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H_2 Optimal Control

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Abstract

An optimization-based approach to linear feedback control system design uses the H_2 norm, or energy of the impulse response, to quantify closed-loop performance. In this entry, an overview of state-space methods for solving H_2 optimal control problems via Riccati equations and matrix inequalities is presented in a continuous-time setting. Both regular and singular problems are considered. Connections to so- called LQR and LQG control problems are also described.

Keywords

Feedback control · H_2 control · Linear matrix inequalities · Linear systems · Riccati equations · State-space methods

Introduction

Modern multivariable control theory based on state-space models is able to handle multi-feedback-loop designs, with the added benefit that design methods derived from it are amenable to computer implementation. Indeed, over the last five decades, a number of multivariable analysis and design have been developed using the state-space description of systems. Of these design tools, H_2 optimal control problems involve minimizing the H_2 norm of the closed-loop transfer function from exogenous disturbance signals to a pertinent controlled output signals of a given plane by appropriate use of an internally stabilizing feedback controller. It was not until the 1990s that a complete solution to the general H_2 optimal control problem began to emerge. To elaborate on this, let us concentrate our discussion on H_2 optimal control for a continuous-time system Σ expressed in the following state-space form:

$$\dot{x} = Ax + Bu + Ew \tag{1}$$

$$y = C_1x + D_{11}u + D_1w \tag{2}$$

$$z = C_2x + D_2u + D_{22}w \tag{3}$$

where x is the state variable; u is the control input; w is the exogenous disturbance input; y is the measurement output; and z is the controlled output. The system Σ is typically an augmented or generalized plant model including weighting functions that reflect design requirements. The H_2 optimal control problem is to find an appropriate control law, relating the control input u to the measured output y, such that when it is applied to the given plant in Eqs. (1), (2) and (3), the resulting closed-loop system is internally

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stable and the H_2 norm of the resulting closed-loop transfer matrix from the disturbance input w to the controlled output z, denoted by $T_{zw}(s)$, is minimized. For a stable transfer matrix $T_{zw}(s)$, the H_2 norm is defined as

$$\|T_{zw}\|_2 = \left(\frac{1}{2\pi} \operatorname{trace}\left[\int_{-\infty}^{\infty} T_{zw}(j\omega) T_{zw}^{\mathrm{H}}(j\omega) d\omega\right]\right)^{\frac{1}{2}} \quad (4)$$

where T_{zw}^{H} is the conjugate transpose of T_{zw}. Note that the H_2 norm is equal to the energy of the impulse response associated with $T_{zw}(s)$ and this is finite only if the direct feedthrough term of the transfer matrix is zero.

It is standard to make the following assumptions on the problem data: $D_{11} = 0$; $D_{22} = 0$; (A, B) is stabilizable; (A, C_1) is detectable. The last two assumptions are necessary for the existence of an internally stabilizing control law. The first assumption can be made without loss of generality via a constant loop transformation. Finally, either the assumption $D_{22} = 0$ can be achieved by a pre-static feedback law or the problem does not yield a solution that has finite H_2 closed-loop norm.

There are two main groups into which all H_2 optimal control problems can be divided. The first group, referred to as regular H_2 optimal control problems, consists of those problems for which the given plant satisfies two additional assumptions:

1. the subsystem from the control input to the controlled output, i.e., (A, B, C_2, D_2), has no invariant zeros on the imaginary axis, and its direct feedthrough matrix, D_2, is injective (i.e., it is tall and of full rank); and
2. the subsystem from the exogenous disturbance to the measurement output, i.e., (A, E, C_1, D_1), has no invariant zeros on the imaginary axis, and its direct feedthrough matrix, D_1, is surjective (i.e., it is fat and of full rank).

Assumption 1 implies that (A, B, C_2, D_2) is left invertible with no infinite zero, and Assumption 2 implies that (A, E, C_1, D_1) is right invertible with no infinite zero. The second, referred to as singular H_2 optimal control problems, consists of those which are not regular.

Most of the research in the literature was expended on regular problems. Also, most of the available textbooks and review articles, see, for example, Anderson and Moore (1989), Bryson and Ho (1975), Fleming and Rishel (1975), Kailath (1974), Kwakernaak and Sivan (1972), Lewis (1986), and Zhou et al. (1996), to name a few, cover predominantly only a subset of regular problems. The singular H_2 control problem with state feedback was studied in Geerts (1989) and Willems et al. (1986). Using different classes of state and measurement feedback control laws, Stoorvogel et al. (1993) studied the general H_2 optimal control problems for the first time. In particular, necessary and sufficient conditions are provided therein for the existence of a solution in the case of state feedback control and in the case of measurement feedback control. Following this, Trentelman and Stoorvogel (1995) explored necessary and sufficient conditions for the existence of an H_2 optimal controller within the context of discrete-time and sampled-data systems. At the same time, Chen et al. (1993) and Chen et al. (1994a) provided a thorough treatment of the H_2 optimal control problem with state feedback controllers. This includes a parameterization and construction of the set of all H_2 optimal controllers and the associated sets of H_2 optimal fixed modes and H_2 optimal fixed decoupling zeros. Also, they provided a computationally feasible design algorithm for selecting an H_2 optimal state feedback controller that places the closed-loop poles at desired locations whenever possible. Furthermore, Chen and Saberi (1993) and Chen et al. (1996) developed the necessary and sufficient conditions for the uniqueness of an H_2 optimal controller. Interested readers are referred to the textbook (Saberi et al. 1995) for a detailed treatment of H_2 optimal control problems in their full generality.

Regular Case

Solving regular H_2 optimal control problems is relatively straightforward. In the case that all of the state variables of the given plant are available for feedback, i.e., $y = x$, and Assumption 1 holds, the corresponding H_2 optimal control problem can be solved in terms of the unique positive semi-definite stabilizing solution $P \geq 0$ of the following algebraic Riccati equation:

$$A^{\mathrm{T}}P + PA + C_2^{\mathrm{T}}C_2 - (PB + C_2^{\mathrm{T}}D_2)(D_2^{\mathrm{T}}D_2)^{-1} (D_2^{\mathrm{T}}C_2 + B^{\mathrm{T}}P) = 0 \quad (5)$$

The H_2 optimal state feedback law is given by

$$u = Fx = -(D_2^{\mathrm{T}}D_2)^{-1}(D_2^{\mathrm{T}}C_2 + B^{\mathrm{T}}P)\, x \quad (6)$$

and the resulting closed-loop transfer matrix from w to z, $T_{zw}(s)$, has the following property:

$$\|T_{zw}\|_2 = \sqrt{\operatorname{trace}(E^{\mathrm{T}}PE)} \quad (7)$$

Note that the H_2 optimal state feedback control law is generally nonunique. A trivial example is the case when $E = 0$, whereby every stabilizing control law is an optimal solution. It is also interesting to note that the closed-loop system comprising the given plant with $y = x$ and the state feedback control law of Eq. (6) has poles at all the stable invariant zeros and all the mirror images of the unstable invariant zeros of (A, B, C_2, D_2) together with some other fixed locations in the left half complex plane. More detailed results about the optimal fixed modes and fixed decoupling zeros for general H_2 optimal control can be found in Chen et al. (1993).

It can be shown that the well-known linear quadratic regulation (LQR) problem can be reformulated as a regular H_2 optimal control problem. For a given plant

$$\dot{x} = Ax + Bu, \quad x(0) = X_0 \quad (8)$$

with (A, B) being stabilizable, the LQR problem is to find a control law $u = Fx$ such that the following performance index is minimized:

$$J = \int_0^\infty (x^{\mathrm{T}}Q_\star x + u^{\mathrm{T}}R_\star u)dt, \quad (9)$$

where $R_\star > 0$ and $Q_\star \geq 0$ with $(A, Q_\star^{\frac{1}{2}})$ being detectable. The LQR problem is equivalent to finding a static state feedback H_2 optimal control law for the following auxiliary plant Σ_{LQR}:

$$\dot{x} = Ax + Bu + X_0 w \quad (10)$$

$$y = x \quad (11)$$

$$z = \begin{pmatrix} 0 \\ Q_\star^{\frac{1}{2}} \end{pmatrix} x + \begin{pmatrix} R_\star^{\frac{1}{2}} \\ 0 \end{pmatrix} u \quad (12)$$

For the measurement feedback case with both Assumptions 1 and 2 being satisfied, the corresponding H_2 optimal control problem can be solved by finding a positive semi-definite stabilizing solution $P \geq 0$ for the Riccati equation given in Eq. (5) and a positive semi-definite stabilizing solution $Q \geq 0$ for the following Riccati equation:

$$QA^{\mathrm{T}} + AQ + EE^{\mathrm{T}} - (QC_1^{\mathrm{T}} + ED_1^{\mathrm{T}}) (D_1D_1^{\mathrm{T}})^{-1}(D_1E^{\mathrm{T}} + C_1Q) = 0 \quad (13)$$

The H_2 optimal measurement feedback law is given by

$$\dot{v} = (A + BF + KC_1)v - Ky, \quad u = Fx \quad (14)$$

where F is as given in Eq. (6) and

$$K = -(QC_1^{\mathrm{T}} + ED_1^{\mathrm{T}})(D_1D_1^{\mathrm{T}})^{-1} \quad (15)$$

In fact, such an optimal control law is unique, and the resulting closed-loop transfer matrix from w to z, $T_{zw}(s)$, has the following property:

$$\|T_{zw}\|_2 = \left\{\operatorname{trace}(E^{\mathrm{T}}PE) + \operatorname{trace}\left[\left(A^{\mathrm{T}}P + PA + C_2^{\mathrm{T}}C_2\right)Q\right]\right\}^{\frac{1}{2}} \quad (16)$$

Similarly, consider the standard LQG problem for the following system

$$\dot{x} = Ax + Bu + G_\star d \quad (17)$$

$$y = Cx + N_\star n, \quad N_\star > 0 \quad (18)$$

$$z = \begin{pmatrix} H_\star x \\ R_\star u \end{pmatrix}, \quad R_\star > 0, \quad w = \begin{pmatrix} d \\ n \end{pmatrix} \tag{19}$$

where x is the state, u is the control, d and n are white noises with identity covariance, and y is the measurement output. It is assumed that (A, B) is stabilizable and (A, C) is detectable. The control objective is to design an appropriate control law that minimizes the expectation of $|z|^2$. Such an LQG problem can be solved via the H_2 optimal control problem for the following auxiliary system Σ_{LQG}, see Doyle (1983):

$$\dot{x} = Ax + Bu + [\, G_\star \quad 0 \,]w \tag{20}$$

$$y = Cx + [\, 0 \quad N_\star \,]w \tag{21}$$

$$z = \begin{pmatrix} H_\star \\ 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ R_\star \end{pmatrix} u \tag{22}$$

H_2 optimal control problem for discrete-time systems can be solved in a similar way via the corresponding discrete-time algebraic Riccati equations. It is worth noting that many works can be found in the literature that deal with solutions to discrete-time algebraic Riccati equations related to optimal control problems; see, for example, Kucera (1972), Pappas et al. (1980), and Silverman (1976), to name a few. It is proven in Chen et al. (1994b) that solutions to the discrete- and continuous-time algebraic Riccati equations for optimal control problems can be unified. More specifically, the solution to a discrete-time Riccati equation can be done through solving an equivalent continuous-time one and vice versa.

Singular Case

As in the previous section, only the key procedure in solving the singular H_2-optimization problem for continuous-time systems is addressed. For the singular problem, it is generally not possible to obtain an optimal solution, except for some situations when the given plant satisfies certain geometric constraints, see, e.g., Chen et al. (1993) and Stoorvogel et al. (1993). It is more feasible to find a suboptimal control law for the singular problem, i.e., to find an appropriate control law such that the H_2 norm of the resulting closed-loop transfer matrix from w to z can be made arbitrarily close to the best possible performance. The procedure given below is to transform the original problem into an H_2 almost disturbance decoupling problem; see Stoorvogel (1992) and Stoorvogel et al. (1993).

Consider the given plant in Eqs. (1), (2) and (3) with Assumption 1 and/or Assumption 2 not satisfied. First, find the largest solution $P \geq 0$ for the following linear matrix inequality

$$F(P) = \begin{pmatrix} A^{\mathrm{T}}P + PA + C_2^{\mathrm{T}}C_2 & PB + C_2^{\mathrm{T}}D_2 \\ B^{\mathrm{T}}P + D_2^{\mathrm{T}}C_2 & D_2^{\mathrm{T}}D_2 \end{pmatrix} \geq 0 \tag{23}$$

and find the largest solution $Q \geq 0$ for

$$G(Q) = \begin{pmatrix} AQ + QA^{\mathrm{T}} + EE^{\mathrm{T}} & QC_1^{\mathrm{T}} + ED_1^{\mathrm{T}} \\ C_1Q + D_1E^{\mathrm{T}} & D_1D_1^{\mathrm{T}} \end{pmatrix} \geq 0 \tag{24}$$

Note that by decomposing the quadruples (A, B, C_2, D_2) and (A, E, C_1, D_1) into various subsystems in accordance with their structural properties, solutions to the above linear matrix inequalities can be obtained by solving a Riccati equation similar to those in Eq. (5) or Eq. (13) for the regular case. In fact, for the regular problem, the largest solution $P \geq 0$ for Eq. (23) and the stabilizing solution $P \geq 0$ for Eq. (5) are identical. Similarly, the largest solution $Q \geq 0$ for Eq. (24) and the stabilizing solution $Q \geq 0$ for Eq. (13) are also the same. Interested readers are referred to Stoorvogel et al. (1993) for more details or to Chen et al. (2004) for a more systematic treatment on the structural decomposition of linear systems and its connection to the solutions of the linear matrix inequalities.

It can be shown that the best achievable H_2 norm of the closed-loop transfer matrix from w to z, i.e., the best possible performance over all internally stabilizing control laws, is given by

$$\gamma_2^\star = \left\{\text{trace}(E^{\mathrm{T}} P E) + \text{trace}\left[\left(A^{\mathrm{T}} P + P A + C_2^{\mathrm{T}} C_2\right) Q\right]\right\}^{\frac{1}{2}} \tag{25}$$

Next, partition

$$F(P) = \begin{pmatrix} C_{\mathrm{P}}^{\mathrm{T}} \\ D_{\mathrm{P}}^{\mathrm{T}} \end{pmatrix} \begin{pmatrix} C_{\mathrm{P}} & D_{\mathrm{P}} \end{pmatrix} \quad \text{and}$$

$$G(Q) = \begin{pmatrix} E_{\mathrm{Q}} \\ D_{\mathrm{Q}} \end{pmatrix} \begin{pmatrix} E_{\mathrm{Q}}^{\mathrm{T}} & D_{\mathrm{Q}}^{\mathrm{T}} \end{pmatrix} \tag{26}$$

where $[C_{\mathrm{P}} \quad D_{\mathrm{P}}]$ and $[E_{\mathrm{Q}}^{\mathrm{T}} \quad D_{\mathrm{Q}}^{\mathrm{T}}]$ are of maximal rank, and then define an auxiliary system Σ_{PQ}:

$$\dot{x}_{\mathrm{PQ}} = A x_{\mathrm{PQ}} + B u + E_{\mathrm{Q}} w_{\mathrm{PQ}} \tag{27}$$

$$y = C_1 x_{\mathrm{PQ}} + D_{\mathrm{Q}} w_{\mathrm{PQ}} \tag{28}$$

$$z_{\mathrm{PQ}} = C_{\mathrm{P}} x_{\mathrm{PQ}} + D_{\mathrm{P}} u \tag{29}$$

It can be shown that the quadruple $(A, B, C_{\mathrm{P}}, D_{\mathrm{P}})$ is right invertible and has no invariant zeros in the open right-half complex plane and the quadruple $(A, E_{\mathrm{Q}}, C_1, D_{\mathrm{Q}})$ is left invertible and has no invariant zeros in the open right-half complex plane. It can also be shown that there exists an appropriate control law such that when it is applied to Σ_{PQ}, the resulting closed-loop system is internally stable and the H_2 norm of the closed-loop transfer matrix from w_{PQ} to z_{PQ} can be made arbitrarily small. Equivalently, H_2 almost disturbance decoupling problem for Σ_{PQ} is solvable.

More importantly, it can further be shown that if an appropriate control law that solves the H_2 almost disturbance decoupling problem for Σ_{PQ}, then it solves the H_2 suboptimal problem for Σ. As such, the solution to the singular H_2 control problem for Σ can be done by finding a solution to the H_2 almost disturbance decoupling problem for Σ_{PQ}. There are vast results available in the literature dealing with disturbance decoupling problems. More detailed treatments can be found in Saberi et al. (1995).

Conclusion

This entry considers the basic solutions to H_2 optimal control problems for continuous-time systems. Both the regular problem and the general singular problem are presented. Readers interested in more details are referred to Saberi et al. (1995) and the references therein for the complete treatment of H_2 optimal control problems and to Chapter 10 of Chen et al. (2004) for the unification and differentiation of H_2 control, H_∞ control, and disturbance decoupling control problems. H_2 optimal control is a mature area and has a long history. Possible future research includes issues on how to effectively utilize the theory in solving real-life problems.

Cross-References

▶ H-Infinity Control
▶ Linear Quadratic Optimal Control
▶ Optimal Control via Factorization and Model Matching
▶ Stochastic Linear-Quadratic Control

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