Problem 8.2

Non-iterative computation of optimal value in H_∞ control

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1 DESCRIPTION OF THE PROBLEM

We consider an n-th order generalized linear system Σ characterized by the following state-space equations:

$$\Sigma : \begin{cases} \dot{x} = A\ x + \ B\ \ u + \ E\ \ w \\ y = C_1\ x + D_{11}\ u + \ D_1\ \ w \\ h = C_2\ x + \ D_2\ \ u + D_{22}\ w \end{cases} \tag{1}$$

where x is the state, u is the control input, w is the disturbance input, y is the measurement output, and h is the controlled output of Σ. For simplicity, we assume that $D_{11} = 0$ and $D_{22} = 0$. We also let Σ_{P} be the subsystem characterized by the matrix quadruple (A, B, C_2, D_2) and Σ_{Q} be the subsystem characterized by (A, E, C_1, D_1).

The standard H_∞ optimal control problem is to find an internally stabilizing proper measurement feedback control law,

$$\Sigma_{\mathrm{cmp}} : \begin{cases} \dot{v} = A_{\mathrm{cmp}}\ v + B_{\mathrm{cmp}}\ y \\ u = C_{\mathrm{cmp}}\ v + D_{\mathrm{cmp}}\ y \end{cases} \tag{2}$$

such that when it is applied to the given plant (1), the H_∞-norm of the resulting closed-loop transfer matrix function from w to h, say $T_{hw}(s)$, is minimized. We note that the H_∞-norm of an asymptotically stable and proper continuous-time transfer matrix $T_{hw}(s)$ is defined as

$$\|T_{hw}\|_\infty := \sup_{\omega \in [0,\infty)} \sigma_{\max}[T_{hw}(j\omega)] = \sup_{\|w\|_2 = 1} \frac{\|h\|_2}{\|w\|_2},$$

where w and h are, respectively, the input and output of $T_{hw}(s)$.

The infimum or the optimal value associated with the H_∞ control problem is defined as

$$\gamma^* := \inf \left\{ \|T_{hw}(\Sigma \times \Sigma_{\rm cmp})\|_\infty \mid \ \Sigma_{\rm cmp} \text{ internally stabilizes } \Sigma \right\}. \quad (3)$$

Obviously, $\gamma^* \geq 0$. In fact, when $\gamma^* = 0$, the problem is reduced to the well-known problem of H_∞ almost disturbance decoupling with measurement feedback and internal stability.

We note that in order to design a meaningful H_∞ control law for the given system (1), the designer should know before hand the infimum γ^*, which represents the best achievable level of disturbance attenuation. Unfortunately, the problem of a noniterative computation of this γ^* for general systems still remains unsolved in the open literature.

2 MOTIVATION AND HISTORY OF THE PROBLEM

Over the last two decades, we have witnessed a proliferation of literature on H_∞ optimal control since it was first introduced by Zames [20]. The main focus of the work has been on the formulation of the problem for robust multivariable control and its solution. Since the original formulation of the H_∞ problem in Zames [20], a great deal of work has been done on finding the solution to this problem. Practically all the research results of the early years involved a mixture of time-domain and frequency-domain techniques including the following: 1) *interpolation approach* (see, e.g., [13]); chenbm2) *frequency domain approach* (see, e.g., [5, 8, 9]); 3) *polynomial approach* (see, e.g., [12]); and 4) *J-spectral factorization approach* (see, e.g., [11]). Recently, considerable attention has been focused on purely *time-domain methods* based on algebraic Riccati equations (ARE) (see, e.g., [6, 7, 10, 15, 16, 17, 18, 19, 21]). Along this line of research, connections are also made between H_∞ optimal control and differential games (see, e.g., [1, 14]).

It is noted that most of the results mentioned above are focusing on finding solutions to H_∞ control problems. Many of them assume that γ^* is known or simply assume that $\gamma^* = 1$. The computation of γ^* in the literature are usually done by certain iteration schemes. For example, in the regular case and utilizing the results of Doyle et al. [7], an iterative procedure for approximating γ^* would proceed as follows: one starts with a value of γ and determines whether $\gamma > \gamma^*$ by solving two "indefinite" algebraic Riccati equations and checking the positive semi-definiteness and stabilizing properties of these solutions. In the case when such positive semi-definite solutions exist and satisfy a *coupling condition*, then we have $\gamma > \gamma^*$ and one simply repeats the above steps using a smaller value of γ. In principle, one can approximate the infimum γ^* to within any degree of accuracy in this manner. However, this search procedure is exhaustive and can be very costly. More significantly, due to the possible high-gain occurrence as γ gets close to γ^*, numerical solutions for these H_∞ AREs can become highly sensitive and ill-conditioned. This difficulty also arises in the *coupling condition*. Namely, as γ decreases, evaluation of the *coupling condition* would generally involve finding eigenvalues of stiff matrices. These numerical difficulties are likely to be more severe for problems associated with the singular case. Thus, in general, the iterative procedure for the computation of γ^* based on AREs is not reliable.

3 AVAILABLE RESULTS

There are quite a few researchers who have attempted to develop procedures for the determination of the value of γ^* without iterations. For example, Petersen [15] has solved the problem for a class of one-block regular case. Scherer [17, 18] has obtained a partial answer for state feedback problem for a larger class of systems by providing a computable candidate value together with algebraically verifiable conditions, and Chen and his co-workers [3, 4] (see also [2]) have developed a noniterative procedures for computing the value of γ^* for a class of systems (singular case) that satisfy certain geometric conditions.

To be more specific, we introduce the following two geometric subspaces of linear systems: Given an n-th order linear system Σ_* characterized by a matrix quadruple (A_*, B_*, C_*, D_*), we define

i. $\mathcal{V}^-(\Sigma_*)$, a weakly unobservable subspace, is the maximal subspace of $\mathbb{R}^n$ which is $(A_* + B_*F_*)$-invariant and contained in $\mathrm{Ker}\,(C_* + D_*F_*)$ such that the eigenvalues of $(A_* + B_*F_*)|\mathcal{V}^-$ are contained in $\mathbb{C}^-$, the open-left complex plane, for some constant matrix F_*; and

ii. $\mathcal{S}^-(\Sigma_*)$, a strongly controllable subspace, is the minimal $(A_* + K_*C_*)$-invariant subspace of $\mathbb{R}^n$ containing $\mathrm{Im}\,(B_* + K_*D_*)$ such that the eigenvalues of the map which is induced by $(A_* + K_*C_*)$ on the factor space $\mathbb{R}^n/\mathcal{S}^-$ are contained in $\mathbb{C}^-$ for some constant matrix K_*.

The problem of noniterative computation of γ^* has been solved by Chen and his co-workers [3, 4] (see also [2]) for a class of systems that satisfy the following conditions:

i. $\mathrm{Im}(E) \subset \mathcal{V}^-(\Sigma_{\mathrm{P}}) + \mathcal{S}^-(\Sigma_{\mathrm{P}})$; and

ii. $\mathrm{Ker}(C_2) \supset \mathcal{V}^-(\Sigma_{\mathrm{Q}}) \cap \mathcal{S}^-(\Sigma_{\mathrm{Q}})$,

together with some other minor assumptions. The work of Chen et al. involves solving a couple of algebraic Riccati and Lyapunov equations. The computation of γ^* is then done by finding the maximum eigenvalue of a resulting constant matrix.

It has been demonstrated by an example in Chen [2] that the noniterative computation of γ^* can be done for a larger class of systems, which do not necessarily satisfy the above geometric conditions. It is believed that there are rooms to improve the existing results.

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