Loop Transfer Recovery For General Nonminimum Phase Discrete Time Systems

Part 2: Design

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I. INTRODUCTION AND PROBLEM STATEMENT

As discussed earlier in Part 1 [3], the basic loop transfer recovery (LTR) problem is concerned with analysing and possibly designing a controller which can achieve the same robustness properties as those of a state feedback controller. To be specific, consider a plant Σ,

$$x(k+1) = Ax(k) + Bu(k) \quad , \quad y(k) = Cx(k) + Du(k) \tag{1}$$

where the state vector $x \in \mathbf{R}^n$, output vector $y \in \mathbf{R}^p$ and input vector $u \in \mathbf{R}^m$. Without loss of generality, assume that $[B' \ D']'$ and $[C \ \ D]$ are of maximal rank. Let us also assume that Σ is stabilizable and detectable. Let the state feedback control law,

$$u = -Fx, \tag{2}$$

be such that (a) the closed-loop system is asymptotically stable, and (b) the open-loop transfer function when the loop is broken at the input point of the plant meets the given frequency dependent specifications. Then $L_t(z)$, $S_t(z)$ and $T_t(z)$, the target loop transfer function, sensitivity and complimentary sensitivity functions are respectively

$$L_t(z) = F\Phi B,$$

$$S_t(z) = [I_m + L_t(z)]^{-1},$$

and

$$T_t(z) = I_m - S_t(z) = [I_m + L_t(z)]^{-1} L_t(z) \tag{3}$$

where $\Phi = (zI_n - A)^{-1}$ and I_m denotes an identity matrix of dimension $m \times m$. We would like to recover $L_t(z)$ using only a measurement feedback controller $\mathbf{C}(z)$. That is, given a target loop transfer function $L_t(z)$ and the plant transfer function P(s),

$$P(z) = C\Phi B + D,$$

we seek to design a controller $\mathbf{C}(z)$ such that the *loop transfer recovery error* $E(z)$,

$$E(z) \equiv L_t(z) - \mathbf{C}(z)P(z), \tag{4}$$

is either exactly or approximately equal to zero in the frequency region of interest while guaranteeing the stability of the resulting closed-loop system.

The notion of achieving exact LTR (ELTR) corresponds to $E(z) = 0$ for all z. In the case of asymptotic recovery, one normally parameterizes the controller $\mathbf{C}(z)$ in terms of a scalar tuning parameter σ and thus obtains a family of controllers $\mathbf{C}(z, \sigma)$. We say asymptotic LTR (ALTR) is achieved if $\mathbf{C}(z, \sigma)P(z) \to L_t(z)$ pointwise in z as $\sigma \to \infty$. Achievability of ALTR enables the designer to choose a member of the family of controllers that corresponds to a particular value of σ which achieves a desired level of recovery.

In Part 1 [3], for general discrete systems, the above LTR problem has been considered and analysed using any of three different observer or estimator based controllers. The estimators considered there are (1) prediction estimator, (2) current estimator and (3) reduced order estimator. Both prediction estimator and current estimator are full order observers. The reduced order estimator is a current estimator but uses the reduced order observer. The prediction estimator estimates the state $x(k+1)$ based on the measurements $y(k)$ up to and including the (k)-th instant, where as the current estimator estimates $x(k+1)$ based on the measurements $y(k+1)$ up to and including the $(k+1)$-th instant. The analysis of Part 1 corresponding to all these three different estimator based controllers unifies it into a single mathematical frame work. The LTR analysis given there focuses on four fundamental issues, (1) the recoverability of a target loop when it is arbitrarily given, (2) the recoverability of a target loop taking into account its specific characteristics, (3) the establishment of necessary and sufficient conditions on the given system so that it has at least one recoverable target loop transfer function or sensitivity function, and (4) the recoverability of a sensitivity function in a specified subspace of the control space. All this analysis of Part 1 shows some fundamental limitations of the given system as a consequence of its structural properties. Also, Part 1 decomposes the so called recovery matrix into two parts, the first one can always be rendered zero while the other in general cannot be rendered zero and hence can be termed as the *recovery error matrix*. The analysis of Part 1 also discovers a multitude of ways in which freedom exists to shape the recovery error matrix in a desired way. Thus it helps to set meaningful design goals at the onset of design.

Part 1 also reveals both similarities as well as fundamental differences that arise in LTR analysis of continuous and discrete time systems. A fundamental difference between continuous time and discrete time systems that should be emphasized is this. In the discrete case, as is well known, in order to preserve stability, all the closed-loop eigenvalues must be restricted to lie within the unit circle in complex plane. This implies that unlike continuous case which permits both finite as well as asymptotically infinite eigenvalue assignment to a closed-loop system, in the discrete case one is restricted to only finite eigenvalue assignment. Because of this, in the continuous case, there exists target loops which are not exactly recoverable, but are asymptotically recoverable by appropriate infinite eigenstructure assignment; on the other hand, in discrete systems, since both asymptotic as well as exact recovery involves only finite eigenstructure assignment, every asymptotically recoverable target loop is also exactly recoverable and vice versa. Thus, in discrete systems, one needs to talk about just recovery rather than emphasizing exact or asymptotic recovery.

In this Part 2 of the paper, we consider design of all three, prediction, current and reduced order estimator based controllers for the purpose of loop transfer recovery. For each one of such controllers, after reviewing from Part 1 the necessary design constraints and the available design freedom, three different design techniques are developed. The first one is an eigenstructure assignment scheme, and the other two are optimization based designs. Eigenstructure assignment method yields a controller design which achieves *any chosen* recovery error matrix among a set of admissible recovery error matrices. On the other hand, one of the optimization based design methods leads to a controller that achieves a recovery error matrix having the infimum H_∞ norm, while the other does the same except it achieves a recovery error matrix having the infimum H_2 norm. The eigenstructure assignment method given here is a special case of the asymptotic time-scale and eigenstructure assignment (ATEA) method introduced in [8] and fully developed in [2] in connection with continuous systems. Since in discrete systems, one does not have the option of assigning the asymptotically infinite eigenvalues, no multiple time-scale structure assignment is feasible. The algorithm of ATEA as in [2] when the option of time-scale structure assignment is removed from it, yields a simple design tool for discrete LTR as well. Regarding optimization based design methods, while partial results are available in the literature based on H_2 norm minimization [5], [11], no methods of H_∞ norm minimization are yet available for discrete systems. This paper develops new H_∞ norm minimization methods, and then streamlines and strengthens the available H_2 norm minimization methods. An important difference between ATEA and optimization based designs is this. ATEA is capable of achieving any admissible recovery error matrix where as optimization based methods always lead to a particular recovery error matrix having the infimum H_∞ or H_2 norm depending on the method used.

As mentioned earlier, in discrete systems, when one talks about recoverability, one need not distinguish between the notions of 'exact' and 'asymptotic' recoverabilities as they both imply one and the same. However, as will be seen in the text, optimization based methods of recovery design some times lead only to suboptimal designs. For the case of recoverable target loops, such suboptimal designs yield asymptotic recovery. To be specific, in H_∞-optimization methods, one normally generates a sequence of observer gains by solving parameterized algebraic Riccati equations. As the parameter tends to a certain value, the corresponding sequence of H_∞ norms of the resulting recovery matrices tends to a limit which is the infimum of the H_∞ norm of the recovery matrix over the set of all possible observer gains. Obviously, for the case when the infimum of H_∞ norm of the recovery matrix is zero, the sequence of observer gains thus obtained lead to a suboptimal design that corresponds to asymptotic recovery.

The conventional LTR design task seeks the recovery over the entire control space. As discussed in Part 1, one can also formulate another generalized design task which seeks the recovery only over a specified subspace of the control space. Such a formulation is meaningful especially when recovery over the entire control space is not feasible. All the three design methods developed here can easily be modified to deal with such a generalized design task.

The paper is organised as follows. Section II reviews the necessary design constraints and the available design freedom. Section III develops the general ATEA method of design. Section IV develops optimization based designs. Here two designs are considered; one minimizes the H_∞ norm of a recovery matrix while the other minimizes the H_2 norm of the same. Section V considers the generalized design task of recovering the target sensitivity and complimentary sensitivity functions over a subspace of the control space. All the previous sections consider the case when the target loop transfer function is specified at the input point of the given system. Section VI reformulates the LTR design when a target loop transfer function is specified at the output point of the given system in terms of

LTR design when a target loop transfer function is specified at the input point of a system dual to the given system. Finally, Section VII draws the conclusions of our work.

As in Part 1, throughout this paper, A' denotes the transpose of A, A^H denotes the complex conjugate transpose of A, I denotes an identity matrix while I_k denotes the identity matrix of dimension $k \times k$. $\lambda(A)$ denotes the set of eigenvalues of A. Similarly, $\sigma_{max}[A]$ and $\sigma_{min}[A]$ respectively denote the maximum and minimum singular values of A. Ker $[V]$ and Im $[V]$ denote respectively the kernel and the image of V. $\mathbf{C}^{\odot}$ denotes the set of complex numbers inside the open unit circle while $\mathbf{C}^{\otimes}$ is the complimentary set of $\mathbf{C}^{\odot}$. Given a discrete transfer function $G(z)$, we define the discrete frequency response $G^*(j\omega)$ as $G(e^{j\omega T})$ where T is the sampling period of the discrete-time system. An asymptotically stable matrix is the one whose eigenvalues are all in $\mathbf{C}^{\odot}$.

While discussing the design procedures, we will always use a generic controller which could be based on any one of the three estimators, prediction, current or reduced order. In that case, as in Part 1, we will always use the following notation :

$$
\begin{aligned}
\mathrm{C}(z) &:= \text{The transfer function of the controller,} \\
L(z) &:= \mathrm{C}(z)P(z) = \text{The achieved loop transfer function,} \\
S(z) &:= [I_m + L(z)]^{-1} = \text{The achieved sensitivity function,} \\
T(z) &:= I_m - S(z) = \text{The achieved complimentary sensitivity} \\
&\qquad \text{function,} \\
E(z) &:= L_t(z) - L(z) = \text{Loop recovery error,} \\
M(z) &:= \text{The recovery matrix (to be defined later on),} \\
M^0(z) &:= \text{A part of the recovery matrix } M(z) \text{ that can be} \\
&\qquad \text{rendered zero,} \\
M^e(z) &:= \text{A part of the recovery matrix } M(z) \text{ that cannot be} \\
&\qquad \text{rendered zero and hence termed as recovery error matrix,} \\
\mathrm{T}^R(\Sigma) &:= \text{The set of either exactly or asymptotically recoverable} \\
&\qquad \text{target loops for } \Sigma.
\end{aligned}
$$

Whenever we have a particular controller in mind, we use appropriate subscripts to distinguish them. Subscripts p, c and r are used respectively to represent prediction, current, and reduced order estimator based controllers. For example, $L_p(z)$, $M_c^s(z)$ and $\mathbf{T}_r^R(\Sigma)$ denote respectively the achieved loop transfer function with prediction estimator based controller, the recovery error matrix when a current estimator based controller is used, and the set of recoverable target loops for Σ using reduced order estimator based controllers.

II. CONTROLLER STRUCTURES — DESIGN CONSTRAINTS AND AVAILABLE FREEDOM

In this section, we will recall three different controller structures as well as their design constraints and freedom available for the purpose of achieving LTR. All the three controllers considered are observer based, but the type of observer (or state estimator) used in each one is structurally different. The estimators considered are (1) prediction estimator, (2) current estimator and (3) reduced order estimator. The structural details of the controllers are as follows.

Prediction estimator based controller :

The dynamic equations of the controller are

$$\begin{cases} \hat{x}(k+1) = A\hat{x}(k) + Bu(k) + K_p[y(k) - C\hat{x}(k) - Du(k)], \\ u(k) = \hat{u}(k) = -F\hat{x}(k), \end{cases} \tag{5}$$

where K_p is the gain chosen so that $A - K_pC$ is asymptotically stable. The transfer function of the controller is

$$\mathbf{C}_p(z) = F[zI_n - A + BF + K_pC - K_pDF]^{-1}K_p. \tag{6}$$

Current Estimator Based Controller :

Let us first rewrite the matrices C and D in the form,

$$C = \begin{bmatrix} C_0 \\ C_1 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} D_0 \\ 0 \end{bmatrix}, \tag{7}$$

where D_0 is of maximal rank, i.e., $\text{rank}(D) = \text{rank}(D_0) = m_0$. Thus, the output y can be partitioned as,

$$\begin{bmatrix} y_0(k) \\ y_1(k) \end{bmatrix} = \begin{bmatrix} C_0 \\ C_1 \end{bmatrix} x(k) + \begin{bmatrix} D_0 \\ 0 \end{bmatrix} u(k).$$

The dynamic equations of the controller are

$$\begin{cases} \hat{x}(k+1) = A\hat{x}(k) + Bu(k) + K_c \left(\begin{bmatrix} y_0(k) \\ y_1(k+1) \end{bmatrix} - C_c\hat{x}(k) - D_c u(k) \right), \\ \hat{u}(k) = u(k) = -F\hat{x}(k), \end{cases} \tag{8}$$

where

$$C_c = \begin{bmatrix} C_0 \\ C_1 A \end{bmatrix} \quad \text{and} \quad D_c = \begin{bmatrix} D_0 \\ C_1 B \end{bmatrix}, \tag{9}$$

and where the gain K_c is chosen so that $A - K_c C_c$ is asymptotically stable. The transfer function from $-u$ to y that results in using the current estimator is then given by

$$\mathbf{C}_c(z) = F\left[zI_n - A + K_c C_c + BF - K_c D_c F\right]^{-1} K_c \begin{bmatrix} I_{m_0} & 0 \\ 0 & zI \end{bmatrix}. \tag{10}$$

Reduced Order Estimator Based Controller :

Again, without any loss of generality but for simplicity of presentation, it is assumed that the matrices C and D are transformed into the form,

$$C = \begin{bmatrix} 0 & C_{02} \\ I_{p-m_0} & 0 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} D_0 \\ 0 \end{bmatrix}. \tag{11}$$

Then Σ can be partitioned as follows,

$$\begin{cases} \begin{pmatrix} x_1(k+1) \\ x_2(k+1) \end{pmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{pmatrix} x_1(k) \\ x_2(k) \end{pmatrix} + \begin{bmatrix} B_{11} \\ B_{22} \end{bmatrix} u(k), \\ \begin{pmatrix} y_0(k) \\ y_1(k) \end{pmatrix} = \begin{bmatrix} 0 & C_{02} \\ I_{p-m_0} & 0 \end{bmatrix} \begin{pmatrix} x_1(k) \\ x_2(k) \end{pmatrix} + \begin{bmatrix} D_0 \\ 0 \end{bmatrix} u(k). \end{cases} \tag{12}$$

Since $y_1 = x_1$ is already available, one needs to estimate only the state variable x_2. Then the dynamic equations of the reduced order estimator based controller are as follows:

$$\begin{cases} v(k+1) = (A_r - K_r C_r)v(k) + (B_r - K_r D_r)u(k) + G_r y(k), \\ u(k) = \hat{u}(k) = -F_1 x_1(k) - F_2 \hat{x}_2(k) = -F_2 v(k) - [0, F_1 + F_2 K_{r1}]y(k) \end{cases} \tag{13}$$

where the gain K_r is chosen such that $A_r - K_r C_r$ is asymptotically stable, and where

$$A_r = A_{22}, \quad B_r = B_{22}, \quad C_r = \begin{bmatrix} C_{02} \\ A_{12} \end{bmatrix}, \quad D_r = \begin{bmatrix} D_0 \\ B_{11} \end{bmatrix}, \tag{14}$$

$$F = [F_1, \quad F_2], \quad K_r = [K_{r0}, \ K_{r1}], \tag{15}$$

$$G_r = [K_{r0}, \ A_{21} - K_{r1}A_{11} + (A_r - K_rC_r)K_{r1}]. \tag{16}$$

The transfer function from $-u$ to y that results in using the reduced order estimator is then given by

$$\begin{aligned} \mathbf{C}_r(z) &= F_2(zI - A_r + K_rC_r + B_rF_2 - K_rD_rF_2)^{-1} \\ &\cdot \Big(G_r - (B_r - K_rD_r)[0, \ F_1 + F_2K_{r1}]\Big) + [0, \ F_1 + F_2K_{r1}]. \end{aligned} \tag{17}$$

A fundamental result of Part 1, namely Lemma 1, rewrites the loop transfer recovery error $E(z)$ between the target loop transfer function $L_t(z)$ and that realised by any one of the above controllers, in terms of a so called *recovery matrix* $M(z)$. That is,

$$E(z) = M(z)[I_m + M(z)]^{-1}(I_m + F\Phi B). \tag{18}$$

The expression for the recovery matrix $M(z)$ depends on the controller used. In particular, we have the following expressions,

$$M_p(z) = F(zI_n - A + K_pC)^{-1}(B - K_pD), \tag{19}$$

$$M_c(z) = F\,(zI_n - A + K_cC_c)^{-1}\,(B - K_cD_c)\,, \tag{20}$$

$$M_r(z) = F_2(zI - A_r + K_rC_r)^{-1}(B_r - K_rD_r). \tag{21}$$

It is easy to see that

$$E^*(j\omega) = 0 \text{ if and only if } M^*(j\omega) = 0$$

for all $\omega \in \Omega$, where Ω is the set of all $0 \leq \mid \omega \mid \leq \pi/T$ for which $L_t^*(j\omega)$ and $L^*(j\omega) = \mathbf{C}^*(j\omega)P^*(j\omega)$ are well defined (i.e., all required inverses exist). This implies that the study of LTR can be cast in terms of the study of the recovery matrix $M(z)$. Also, since the expression for $M(z)$ for each controller is structurally similar to those of others, one can unify the LTR analysis and design involving all three different controllers into a single mathematical framework. This is done by defining the auxiliary systems Σ_c and Σ_r which are respectively characterised by the matrix quadruples (A, B, C_c, D_c) and (A_r, B_r, C_r, D_r).

In view of the above discussion, in order to determine the available design freedom for each controller, one needs to study an appropriate recovery matrix $M(z)$. Such a study has been undertaken in Part 1. It is shown there that the recovery matrix $M(z)$ can be decomposed into two parts, $M^0(z)$ that can always be rendered zero and $M^e(z)$ that cannot in general be rendered zero. As such $M^e(z)$ is termed as *recovery error matrix*. Let us present here a brief summary of the analysis given in Part 1. As in Part 1, our general discussion is always in terms of the given system Σ and the prediction estimator based controller. The details for other two controllers are presented only when they need to be emphasized.

Assuming that $A - K_pC$ is nondefective, one can expand the recovery matrix $M_p(z)$ in a dyadic form,

$$M_p(z) = \sum_{i=1}^{n} \frac{R_i}{z - \lambda_i} \tag{22}$$

where the residue R_i is given by

$$R_i = FW_iV_i^{H}[B - K_pD]. \tag{23}$$

Here W_i and V_i are respectively the right and left eigenvectors associated with an eigenvalue λ_i of $A - K_pC$ and they are scaled so that $WV^{H} = V^{H}W = I_n$ where

$$W = [W_1, W_2, \cdots, W_n] \quad \text{and} \quad V = [V_1, V_2, \cdots, V_n]. \tag{24}$$

To review what can and what cannot be rendered zero, let us partition $M_p(z)$ into three parts, each part having a particular type of characteristics,

$$M_p(z) = M_p^-(z) + M_p^b(z) + M_p^e(z), \tag{25}$$

where

$$M_p^-(z) = \sum_{i=1}^{n_a^-} \frac{R_i^-}{z - \lambda_i^-}, \quad M_p^b(z) = \sum_{i=1}^{n_b} \frac{R_i^b}{z - \lambda_i^b},$$

and

$$M_p^e(z) = \sum_{i=1}^{n_a^+ + n_c + n_f} \frac{R_i^e}{z - \lambda_i^e}.$$

In the above partition, appropriate superscripts $-$, b, and e are added to R_i and λ_i in order to associate them respectively with $M_p^-(z)$, $M_p^b(z)$, and $M_p^e(z)$. Next, define the following sets where $n_e = n_a^+ + n_c + n_f$:

$$\Lambda^- = \{\lambda_i^- \,;\, i{=}1 \text{ to } n_a^-\},\ V^- = \{V_i^- \,;\, i{=}1 \text{ to } n_a^-\},\ W^- = \{W_i^- \,;\, i{=}1 \text{ to } n_a^-\}$$

$$\Lambda^b = \{\lambda_i^b \,;\, i{=}1 \text{ to } n_b\},\quad V^b = \{V_i^b \,;\, i{=}1 \text{ to } n_b\},\quad W^b = \{W_i^b \,;\, i{=}1 \text{ to } n_b\}$$

$$\Lambda^e = \{\lambda_i^e \,;\, i{=}1 \text{ to } n_e\},\quad V^e = \{V_i^e \,;\, i{=}1 \text{ to } n_e\},\quad W^e = \{W_i^e \,;\, i{=}1 \text{ to } n_e\}.$$

We now proceed to describe in detail the necessary design constraints and the available design freedom in assigning an appropriate eigenstructure to $A - K_pC$. We do this by considering one part of $M_p(z)$ at a time.

Discussion on $M_p^-(z)$: Consider an arbitrary target loop transfer function $L_t(z)$. The term $M_p^-(z)$ can identically be rendered zero. To accomplish this, the set of n_a^- eigenvalues Λ^- and the corresponding set of left eigenvectors V^- of $A - K_pC$ must be selected to coincide respectively with the set of plant minimum phase invariant zeros and the corresponding left state zero directions of Σ.

Discussion on $M_p^b(z)$: Consider an arbitrary target loop transfer function $L_t(z)$. The term $M_p^b(z)$ can identically be rendered zero. To accomplish this, the set of n_b eigenvalues Λ^b can be assigned arbitrarily in $\mathbb{C}^{\odot}$, while the corresponding set of left eigenvectors V^b of $A - K_pC$ is in the null space of matrix $[B - K_pD]'$.

Discussion on $M_p^e(z)$: In general, for an arbitrary target loop transfer function $L_t(z)$, it cannot be rendered zero either asymptotically or otherwise by any assignment of Λ^e and the associated sets of right and left eigenvectors, W^e and V^e. Note also that the sets of eigenvectors W^e must span the subspace $\mathcal{S}^-(\Sigma)$ [3].

Since both $M_p^-(z)$ and $M_p^b(z)$ can be rendered zero, for future use, we can combine them into one term,

$$M_p^0(z) = M_p^-(z) + M_p^b(z).$$

We define likewise, $\Lambda^0 = \Lambda^- \cup \Lambda^b$, $W^0 = W^- \cup W^b$, $V^0 = V^- \cup V^b$. Similarly, we define the set of residues corresponding to the eigenvalues in Λ^0 as R^0, and the one corresponding to the eigenvalues in Λ^e as R^e. Thus $M_p(z)$ can be rewritten as

$$M_p(z) = M_p^0(z) + M_p^e(z). \tag{26}$$

To summarize the above development, $M_p(z)$ can essentially be decomposed into two parts, $M_p^0(z)$ and $M_p^e(z)$. The first part $M_p^0(z)$ is dependent on Λ^0 a set of eigenvalues, and R^0 the corresponding set of residues. R^0 in turn depends on the sets of right and left eigenvectors, W^0 and V^0. $M_p^0(z)$ can always be rendered zero by choosing appropriately Λ^0, W^0 and

V^0. On the other hand, the second part $M_p^e(z)$ cannot be rendered zero in general for an arbitrary target loop transfer function. Hence, $M_p^e(z)$ can be termed as the ***recovery error matrix***. This recovery error matrix $M_p^e(z)$ is parameterized in terms of Λ^e and R^e, where R^e in turn is parameterized in terms of W^e and V^e. There is complete freedom in choosing the set of eigenvalues Λ^e so that its elements are all with in the unit circle in complex plane, where as the sets of eigenvectors W^e and V^e have to satisfy the well known eigenvector assignment constraints [6]. Also, W^e must span the subspace $\mathcal{S}^-(\Sigma)$. Although W^e and V^e have to satisfy certain constraints, there exists a considerable amount of freedom in selecting them. As such, one can shape the recovery error matrix $M_p^e(z)$ by selecting appropriately Λ^e, W^e and V^e. In other words, for every given system and for each type of controller, there exists a set of admissible recovery error matrices, and such a set can be denoted as $\mathcal{M}^e(\Sigma)$. Thus, notationally, $\mathcal{M}_p^e(\Sigma)$, $\mathcal{M}_c^e(\Sigma)$ and $\mathcal{M}_r^e(\Sigma)$ are respectively the admissible sets of recovery error matrices for prediction, current and reduced order estimator based controllers. Now proceeding with our general discussion for a prediction estimator based controller, one then naturally seeks a design method which leads to a chosen recovery error matrix $M_p^e(z)$ among the set of admissible recovery error matrices $\mathcal{M}_p^e(\Sigma)$. In the following section, we will give an eigenstructure assignment design method capable of achieving any chosen $M_p^e(z) \in \mathcal{M}_p^e(\Sigma)$. In Section IV, we will describe two optimization based design methods, one method leads to a design that yields the infimum H_∞ norm of the recovery error matrix, while the other yields the infimum H_2 norm. We emphasize that the eigenstructure assignment design method can lead to any chosen recovery error matrix, where as the optimization based design methods yield a particular recovery error matrix having either the infimum H_∞ norm or H_2 norm depending on the method used.

The above discussion pertains to the case where the target loop transfer function $L_t(z) = F\Phi B$ is arbitrarily specified. However, as stated in Theorem 5 of Part 1, when specific properties of $L_t(z)$ are taken into account, one can render the recovery error matrix zero provided the given system satisfies the following conditions depending on the controller used:

(1) For a prediction estimator based controller, the condition is that $\mathcal{S}^-(\Sigma) \subseteq \operatorname{Ker}(F)$.

(2) For a current estimator based controller, the condition is that $\mathcal{S}^-(\Sigma) \cap \{x \mid Cx \in \operatorname{Im}(D)\} \subseteq \operatorname{Ker}(F)$.

(3) For a reduced order estimator based controller, the condition is that $\mathcal{S}^{-}(\Sigma) \cap \{x \mid Cx \in \operatorname{Im}(D)\} \subseteq \operatorname{Ker}(F)$.

Thus under the above conditions, the set of admissible recovery error matrices for a given target loop transfer function contains an element which is identically zero for all z in complex plane. In that case, either eigenstructure assignment method of design or optimization based methods of design can achieve exact loop transfer recovery.

III. DESIGN BY EIGENSTRUCTURE ASSIGNMENT

For continuous time systems, we developed earlier a time-scale and eigenstructure assignment (ATEA) method of LTR design which is capable of exploiting all the available design freedom [8], [2]. In discrete time systems, as explained earlier, one does not deal with asymptotically infinite eigenvalues and as such there is no feasibility of assigning a multiple time scale structure to controller dynamics as in the case of continuous systems. That is, in discrete time systems, one can assign only a finite eigenstructure. As such, the design procedure we propose here is a special case of ATEA in which the option of assigning a chosen time scale structure is removed and hence is some what simpler than that for continuous systems. Although there is no time scale structure assignment, since the method proposed here is a special case of ATEA, we still call it as ATEA design. The present ATEA design method does not call for parameterizing the gain K in terms of a tunable parameter.

The input parameters to ATEA design are the sets of eigenvalues Λ^b and Λ^s, and the residue set R^s which can equivalently be specified in terms of the right and left eigenvectors W^s and V^s. Also, R^0 is to be rendered zero so that $M_p^0(z) = 0$. Note that Λ^b and Λ^s in addition to Λ^- form the eigenvalues of the observer dynamic matrix. Furthermore Λ^s and R^s shape the recovery error matrix $M_p^s(z)$ as desired. Thus the prescription of Λ^s and R^s is equivalent to prescribing a desired $M_p^s(z) \in \mathcal{M}_p^s(\Sigma)$. We now give a step by step ATEA design method of obtaining the observer gain K_p which when used in prediction estimator based controller leads to the prescribed recovery error matrix $M_p^s(z)$. The following steps of the ATEA design algorithm assume that the given system Σ has already been transformed to the form of s.c.b (see, Section III of Part 1 [3]).

Step 1 : This step deals with the assignment of eigenstructure to the subsystem (9) of Part 1. Choose a gain K_p^b such that $\lambda(A_{bb} - K_p^b C_b)$ coincides with the specified set Λ^b. Note that the existence of such a K_p^b is guaranteed by Property 2 of Section III of Part 1 [3]. We also note that the eigenvectors of $A_{bb} - K_p^b C_b$ can be assigned in any chosen way consistent with the freedom available in assigning them [6]. Owing to the properties of s.c.b, ATEA design always results in an eigenvector set V^b corresponding to the eigenvalues Λ^b of the observer, in the null space of $(B - K_p D)'$ so that $M_p^b(z) = 0$.

Step 2 : This step deals with the assignment of eigenstructure to the subsystems (8), (10) and (12) of Part 1. Let A^e and C^e be defined as

$$A^e = \begin{bmatrix} A_{aa}^+ & 0 & L_{af}^+ C_f \\ B_c E_{ca}^+ & A_{cc} & L_{cf} C_f \\ B_f E_a^+ & B_f E_c & A_f \end{bmatrix}, \qquad C^e = \begin{bmatrix} C_{0a}^+ & C_{0c} & C_{0f} \\ 0 & 0 & C_f \end{bmatrix}. \tag{27}$$

The design specifications utilised here are Λ^e and W^e. In view of the s.c.b, W^e is of the form,

$$W^e = [0 \quad (W_{a+}^e)^{\mathrm{H}} \quad 0 \quad (W_c^e)^{\mathrm{H}} \quad (W_f^e)^{\mathrm{H}}]^{\mathrm{H}}.$$

Let $W_e^e = [(W_{a+}^e)^{\mathrm{H}} \quad (W_c^e)^{\mathrm{H}} \quad (W_f^e)^{\mathrm{H}}]^{\mathrm{H}}$. Now select a gain K_p^e such that $\lambda(A^e - K_p^e C^e)$ and the set of right eigenvectors of $A^e - K_p^e C^e$ coincide with the specified Λ^e and W_e^e. Again note that the existence of such a K_p^e is guaranteed by Property 2 of Section III of Part 1. Let us next partition K_p^e as

$$K_p^e = \begin{bmatrix} K_p^{a0+} & K_p^{a1+} \\ K_p^{c0} & K_p^{c1} \\ K_p^{f0} & K_p^{f1} \end{bmatrix}.$$

Step 3 : In this step, K_p^b and K_p^e calculated in Steps 1 and 2 are put together into a composite matrix. Let

$$K_p = \Gamma_1 \begin{bmatrix} B_{0a}^- & L_{af}^- & L_{ab}^- \\ B_{0a}^+ + K_p^{a0+} & K_p^{a1+} & L_{ab}^+ \\ B_{0b} & L_{bf} & K_p^b \\ B_{0c} + K_p^{c0} & K_p^{c1} & L_{cb} \\ B_{0f} + K_p^{f0} & K_p^{f1} & 0 \end{bmatrix} \Gamma_2^{-1}. \tag{28}$$

We have the following theorem.

Theorem 1. *Consider a gain as given by (28). Then we have the following properties:*

1. *The eigenvalues of $A - K_p C$ are given by Λ^-, Λ^b and Λ^e.*
2. *The achieved recovery error matrix coincides with the specified $M_p^e(z)$ for all z in $\mathbf{C}$.*

Proof : It follows from the properties of s.c.b and some simple algebra. ∎

Example 1 : Consider a non-strictly proper discrete-time system Σ with sampling period $T = 1$, and characterised by

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0.5 & 0 \\ 1 & 0 & 0 & 1.5 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix},$$

$$C = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \qquad D = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Let the target loop $L_t(z)$ and thus the target sensitivity function $S_t(z)$ be specified by

$$F = \begin{bmatrix} 0.00 & 0.0000 & 1 & 0.000 \\ 1.25 & 0.8333 & 1 & 2.875 \end{bmatrix}.$$

It is simple to verify that the given system Σ is invertible, i.e., $n_b = n_c = 0$, with two infinite zeros of order 2 and two invariant zeros at $\{-0.5, 1.5\}$. It can also be verified that the target loop specified by the given F is not recoverable by any of the three controllers being considered in this paper. Let Λ^e and W^e along with the corresponding recovery error matrix $M^e(z)$ be as given below depending on the controller used:

Prediction Estimator Based Controller :

$$\Lambda^e = \{-0.1, 0, 0.1\} \ , \ W^e = \begin{bmatrix} -0.3684 & -0.3478 & 0.3276 \\ -0.5157 & -0.5217 & 0.5241 \\ 0.0000 & 0.0000 & 0.0000 \\ -0.7735 & -0.7790 & 0.7861 \end{bmatrix}$$

and

$$M_p^e(z) = \frac{1}{z^3 - 0.01z} \begin{bmatrix} 0 & 0 \\ 0 & 0.8333z^2 + 1.25z - 13.41 \end{bmatrix}.$$

Current Estimator Based Controller :

$$\Lambda^s = \{-0.1, 0.1, \} \ , \ W^s = \begin{bmatrix} 0.0000 & 0.0000 \\ -0.5812 & -0.5300 \\ 0.0000 & 0.0000 \\ -0.8137 & -0.8480 \end{bmatrix}$$

and

$$M_c^s(z) = \frac{1}{z^2 - 0.01} \begin{bmatrix} 0 & 0 \\ 0 & 0.8333z - 7.69 \end{bmatrix}.$$

Reduced Order Estimator Based Controller :

$$\Lambda^s = \{-0.1, 0.1, \} \ , \ W^s = \begin{bmatrix} -0.5812 & -0.5300 \\ 0.0000 & 0.0000 \\ -0.8137 & -0.8480 \end{bmatrix}$$

and

$$M_r^s(z) = \frac{1}{z^2 - 0.01} \begin{bmatrix} 0 & 0 \\ 0 & 0.8333z - 7.69 \end{bmatrix}.$$

Then using ATEA algorithm, we obtain the following controllers.

Prediction Estimator Based Controller :

$$\hat{x}(k+1) = \begin{bmatrix} -1.50 & 1.0000 & 0.0 & 0.000 \\ -3.49 & -0.8333 & 0.0 & -1.875 \\ 0.00 & 0.0000 & -0.5 & 0.000 \\ -3.36 & 0.0000 & 0.0 & 1.500 \end{bmatrix} \hat{x}(k) + \begin{bmatrix} 0 & 1.50 \\ 0 & 2.24 \\ 1 & 1.00 \\ 0 & 4.36 \end{bmatrix} y(k),$$

$$-u(k) = \begin{bmatrix} 0.00 & 0.0000 & 1 & 0.000 \\ 1.25 & 0.8333 & 1 & 2.875 \end{bmatrix} \hat{x}(k).$$

The eigenvalues of the above prediction estimator are at $\{-0.5, -0.1, 0, 0.1\}$ while the achieved recovery error matrix M_p^s coincides with the one specified.

Current Estimator Based Controller :

$$v(k+1) = \begin{bmatrix} 0.00 & 0.0000 & 0.0 & 0.000 \\ -1.25 & -2.3333 & 0.0 & -1.875 \\ 1.00 & 0.0000 & -0.5 & 0.000 \\ 1.00 & -2.2400 & 0.0 & 1.500 \end{bmatrix} v(k) + \begin{bmatrix} 0 & 0.00 \\ 0 & -8.95 \\ 1 & 1.00 \\ 0 & 1.00 \end{bmatrix} y(k),$$

$$-u(k) = \begin{bmatrix} 0.00 & 0.0000 & 1 & 0.000 \\ 1.25 & 0.8333 & 1 & 2.875 \end{bmatrix} v(k) + \begin{bmatrix} 0 & 0.00 \\ 0 & 8.94 \end{bmatrix} y(k).$$

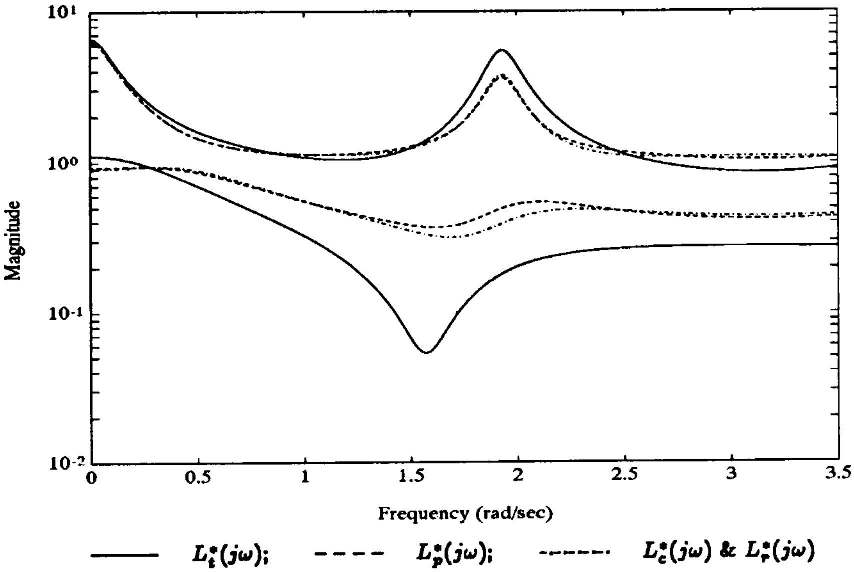


Figure 1: The max. and min. singular values of target and achieved loops.

The eigenvalues of the above current estimator are at $\{-0.5, -0.1, 0, 0.1\}$ while the achieved recovery error matrix M_c^s coincides with the one specified.

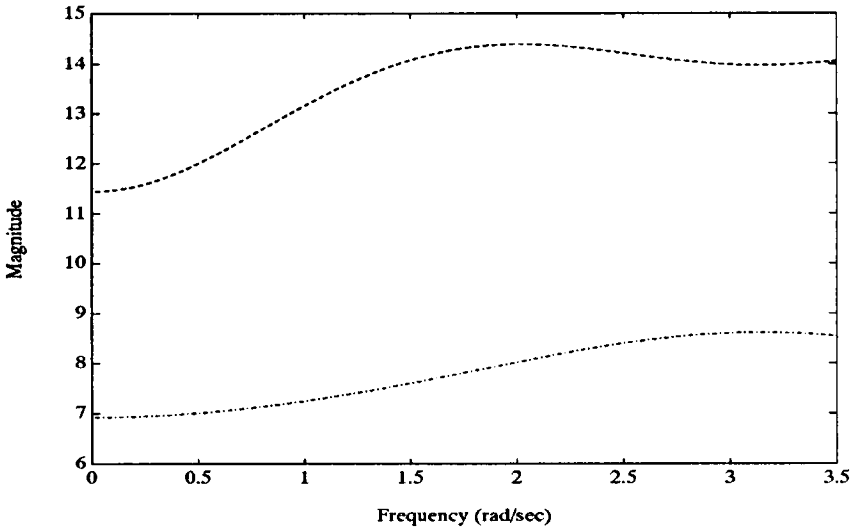
Reduced Order Estimator Based Controller :

$$v(k+1) = \begin{bmatrix} -2.3333 & 0.0 & -1.875 \\ 0.0000 & -0.5 & 0.000 \\ -2.2400 & 0.0 & 1.500 \end{bmatrix} v(k) + \begin{bmatrix} 0 & -8.95 \\ 1 & 1.00 \\ 0 & 1.00 \end{bmatrix} y(k),$$

$$-u(k) = \begin{bmatrix} 0.0000 & 1 & 0.000 \\ 0.8333 & 1 & 2.875 \end{bmatrix} v(k) + \begin{bmatrix} 0 & 0.00 \\ 0 & 8.94 \end{bmatrix} y(k).$$

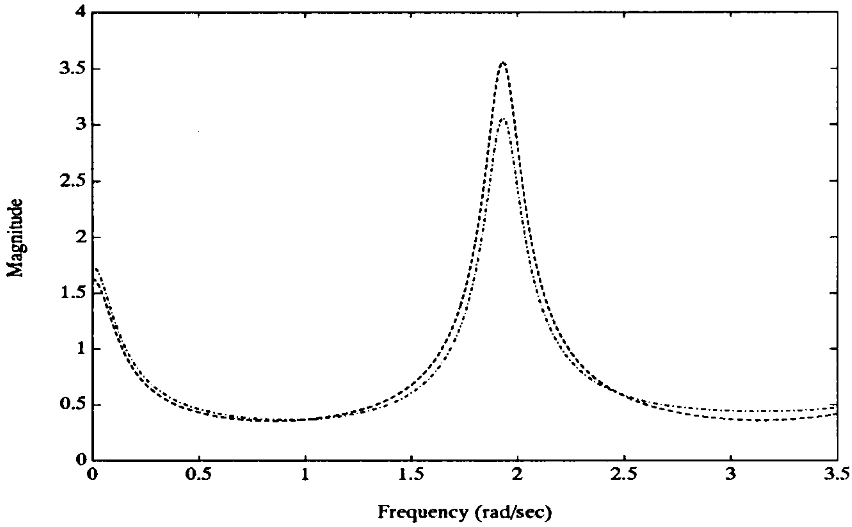
The eigenvalues of the above reduced order estimator are at $\{-0.5, -0.1, 0.1\}$ while the achieved recovery error matrix M_r^s coincides with the one specified.

The plots of maximum and minimum singular values of the target and the achieved loop transfer function via all the three controllers are shown in Figure 1. Also, the plots of maximum singular values of $M_p^*(j\omega)$, $M_c^*(j\omega)$ and $M_r^*(j\omega)$ are shown in Figure 2, while the plots of maximum singular



- - - - $\sigma_{max}[M_p^*(j\omega)]$; -·-·-·- $\sigma_{max}[M_c^*(j\omega)]$ & $\sigma_{max}[M_r^*(j\omega)]$

Figure 2: The max. singular values of recovery matrices.



- - - - $\sigma_{max}[E_p^*(j\omega)]$; -·-·-·- $\sigma_{max}[E_c^*(j\omega)]$ & $\sigma_{max}[E_r^*(j\omega)]$

Figure 3: The max. singular values of loop transfer recovery errors.

values of $E_p^*(j\omega)$, $E_c^*(j\omega)$ and $E_r^*(j\omega)$ are shown in Figure 3. Clearly, as expected, the above current and reduced order estimators yield the same performance in the sense that the achieved recovery is same. □

IV. OPTIMIZATION BASED DESIGN METHODS

As is clear from Section II, the whole notion of LTR is to render the recovery matrix $M_p(z) = F(zI_n - A + K_pC)^{-1}(B - K_pD)$ small in some sense or other. The ATEA design method of Section III views this task from the perspective of eigenstructure assignment to the controller dynamic matrix. It enables us to design a controller which achieves any specified recovery error matrix among a set of such admissible matrices. An alternative method, as in the case of continuous systems, is to view the controller design for LTR as finding an observer gain K_p which minimizes some (say, either H_∞ or H_2) norm of $M_p(z)$. That is, one can cast the LTR design as a straightforward mathematical optimization problem. A suboptimal or optimal solution to such an optimization problem provides the needed observer gain. Note that when LTR problem is formulated as an optimization problem of minimizing some norm of $M_p(z)$ by appropriate selection of K_p, the optimization method apparently renders $M_p^0(z)$ zero while minimizing the specified norm of $M_p^e(z)$. In contrast to this, eigenstructure assignment method is flexible, and is capable of yielding any recovery error matrix $M_p^e(z) \in \mathcal{M}_p^e(\Sigma)$ while rendering $M_p^0(z)$ zero.

Goodman [5] is the first person who formulated earlier the LTR problem for discrete systems as a H_2 minimization problem of the recovery matrix $M_p(z)$. He considered only strictly proper square invertible minimum phase systems having infinite zeros of order one. Recently, Zhang and Freudenberg [11] considered square strictly proper nonminimum phase systems. They develop explicit expressions for the resulting recovery error matrix and the sensitivity function when prediction as well as current estimator based controllers are used and when optimization is used to minimize the H_2 norm of the recovery matrix. The optimization procedure used by [11] follows along the same lines as that for continuous systems as in [4]. It turns out, as in the continuous case [2], that the controller design based on optimization procedures for LTR of general discrete systems, can be cast as an optimal state feedback design for an auxiliary system related to the given one. Then, following a mass of existing literature on such optimal state feedback designs, especially for continuous systems, one can develop in a straightforward way optimisation based design procedures for LTR of general discrete systems. This is what we pursue in this section.

To proceed with, we consider the following auxiliary system,

$$\Sigma^{au1}: \begin{cases} x(k+1) = A'x(k) + C'u(k) + F'w(k), \\ y(k) \quad = x(k), \\ z(k) \quad = B'x(k) + D'u(k). \end{cases} \tag{29}$$

Here w is treated as an exogenous disturbance input to Σ^{au1} and u is the controlling input while the variable z is considered as the controlled output. Then it is trivial to verify that the closed loop transfer function from w to y of Σ^{au1} under a state feedback law $u(k) = -K_p' x(k)$ is given by

$$M'(z) = (B' - D'K_p')(zI_n - A' + C'K_p')^{-1}F'. \tag{30}$$

Hence, the minimisation of $M_p(z)$ over all the possible stabilising gains K_p is equivalent to the minimisation of $M'(z)$ over all the stabilising state feedback control laws for Σ^{au1}. As such the design of observer based controllers for LTR is translated to an optimal state feedback controller design.

A. H_∞-optimization Based Algorithm

Throughout this section, we assume that the given system Σ characterized by (A, B, C, D) has no invariant zeros on the unit circle. Denoting γ^* as the infimum of $\|M_p(z)\|_\infty$ over all possible stabilising gains K_p, we present here a basic algorithm of computing the gain matrix $K_p(\gamma)$ such that the resulting H_∞-norm of the recovery matrix $M_p(z,\gamma)$, is less than a priori given desired scalar $\gamma > \gamma^*$. The algorithm is as follows:

Step 1 : At first we compute nonsingular transformations U and V such that

$$UD'V = \begin{bmatrix} I_{m_0} & 0 \\ 0 & 0 \end{bmatrix}.$$

Then partition

$$C'V = [\,\tilde{B}_0 \quad \tilde{B}_1\,] \quad \text{and} \quad UB' = \begin{bmatrix} \tilde{C}_0 \\ \tilde{C}_1 \end{bmatrix}.$$

Following the procedure of constructing a special coordinate basis (s.c.b), see Section III of Part 1 [3], one can readily calculate three nonsingular transformations Γ_s, Γ_i and Γ_o such that

$$\Gamma_s^{-1}(A' - \tilde{B}_0\tilde{C}_0)\Gamma_s = \begin{bmatrix} A_{aa}^- & 0 & L_{ab}^- C_b & 0 & L_{af}^- C_f \\ 0 & A_{aa}^+ & L_{ab}^+ C_b & 0 & L_{af}^+ C_f \\ 0 & 0 & A_{bb} & 0 & L_{bf} C_f \\ B_c E_{ca}^- & B_c E_{ca}^+ & L_{cb} C_b & A_{cc} & L_{cf} C_f \\ B_f E_{fa}^- & B_f E_{fa}^+ & B_f E_{fb} & B_f E_{fc} & A_f \end{bmatrix}, \tag{31}$$

$$\Gamma_s^{-1}[\,\tilde{B}_0 \quad \tilde{B}_1\,]\Gamma_i = \begin{bmatrix} B_{0a}^- & 0 & 0 \\ B_{0a}^+ & 0 & 0 \\ B_{0b} & 0 & 0 \\ B_{0c} & 0 & B_c \\ B_{0f} & B_f & 0 \end{bmatrix}, \quad \Gamma_s^{-1}F' = \begin{bmatrix} E_a^- \\ E_a^+ \\ E_b \\ E_c \\ E_f \end{bmatrix}, \tag{32}$$

$$\Gamma_o^{-1}\begin{bmatrix} \tilde{C}_0 \\ \\ \tilde{C}_1 \end{bmatrix}\Gamma_s = \begin{bmatrix} C_{0a}^- & C_{0a}^+ & C_{0b} & C_{0c} & C_{0f} \\ 0 & 0 & 0 & 0 & C_f \\ 0 & 0 & C_b & 0 & 0 \end{bmatrix}, \tag{33}$$

and

$$\Gamma_o^{-1}\begin{bmatrix} I_{m_0} & 0 \\ \\ 0 & 0 \end{bmatrix}\Gamma_i = \begin{bmatrix} I_{m_0} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{34}$$

Next, we define a subsystem Σ^{au2} of the above dual system as,

$$\Sigma^{au2} : \begin{cases} x(k+1) = Ax(k) + Bu(k) + Ew(k), \\ y(k) \quad = x(k), \\ z(k) \quad = Cx(k) + Du(k), \end{cases} \tag{35}$$

where

$$A = \begin{bmatrix} A_{aa}^- & 0 & L_{ab}^- C_b & L_{af}^- C_f \\ 0 & A_{aa}^+ & L_{ab}^+ C_b & L_{af}^+ C_f \\ 0 & 0 & A_{bb} & L_{bf} C_f \\ B_f E_{fa}^- & B_f E_{fa}^+ & B_f E_{fb} & A_f \end{bmatrix}, \quad B = \begin{bmatrix} B_{0a}^- & 0 \\ B_{0a}^+ & 0 \\ B_{0b} & 0 \\ B_{0f} & B_f \end{bmatrix}, \tag{36}$$

and

$$E = \begin{bmatrix} E_a^- \\ E_a^+ \\ E_b \\ E_f \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & C_f \\ 0 & 0 & C_b & 0 \end{bmatrix}, \quad D = \begin{bmatrix} I_{m_0} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}. \tag{37}$$

Here we note that the system characterised by (A, B, C, D) is left invertible and has no invariant zeros on the unit circle if and only if Σ characterized by (A, B, C, D) has no invariant zeros on the unit circle.

Step 2 : Solve the Riccati equation,

$$P = A'PA + C'C - \begin{bmatrix} B'PA + D'C \\ E'PA \end{bmatrix}' \cdot \begin{bmatrix} D'D + B'PB & B'PE \\ E'PB & -\gamma^2 I + E'PE \end{bmatrix}^{-1} \begin{bmatrix} B'PA + D'C \\ E'PA \end{bmatrix},$$

for a positive semi-definite P which satisfies the conditions:

$$D'D + B'PB > 0 \quad \text{and} \quad I - E'PE/\gamma^2 > 0$$

and

$$\lambda \left\{ A - [B \quad E] \begin{bmatrix} D'D + B'PB & B'PE \\ E'PB & -\gamma^2 I + E'PE \end{bmatrix}^{-1} \begin{bmatrix} B'PA + D'C \\ E'PA \end{bmatrix} \right\} \subset \mathbf{C}^{\odot}.$$

We note that such a P always exists and is unique since (A, B, C, D) is left invertible and has no invariant zeros on the unit circle [10].

Step 3 : Compute

$$F_1 = [B'PB + D'D + B'PE(\gamma^2 I - E'PE)^{-1} E'PB]^{-1} \cdot [B'PA + D'C + B'PE(\gamma^2 I - E'PE)^{-1} E'PA]$$

and partition it as

$$F_1 = \begin{bmatrix} F_{a0}^- & F_{a0}^+ & F_{b0} & F_{f0} \\ F_{af}^- & F_{af}^+ & F_{bf} & F_{ff} \end{bmatrix}.$$

Then let

$$F(\gamma) = V\Gamma_i \begin{bmatrix} C_{0a}^- + F_{a0}^- & C_{0a}^+ + F_{a0}^+ & C_{0b} + F_{b0} & C_{0c} & C_{0f} + F_{f0} \\ F_{af}^- & F_{af}^+ & F_{bf} & E_{fc} & F_{ff} \\ 0 & 0 & 0 & F_{cc} & 0 \end{bmatrix} \Gamma_s^{-1}$$

where F_{cc} is such that $\lambda(A_{cc} - B_c F_{cc}) \in \mathbf{C}^{\odot}$. Next, choose $K_p(\gamma)$ as

$$K_p(\gamma) = F'(\gamma). \tag{38}$$

We have the following theorem.

Theorem 2. *Let $K_p(\gamma)$ be computed as in (38) and let $M_p(z,\gamma)$ be the resulting recovery matrix. Then, $\|M_p(z,\gamma)\|_\infty$ is strictly less than γ, and tends to γ^* as $\gamma \to \gamma^*$.*

Proof : See Appendix A. ∎

Remark 1. *Note that once the estimator gains $K_p(\gamma^*)$, $K_c(\gamma^*)$ and $K_r(\gamma^*)$ are calculated from the above design procedure, the corresponding recovery error matrices can easily be calculated from the expressions (19), (20) and (21), i.e.,*

$$M_p^*(z) = F[zI_n - A + K_p(\gamma^*)C]^{-1}[B - K_p(\gamma^*)D], \tag{39}$$

$$M_c^*(z) = F\left[zI_n - A + K_c(\gamma^*)C_c\right]^{-1}\left[B - K_c(\gamma^*)D_c\right], \tag{40}$$

$$M_r^*(z) = F_2[zI - A_r + K_r(\gamma^*)C_r]^{-1}[B_r - K_r(\gamma^*)D_r]. \tag{41}$$

Moreover, these recovery error matrices have the least H_∞ norm among the sets of the corresponding admissible recovery error matrices.

Example 2 : Consider a discrete-time system Σ given in Astrom et al [1] with sampling period $T = 1$, and characterized by

$$A = \begin{bmatrix} 1.1036 & 1 & 0 \\ -0.4060 & 0 & 1 \\ 0.0498 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0.0803 \\ 0.1544 \\ 0.0179 \end{bmatrix},$$

$$C = [1 \quad 0 \quad 0], \qquad D = 0.$$

Let the target loop $L_t(z)$ and the target sensitivity function $S_t(z)$ be specified by

$$F = [7.1222 \quad 7.5293 \quad 2.7373].$$

It is simple to verify that the given system Σ is invertible with one infinite zero of order one and two invariant zeros at $\{-1.7989, -0.1239\}$. It can also be verified that the target loop specified by the given F is not recoverable either by a prediction or by a current or by a reduced order estimator based controller. The following are the prediction, current and reduced order estimator based controllers obtained by the H_∞-optimization based algorithm. All these controllers achieve the infimum of the H_∞-norm of the corresponding recovery matrices.

Prediction Estimator Based Controller :

$$\hat{x}(k+1) = \begin{bmatrix} -1.5371 & 0.3954 & -0.2198 \\ -1.2039 & -1.1625 & 0.5774 \\ -0.1275 & -0.1348 & -0.0490 \end{bmatrix} \hat{x}(k) + \begin{bmatrix} 2.0688 \\ -0.3017 \\ 0.0498 \end{bmatrix} y(k),$$

$$-u(k) = [7.1222 \quad 7.5293 \quad 2.7373]\,\hat{x}(k).$$

The eigenvalues of the above prediction estimator are placed at $\{0, -0.1239, -0.8413\}$ while the resulting recovery error matrix $M_p^e(z)$ is given by,

$$M_p^e(z) = \frac{1.7834z + 2.1198}{z^2 + 0.8413z}.$$

Current Estimator Based Controller :

$$v(k+1) = \begin{bmatrix} 0.0000 & 0.0000 & 0.0000 \\ -1.5716 & -1.2115 & 0.6046 \\ -0.0777 & -0.1348 & -0.0490 \end{bmatrix} + \begin{bmatrix} 0.0000 \\ -1.7217 \\ -0.0944 \end{bmatrix} y(k),$$

$$-u(k) = [7.1222 \quad 7.5293 \quad 2.7373]\,v(k) + 8.0552y(k).$$

The eigenvalues of the above current estimator are placed at $\{0, 0, -0.1239\}$ while the resulting recovery error matrix $M_c^e(z)$ is given by,

$$M_c^e(z) = \frac{1.1366}{z}.$$

Reduced Order Estimator Based Controller :

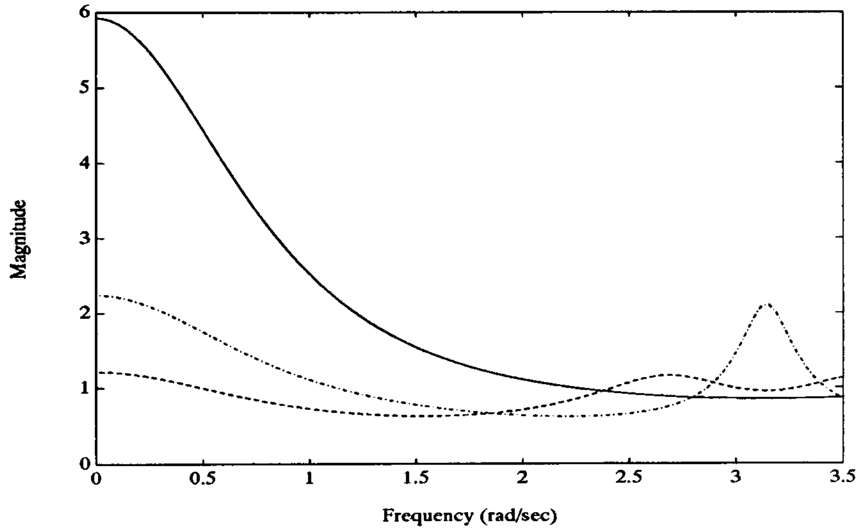
$$v(k+1) = \begin{bmatrix} -1.2115 & 0.6046 \\ -0.1348 & -0.0490 \end{bmatrix} v(k) + \begin{bmatrix} -1.7217 \\ -0.0944 \end{bmatrix} y(k),$$

$$-u(k) = [7.5293 \quad 2.7373]\,v(k) + 8.0553y(k).$$

The eigenvalues of the above reduced order estimator are placed at $\{0, -0.1239\}$ while the resulting recovery error matrix $M_r^e(z)$ is given by,

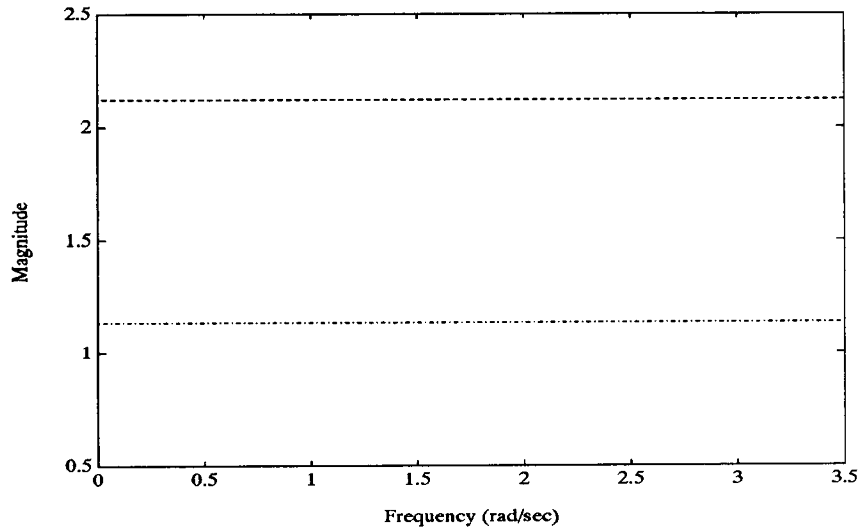
$$M_r^e(z) = \frac{1.1366}{z}.$$

The plots of singular values of the target and the achieved loop transfer function via all the three controllers are shown in Figure 4. Also, the plots of singular values of $M_p^*(j\omega)$, $M_c^*(j\omega)$ and $M_r^*(j\omega)$ are shown in Figure 5, while the plots of singular values of $E_p^*(j\omega)$, $E_c^*(j\omega)$ and $E_r^*(j\omega)$ are shown in Figure 6. Clearly, for this example, the above current and reduced order estimators yield the same performance in the sense that the achieved recovery is same. □



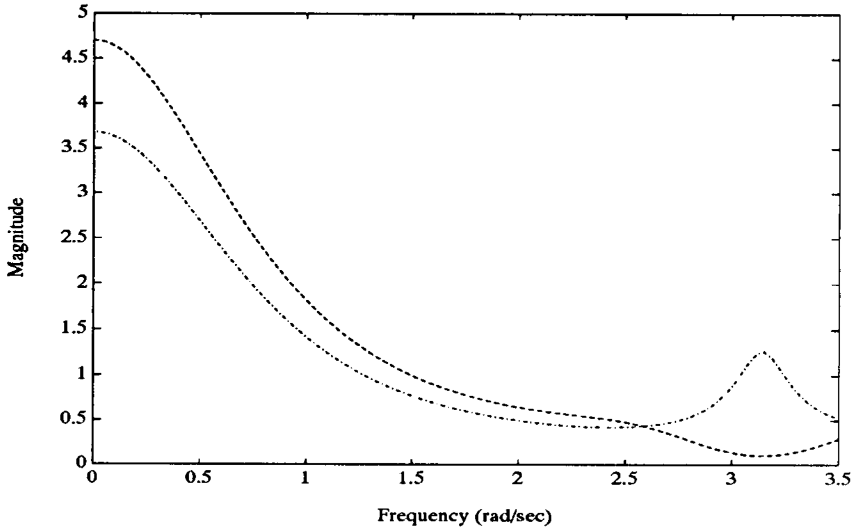
—— $|L_t^*(j\omega)|$; - - - - $|L_p^*(j\omega)|$; -·-·-·- $|L_c^*(j\omega)|$ & $|L_r^*(j\omega)|$

Figure 4: The singular values of target and achieved loops.



- - - - $|M_p^*(j\omega)|$; -·-·-·- $|M_c^*(j\omega)|$ & $|M_r^*(j\omega)|$

Figure 5: The singular values of recovery matrices.



---- $|E_p^*(j\omega)|$; ------- $|E_c^*(j\omega)|$ & $|E_r^*(j\omega)|$

Figure 6: The singular values of loop transfer recovery errors.

B. H_2-optimization Based Algorithm

It is well-known in the literature that the solution of H_2-optimization problem is equivalent to the solution of H_∞-optimization where in γ is set to ∞. Utilizing this fact, in this subsection, we proceed to give an algorithm that minimizes the H_2-norm of $M_p(z)$ over all possible stabilizing gain matrices K_p. To do so, as in the previous subsection, we assume that the given system Σ characterized by (A, B, C, D) has no invariant zeros on the unit circle.

Step 1 : Transform the dual system of (A, B, C, D) in the form of s.c.b, and construct an auxiliary system Σ^{au2} as in (35).

Step 2 : Solve the Riccati equation,

$$P = A'PA + C'C - (B'PA + D'C)'(D'D + B'PB)^{-1}(B'PA + D'C),$$

for a positive semi-definite P which satisfies the conditions:

$$D'D + B'PB > 0$$

and

$$\lambda\{A - B(D'D + B'PB)^{-1}(B'PA + D'C)\} \subset \mathbf{C}^{\odot}.$$

Again, we note that such a P always exists and is unique.

Step 3 : Partition,

$$[B'PB + D'D]^{-1}[B'PA + D'C] = \begin{bmatrix} F_{a0}^{-} & F_{a0}^{+} & F_{b0} & F_{f0} \\ F_{af}^{-} & F_{af}^{+} & F_{bf} & F_{ff} \end{bmatrix}.$$

Step 4 : Let

$$F = V\Gamma_i \begin{bmatrix} C_{0a}^{-} + F_{a0}^{-} & C_{0a}^{+} + F_{a0}^{+} & C_{0b} + F_{b0} & C_{0c} & C_{0f} + F_{f0} \\ F_{af}^{-} & F_{af}^{+} & F_{bf} & E_{fc} & F_{ff} \\ 0 & 0 & 0 & F_{cc} & 0 \end{bmatrix} \Gamma_s^{-1} \tag{42}$$

where F_{cc} is such that $\lambda(A_{cc} - B_c F_{cc}) \in \mathbf{C}^{\odot}$. Next, choose K_p as

$$K_p = F'. \tag{43}$$

We have the following theorem.

Theorem 3. *Let K_p be computed as in (43) and let $M_p(z)$ be the resulting recovery matrix. Then, $\|M_p(z)\|_{H_2}$ is the infimum among all the possible ones.*

Proof : It follows from the well-known relationship between H_∞- and H_2-optimizations. ∎

Remark 2. *Consider the case when the system characterized by the matrix quadruple (A, B, C, D) is invertible with $D = 0$ and $\det(CB) \neq 0$. Then, as determined first by Goodman [5], the gain matrix of (43) reduces to $K_p = [(CB)^{-1}CA]'$ if the system characterized by (A, B, C, D) is of minimum phase. On the other hand, the gain matrix of (43) reduces to $K_p = [(C_m B)^{-1} C_m A]'$ if the system characterized by (A, B, C, D) is of nonminimum phase, where C_m is the minimum phase counterpart of C in the all-pass factorization of (A, B, C, D) [11].*

Remark 3. *Consider a special case of a strictly proper square invertible system with all its infinite zeros of order one. For this special case, and when prediction and current estimator based controllers are used, and moreover*

when the target loop transfer function is specified by breaking the loop at the output point of the given system, Zhang and Freudenberg [11] earlier developed closed-form expressions for the resulting recovery errors when observer gain matrices are calculated using their H_2-optimization procedure. It can easily be shown that for the special cases considered by Zhang and Freudenberg [11], the resulting recovery error matrices when our design procedures are used, are indeed equivalent to the ones given by Zhang and Freudenberg.

Example 3 : Consider the system and the target loop given in Example 2. The following are the prediction, current and reduced order estimator based controllers obtained by the H_2-optimization based algorithm. All these controllers achieve the infimum of the H_2-norm of the corresponding recovery matrices.

Prediction Estimator Based Controller :

$$\hat{x}(k+1) = \begin{bmatrix} -1.2517 & 0.3954 & -0.2198 \\ -1.1686 & -1.1625 & 0.5774 \\ -0.1275 & -0.1348 & -0.0490 \end{bmatrix} \hat{x}(k) + \begin{bmatrix} 1.7834 \\ -0.3371 \\ 0.0498 \end{bmatrix} y(k),$$

$$-u(k) = [7.1222 \quad 7.5293 \quad 2.7373]\, \hat{x}(k).$$

The eigenvalues of the above prediction estimator are placed at $\{0, -0.1239, -0.5559\}$ while the resulting recovery error matrix $M_p^e(z)$ is given by,

$$M_p^e(z) = \frac{1.7834z + 1.7954}{z^2 + 0.5559z}.$$

Current Estimator Based Controller :

$$v(k+1) = \begin{bmatrix} 0.0000 & 0.0000 & 0.0000 \\ -1.8671 & -1.4313 & 0.7268 \\ -0.1143 & -0.1620 & -0.0339 \end{bmatrix} v(k) + \begin{bmatrix} 0.0000 \\ -2.7901 \\ -0.2268 \end{bmatrix} y(k),$$

$$-u(k) = [7.1222 \quad 7.5293 \quad 2.7373]\, v(k) + 12.4294y(k).$$

The eigenvalues of the above current estimator are placed at $\{0, -0.1239, -0.5559\}$ while the resulting recovery error matrix $M_c^e(z)$ is given by,

$$M_c^e(z) = \frac{0.7854}{z + 0.5559}.$$

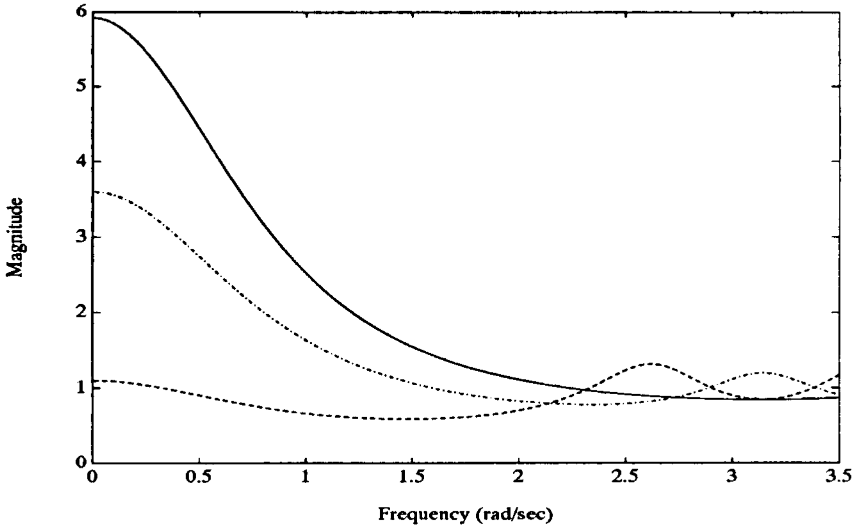


Figure 7: The singular values of target and achieved loops.

Reduced Order Estimator Based Controller :

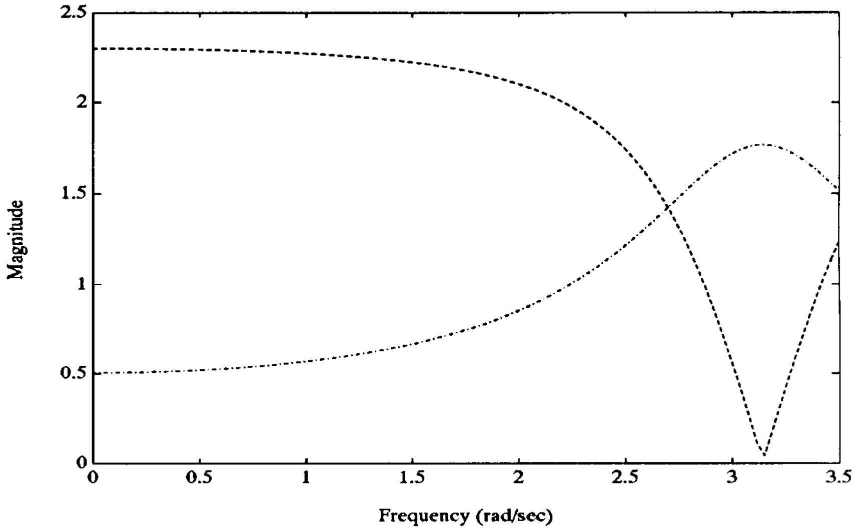
$$v(k+1) = \begin{bmatrix} -1.4313 & 0.7268 \\ -0.1620 & -0.0339 \end{bmatrix} v(k) + \begin{bmatrix} -2.7901 \\ -0.2268 \end{bmatrix} y(k),$$

$$-u(k) = [\,7.5293 \quad 2.7373\,]\, v(k) + 12.4294 y(k).$$

The eigenvalues of the above reduced order estimator are placed at $\{-0.1239, -0.5559\}$ while the resulting recovery error matrix $M_r^*(z)$ is given by,

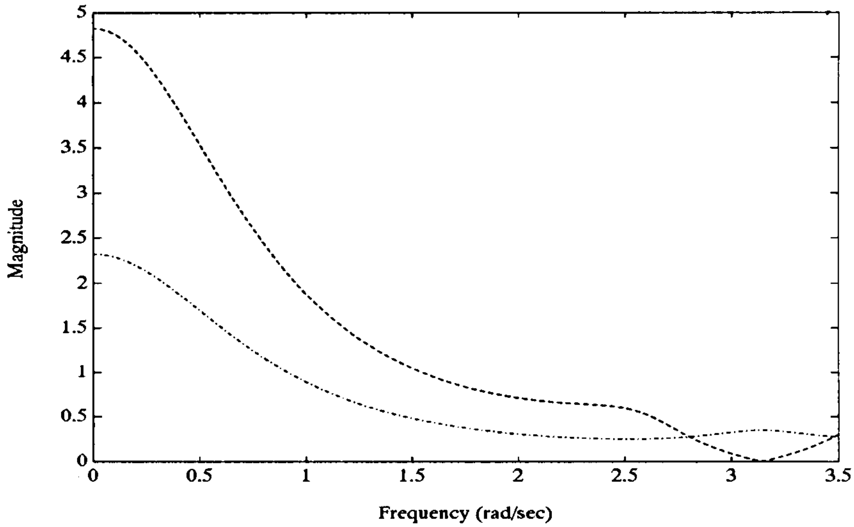
$$M_r^*(z) = \frac{0.7854}{z + 0.5559}.$$

The plots of singular values of the target and the achieved loop transfer function via all the three controllers are shown in Figure 7. Also, the plots of singular values of $M_p^*(j\omega)$, $M_c^*(j\omega)$ and $M_r^*(j\omega)$ are shown in Figure 8, while the plots of singular values of $E_p^*(j\omega)$, $E_c^*(j\omega)$ and $E_r^*(j\omega)$ are shown in Figure 9. Clearly, for this example, the above current and reduced order estimators yield the same performance in the sense that the achieved recovery is same. □



---- $|M_p^*(j\omega)|$; ------- $|M_c^*(j\omega)|$ & $|M_r^*(j\omega)|$

Figure 8: The singular values of recovery matrices.



---- $|E_p^*(j\omega)|$; ------- $|E_c^*(j\omega)|$ & $|E_r^*(j\omega)|$

Figure 9: The singular values of loop transfer recovery errors.

Example 4 : Consider a non-strictly proper discrete-time system Σ with sampling period $T = 1$, and characterised by

$$A = \begin{bmatrix} -2.0 & 1.0 & 1 & -1 \\ 1.0 & -0.5 & 0 & 0 \\ 1.0 & 0.0 & 0 & 0 \\ 2.5 & 0.0 & 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$C = \begin{bmatrix} 0 & -0.25 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Let the target loop $L_t(z)$ and target sensitivity function $S_t(z)$ be specified by

$$F = \begin{bmatrix} 1 & -1 & 1 & 0 \\ -1 & 1 & 1 & 0 \end{bmatrix}.$$

It is simple to verify that the given system Σ is invertible with one infinite zero of order one and three invariant zeros at $\{-0.25, 0, 2\}$. It can also be verified that

$$\mathcal{S}^-(\Sigma) = \text{span} \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \mathcal{S}^-(\Sigma) \cap \{x \,|\, Cx \in \text{Im}\,(D)\} = \text{span} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Hence, the target loop specified by the given F is not recoverable by prediction estimator based controller, but it is recoverable either by current or by reduced order estimator based controllers. The following are the prediction, current and reduced order estimator based controllers obtained by the H_2-optimization based algorithm. Again, all these controllers achieve the infimum of the H_2-norm of the corresponding recovery matrices.

Prediction Estimator Based Controller :

$$\hat{x}(k+1) = \begin{bmatrix} -0.5 & 0.00 & 0 & -1 \\ 0.0 & -0.25 & 0 & 0 \\ 0.0 & 0.00 & 0 & 0 \\ 3.0 & 0.00 & 0 & 2 \end{bmatrix} \hat{x}(k) + \begin{bmatrix} 0 & -0.5 \\ 1 & 1.0 \\ 0 & 1.0 \\ 0 & -0.5 \end{bmatrix} y(k).$$

$$-u(k) = \begin{bmatrix} 1 & -1 & 1 & 0 \\ -1 & 1 & 1 & 0 \end{bmatrix} y(k).$$

The eigenvalues of the above prediction estimator are placed at $\{0, 0, -0.25, 0.5\}$ while the resulting recovery error matrix $M_p^e(z)$ is given by,

$$M_p^e(z) = \frac{1}{z^2 - 0.5z} \begin{bmatrix} 0 & z-2 \\ 0 & -(z-2) \end{bmatrix}.$$

Current Estimator Based Controller :

$$v(k+1)=\begin{bmatrix}0 & 0.00 & 0 & 0.0\\ 1 & -0.25 & 0 & 0.0\\ 1 & 0.00 & 0 & 0.0\\ 1 & 0.00 & 0 & 0.5\end{bmatrix}v(k)+\begin{bmatrix}0 & 0.00\\ 1 & 1.00\\ 0 & 1.00\\ 0 & 0.25\end{bmatrix}y(k),$$

$$-u(k)=\begin{bmatrix}1 & -1 & 1 & 0\\ -1 & 1 & 1 & 0\end{bmatrix}v(k)+\begin{bmatrix}0 & 1\\ 0 & -1\end{bmatrix}y(k).$$

The eigenvalues of the above current estimator are placed at $\{0,\ 0,\ -0.25,\ 0.5\}$ while the resulting recovery error matrix $M_c^*(z)\equiv 0$.

Reduced Order Estimator Based Controller :

$$v(k+1)=\begin{bmatrix}-0.25 & 0 & 0.0\\ 0.00 & 0 & 0.0\\ 0.00 & 0 & 0.5\end{bmatrix}v(k)+\begin{bmatrix}1 & 1.00\\ 0 & 1.00\\ 0 & 0.25\end{bmatrix}y(k).$$

$$-u(k)=\begin{bmatrix}-1 & 1 & 0\\ 1 & 1 & 0\end{bmatrix}v(k)+\begin{bmatrix}0 & 1\\ 0 & -1\end{bmatrix}y(k).$$

The eigenvalues of the above reduced order estimator are placed at $\{-0.25,\ 0,\ 0.5\}$ while the resulting recovery error matrix $M_r^*(z)\equiv 0$.

The plots of maximum and minimum singular values of the target and the achieved loop transfer function via all the three controllers are shown in Figure 10. Also, the plot of maximum singular value of $M_p^*(j\omega)$is shown in Figure 11, while the plots of maximum singular value of $E_p^*(j\omega)$ is shown in Figure 12. Clearly, for this example, the above current and reduced order estimators yield exact recovery. □

V. DESIGN FOR RECOVERY OVER A SPECIFIED SUBSPACE

Sections III and IV consider the conventional LTR design problem which seeks the recovery over the entire control space. In this section, given a subspace $\mathcal{S}$ of $\mathbf{R}^m$, the interest is in designing a controller so that the achieved and target sensitivity and complimentary sensitivity functions projected onto the subspace $\mathcal{S}$ match each other. The conditions under which such a design is possible are given in Part 1. To recapitulate these conditions, let

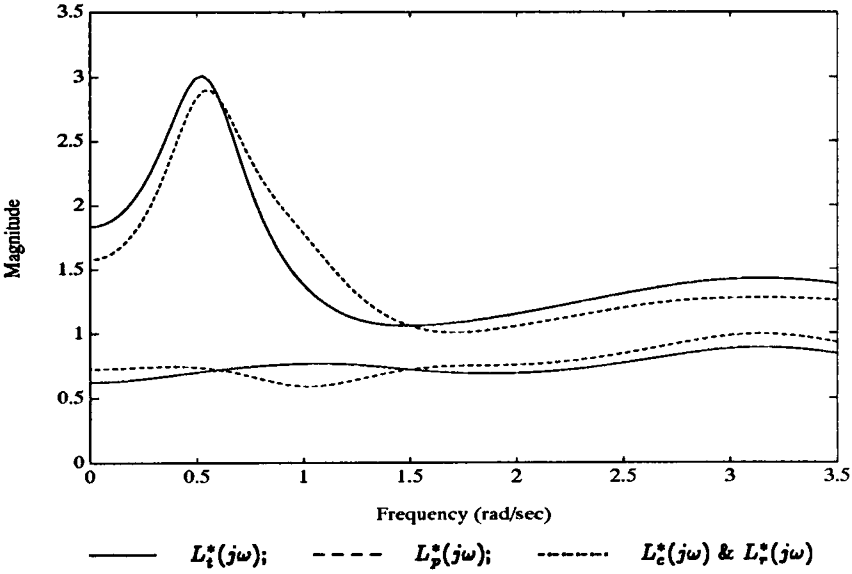


Figure 10: The max. and min. singular values of target and achieved loops.

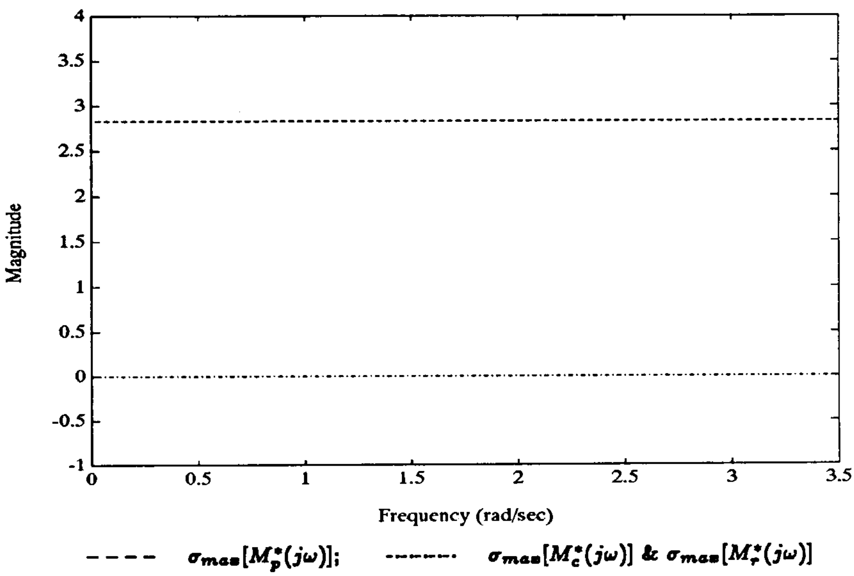
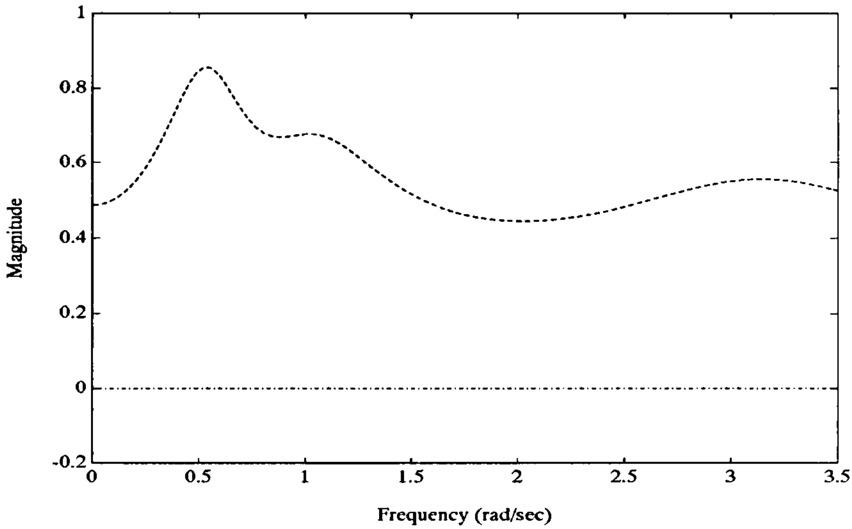


Figure 11: The max. singular values of recovery matrices.



---- $\sigma_{max}[E_p^*(j\omega)]$; ······· $\sigma_{max}[E_c^*(j\omega)]$ & $\sigma_{max}[E_r^*(j\omega)]$

Figure 12: The max. singular values of loop transfer recovery errors.

V^s be a matrix whose columns form an orthogonal basis of $\mathcal{S} \in \mathbf{R}^m$. Assume that the columns of V^s are scaled so that the norm of each column is unity. Let $P^s = V^s(V^s)'$ be the unique orthogonal projection matrix onto $\mathcal{S}$. Then, define three auxiliary systems Σ_p^s, Σ_c^s and Σ_r^s characterized, respectively, by the quadruples (A, BV^s, C, DV^s), (A, BV^s, C_c, D_cV^s) and $(A_r, B_rV^s, C_r, D_rV^s)$. Then the analysis given in Part 1 implies that any admissible and arbitrarily specified sensitivity function (i.e., when F is specified arbitrarily) is recoverable in $\mathcal{S}$ if and only if the following condition is satisfied depending upon the controller used.

(1) Prediction estimator based controller :

Any arbitrary admissible sensitivity function is recoverable if and only if the auxiliary system Σ_p^s is left invertible and of minimum phase with no infinite zeros (i.e., DV^s is of maximal rank).

(2) Current estimator based controller :

Any arbitrary admissible sensitivity function is recoverable if and only if the auxiliary system Σ_c^s is left invertible and of minimum phase with no infinite zeros (i.e., D_cV^s is of maximal rank).

(3) Reduced order estimator based controller :

Any arbitrary admissible sensitivity function is recoverable if and only if the auxiliary system Σ_r^s is left invertible and of minimum phase with no infinite zeros (i.e., $D_r V^s$ is of maximal rank).

The above results are concerned with the recovery of sensitivity function when F is arbitrary or unknown. As is done in Part 1, one can also formulate the recovery conditions for a known F as follows. A known admissible sensitivity function (i.e., when F is known) is recoverable in $\mathcal{S}$ if and only if the following condition is satisfied depending on the controller used.

1. For a prediction estimator based controller, the condition is that $\mathcal{S}^-(\Sigma_p^s) \subseteq \operatorname{Ker}(F)$.
2. For a current estimator based controller, the condition is that $\mathcal{S}^-(\Sigma_c^s) \subseteq \operatorname{Ker}(F)$.
3. For a reduced order estimator based controller, the condition is that $\begin{pmatrix} 0 \\ I \end{pmatrix} \mathcal{S}^-(\Sigma_r^s) \subseteq \operatorname{Ker}(F)$.

Remark 4. *If the given system Σ is strictly proper, i.e., $D = 0$, then it is simple to verify that*

$$\mathcal{S}^-(\Sigma_c^s) = \begin{pmatrix} 0 \\ I \end{pmatrix} \mathcal{S}^-(\Sigma_s^r) = \mathcal{S}^-(\Sigma_p^s) \cap \{ x \mid Cx \in \operatorname{Im}(DV^s) \}.$$

This is not true in general for non-strictly proper systems.

Thus the task of designing a controller for recovery in a subspace collapses to the same task discussed in Sections III and IV except that one needs to use the auxiliary systems Σ_p^s, Σ_c^s and Σ_r^s respectively in place of Σ_p, Σ_c and Σ_r. The following example illustrates this.

Example 5 : Consider a non-strictly proper discrete-time system Σ with sampling period $T = 1$, and characterized by

$$A = \begin{bmatrix} 1 & 4 & 1 & 1 & 1 \\ 4 & 4 & 4 & 4 & 4 \\ 1 & 4 & 5 & 0 & 0 \\ 1 & 4 & 0 & 3 & 0 \\ 1 & 4 & 0 & 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$C = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Let the target loop $L_t(z)$ and target sensitivity function $S_t(z)$ be specified by

$$F = \begin{bmatrix} 1.0000 & 4.0000 & 5.0000 & 0.0000 & 0.0000 \\ 0.5795 & 3.6504 & 1.0000 & -0.4554 & 1.7594 \\ 5.1676 & 8.3372 & 4.0000 & 9.5197 & 2.4978 \end{bmatrix}.$$

It is simple to verify that the given system Σ has two infinite zeros of order one and three invariant zeros at $\{2, 3, 4\}$. It can also be verified that the target loop specified by the given F is not recoverable either by a prediction or by a current or by a reduced order estimator based controller. Let us consider a subspace S spanned by

$$V^s = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

It is simple to verify that the given target sensitivity function is recoverable in S using either a current or a reduced order estimator based controller. The following current and reduced order estimator based controllers obtained by ATEA achieve such a recovery.

Current Estimator Based Controller :

$$v(k+1) = \begin{bmatrix} 0.0000 & 0.0000 & 0 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0 & 0.0000 & 0.0000 \\ 9.0027 & 33.7281 & 4 & 37.8334 & -10.2964 \\ -5.3994 & -19.7721 & 0 & -27.2535 & 8.2335 \\ 1.8485 & 7.1520 & 0 & 4.0114 & 0.9083 \end{bmatrix} v(k) + \begin{bmatrix} 0 & 0.0000 & 0.0000 \\ 0 & 0.0000 & 0.0000 \\ 1 & 9.0027 & -153.7028 \\ 0 & -5.3994 & 135.5881 \\ 0 & 1.8485 & -14.1746 \end{bmatrix} y(k),$$

$$-u(k) = \begin{bmatrix} 1.0000 & 4.0000 & 5.0000 & 0.0000 & 0.0000 \\ 0.5795 & 3.6504 & 1.0000 & -0.4554 & 1.7594 \\ 5.1676 & 8.3372 & 4.0000 & 9.5197 & 2.4978 \end{bmatrix} v(k) + \begin{bmatrix} 0 & 1.0000 & 38.2712 \\ 0 & 0.5795 & 14.2796 \\ 0 & 5.1676 & -14.6080 \end{bmatrix} y(k).$$

The eigenvalues of the above current estimator are placed at {0, 0.1, 0.2, 0.3}.

Reduced Order Estimator Based Controller :

$$v(k+1)=\begin{bmatrix}4 & 40.9010 & -11.1312\\ 0 & -30.6150 & 9.1483\\ 0 & 4.7193 & 0.7156\end{bmatrix}v(k)+\begin{bmatrix}1 & 9.6516 & -192.8259\\ 0 & -6.1104 & 171.8538\\ 0 & 1.9983 & -20.4207\end{bmatrix}y(k)$$

$$-u(k)=\begin{bmatrix}5 & 0.0000 & 0.0000\\ 1 & -0.4554 & 1.7594\\ 4 & 9.5197 & 2.4978\end{bmatrix}v(k)+\begin{bmatrix}0 & 1.0000 & 41.0500\\ 0 & 0.5795 & 15.3384\\ 0 & 5.1676 & -17.8622\end{bmatrix}y(k).$$

The eigenvalues of the above reduced order estimator are placed at {0, 0.1, 0.2}. □

VI. LTR DESIGN FOR OUTPUT BREAK POINT

All the previous sections consider the case when the target loop transfer function is specified at the input point of the given system. Let us now consider the LTR design when a target loop transfer function $L_t(z) = C(zI - A)^{-1}K$ is specified at the output point of the given system (A, B, C, D). Let $\lambda(A - KC) \in \mathbf{C}^{\odot}$. Then as is discussed in Section VI of Part 1 [3], the above design can be reformulated as the loop transfer recovery problem when a fictitious target loop transfer function $L_d(z) = F_d(zI - A_d)^{-1}B_d$ is specified at the input point of a fictitious dual system Σ_d characterized by (A_d, B_d, C_d, D_d) where $A_d = A'$, $B_d = C'$, $C_d = B'$, $D_d = D'$, and $F_d = K'$. Now, to come up with a controller $\mathbf{C}_d(z)$ for Σ_d, one can utilize any one of the three controllers, namely, the prediction, current and reduced order estimator based controllers. Moreover, the design can be accomplished using any one of the methods developed earlier, namely, the eigenstructure assignment method, the H_∞- or the H_2-optimization based designs. Finally, one needs simply to implement the controller $\mathbf{C}(z) = \mathbf{C}_d'(z)$ to achieve the needed design for the given system.

VII. CONCLUSIONS

This Part 2 of the paper considers three design methods for LTR of general discrete time systems. The first one is an eigenstructure assignment scheme, and the other two are optimization based designs. Eigenstructure assignment method yields a controller design which achieves *any chosen* recovery error matrix among a set of admissible recovery error matrices. On the other hand, one of the optimization based design methods leads to a controller that achieves a recovery error matrix having the infimum H_∞ norm, while the other does the same except it achieves a recovery error matrix having the infimum H_2 norm. Any controller, whether it is prediction or current or reduced order estimator based, can be designed by using any one of the above three design methods. Once the estimator gain is known, the corresponding recovery error matrices can explicitly be calculated in closed form. Besides the conventional LTR design problem which is concerned with the recovery over the entire control space, another generalized recovery design problem where the concern is with the recovery over a specified subspace of the control space is also considered. All the design methods developed here are implemented in a 'Matlab' software package. A number of design examples illustrate several aspects of eigenstructure assignment design as well as optimization based designs.

Acknowledgement

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A. Proof of Theorem 2

We need to introduce the following lemma first in order to prove Theorem 2.

Lemma 1. *The following two statements are equivalent:*

1. *There exists an internally stabilising static state feedback law $u(k) = -K_p^\prime x(k)$ such that the closed-loop transfer function from w to z of Σ^{au1} has an H_∞-norm less than 1.*

2. *There exists an internally stabilising static state feedback law $u(k) = -Fx(k)$ such that the closed-loop transfer function from w to z of Σ^{au2} has an H_∞-norm less than 1.*

Proof of Lemma 1: Without loss of generality, we assume that the system characterised by the quadruple (A', C', B', D') is in the form of s.c.b. Now, let us assume the first statement is true, i.e., there exists a state feedback law $u(k) = -K'x(k)$ such that the resulting closed-loop system of Σ^{au1} is asymptotically stable and the transfer function from w to z has an H_∞-norm less than 1. Partitioning

$$K' = \begin{bmatrix} F_{a0}^- + C_{0a}^- & F_{a0}^+ + C_{0a}^+ & F_{b0} + C_{0b} & F_{c0} + C_{0c} & F_{f0} + C_{0f} \\ F_{af}^- & F_{af}^+ & F_{bf} & F_{cf} & F_{ff} \\ F_{ac}^- & F_{ac}^+ & F_{bc} & F_{cc} & F_{fc} \end{bmatrix},$$

it is trivial to verify that the closed-loop system of Σ^{au1} under the state feedback law $u = -K'x$ is equivalent to the closed-loop system of Σ^{au2} with the following dynamical state feedback,

$$\begin{cases} x_c(k+1) = (A_{cc} - B_c F_{cc}) x_c(k) \\ \quad +[B_c(E_{ca}^- - F_{ac}^-),\ B_c(E_{ca}^+ - F_{ac}^+),\ L_{cb}C_b - B_c F_{bc},\ L_{cf}C_f - B_c F_{fc}] x(k), \\ u(k) = -\begin{bmatrix} F_{c0} \\ F_{cf} \end{bmatrix} x_c(k) - \begin{bmatrix} F_{a0}^- & F_{a0}^+ & F_{b0} & F_{f0} \\ F_{af}^- & F_{af}^+ & F_{bf} & F_{ff} \end{bmatrix} x(k). \end{cases} \tag{44}$$

Obviously, the resulting closed-loop system of Σ^{au2} with the dynamical state feedback law (44) is asymptotically stable and the transfer function from w to z has an H_∞-norm less than 1. Then it follows from Theorem 9.2 of Stoorvogel [10] that there exists a symmetric matrix $P \geq 0$ such that the following conditions hold:

1. $V > 0$ and $R > 0$, where

$$V := DD + B'PB,$$
$$R := I - E'PE + E'PBV^{-1}B'PE.$$

   This implies that the matrix $G(P)$ is invertible, where

$$G(P) := \begin{bmatrix} D'D + B'PB & \tilde{B}'PE \\ E'PB & E'PE - I \end{bmatrix}. \tag{45}$$

2. P satisfies the following discrete time ARE:

$$P = A'PA + C'C - \begin{pmatrix} B'PA + D'C \\ E'PA \end{pmatrix}' G(P)^{-1} \begin{pmatrix} B'PA + D'C \\ E'PA \end{pmatrix}. \tag{46}$$

3. The matrix A_{cl} has all its eigenvalues inside the unit circle, where

$$A_{cl} := A - [B \quad E]\, G(P)^{-1} \begin{pmatrix} B'PA + D'C \\ E'PA \end{pmatrix}. \tag{47}$$

Now let us define a cost function,

$$\mathcal{J}(0) := \sup_{w} \inf_{u^+} \{ \|z_{u,w}\|^2 - \|w\|^2 \}, \tag{48}$$

where $u^+ := u|_{[1,\infty)}$. Moreover, we impose an additional constraint $u(0) = 0$. Then the dynamic state feedback induces in closed loop a mapping from w to u, say f. In other words $u = f(w)$. Because we have $x(0) = 0$ we found that $u(0) = [f(w)](0) = 0$. It is clear that

$$\begin{aligned} \mathcal{J}(0) &\le \sup_{w} \{ \|z_{f(w),w}\|^2 - \|w\|^2 \} \\ &\le \sup_{w} \{ (\gamma^2 - 1) \|w\|^2 \} \\ &= 0, \end{aligned}$$

where $\gamma < 1$ is the closed-loop H_∞ norm we obtained via the *dynamic* state feedback control law (44). Also note that the supremum in (48) is finite and is only attained by 0. We know that

$$\sup_{w^+} \inf_{u^+} \{ \|z_{u^+,w^+}\|^2 - \|w^+\|^2 = x'(1) P x(1),$$

where $w^+ := w|_{[1,\infty)}$, as an optimization problem on $[1,\infty)$. Therefore we have

$$\mathcal{J}(0) = \sup_{w(0)} \{ \|z(0)\|^2 - \|w(0)\|^2 + x'(1) P x(1). \tag{49}$$

Moreover, the maximum is uniquely attained by $w(0) = 0$ (uniqueness stems from the uniqueness in maximization in (48)), (49) can be rewritten in the following form

$$\sup_{w(0)} \{ w'(0) [E'PE - I] w(0) \}.$$

Then boundedness and uniqueness of maximum imply

$$E'PE - I < 0.$$

By Theorem 9.4 of Stoorvogel [10] that there exists an internally stabilizing static state feedback law $u(k) = -Fx(k)$ such that the closed-loop transfer function from w to z of Σ^{au2} has an H_∞-norm less than 1.

Conversely, let us assume that the second statement is true, i.e., there exists a state feedback law $u(k) = -Fx(k)$ such that the resulting closed-loop system of Σ^{au2} is asymptotically stable and the transfer function from w to z has an H_∞-norm less than 1. Again, let us partition F as

$$F = \begin{bmatrix} F_{a0}^- & F_{a0}^+ & F_{b0} & F_{f0} \\ F_{af}^- & F_{af}^+ & F_{bf} & F_{ff} \end{bmatrix}.$$

Choosing

$$K' = \begin{bmatrix} F_{a0}^- + C_{0a}^- & F_{a0}^+ + C_{0a}^+ & F_{b0} + C_{0b} & C_{0c} & F_{f0} + C_{0f} \\ F_{af}^- & F_{af}^+ & F_{bf} & E_{fc} & F_{ff} \\ 0 & 0 & 0 & F_{cc} & 0 \end{bmatrix},$$

where F_{cc} is such that $\lambda(A_{cc} - B_c F_{cc}) \in \mathbf{C}^{\odot}$, it is simple to verify that the closed-loop system of Σ^{au1} under the state feedback law $u(k) = -K'x(k)$ is asymptotically stable and the transfer function from w to z has an H_∞-norm less than 1. □

Remark 5. *The above lemma is a generalization of a state feedback H_∞-optimization problem for discrete-time systems given in [10].*

Now, the proof of Theorem 2 follows directly from the above lemma and the results of Stoorvogel [10]. ■

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