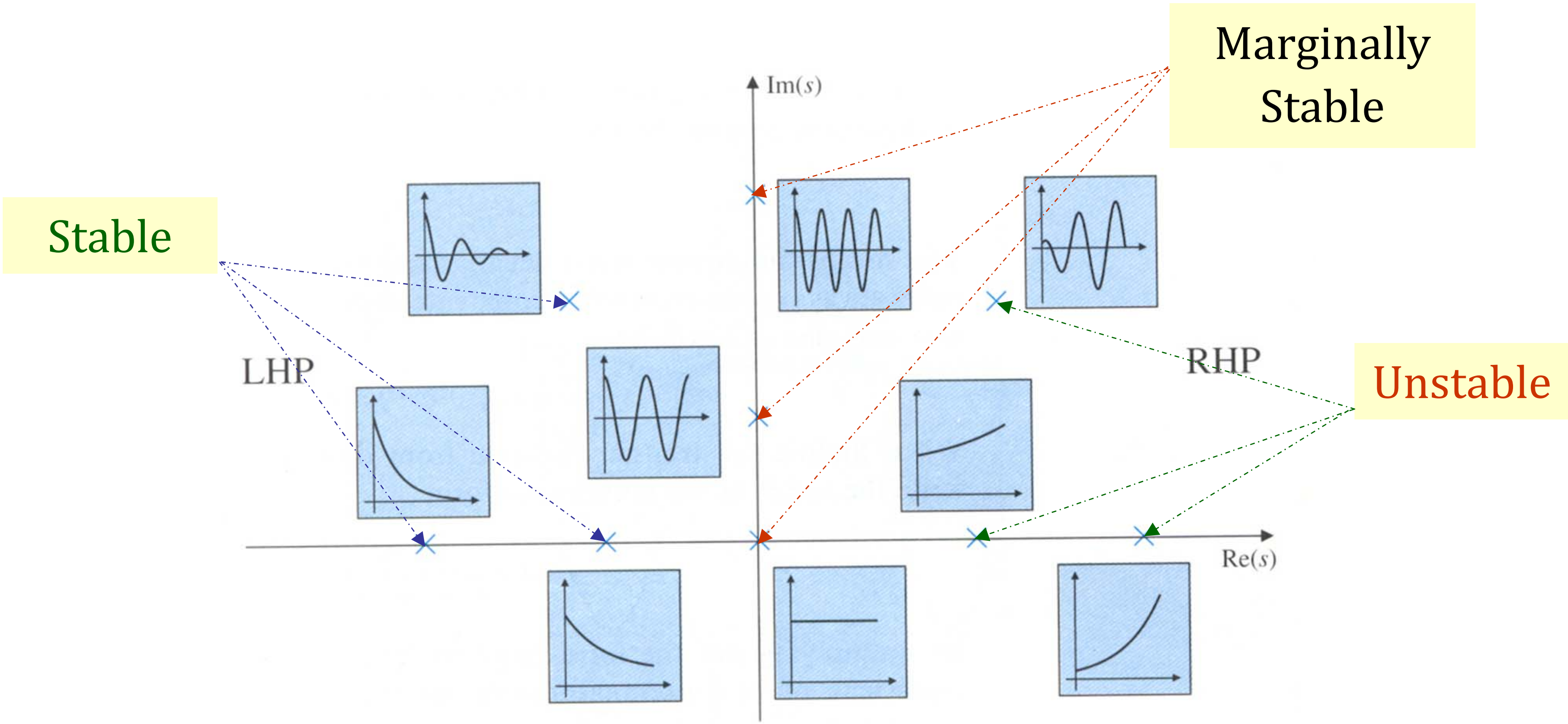
where the first term is the response due to the initial state, x_0, and the second term is the response excited by the external control force, $u(t)$.

Lastly, it is simple to see that the corresponding output response of the system (3.1.1) is given as:

$$y(t) = C\mathrm{e}^{At} x_0 + \int_0^t C\mathrm{e}^{A(t-\tau)} B u(\tau) d\tau + D u(t), \quad t \geq 0. \tag{3.2.13}$$

System Stability

A linear time-invariant system is said to be asymptotically stable if all its closed-loop poles are located on the left-half complex plane (LHP), unstable if at least of its poles are on the right-half plane (RHP)…



Controllability and Observability

Theorem 3.4.2. *The given system Σ of (3.1.1) is controllable if and only if*

$$\text{rank}\,(Q_c) = n, \qquad (3.4.11)$$

where

$$Q_c := [\, B \quad AB \quad \cdots \quad A^{n-1}B \,] \qquad (3.4.12)$$

is called the controllability matrix *of Σ.*

Theorem 3.4.3. *The given system Σ of (3.1.1) is controllable if and only if, for every eigenvalue of A, λ_i, $i = 1, 2, \ldots, n$,*

$$\text{rank}\,[\, \lambda_i I - A \quad B \,] = n. \qquad (3.4.21)$$

Definition 3.4.2. *The given system Σ of (3.1.1) is said to be* stabilizable *if all its uncontrollable modes are asymptotically stable. Otherwise, Σ is said to be* unstabilizable.

Theorem 3.4.6. *The given system Σ of (3.1.1) is observable if and only if either one of the following statements is true:*

1. *The* observability matrix *of* Σ,

$$Q_o := \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \tag{3.4.27}$$

is of full rank, i.e., $\text{rank}\,(Q_o) = n$.

2. *For every eigenvalue of* A, λ_i, $i = 1, 2, \ldots, n$,

$$\text{rank} \begin{bmatrix} \lambda_i I - A \\ C \end{bmatrix} = n. \tag{3.4.28}$$

Definition 3.4.4. *The given system Σ of (3.1.1) is said to be* detectable *if all its unobservable modes are asymptotically stable. Otherwise, Σ is said to be* undetectable.

System Invertibility

Recall the given system (3.1.1), which has a transfer function

$$H(s) = C(sI - A)^{-1}B + D. \tag{3.5.1}$$

Definition 3.5.1. *Consider the linear time-invariant system Σ of (3.1.1). Then,*

1. *Σ is said to be* left invertible *if there exists a rational matrix function of s, say $L(s)$, such that*
$$L(s)H(s) = I_m. \tag{3.5.2}$$

2. *Σ is said to be* right invertible *if there exists a rational matrix function of s, say $R(s)$, such that*
$$H(s)R(s) = I_p. \tag{3.5.3}$$

3. *Σ is said to be* invertible *if it is both left and right invertible.*

4. *Σ is said to be* degenerate *if it is neither left nor right invertible.*

A square system is not necessarily invertible…

Example 3.5.1. Consider a system Σ of (3.1.1) characterized by

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}, \tag{3.5.4}$$

and

$$C = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \tag{3.5.5}$$

Note that both matrices B and C are of full rank. It is controllable and observable, and has a transfer function:

$$H(s) = \frac{1}{s^3 - 3s^2 + s} \begin{bmatrix} (s-1)^2 & s-1 \\ s-1 & 1 \end{bmatrix}. \tag{3.5.6}$$

Clearly, although square, it is a degenerate system as the determinant of $H(s)$ is identical to zero.

Normal Rank and Invariant Zeros

Definition 3.6.1. *Consider the given system* Σ *of (3.1.1). The* normal rank *of its transfer function* $H(s) = C(sI - A)^{-1}B + D$, *or in short,* $\text{normrank}\{H(s)\}$, *is defined as*

$$\text{normrank}\,\{H(s)\} = \max\left\{\text{rank}\,[H(\lambda)]\,\middle|\,\lambda \in \mathbb{C}\right\}. \qquad (3.6.2)$$

Definition 3.6.2. *Consider the given system* Σ *of (3.1.1). A scalar* $\beta \in \mathbb{C}$ *is said to be an* invariant zero *of* Σ *if*

$$\text{rank}\,\{P_\Sigma(\beta)\} < n + \text{normrank}\,\{H(s)\}. \qquad (3.6.4)$$

Here

$$P_\Sigma(s) := \begin{bmatrix} sI - A & -B \\ C & D \end{bmatrix}$$

which is known as the so-called Rosenbrock system matrix.

Howard H. Rosenbrock
1920–2010

System Invariant Structural Indices (Infinite Zeros, etc…)

In what follows, however, we will introduce the well-known *Kronecker canonical form* for the system matrix $P_\Sigma(s)$, which is able to display the invariant zero structure, invertibility structures and infinite zero structure of Σ altogether. Although it is not a simple task (it is actually a pretty difficult task for systems with a high dynamical order), it can be shown (see Gantmacher [56]) that there exist nonsingular transformations U and V such that $P_\Sigma(s)$ can be transformed into the following form:

$$U P_\Sigma(s) V = \begin{bmatrix} \text{blkdiag}\{sI - J, L_{l_1}, \ldots, L_{l_{p_b}}, R_{r_1}, \ldots, R_{r_{m_c}}, I - sH, I_{m_0}\} & 0 \\ 0 & 0 \end{bmatrix}, \tag{3.6.11}$$

where 0 is a zero matrix corresponding to the redundant system inputs and outputs, if any; J is in Jordan canonical form, and $sI - J$ has the following $\sum_{i=1}^{\delta} \tau_i$ pencils as its diagonal blocks,

$$sI_{n_{\beta_i,j}} - J_{n_{\beta_i,j}}(\beta_i) := \begin{bmatrix} s - \beta_i & -1 & & \\ & \ddots & \ddots & \\ & & s - \beta_i & -1 \\ & & & s - \beta_i \end{bmatrix}, \tag{3.6.12}$$

$j = 1, 2, \ldots, \tau_i$ and $i = 1, 2, \ldots, \delta$; and L_{l_i}, $i = 1, 2, \ldots, p_\mathrm{b}$, is an $(l_i + 1) \times l_i$ bidiagonal pencil given by

$$L_{l_i} := \begin{bmatrix} -1 & & \\ s & \ddots & \\ & \ddots & -1 \\ & & s \end{bmatrix}, \tag{3.6.13}$$

R_{r_i}, $i = 1, 2, \ldots, m_\mathrm{c}$, is an $r_i \times (r_i + 1)$ bidiagonal pencil given by

$$R_{r_i} := \begin{bmatrix} s & -1 & & \\ & \ddots & \ddots & \\ & & s & -1 \end{bmatrix}, \tag{3.6.14}$$

H is nilpotent and in Jordan form, and $I - sH$ has the following m_d pencils as its diagonal blocks,

$$I_{q_i+1} - sJ_{q_i+1}(0) := \begin{bmatrix} 1 & -s & & \\ & \ddots & \ddots & \\ & & 1 & -s \\ & & & 1 \end{bmatrix}, \quad q_i > 0, \; i = 1, 2, \ldots, m_\mathrm{d}, \tag{3.6.15}$$

and finally m_0 in I_{m_0} is the rank of D, *i.e.*, $m_0 = \mathrm{rank}\,(D)$.

Definition 3.6.3. *Consider the given system* Σ *of (3.1.1) whose system matrix* $P_\Sigma(s)$ *has a Kronecker form as in (3.6.11) to (3.6.15). Then,*

1. β_i *is said to be an* invariant zero *of* Σ *with a* geometric multiplicity *of* τ_i and an algebraic multiplicity of $\sum_{j=1}^{\tau_i} n_{\beta_i,j}$. *It has a zero structure*

$$S^\star_{\beta_i}(\Sigma) := \{n_{\beta_i,1}, n_{\beta_i,2}, \ldots, n_{\beta_i,\tau_i}\}. \tag{3.6.16}$$

   β_i *is said to be a* simple invariant zero *if* $n_{\beta_i,1} = \cdots = n_{\beta_i,\tau_i} = 1$.

2. *The* left invertibility structure *of* Σ *is defined as*

$$S^\star_{\mathrm{L}}(\Sigma) := \{l_1, l_2, \ldots, l_{p_\mathrm{b}}\}. \tag{3.6.17}$$

3. *The* right invertibility structure *of* Σ *is defined as*

$$S^\star_{\mathrm{R}}(\Sigma) := \{r_1, r_2, \ldots, r_{m_\mathrm{c}}\}. \tag{3.6.18}$$

4. *Finally,* m_0 *is the number of the infinite zeros of* Σ *of order* 0. *The* infinite zero structure *of* Σ *of order higher than* 0 *is defined as:*

$$S^\star_\infty(\Sigma) := \{q_1, q_2, \ldots, q_{m_\mathrm{d}}\}. \tag{3.6.19}$$

   We say that Σ *has* m_d *infinite zeros of order* $q_1, q_2, \ldots, q_{m_\mathrm{d}}$, *respectively. If* $q_1 = \cdots = q_{m_d}$ *and* $m_0 = 0$, *then* Σ *is said to be of* uniform rank q_1. *On the other hand, if* $m_0 > 0$ *and* $S^\star_\infty(\Sigma) = \emptyset$, *then* Σ *is said to be of uniform rank* 0.

Everything about a linear system is characterized by these indices. Control performance is fully determined by these structural properties.

Example 3.6.1. Consider a system Σ of (3.1.1) characterized by

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \tag{3.6.20}$$

$$C = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{3.6.21}$$

It can be shown (using the technique to be given later in Section 5.6 of Chapter 5) that with the following transformations

$$U = \cdots \qquad V = \cdots$$

the Kronecker canonical form of Σ is given as follows:

$$UP_\Sigma(s)V = \left[\begin{array}{cc|cc|cc|cc|ccc}
s-1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & s-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & s & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & s & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & s & -1 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 1 & -s & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -s & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -s \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right].$$

Thus, we have $S_1^\star(\Sigma) = \{2\}$, $S_L^\star(\Sigma) = \{2\}$, $S_R^\star(\Sigma) = \{1\}$, $S_\infty^\star(\Sigma) = \{1, 2\}$, *i.e.*, Σ has a nonsimple invariant zero at $s = 1$, and two infinite zeros of order 1 and 2, respectively. Σ is degenerate as both $S_L^\star(\Sigma)$ and $S_R^\star(\Sigma)$ are nonempty.

Leopold Kronecker
1823–1891
German Mathematician

Felix Gantmacher
1908–1964
Soviet Mathematician

Geometric Subspaces

The weakly unobservable subspace

Definition 3.7.1. *Consider the continuous-time system Σ of (3.1.1). An initial state of Σ, $x_0 \in \mathbb{R}^n$, is called* weakly unobservable *if there exists an input signal $u(t)$ such that the corresponding system output $y(t) = 0$ for all $t \geq 0$. The subspace formed by the set of all weakly unobservable points of Σ is called the* weakly unobservable subspace *of Σ and is denoted by $\mathcal{V}^*(\Sigma)$.*

The following lemma shows that any state trajectory of Σ starting from an initial condition in $\mathcal{V}^*(\Sigma)$ with a control input that produces an output $y(t) = 0$, $t \geq 0$, will always stay inside the weakly unobservable subspace, $\mathcal{V}^*(\Sigma)$.

Lemma 3.7.1. *Let x_0 be an initial state of Σ with $x_0 \in \mathcal{V}^*(\Sigma)$ and u be an input such that the corresponding system output $y(t) = 0$ for all $t \geq 0$. Then the resulting state trajectory $x(t) \in \mathcal{V}^*(\Sigma)$ for all $t \geq 0$.*

Theorem 3.7.1. *The weakly unobservable subspace of Σ, $\mathcal{V}^*(\Sigma)$, is equivalent to the largest subspace $\mathcal{V}$ that satisfies either one of the following conditions:*

1. $\begin{bmatrix} A \\ C \end{bmatrix} \mathcal{V} \subset (\mathcal{V} \times 0) + \operatorname{im} \left\{ \begin{bmatrix} B \\ D \end{bmatrix} \right\}$.

2. *There exists an $F \in \mathbb{R}^{m \times n}$ such that $(A+BF)\mathcal{V} \subset \mathcal{V}$ and $(C+DF)\mathcal{V} = 0$.*

Using the result of Theorem 3.7.1, we can further define the stable and the unstable weakly unobservable subspaces of Σ.

Definition 3.7.2. *Consider a system Σ characterized by a quadruple (A, B, C, D). Then we define $\mathcal{V}^{\mathrm{x}}(\Sigma)$ to be the largest subspace $\mathcal{V}$ that satisfies*

$$(A + BF)\mathcal{V} \subset \mathcal{V}, \quad (C + DF)\mathcal{V} = 0, \tag{3.7.8}$$

and the eigenvalues of $(A+BF)|\mathcal{V}$ are contained in $\mathbb{C}^{\mathrm{x}} \subset \mathbb{C}$ for some $F \in \mathbb{R}^{n \times m}$. Obviously, $\mathcal{V}^{\mathrm{x}} = \mathcal{V}^*$ if $\mathbb{C}^{\mathrm{x}} = \mathbb{C}$. We further define $\mathcal{V}^- := \mathcal{V}^{\mathrm{x}}$ if $\mathbb{C}^{\mathrm{x}} = \mathbb{C}^- \cup \mathbb{C}^0$, and $\mathcal{V}^+ := \mathcal{V}^{\mathrm{x}}$ if $\mathbb{C}^{\mathrm{x}} = \mathbb{C}^+$.

The strongly controllable subspace

Next introduce the strongly controllable subspace of Σ, $\mathcal{S}(\Sigma)$. $\mathcal{S}$ and $\mathcal{V}$ are dual in the sense that $\mathcal{V}^{\mathrm{x}}(\Sigma^\star) = \mathcal{S}^{\mathrm{x}}(\Sigma)^\perp$, where $\Sigma^\star$ is characterized by the quadruple (A', C', B', D'). The physical interpretation of $\mathcal{S}$ is rather abstract and can be found in Trentelman *et al.* [141].

Definition 3.7.4. *Consider a system Σ characterized by a quadruple (A, B, C, D). Then we define the strongly controllable subspace of Σ, $\mathcal{S}^{\mathrm{x}}(\Sigma)$, to be the smallest subspace $\mathcal{S}$ that satisfies*

$$(A + KC)\mathcal{S} \subset \mathcal{S}, \quad \mathrm{im}\,(B + KD) \subset \mathcal{S}, \tag{3.7.9}$$

and the eigenvalues of the map that is induced by $A + KC$ on the factor space $\mathbb{R}/\mathcal{S}$ are contained in $\mathbb{C}^{\mathrm{x}} \subset \mathbb{C}$ for some $K \in \mathbb{R}^{p \times n}$. We let $\mathcal{S}^* := \mathcal{S}^{\mathrm{x}}$ if $\mathbb{C}^{\mathrm{x}} = \mathbb{C}$, $\mathcal{S}^- := \mathcal{S}^{\mathrm{x}}$ if $\mathbb{C}^{\mathrm{x}} = \mathbb{C}^- \cup \mathbb{C}^0$, and $\mathcal{S}^+ := \mathcal{S}^{\mathrm{x}}$ if $\mathbb{C}^{\mathrm{x}} = \mathbb{C}^+$.

Example 3.7.1. Let us re-consider the system Σ with (A, B, C, D) being given in Example 3.6.1. It can be verified that the various geometric subspaces of Σ are given as:

The largest weakly unobservable subspace, which can be made unstable

$$\mathcal{V}^*(\Sigma) = \mathcal{V}^+(\Sigma) = \operatorname{im}\left\{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right\},\quad \mathcal{V}^-(\Sigma) = \operatorname{im}\left\{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right\},$$

The largest weakly unobservable subspace, which can be made stable

The largest strongly controllable subspace, which can be made unstable

$$\mathcal{S}^*(\Sigma) = \mathcal{S}^+(\Sigma) = \operatorname{im}\left\{\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right\},$$

$$\mathcal{S}^-(\Sigma) = \operatorname{im}\left\{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}\right\},$$

The largest strongly controllable subspace, which can be made stable

The controllable weakly unobservable subspace and the distributionally weakly unobservable subspace

Intuitively, it is pretty clear from the definitions that the controllable weakly unobservable subspace is a subspace of the weakly unobservable subspace that is inside the strongly controllable subspace, *i.e.*,

$$\mathcal{R}^*(\Sigma) = \mathcal{V}^*(\Sigma) \cap \mathcal{S}^*(\Sigma). \tag{3.7.10}$$

This indeed turns out to be the case (see, *e.g.*, Trentelman *et al.* [141] for the detailed proof). Another popular subspace (paired with $\mathcal{R}^*$) is called the *distributionally weakly unobservable subspace* (denoted by $\mathcal{N}^*$) and is equivalent to the sum of the weakly unobservable subspace and the strongly controllable subspace, *i.e.*,

$$\mathcal{N}^*(\Sigma) = \mathcal{V}^*(\Sigma) + \mathcal{S}^*(\Sigma). \tag{3.7.11}$$

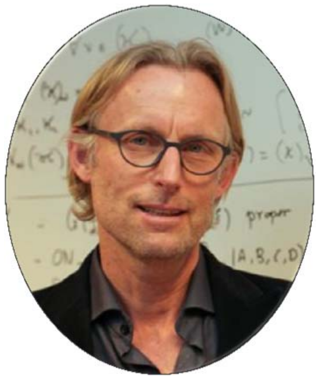
Finally, we define two more geometric subspaces of Σ, which were originally introduced by Scherer [124,125] for tackling H_∞ almost disturbance decoupling problems.

Definition 3.7.5. *For the given system Σ of (3.1.1) and for any $\lambda \in \mathbb{C}$, we define*

$$\mathcal{V}_\lambda(\Sigma) := \left\{ \zeta \in \mathbb{C}^n \;\middle|\; \exists\, \omega \in \mathbb{C}^m \;:\; 0 = \begin{bmatrix} A - \lambda I & B \\ C & D \end{bmatrix} \begin{pmatrix} \zeta \\ \omega \end{pmatrix} \right\}, \qquad (3.7.12)$$

$$\mathcal{S}_\lambda(\Sigma) := \left\{ \zeta \in \mathbb{C}^n \;\middle|\; \exists\, \omega \in \mathbb{C}^{n+m} \;:\; \begin{pmatrix} \zeta \\ 0 \end{pmatrix} = \begin{bmatrix} A - \lambda I & B \\ C & D \end{bmatrix} \omega \right\}. \qquad (3.7.13)$$

$\mathcal{V}_\lambda(\Sigma)$ and $\mathcal{S}_\lambda(\Sigma)$ are associated with the state zero directions of Σ if λ is an invariant zero of Σ. Clearly, $\mathcal{S}_\lambda(\Sigma) = \mathcal{V}_{\bar{\lambda}}(\Sigma^\star)^\perp$.



Harry Trentelman
University of Groningen

Carsten Scherer
University of Stuttgart

Theorem 3.8.1. *Consider a given system Σ characterized by a matrix quadruple (A, B, C, D). Also, consider a state feedback gain matrix $F \in \mathbb{R}^{m \times n}$. Then, Σ_F as characterized by the quadruple $(A + BF, B, C + DF, D)$ has the following properties:*

1. *Σ_F is a controllable (stabilizable) system if and only if Σ is a controllable (stabilizable) system;*
2. *The normal rank of Σ_F is equal to that of Σ;*
3. *The invariant zero structure of Σ_F is the same as that of Σ;*
4. *The infinite zero structure of Σ_F is the same as that of Σ;*
5. *Σ_F is (left or right) invertible or degenerate if and only if Σ is (left or right) invertible or degenerate;*
6. *$\mathcal{V}^x(\Sigma_F) = \mathcal{V}^x(\Sigma)$ and $\mathcal{S}^x(\Sigma_F) = \mathcal{S}^x(\Sigma)$;*
7. *$\mathcal{R}^*(\Sigma_F) = \mathcal{R}^*(\Sigma)$ and $\mathcal{N}^*(\Sigma_F) = \mathcal{N}^*(\Sigma)$; and*
8. *$\mathcal{V}_\lambda(\Sigma_F) = \mathcal{V}_\lambda(\Sigma)$ and $\mathcal{S}_\lambda(\Sigma_F) = \mathcal{S}_\lambda(\Sigma)$.*

Theorem 3.8.2. *Consider a given system* Σ *characterized by a matrix quadruple* (A, B, C, D). *Also, consider an output injection gain matrix* $K \in \mathbb{R}^{n \times p}$. *Then,* $\Sigma_{\rm K}$ *as characterized by the quadruple* $(A+KC, B+KD, C, D)$ *has the following properties:*

1. $\Sigma_{\rm K}$ *is an observable (detectable) system if and only if* Σ *is an observable (detectable) system;*

2. *The normal rank of* $\Sigma_{\rm K}$ *is equal to that of* Σ;

3. *The invariant zero structure of* $\Sigma_{\rm K}$ *is the same as that of* Σ;

4. *The infinite zero structure of* $\Sigma_{\rm K}$ *is the same as that of* Σ;

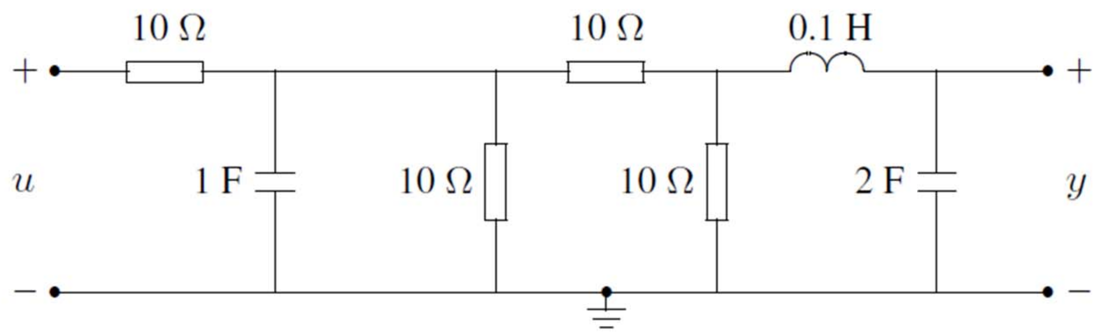
5. $\Sigma_{\rm K}$ *is (left or right) invertible or degenerate if and only if* Σ *is (left or right) invertible or degenerate;*

6. $\mathcal{V}^{\rm x}(\Sigma_{\rm K}) = \mathcal{V}^{\rm x}(\Sigma)$ *and* $\mathcal{S}^{\rm x}(\Sigma_{\rm K}) = \mathcal{S}^{\rm x}(\Sigma)$;

7. $\mathcal{R}^*(\Sigma_{\rm K}) = \mathcal{R}^*(\Sigma)$ *and* $\mathcal{N}^*(\Sigma_{\rm K}) = \mathcal{N}^*(\Sigma)$; *and*

8. $\mathcal{V}_\lambda(\Sigma_{\rm K}) = \mathcal{V}_\lambda(\Sigma)$ *and* $\mathcal{S}_\lambda(\Sigma_{\rm K}) = \mathcal{S}_\lambda(\Sigma)$.

Exercise 3.1. Consider an electric network shown in the circuit below with its input, u, being a voltage source, and output, y, being the voltage across the 2 F capacitor. Assume that the initial voltages across the 1 F and 2 F capacitors are 1 V and 2 V, respectively, and that the inductor is initially uncharged.



Circuit for Exercise 3.1.

(a) Derive the state and output equations of the network.

(b) Find the unit step response of the network.

(c) Find the unit impulse response of the network.

(d) Determine the stability of the network.

(e) Determine the controllability and observability of the network.

(f) Determine the invertibility of the network.

(g) Determine the finite and infinite zero structures of the network.

Exercise 3.2. Given

$$e^{At} = \begin{bmatrix} -e^{-t} + \alpha e^{-2t} & -e^{-t} + \beta e^{-2t} \\ 2e^{-t} - 2e^{-2t} & 2e^{-t} - e^{-2t} \end{bmatrix},$$

determine the values of the scalars α and β, and the matrices A and A^{100}.

Exercise 3.5. Consider an uncontrollable system, $\dot{x} = Ax + Bu$, with $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$. Assume that

$$\mathrm{rank}\,(Q_c) = \mathrm{rank}\,([\,B \quad AB \quad \cdots \quad A^{n-1}B\,]) = r < n.$$

Let $\{q_1, q_2, \ldots, q_r\}$ be a basis for the range space of the controllability matrix, Q_c, and let $\{q_{r+1}, \ldots, q_n\}$ be any vectors such that

$$T = [\,q_1 \quad q_1 \quad \cdots \quad q_r \quad q_{r+1} \quad \cdots \quad q_n\,]$$

is nonsingular. Show that the state transformation

$$x = T\tilde{x} = T\begin{pmatrix} \tilde{x}_c \\ \tilde{x}_{\bar{c}} \end{pmatrix}, \quad \tilde{x}_c \in \mathbb{R}^r, \quad \tilde{x}_{\bar{c}} \in \mathbb{R}^{n-r},$$

transforms the given system into the form

$$\begin{pmatrix} \dot{\tilde{x}}_c \\ \dot{\tilde{x}}_{\bar{c}} \end{pmatrix} = \begin{bmatrix} A_{cc} & A_{c\bar{c}} \\ 0 & A_{\bar{c}\bar{c}} \end{bmatrix} \begin{pmatrix} \tilde{x}_c \\ \tilde{x}_{\bar{c}} \end{pmatrix} + \begin{bmatrix} B_c \\ 0 \end{bmatrix} u,$$

where (A_{cc}, B_c) is controllable. Show that the uncontrollable modes of the system are given by $\lambda(A_{\bar{c}\bar{c}})$.

Exercise 3.7. Verify the result of Exercise 3.5 for the following systems:

$$\dot{x} = \begin{bmatrix} 5 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ -2 & 0 & 2 & -2 \\ -1 & -1 & -1 & 3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \end{bmatrix} u,$$

and

$$\dot{x} = \begin{bmatrix} -3 & -3 & 1 & 0 \\ 26 & 36 & -3 & -25 \\ 30 & 39 & -2 & -27 \\ 30 & 43 & -3 & -32 \end{bmatrix} x + \begin{bmatrix} 3 & 3 \\ -2 & -1 \\ 0 & 3 \\ 0 & 1 \end{bmatrix} u.$$

Exercise 3.17. Given a linear system

$$\dot{x} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} u, \quad y = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} x,$$

show that it is invertible, controllable and observable. Also, show that it has two infinite zeros of order 1 (and thus has a normal rank equal to 2), and has one invariant zero at $s = 1$ with a geometric multiplicity of 2 and an algebraic multiplicity of 2. Verify that such an invariant zero is also a blocking zero of the system.