Loop Transfer Recovery For General Nonminimum Phase Discrete Time Systems

Part 1: Analysis

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I. INTRODUCTION

In recent years, a method of multivariable feedback control system design using so called LQG/LTR techniques has gained significance. As is known, many performance and robust stability objectives can be cast in terms of maximum magnitude or maximum singular values of some particular *closed-loop* transfer functions, e.g., sensitivity and complementary sensitivity functions at certain points in a closed-loop. Such magnitude or singular value requirements on some *closed-loop* transfer functions can be directly determined by corresponding singular values of certain related *open-loop* transfer functions. Thus equivalently, the design specifications can be prescribed in terms of some *open-loop* transfer functions. In prescribing the *open-loop* transfer functions, the point at which loop is broken can be the input or output or any arbitrary point of the given plant. Here we deal with the situation when the loop is broken at the input point of the plant. Then the design methodology of LQG/LTR can essentially be partitioned into two steps. The first step involves "loop shaping" utilizing a state feedback control law so that the resulting open-loop transfer function when the loop is broken at the input point of the plant meets the design specifications. The resulting open-loop transfer function is called the target loop transfer function. The second step, called loop transfer recovery (LTR), involves the design of an output feedback control law such that the resulting open-loop transfer function would be either exactly or approximately the same as the target open-loop transfer function. In other words, the idea of LTR is to come up with a measurement feedback compensator, typically observer based, to recover either exactly or asymptotically a specific open-loop transfer function prescribed in terms of a state feedback gain.

Ever since the seminal works of [14] and [9], LTR has been the subject of a number of authors including [2], [3], [4], [5], [6], [7], [10], [12], [13], [15], [17], [18], [19], [21], [23], [24], [25], [29], [31], [32], [34] and [35]. Both continuous and discrete systems have been treated earlier. Recently Chen, Saberi and Sannuti in [6] analysed in depth the mechanism of LTR for continuous systems. The analysis given there considers four main issues. The first issue is concerned with what can and what cannot be achieved for a given system and for an arbitrarily specified target loop transfer function. On the other hand, the second issue is concerned with the development of necessary or/and sufficient conditions a target loop has to satisfy so that it can either exactly or asymptotically be recovered for a given system while the third issue is concerned with the development of necessary or/and sufficient conditions on a given system such that it has at least one recoverable target loop. The fourth issue deals with a generalisation of all the above three issues when recovery is required over a subspace of the control space. It concerns with generalising the traditional LTR concept to sensitivity recovery over a subspace and deals with method(s) to test whether projections of target and achievable sensitivity and complementary sensitivity functions onto a given subspace match each other or not. Such an analysis pinpoints the limitations of the given system for the recovery of arbitrarily specified target loops via either current or prediction estimator based controllers. These limitations are the consequences of the structural properties (i.e., finite and infinite zero structure, and invertibility) of the given system. Also, the conditions required on a target loop transfer function for its recoverability, turn out to be constraints on its finite and infinite zero structure as related to the corresponding structure of the given system. Furthermore, the analysis given in [6] discovers a multitude of ways in which freedom exists to shape the loops in a desired way as close as possible to the target shapes. Also, possible pole zero cancellations between the eigenvalues of the controller and the input or/and output decoupling zeros of the given system are characterized. Next, regarding the design of controllers for LTR, [7] developed three methods of observer based controller design. The first method is an asymptotic time-scale and eigenstructure assignment (ATEA) method while the other two are optimization based, one minimizing the H_∞ norm and the other H_2 norm of a so called recovery matrix. The analysis as well as design methods as given in [6] and [7] are fairly complete for general nonminimum phase nonstrictly proper plants of continuous type.

In contrast to the continuous systems, the results available for discrete systems are relatively few. In order to facilitate the discussion of the available results, let us first recall that for discrete systems there exist three different types of observer based controllers; namely, 'prediction estimator', and full or reduced order type 'current estimator' based controllers. In the case of continuous systems, as is well known, any arbitrary target loop transfer function is asymptotically recoverable provided that the given system is left invertible and of minimum phase. However, this is not necessarily so for discrete systems as discussed first by Goodman [12]. Using prediction estimator based controllers, Goodman characterized the so called recovery matrix and showed that in general it cannot be rendered zero even for square minimum phase strictly proper systems. He showed further that prediction estimator when its gain is calculated via Kalman filter formalism in which the covariance of the fictitious input noise is arbitrarily increased to infinity, minimizes the H_2 norm of the recovery matrix. Later on, Maciejowski [15] continued the study of LTR for square minimum phase strictly proper discrete systems using current estimators. Although Maciejowski studied the recovery at the output point of the plant, his results when translated to recovery at the input point of the plant, imply that recovery of an arbitrarily specified target loop transfer function is possible for the class of systems he considered, namely, strictly proper square minimum phase systems having only infinite zeros of order one. Also, Maciejowski [15] as well as Ishihara and Takeda [13] observe that it is impossible in general to have either exact or asymptotic LTR when the plant is either nonminimum phase or when prediction estimator is used even if it has all infinite zeros of order one. Realizing that in general LTR for discrete systems is not feasible, Niemann and Sogaard-Andersen [19] consider square strictly proper systems with a prediction estimator, and develop a parameterization of exactly recoverable target loop transfer functions in terms of system zeros and associated zero directions. Recently, Zhang and Freudenberg [35], considering only square strictly proper plants having only infinite zeros of order one, study the LTR mechanism at the output point of the plant. They develop explicit expressions for the recovery error and the resulting sensitivity function when prediction as well as current estimator based controllers are used and when optimization is used to minimize the H_2 norm of a recovery matrix (for precise definition of recovery matrix, see Lemma 1). The analysis of LTR done so far on discrete systems, as summarized above, although presents some glimpses of what is happening in some special cases, it does not reveal a total picture of LTR mechanism for general discrete systems. In fact, it is fair to say that no systematic analysis of all the issues involved in LTR exists to date for general discrete systems, and whatever is available is far away from being complete. For example, as pointed out by Maciejowski [15], most practical discrete systems have a direct feed through from inputs to outputs and thus are non-strictly proper. Yet no work to date deals with non-strictly proper discrete systems. Similarly, as shown in Astrom et al [1], sampling of continuous systems most often introduces unstable invariant zeros in the resulting discrete systems.

Yet, even for strictly proper discrete systems, the results showing the effects of unstable invariant zeros on LTR are to a great extent incomplete; just to mention a few, no characterization of recoverable target loops of a given system exists, similarly analysis for recovery in any given subspace of control space is nonexistent. Similarly, regarding design for LTR, while partial results are available based on minimization of H_2 norm of a certain recovery matrix [12], [35], no methods of H_∞ norm minimisation and eigenstructure assignment are yet available. Thus the intent of this two part paper is to present both systematic analysis as well as design tools for LTR of general nonminimum phase nonstrictly proper discrete plants. Part 1 of the paper deals with complete analysis. All the four issues mentioned earlier in connection with continuous systems are reexamined for discrete systems. The analysis and the method of presentation given in Part 1 unifies the discussion regarding all the three controllers, namely, 'prediction estimator' based, and full or reduced order 'current estimator' based controllers. Part 2 of the paper [8] deals with design where both eigenstructure assignment method and optimization based methods in which either H_∞ or H_2 norm of certain recovery matrix is minimized, are developed.

The analysis and design aspects presented in this two part paper reveal both similarities as well as fundamental differences between continuous and discrete systems. One fundamental difference which we want to emphasize here is this. In discrete systems, as is well known, in order to preserve stability, all the closed-loop eigenvalues must be restricted to lie within the unit circle in complex plane. This implies that unlike continuous case which permits both finite as well as asymptotically infinite eigenvalue assignment, in the discrete case one is restricted to only finite eigenvalue assignment. This restriction leads to several important differences in connection with LTR between continuous and discrete systems. To quote one such difference, let us recall that asymptotic recovery in the case of continuous systems allows assignment of both asymptotically finite as well as infinite observer eigenvalues by using high observer gains, where as exact recovery allows only assignment of finite observer eigenvalues. Thus, in continuous systems, there exists target loops which are only asymptotically recoverable but not exactly recoverable. On the other hand, in discrete systems, since both asymptotic as well as exact recovery involves only finite eigenvalue assignment, every asymptotically recoverable target loop is also exactly recoverable. This implies that one needs to talk about just recovery rather than emphasising exact or asymptotic recovery. However, in optimisation based design methods, such as H_∞ norm minimisation, one some times ends up in suboptimal designs which correspond to only asymptotic recovery. In that case, a distinction can be made between exact and asymptotic recovery.

Throughout the paper, A' denotes the transpose of A, A^H denotes the complex conjugate transpose of A, I denotes an identity matrix while I_k denotes the identity matrix of dimension $k \times k$. $\lambda(A)$ and $\mathrm{Re}[\lambda(A)]$ respectively denote the set of eigenvalues and real parts of eigenvalues of A. Similarly, $\sigma_{max}[A]$ and $\sigma_{min}[A]$ respectively denote the maximum and minimum singular values of A. Ker $[V]$ and Im $[V]$ denote respectively the kernel and the image of V. $\mathbf{C}^{\odot}$ denotes the set of complex numbers inside the open unit circle while $\mathbf{C}^{\otimes}$ is the complementary set of $\mathbf{C}^{\odot}$. Also, $\mathcal{R}_p$ denotes the sub-ring of all proper rational functions of z while the set of matrices of dimension $l \times q$ whose elements belong to $\mathcal{R}_p$ is denoted by $\mathcal{M}^{l\times q}(\mathcal{R}_p)$. Given a discrete transfer function $G(z)$, we define the discrete frequency response $G^*(j\omega)$ as $G(e^{j\omega T})$ where T is the sampling period of the discrete-time system. An asymptotically stable matrix is the one whose eigenvalues are all in $\mathbf{C}^{\odot}$.

II. PROBLEM FORMULATION

In this section, we formulate the LTR problem in precise mathematical terms. Let us consider a nonstrictly proper discrete-time system Σ,

$$x(k+1) = Ax(k) + Bu(k)\ ,\ \ y(k) = Cx(k) + Du(k) \tag{1}$$

where the state vector $x \in \mathbf{R}^n$, output vector $y \in \mathbf{R}^p$ and input vector $u \in \mathbf{R}^m$. Without loss of generality, assume that $[B', D']'$ and $[C, D]$ are of maximal rank. Let us also assume that Σ is stabilizable and detectable. Let F be a full state feedback gain matrix such that (a) the closed-loop system is asymptotically stable, i.e., eigenvalues of $A - BF$ lie inside the unit circle, and (b) the open-loop transfer function when the loop is broken at the input point of the given system meets the given frequency dependent specifications. The state feedback control is

$$u(k) = -Fx(k) \tag{2}$$

and the loop transfer function evaluated when the loop is broken at the input point of the given system, the so called target loop transfer function, is

$$L_t(z) = F\Phi B \tag{3}$$

where $\Phi = (zI_n - A)^{-1}$. The corresponding target sensitivity and complementary sensitivity functions are

$$S_t(z) = [I_m + F\Phi B]^{-1} \quad \text{and} \quad T_t(z) = I_m - S_t(z). \tag{4}$$

Arriving at an appropriate value for F is concerned with the issue of loop shaping which is an engineering art and often includes the use of linear quadratic regulator (LQR) design in which the cost matrices are used as free design parameters to generate the target loop transfer function $L_t(z)$ and thus the desired sensitivity and complementary sensitivity functions. The next step of design is to recover the target loop using only a measurement feedback controller. This is the problem of loop transfer recovery (LTR) and is the focus of this paper. To explain it clearly, consider the configuration of Figure 1 where $\mathbf{C}(z)$ and $P(z)$,

$$P(z) = C\Phi B + D,$$

are respectively the transfer functions of a controller and of the given system. Given $P(z)$ and a target loop transfer function $L_t(z)$, one seeks then to design a $\mathbf{C}(z)$ such that the loop recovery error $E(z)$,

$$E(z) \equiv L_t(z) - \mathbf{C}(z)P(z),$$

is either exactly or approximately equal to zero in the frequency region of interest while guaranteeing the stability of the resulting closed-loop system. The notion of achieving exact LTR (ELTR) corresponds to $E(z) = 0$ for all z. In the case of asymptotic recovery, one normally parameterizes the controller $\mathbf{C}(z)$ in terms of a scalar tuning parameter σ and thus obtains a family of controllers $\mathbf{C}(z,\sigma)$. We say asymptotic LTR (ALTR) is achieved if $\mathbf{C}(z,\sigma)P(z) \to L_t(z)$ pointwise in z as $\sigma \to \infty$. Achievability of ALTR enables the designer to choose a member of the family of controllers that corresponds to a particular value of σ which achieves a desired level of recovery. We now consider the following definitions in order to impart precise meanings to ELTR and ALTR:

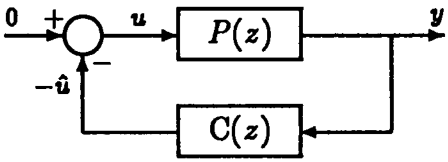

Figure 1: Plant—Controller closed-loop configuration.

Definition 1. *The set of admissible target loops* $\mathbf{T}(\Sigma)$ *of a system* Σ *is defined by*

$$\mathbf{T}(\Sigma) = \{L_t(z) \in \mathcal{M}^{m\times m}(\mathcal{R}_p) \,|\, L_t(z) = F\Phi B \text{ and } \lambda(A - BF) \in \mathbf{C}^{\odot}\}.$$

Definition 2. *A target loop transfer function* $L_t(z) \in \mathbf{T}(\Sigma)$ *is said to be exactly recoverable (ELTR) if there exists a* $\mathrm{C}(z) \in \mathcal{M}^{m\times p}(\mathcal{R}_p)$ *such that (i) the closed-loop system comprising of* $\mathrm{C}(z)$ *and* $P(z)$ *as in the configuration of Figure 1 is asymptotically stable, and (ii)* $\mathrm{C}(z)P(z) = L_t(z)$.

Definition 3. *A target loop transfer function* $L_t(z) \in \mathbf{T}(\Sigma)$ *is said to be asymptotically recoverable (ALTR) if there exists a parameterized family of controllers* $\mathrm{C}(z,\sigma) \in \mathcal{M}^{m\times p}(\mathcal{R}_p)$*, where* σ *is a scalar parameter taking positive values, such that (i) the closed-loop system comprising of* $\mathrm{C}(z,\sigma)$ *and* $P(z)$ *as in the configuration of Figure 1 is asymptotically stable for all* $\sigma > \sigma^*$*, where* $0 \le \sigma^* < \infty$*, and (ii)* $\mathrm{C}(z,\sigma)P(z) \rightarrow L_t(z)$ *pointwise in* z *as* $\sigma \rightarrow \infty$*. Moreover, the limits, as* $\sigma \rightarrow \infty$*, of all the eigenvalues of the closed-loop system should remain in* $\mathbf{C}^{\odot}$.

As mentioned earlier, it turns out that for discrete systems in contrast with continuous systems, every asymptotically recoverable target loop can also be exactly recoverable and vice versa. One might then wonder why one needs to distinguish between ELTR and ALTR. This is perhaps, as will be seen in Part 2 of the paper, even for the case when ELTR can be achieved, some optimization based design methods, such as H_∞ norm minimization, typically end up in suboptimal designs which correspond to asymptotic recovery. But in this Part 1 of the paper which is mainly concerned with analysis, we will not hereafter distinguish between the notions of exact and asymptotic recovery. Also, we will not parameterize a controller in terms of a tunable parameter σ in an attempt to achieve whatever can be achieved asymptotically rather than exactly. We maintain that such a parameterization can always be done if one chooses so. We have the following additional definitions.

Definition 4. *A target loop transfer function $L_t(z)$ belonging to $\mathbf{T}(\Sigma)$ is said to be recoverable if $L_t(z)$ is either exactly or asymptotically recoverable.*

Definition 5. *The set of recoverable target loops for the given system Σ is denoted by $\mathbf{T}^{R}(\Sigma)$* †.

Next, in view of the definition of sensitivity function $S_t(z)$ as in (4), it is simple to note that the recovery of target loop $L_t(z)$ implies the recovery of $S_t(z)$ and vice versa.

As mentioned in introduction, the purpose of this Part 1 paper is to do in depth analysis of LTR mechanism in general discrete systems. As in the case of continuous systems, the analysis of LTR mechanism carried out here concentrates on four fundamental issues. The first issue is concerned with what can and what cannot be achieved for a given system without taking into account any specific target loop transfer function, i.e., the target loop transfer function is considered as arbitrarily given. On the other hand, the second issue is concerned with the development of necessary or/and sufficient conditions a target loop has to satisfy so that it can either exactly or asymptotically be recovered for a given system. The third issue deals with the development of necessary or/and sufficient conditions on a given system such that it has at least one recoverable target loop. The fourth issue concerns with a generalisation of all the above three issues when recovery is required over a subspace of the control space. To be exact, it concerns with generalizing the traditional LTR concept to sensitivity recovery over a subspace and deals with method(s) to test whether projections of target and achievable sensitivity and complementary sensitivity functions onto a given subspace match each other or not. As in the case of continuous systems, the analysis presented here shows some fundamental limitations of

† In continuous time systems, we defined three sets; the set of exactly recoverable target loops $\mathbf{T}^{ER}(\Sigma)$, the set of recoverable target loops $\mathbf{T}^{R}(\Sigma)$, and the set of target loops which are recoverable but not exactly recoverable $\mathbf{T}^{AR}(\Sigma)$. But in discrete case, owing to the fact that every asymptotically recoverable target loop is exactly recoverable and vice versa, we need to define only the set of recoverable target loops $\mathbf{T}^{R}(\Sigma)$.

the given system as a consequence of its structural properties, namely finite and infinite zero structure and invertibility. It also discovers a multitude of ways in which freedom exists to shape the recovery error in a desired way.

The rest of the paper is organized as follows. Section III recalls a special coordinate basis (s.c.b) of [27] and [28] which displays explicitly both the finite as well as infinite zero structure of the given system. In Section IV, the structural details of three observer based controllers, namely prediction, full and reduced order current estimator based controllers, are discussed. Also, in Section IV, some preliminary analysis is given showing that the required LTR analysis for all the three controllers considered here can be unified into a single mathematical frame work. Section V deals with all the issues of LTR analysis, while Section VI dualizes the results of Section V for the case when the target loops are specified at the plant output point. Finally, Section VII draws the conclusions of our work.

III. PRELIMINARIES

As in the case of LTR analysis of continuous systems, finite and infinite zero structures of both the given discrete system and the target loop transfer function play a dominant role in the recovery analysis as well as design. Keeping this in mind, we recall in this section a special coordinate basis (s.c.b) of a linear time invariant system [27], [28]. Such a s.c.b has a distinct feature of explicitly displaying the finite and infinite zero structure of a given system. Consider the system Σ characterized by (A, B, C, D). It is simple to verify that there exist non-singular transformations U and V such that

$$UDV = \begin{bmatrix} I_{m_0} & 0 \\ 0 & 0 \end{bmatrix}, \tag{5}$$

where m_0 is the rank of matrix D. Hence hereafter, without loss of generality, it is assumed that matrix D has the form given on the right hand side of (5).

One can now rewrite the system of (1) as,

$$\begin{cases} x(k+1) = A\, x(k) + [B_0 \;\; B_1] \begin{pmatrix} u_0(k) \\ u_1(k) \end{pmatrix}, \\ \begin{pmatrix} y_0(k) \\ y_1(k) \end{pmatrix} = \begin{bmatrix} C_0 \\ C_1 \end{bmatrix} x(k) + \begin{bmatrix} I_{m_0} & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} u_0(k) \\ u_1(k) \end{pmatrix}, \end{cases} \tag{6}$$

where the matrices B_0, B_1, C_0 and C_1 have appropriate dimensions. In what follows, whenever there is no ambiguity, in order to avoid the notational clutter, the running time index k will be omitted. We have the following theorem.

Theorem 1 (SCB). *Consider the system Σ characterised by the matrix quadruple (A, B, C, D). There exist nonsingular transformations Γ_1, Γ_2 and Γ_3, an integer $m_f \leq m - m_0$, and integer indexes q_i, $i = 1$ to m_f, such that*

$$x = \Gamma_1 \tilde{x} \ , \ y = \Gamma_2 \tilde{y} \ , \ u = \Gamma_3 \tilde{u}$$

$$\tilde{x} = [x_a', \, x_b', \, x_c', \, x_f']' \ , \ x_a = [(x_a^-)', (x_a^+)']'$$

$$x_f = [x_1', \, x_2', \, \cdots, x_{m_f}']'$$

$$\tilde{y} = [y_0', y_f', y_b']' \ , \ y_f = [y_1, \, y_2, \, \cdots, y_{m_f}]'$$

$$\tilde{u} = [u_0', u_f', \, u_c']' \ , \ u_f = [u_1, \, u_2, \, \cdots, u_{m_f}]'$$

and

$$x_a^-(k+1) = A_{aa}^- x_a^-(k) + B_{0a}^- y_0(k) + L_{af}^- y_f(k) + L_{ab}^- y_b(k) \tag{7}$$

$$x_a^+(k+1) = A_{aa}^+ x_a^+(k) + B_{0a}^+ y_0(k) + L_{af}^+ y_f(k) + L_{ab}^+ y_b(k) \tag{8}$$

$$x_b(k+1) = A_{bb} x_b(k) + B_{0b} y_0(k) + L_{bf} y_f(k) \ , \ y_b = C_b x_b \tag{9}$$

$$\begin{aligned} x_c(k+1) = & A_{cc} x_c(k) + B_c u_c(k) + B_{0c} y_0(k) \\ & + L_{cb} y_b(k) + L_{cf} y_f(k) + B_c [E_{ca}^- x_a^-(k) + E_{ca}^+ x_a^+(k)] \end{aligned} \tag{10}$$

$$y_0 = C_{0a}^- x_a^- + C_{0a}^+ x_a^+ + C_{0b} x_b + C_{0c} x_c + C_{0f} x_f + u_0 \tag{11}$$

and for each $i = 1$ to m_f,

$$\begin{aligned} x_i(k+1) = & A_{q_i} x_i(k) + L_{i0} y_0(k) + L_{if} y_f(k) + B_{q_i} \Big[u_i(k) \\ & + E_{ia} x_a(k) + E_{ib} x_b(k) + E_{ic} x_c(k) + \sum_{j=1}^{m_f} E_{ij} x_j(k) \Big] \end{aligned} \tag{12}$$

$$y_i = C_{q_i} x_i \ , \ y_f = C_f x_f. \tag{13}$$

Here the states x_a^-, x_a^+, x_b, x_c and x_f are respectively of dimension n_a^-, n_a^+, n_b, n_c and $n_f = \sum_{i=1}^{m_f} q_i$ while x_i is of dimension q_i for each

$i = 1$ to m_f. The control vectors u_0, u_f and u_c are respectively of dimension m_0, m_f and $m_c = m - m_0 - m_f$ while the output vectors y_0, y_f and y_b are respectively of dimension $p_0 = m_0$, $p_f = m_f$ and $p_b = p - p_0 - p_f$. The matrices A_{q_i}, B_{q_i} and C_{q_i} have the following form:

$$A_{q_i} = \begin{bmatrix} 0 & I_{q_i-1} \\ 0 & 0 \end{bmatrix}, \quad B_{q_i} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C_{q_i} = [1,\ 0,\ \cdots,\ 0]. \tag{14}$$

(Obviously for the case when $q_i = 1$, we have $A_{q_i} = 0$, $B_{q_i} = 1$ and $C_{q_i} = 1$.) Furthermore, we have $\lambda(A_{aa}^-) \in \mathbf{C}^{\odot}$, $\lambda(A_{aa}^+) \in \mathbf{C}^{\otimes}$, the pair (A_{cc}, B_c) is controllable and the pair (A_{bb}, C_b) is observable. Also, assuming that the variables x_i, $i = 1$ to m_f, are arranged such that $q_i \leq q_{i+1}$, the matrix L_{if} has the particular form,

$$L_{if} = [L_{i1},\ L_{i2},\ \cdots,\ L_{i\,i-1},\ 0,\ 0,\ \cdots,\ 0].$$

Also, the last row of each L_{if} is identically zero.

Proof : This follows from Theorem 2.1 of [27] and [28]. ∎

We can rewrite the s.c.b given by Theorem 1 in a more compact form,

$$\tilde{A} := \Gamma_1^{-1}(A - B_0C_0)\Gamma_1 = \begin{bmatrix} A_{aa}^- & 0 & L_{ab}^-C_b & 0 & L_{af}^-C_f \\ 0 & A_{aa}^+ & L_{ab}^+C_b & 0 & L_{af}^+C_f \\ 0 & 0 & A_{bb} & 0 & L_{bf}C_f \\ B_cE_{ca}^- & B_cE_{ca}^+ & L_{cb}C_b & A_{cc} & L_{cf}C_f \\ B_fE_a^- & B_fE_a^+ & B_fE_b & B_fE_c & A_f \end{bmatrix},$$

$$\tilde{B} := \Gamma_1^{-1}[\,B_0 \quad B_1\,]\Gamma_3 = \begin{bmatrix} B_{0a}^- & 0 & 0 \\ B_{0a}^+ & 0 & 0 \\ B_{0b} & 0 & 0 \\ B_{0c} & 0 & B_c \\ B_{0f} & B_f & 0 \end{bmatrix},$$

$$\tilde{C} := \Gamma_2^{-1}\begin{bmatrix} C_0 \\ C_1 \end{bmatrix}\Gamma_1 = \begin{bmatrix} C_{0a}^- & C_{0a}^+ & C_{0b} & C_{0c} & C_{0f} \\ 0 & 0 & 0 & 0 & C_f \\ 0 & 0 & C_b & 0 & 0 \end{bmatrix},$$

$$\tilde{D} := \Gamma_2^{-1}D\Gamma_3 = \begin{bmatrix} I_{m_0} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

In what follows, we state some important properties of the s.c.b which are pertinent to our present work.

Property 1. *The given system Σ is right invertible if and only if x_b and hence y_b are nonexistent ($n_b = 0$, $p_b = 0$), left invertible if and only if x_c and hence u_c are nonexistent ($n_c = 0$, $m_c = 0$), invertible if and only if both x_b and x_c are nonexistent. Moreover, Σ is degenerate if and only if it is neither left nor right invertible.*

Property 2. *We note that (A_{bb}, C_b) and (A_{q_i}, C_{q_i}) form observable pairs. Unobservability could arise only in the variables x_a and x_c. In fact, the system Σ is observable (detectable) if and only if (A_{obs}, C_{obs}) is an observable (detectable) pair, where*

$$A_{obs} = \begin{bmatrix} A_{aa} & 0 \\ B_c E_{ca} & A_{cc} \end{bmatrix}, \quad A_{aa} = \begin{bmatrix} A_{aa}^- & 0 \\ 0 & A_{aa}^+ \end{bmatrix}, \quad C_{obs} = \begin{bmatrix} C_{0a} & C_{0c} \\ E_a & E_c \end{bmatrix},$$

$$C_{0a} = [C_{0a}^-, C_{0a}^+], \quad E_a = [E_a^-, E_a^+], \quad E_{ca} = [E_{ca}^-, E_{ca}^+].$$

Similarly, (A_{cc}, B_c) and (A_{q_i}, B_{q_i}) form controllable pairs. Uncontrollability could arise only in the variables x_a and x_b. In fact, Σ is controllable (stabilizable) if and only if (A_{con}, B_{con}) is a controllable (stabilizable) pair, where

$$A_{con} = \begin{bmatrix} A_{aa} & L_{ab} C_b \\ 0 & A_{bb} \end{bmatrix}, \quad B_{con} = \begin{bmatrix} B_{0a} & L_{af} \\ B_{0b} & L_{bf} \end{bmatrix},$$

$$B_{0a} = \begin{bmatrix} B_{0a}^- \\ B_{0a}^+ \end{bmatrix}, \quad L_{ab} = \begin{bmatrix} L_{ab}^- \\ L_{ab}^+ \end{bmatrix}, \quad L_{af} = \begin{bmatrix} L_{af}^- \\ L_{af}^+ \end{bmatrix}.$$

Property 3. *Invariant zeros of Σ are the eigenvalues of A_{aa}. Moreover, the minimum phase (or stable) and the nonminimum phase (or unstable) invariant zeros of Σ are the eigenvalues of A_{aa}^- and A_{aa}^+, respectively.*

If all the invariant zeros of a system Σ are in $\mathbf{C}^{\odot}$, i.e., if all the invariant zeros of Σ are stable, then we say Σ is of minimum phase, otherwise Σ is said to be of nonminimum phase.

There are interconnections between the s.c.b and various invariant and almost invariant geometric subspaces. To show these interconnections, we introduce the following geometric subspaces of Σ.

Definition 6. *For a given system Σ characterized by a matrix quadruple (A, B, C, D), we define*

1. *$\mathcal{V}^g(\Sigma)$ to be the maximal subspace of $\mathbf{R}^n$ which is $(A-BF)$-invariant and contained in $\operatorname{Ker}(C-DF)$ such that the eigenvalues of $(A-BF)|\mathcal{V}^g$ are contained in $\mathbf{C}_g \subseteq \mathbf{C}$ for some F.*

2. *$\mathcal{S}^g(\Sigma)$ to be the minimal $(A-KC)$-invariant subspace of $\mathbf{R}^n$ containing in $\operatorname{Im}(B-KD)$ such that the eigenvalues of the map which is induced by $(A-KC)$ on the factor space $\mathbf{R}^n/\mathcal{S}^g$ are contained in $\mathbf{C}_g \subseteq \mathbf{C}$ for some K.*

For the cases that $\mathbf{C}_g = \mathbf{C}$, $\mathbf{C}_g = \mathbf{C}^{\odot}$ and $\mathbf{C}_g = \mathbf{C}^{\otimes}$, we replace the index g in $\mathcal{V}^g$ and $\mathcal{S}^g$ by '$*$', '$-$' and '$+$', respectively.

Various components of the state vector of s.c.b have the following geometrical interpretations.

Property 4.

1. *$x_a^- \oplus x_a^+ \oplus x_c$ spans $\mathcal{V}^*(\Sigma)$.*
2. *$x_a^- \oplus x_c$ spans $\mathcal{V}^-(\Sigma)$.*
3. *$x_a^+ \oplus x_c$ spans $\mathcal{V}^+(\Sigma)$.*
4. *$x_c \oplus x_f$ spans $\mathcal{S}^*(\Sigma)$.*
5. *$x_a^- \oplus x_c \oplus x_f$ spans $\mathcal{S}^+(\Sigma)$.*
6. *$x_a^+ \oplus x_c \oplus x_f$ spans $\mathcal{S}^-(\Sigma)$.*

IV. DIFFERENT CONTROLLER STRUCTURES

In this section, we consider three different controller structures used commonly in discrete systems. All three controllers are observer based, but the type of observer (or state estimator) used in each one is structurally different. The estimators considered here are (1) prediction estimator, (2) current estimator and (3) reduced order estimator. Both prediction estimator and current estimator are full order observers. The reduced order estimator is a current estimator but uses the reduced order observer. The prediction estimator estimates the state $x(k+1)$ based on the measurements $y(k)$ up to and including the (k)-th instant, where as the current estimator estimates $x(k+1)$ based on the measurements $y(k+1)$ up to and including the $(k+1)$-th instant. Since in the prediction estimator based controller, the current estimated value of control does not depend on the most current value of the measurement, it might not be as accurate as it could be in the current estimator based controller. However, the prediction estimator based controller could avail itself the entire sampling period to do the required computations and hence is commonly used when the needed computations are excessive. In contrast, when the needed computations can be done in a short time compared to the sampling period, current estimator based controller can easily be used. We note that prediction estimator forces an inherent time delay which otherwise is absent in the structure of controller. As can be expected, the three different controllers have different capabilities regarding LTR; but as will be seen shortly there exists a common mathematical machinery to analyse them under a single frame work. In the sections to follow, we will systematically do LTR analysis using a generic controller which could be any one of these three controllers. In such an analysis, we shall use the following notation :

$$\begin{aligned}
\mathrm{C}(z) &:= \text{The transfer function of the controller,} \\
L(z) &:= \mathrm{C}(z)P(z) = \text{The achieved loop transfer function,} \\
S(z) &:= [I_m + L(z)]^{-1} = \text{The achieved sensitivity function,} \\
T(z) &:= I_m - S(z) = \text{The achieved complimentary sensitivity} \\
&\quad\ \text{function,} \\
E(z) &:= L_t(z) - L(z) = \text{Loop recovery error,} \\
M(z) &:= \text{The recovery matrix (to be defined later on),} \\
M^0(z) &:= \text{A part of the recovery matrix } M(z) \text{ that can be} \\
&\quad\ \text{rendered zero,} \\
M^e(z) &:= \text{A part of the recovery matrix } M(z) \text{ that cannot be} \\
&\quad\ \text{rendered zero and hence termed as recovery error matrix,} \\
\mathrm{T}^{\mathrm{R}}(\Sigma) &:= \text{The set of either exactly or asymptotically recoverable} \\
&\quad\ \text{target loops for } \Sigma.
\end{aligned}$$

The above notation applies to a generic controller; however, whenever we refer to a particular controller, we use appropriate subscripts to identify them. Subscripts p, c and r are used respectively to represent prediction, current, and reduced order estimator based controllers. For example, $L_p(z)$, $M_c^e(z)$ and $\mathrm{T}_r^{\mathrm{R}}(\Sigma)$ denote respectively the achieved loop transfer function with a prediction estimator based controller, the recovery error matrix when a current estimator based controller is used, and the set of recoverable target loops for Σ using a reduced order estimator based controller.

We now proceed to give the structural details of the controllers considered here.

Prediction Estimator Based Controller :

The dynamic equations of the controller are

$$\begin{cases} \hat{x}(k+1) = A\hat{x}(k) + Bu(k) + K_p[y(k) - C\hat{x}(k) - Du(k)], \\ u(k) = \hat{u}(k) = -F\hat{x}(k), \end{cases} \tag{15}$$

where K_p is the gain chosen so that $A - K_pC$ is asymptotically stable. The transfer function of the controller is

$$\mathbf{C}_p(z) = F[zI_n - A + BF + K_pC - K_pDF]^{-1}K_p. \tag{16}$$

Current Estimator Based Controller :

Here without loss of generality, we assume that the matrices C and D are in the form,

$$C = \begin{bmatrix} C_0 \\ C_1 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} D_0 \\ 0 \end{bmatrix}, \tag{17}$$

where D_0 is of maximal rank, i.e., $\text{rank}(D) = \text{rank}(D_0) = m_0$. Thus, the output y can be partitioned as,

$$\begin{bmatrix} y_0(k) \\ y_1(k) \end{bmatrix} = \begin{bmatrix} C_0 \\ C_1 \end{bmatrix} x(k) + \begin{bmatrix} D_0 \\ 0 \end{bmatrix} u(k).$$

The dynamic equations of the controller are

$$\begin{cases} \hat{x}(k+1) = A\hat{x}(k) + Bu(k) + K_c\left(\begin{bmatrix} y_0(k) \\ y_1(k+1) \end{bmatrix} - C_c\hat{x}(k) - D_cu(k)\right), \\ \hat{u}(k) = u(k) = -F\hat{x}(k), \end{cases} \tag{18}$$

where

$$C_c = \begin{bmatrix} C_0 \\ C_1A \end{bmatrix} \quad \text{and} \quad D_c = \begin{bmatrix} D_0 \\ C_1B \end{bmatrix}, \tag{19}$$

and where the gain K_c is chosen so that $A - K_cC_c$ is asymptotically stable. The transfer function from $-u$ to y that results in using the current estimator is then given by

$$\mathbf{C}_c(z) = F\left[zI_n - A + K_cC_c + BF - K_cD_cF\right]^{-1} K_c \begin{bmatrix} I_{m_0} & 0 \\ 0 & zI_{p-m_0} \end{bmatrix}. \tag{20}$$

Perhaps, some explanation regarding the structure of the current estimator (18) is in order. It is a generalisation for nonstrictly proper systems of the existing current estimator given in Franklin et al [11]. Here we note that $y_0(k)$ and $y_1(k)$ together form the measurement vector $y(k)$. In view of (17), $y_0(k)$ depends on the control $u(k)$ explicitly, where as $y_1(k)$ does not depend on any control at the instant k. In order that the controller be physically realisable, in arriving at $\hat{x}(k+1)$, the current estimator utilizes $y_1(k+1)$ which is a part of the measurement at instant $k+1$, and $y_0(k)$ which is a part of the measurement at instant k. If the given system Σ is strictly proper, $y_0(k)$ is nonexistent and the current estimator (18) coalesces with that given in Franklin *et al* [11], Maciejowski [15] and Zhang and Freudenberg [35].

Reduced Order Estimator Based Controller :

Again, without any loss of generality but for simplicity of presentation, it is assumed that the matrices C and D are transformed into the form,

$$C = \begin{bmatrix} 0 & C_{02} \\ I_{p-m_0} & 0 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} D_0 \\ 0 \end{bmatrix}. \tag{21}$$

Then Σ can be partitioned as follows,

$$\begin{cases} \begin{pmatrix} x_1(k+1) \\ x_2(k+1) \end{pmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{pmatrix} x_1(k) \\ x_2(k) \end{pmatrix} + \begin{bmatrix} B_{11} \\ B_{22} \end{bmatrix} u(k), \\ \begin{pmatrix} y_0(k) \\ y_1(k) \end{pmatrix} = \begin{bmatrix} 0 & C_{02} \\ I_{p-m_0} & 0 \end{bmatrix} \begin{pmatrix} x_1(k) \\ x_2(k) \end{pmatrix} + \begin{bmatrix} D_0 \\ 0 \end{bmatrix} u(k). \end{cases} \tag{22}$$

We observe that $y_1 = x_1$ is already available and need not be estimated. Thus we need to estimate only the state variable x_2. We first rewrite the state equation for x_1 in terms of the output y_1 and state x_2 as follows,

$$y_1(k+1) = A_{11}y_1(k) + A_{12}x_2(k) + B_{11}u(k). \tag{23}$$

Since $y_1(k+1)$ and $y_1(k)$ are known, (23) can be rewritten as

$$\tilde{y}_1(k) = A_{12}x_2(k) + B_{11}u(k) = y_1(k+1) - A_{11}y_1(k). \tag{24}$$

Thus, observation of x_2 is made via (24) as well as by

$$y_0(k) = C_{02}x_2(k) + D_0u(k).$$

Now, a reduced order system suitable for estimating the state x_2 is given by

$$\begin{cases} x_2(k+1) = A_r\, x_2(k) + B_r\, u(k) + A_{21}\, y_1(k), \\ \begin{pmatrix} y_0(k) \\ \tilde{y}_1(k) \end{pmatrix} = y_r(k) = C_r\, x_2(k) + D_r\, u(k) \end{cases} \tag{25}$$

where

$$A_r = A_{22}\ ,\ B_r = B_{22}\ ,\ C_r = \begin{bmatrix} C_{02} \\ A_{12} \end{bmatrix}\ ,\ D_r = \begin{bmatrix} D_0 \\ B_{11} \end{bmatrix}. \tag{26}$$

Based on equation (25), we can construct a reduced order estimate of the state x_2 as,

$$\hat{x}_2(k+1) = A_r \hat{x}_2(k) + B_r u(k) + A_{21} y_1(k) + K_r \left[y_r(k) - C_r \hat{x}_2(k) - D_r u(k) \right], \tag{27}$$

where K_r, the reduced order estimator gain matrix, is chosen such that $A_r - K_r C_r$ is asymptotically stable. Since $\hat{x}_2(k+1)$ depends on the measurement $y_1(k+1)$, the reduced order estimator (27) belongs to the class of current estimators. For the purpose of implementing it, (27) can be rewritten by partitioning $K_r = [K_{r0},\ K_{r1}]$ in conformity with y_0 and $\tilde{y}_1$ and by defining the following variable $v(k)$,

$$v(k) = \hat{x}_2(k) - K_{r1} y_1(k). \tag{28}$$

Then the reduced order estimator based controller is given by

$$\begin{cases} v(k+1) = (A_r - K_r C_r) v(k) + (B_r - K_r D_r) u(k) + G_r y(k), \\ u(k) = \hat{u}(k) = -F_1 x_1(k) - F_2 \hat{x}_2(k) = -F_2 v(k) - [0,\ F_1 + F_2 K_{r1}] y(k), \end{cases} \tag{29}$$

where

$$F = [\,F_1,\quad F_2\,], \qquad G_r = [K_{r0},\ A_{21} - K_{r1} A_{11} + (A_r - K_r C_r) K_{r1}]. \tag{30}$$

The transfer function from $-u$ to y that results in using the reduced order estimator is then given by

$$\mathbf{C}_r(z) = F_2 [zI - A_r + K_r C_r + B_r F_2 - K_r D_r F_2]^{-1} \cdot \Big(G_r - (B_r - K_r D_r)[0,\ F_1 + F_2 K_{r1}] \Big) + [0,\ F_1 + F_2 K_{r1}]. \tag{31}$$

Proposition 1. *For the case when Σ is right invertible and the matrix D is of maximal rank, all the controllers considered here, namely, the prediction, current and reduced order estimator based controllers, are one and the same.*

Proof : When Σ is right invertible and matrix D is of maximal rank, we note that in current estimator, $D = D_0$, C_1 is empty, $C_c = C_0 = C$ and $D_c = D_0 = D$. On the other hand, in reduced order estimator, we have $A_{22} = A$, $B_{22} = B$, $C_{02} = C$, $A_r = A_{22} = A$, $B_r = B_{22} = B$, $C_r = C_{02} = C$, $D_r = D_0 = D$. Using these facts, it is easy to verify the above proposition. ∎

We now proceed to do some preliminary analysis of the loop recovery error $E(z)$. It turns out that the expression, $E(z) = L_t(z) - L(z)$, is not well suited for loop transfer recovery analysis. Realising this, for the class of systems he considered, Goodman [12] related $E(z)$ to a matrix $M(z)$, hereafter called the recovery matrix. The following lemma generalises Goodman's result for general nonstrictly proper systems and for all the three controllers considered here.

Lemma 1. *Let a system Σ be stabilizable and detectable. Also, let $L_t(z) = F\Phi B$ be an admissible target loop, i.e., $L_t(z) \in \mathbf{T}(\Sigma)$. Then the loop recovery error $E(z)$ between the target loop transfer function $L_t(z)$ and that realised by any one of the controllers described earlier, can be written in the form,*

$$E(z) = M(z)[I_m + M(z)]^{-1}(I_m + F\Phi B). \tag{32}$$

Furthermore, for all $\omega \in \Omega$,

$$E^*(j\omega) = 0 \text{ if and only if } M^*(j\omega) = 0$$

where Ω is the set of all $0 \leq |\omega| \leq \pi/T$ for which $L_t^*(j\omega)$ and $L^*(j\omega) = \mathbf{C}^*(j\omega)P^*(j\omega)$ are well defined (i.e., all the required inverses exist). The expression for the recovery matrix $M(z)$ depends on the controller used. In particular, for each one of the controllers considered earlier, we have the following expressions,

$$M_p(z) = F(zI_n - A + K_pC)^{-1}(B - K_pD), \tag{33}$$

$$M_c(z) = F\left(zI_n - A + K_cC_c\right)^{-1}\left(B - K_cD_c\right), \tag{34}$$

$$M_r(z) = F_2(zI - A_r + K_rC_r)^{-1}(B_r - K_rD_r). \tag{35}$$

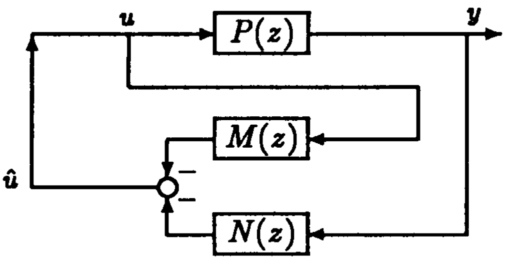

Figure 2: Plant and controller configuration.

Proof : See Appendix A. ∎

A physical interpretation of the the recovery matrix $M(z)$ can be given. To do so, one can view the controller as a device having two inputs, (1) the plant input u and (2) the plant output y as shown in Figure 2. Then, $-M(z)$ is the transfer function from the plant input point to the controller output point while $\tilde{M}(z)$ is the transfer function from the plant input point to the estimated state $\hat{x}$. That is, one can write

$$\hat{U}(z) = -F\hat{X}(z) = -M(z)U(z) - N(z)Y(z), \tag{36}$$

and

$$\hat{X}(z) = \tilde{M}(z)U(z) + \tilde{N}(z)Y(z). \tag{37}$$

Here, depending on the controller used, the expressions for $M(z)$ are as in (33), (34) and (35). Also, $\tilde{M}(z)$ is such that $M(z) = F\tilde{M}(z)$. Moreover, depending on the controller used, the expressions for $N(z)$ and $\tilde{N}(z)$ are as given below.

$$N_p(z) = F(zI - A + K_pC)^{-1}K_p \ , \ N_p(z) = F\tilde{N}_p(z)\,, \tag{38}$$

$$N_c(z) = (zI - A + K_cC_c)^{-1}K_c \begin{bmatrix} I & 0 \\ 0 & zI \end{bmatrix} \ , \ N_c(z) = F\tilde{N}_c(z)\,,$$

$$N_r(z) = F_2(zI - A_r + K_rC_r)^{-1}G_r + [0,\, F_1 + F_2K_{r1}] \ , \ N_r(z) = F\tilde{N}_r(z)\,.$$

In view of the above expressions, Lemma 1 implies that whenever LTR is achieved by the controller, the controller output does not entail any feedback from the plant input point. On the other hand, the state estimate $\hat{x}$ in general depends on the plant input. The significance of Lemma 1 can be seen in two ways. It converts the LTR analysis problem into a study of conditions under which the recovery matrix $M(z)$ can be rendered zero. Also, it unifies the study of $M(z)$ for all the three controllers into a single mathematical framework. To see this explicitly, let us define the auxiliary systems Σ_c and Σ_r which are respectively characterized by the matrix quadruples (A, B, C_c, D_c) and(A_r, B_r, C_r, D_r). Then we have the following observation.

Observation 1.

1. *The LTR mechanism for a given system Σ using a current estimator based controller can be studied using the auxiliary system Σ_c and a a prediction estimator based controller constructed for it.*

2. *The LTR mechanism for a given system Σ using a reduced order estimator based controller can be studied using the auxiliary system Σ_r and a prediction estimator based controller constructed for it where in F_2 takes the place of F.*

In view of Lemma 1 and Observation 1, our study of LTR for all the three controllers is unified and reduces to the study of an appropriate recovery matrix $M(z)$. In order to further cement such a unification, we need to investigate the relationship between the structural properties of Σ_c and Σ, as well as between those of Σ_r and Σ. The following propositions delineates such relationships.

Proposition 2.

1. *Σ_c is of (non-) minimum phase if and only if Σ is of (non-) minimum phase.*

2. *Σ_c is stabilizable and detectable if and only if Σ is stabilizable and detectable.*

3. *Invariant zeros of Σ_c contain invariant zeros of Σ and $z = 0$.*

4. *Orders of infinite zeros of Σ_c are reduced by one from those of Σ.*

5. *Σ_c is left invertible if and only if Σ is left invertible.*

6. $\mathcal{V}^+(\Sigma_c) = \mathcal{V}^+(\Sigma)$.

7. $\mathcal{S}^-(\Sigma_c) = \mathcal{S}^-(\Sigma) \cap \{x \mid Cx \in \mathrm{Im}\,(D)\}$.

8. *$\mathcal{S}^-(\Sigma_c) = \emptyset$ if and only if Σ is left invertible and of minimum phase with no infinite zeros of order higher than one.*

Proof : See Appendix B. ∎

Proposition 3.

1. *Σ_r is of (non-) minimum phase if and only if Σ is of (non-) minimum phase.*

2. *Σ_r is detectable if and only if Σ is detectable.*

3. *Invariant zeros of Σ_r are the same as those of Σ.*

4. *Orders of infinite zeros of Σ_r are reduced by one from those of Σ.*

5. *Σ_r is left invertible if and only if Σ is left invertible.*

6. $\begin{pmatrix} 0_{(p-m_0)\times(n-p+m_0)} \\ I_{(n-p+m_0)} \end{pmatrix} \mathcal{V}^+(\Sigma_r) = \mathcal{V}^+(\Sigma).$

7. $\begin{pmatrix} 0_{(p-m_0)\times(n-p+m_0)} \\ I_{(n-p+m_0)} \end{pmatrix} \mathcal{S}^-(\Sigma_r) = \mathcal{S}^-(\Sigma) \cap \{x \mid Cx \in \operatorname{Im}(D)\}.$

8. *$\mathcal{S}^-(\Sigma_r) = \emptyset$ if and only if Σ is left invertible and of minimum phase with no infinite zeros of order higher than one.*

Proof : See [26]. ∎

Remark 1. *For a left invertible minimum phase system Σ with $D = 0$, it is simple to see that*

$$\mathcal{S}^-(\Sigma_c) = \mathcal{S}^-(\Sigma_r) = \mathcal{S}^-(\Sigma) \cap \{x \mid Cx \in \operatorname{Im}(D)\} = \emptyset$$

if and only if CB is of maximal rank. Also, for a nonstrictly proper SISO system Σ,

$$\mathcal{S}^-(\Sigma) = \mathcal{S}^-(\Sigma_c) = \mathcal{S}^-(\Sigma_r) = \mathcal{S}^-(\Sigma) \cap \{x \mid Cx \in \operatorname{Im}(D)\} = \emptyset$$

if and only if it is of minimum phase.

V. GENERAL LTR ANALYSIS

This section deals with the general analysis of LTR mechanism using any one of the three controllers discussed in the last section. Notationally, in all our general discussions here, we deal with the given system Σ characterized by the quadruple (A, B, C, D) and the prediction estimator based controller in which K_p is the observer gain. In view of Observation 1, all the general discussions presented here can be particularised to current and reduced order estimator based controllers with appropriate notational changes. In all our main theorems, we will however explicitly point out the capabilities of each controller as they could be different for each case.

As is evident from Lemma 1, the nucleus of LTR analysis is the study of $M_p(z)$ to ascertain how and when it can or cannot be rendered zero. The required study of $M_p(z)$ can be undertaken in two ways, with or without the prior knowledge of F that prescribes the target loop transfer function $L_t(z)$. Note that the study of $M_p(z)$ without the prior knowledge of F imitates the traditional LQG design philosophy in which the two tasks of obtaining F and K_p are separated. Keeping this in mind, our goal in the first subsection to follow is to study $M_p(z)$ without taking into account any specific characteristics of F. The second subsection, devoted to LTR analysis while taking into account appropriate characteristics of F, complements the analysis of the first subsection. Decomposing $M_p(z)$ as $F\tilde{M}_p(z)$, the study of $M_p(z)$ without knowing F is the same one as the study of $\tilde{M}_p(z)$. A detailed study of $\tilde{M}_p(z)$ leads to a fundamental Lemma 2 involving with an eigenstructure assignment to the observer dynamic matrix $A - K_pC$ by an appropriate design of K_p. This Lemma 2 reveals the limitations of the given system as a consequence of its structural properties in recovering an arbitrary target loop transfer function via the given controller structure. Thus it leads to Theorem 2 which, for each controller, shows under what conditions on Σ the set of recoverable target loops $\mathrm{T}^{\mathrm{R}}(\Sigma)$ is equal to the set of admissible target loops $\mathrm{T}(\Sigma)$. Most of the results available so far in the literature can then be seen to be special cases of Theorem 2. Also, Lemma 2 enables one to decompose $\tilde{M}_p(z)$ into two essential parts, $\tilde{M}_p^0(z)$ and $\tilde{M}_p^e(z)$. The first part $\tilde{M}_p^0(z)$ can be rendered zero by an appropriate eigenstructure assignment to $A - K_pC$, while the second part $\tilde{M}_p^e(z)$ in general cannot be rendered zero, by any means, although our analysis of $\tilde{M}_p^e(z)$ reveals a multitude of ways by which it can be shaped. The decomposition of $\tilde{M}_p(z)$ into two parts and the subsequent analysis of each part forms the core of entire analysis given throughout this paper. In particular, it leads to several important results given in this section. For example, Theorem 3 characterizes the loop transfer function as well as the sensitivity and complementary sensitivity functions achievable by the considered controller. On the other hand, Theorem 4 shows the subspace $\mathcal{S}^e \in \mathbf{R}^m$ in which $\tilde{M}_p^e(z)$ can be rendered zero, i.e, the projections of the target and achievable sensitivity and complementary sensitivity functions onto $\mathcal{S}^e$ can match each other. Next, in Subsection B, Theorem 5 develops the necessary and sufficient conditions a target loop transfer function $L_t(z)$ has to satisfy so that it can be recoverable for the given system Σ. On the other hand, Theorem 6 develops the necessary and sufficient conditions on Σ so that it has at least one recoverable target loop transfer function. Subsection C generalizes the results of Subsections A and B when recovery is important over a prescribed subspace of the control space. Furthermore, our analysis in this section reveals the mechanism of pole-zero cancellation between the controller eigenvalues and the input or output decoupling zeros of Σ.

A. Recovery Analysis For An Arbitrary Target Loop

In this subsection, we consider that the target loop transfer function $L_t(z) = F\Phi B$ is arbitrary. That is, we do not take into account any specific characteristics of $L_t(z)$ in analyzing the LTR mechanism. As mentioned before, we will focus our attentions on the prediction estimator based controller with gain K_p. Then, as implied by Lemma 1, $\tilde{M}_p(z)$ as given below forms the basis of our study,

$$\tilde{M}_p(z) = (zI_n - A + K_pC)^{-1}(B - K_pD). \tag{39}$$

It is evident that the gain K_p is the only free design parameter in $\tilde{M}_p(z)$. First of all, in order to guarantee the closed-loop stability, K_p must be such that $A - K_pC$ is an asymptotically stable matrix. The remaining freedom in choosing K_p can then be used for the purpose of achieving LTR. We note that exact loop transfer recovery (ELTR) is possible for an arbitrary F if and only if

$$\tilde{M}_p^*(j\omega) = (e^{j\omega T}I_n - A + K_pC)^{-1}(B - K_pD) \equiv 0.$$

However, due to the nonsingularity of $(e^{j\omega T}I_n - A + K_pC)^{-1}$, the fact that $\tilde{M}_p^*(j\omega) \equiv 0$ implies that $B - K_pD \equiv 0$. Hence, rendering all the parts of $\tilde{M}_p^*(j\omega)$ zero is possible only for a very restrictive class of systems. In general only certain parts of $\tilde{M}_p^*(j\omega)$ can be rendered zero. To proceed with our analysis, for clarity of presentation we will temporarily assume that $A - K_pC$ is nondefective. This allows us to expand $\tilde{M}_p(z)$ and hence $M_p(z)$ in a dyadic form,

$$\tilde{M}_p(z) = \sum_{i=1}^{n} \frac{\tilde{R}_i}{z - \lambda_i} \tag{40}$$

where the residue $\tilde{R}_i$ is given by

$$\tilde{R}_i = W_i V_i^{\mathrm{H}} [B - K_p D]. \tag{41}$$

Here W_i and V_i are respectively the right and left eigenvectors associated with an eigenvalue λ_i of $A - K_pC$ and they are scaled so that $WV^{\mathrm{H}} = V^{\mathrm{H}}W = I_n$ where

$$W = [W_1, W_2, \cdots, W_n] \quad \text{and} \quad V = [V_1, V_2, \cdots, V_n]. \tag{42}$$

Remark 2. *The assumption that K_p is selected so that $A - K_pC$ is nondefective is not essential. However, it simplifies our presentation. A removal of this condition necessitates the use of generalized right and left eigenvectors of $A - K_pC$ instead of the right and left eigenvectors W_i and V_i and consequently the expansion of $\tilde{M}_p(z)$ requires a double summation in place of the single summation used in (40).*

We are looking for conditions under which the i-th term of $\tilde{M}_p(z)$ in (40) can be made zero for each $i = 1$ to n. There is only one possibility in discrete-time LTR to do so, namely, assigning λ_i to any location in $\mathbf{C}^{\odot}$ while simultaneously rendering the corresponding residue $\tilde{R}_i$ zero. In other words, such a possibility corresponds to appropriate finite eigenstructure assignment to $A - K_pC$ to render $\tilde{R}_i$ zero. In continuous systems, besides the above possibility, there exists another possibility, namely, assigning λ_i asymptotically large in the negative half s-plane so that a term of the type

$$\frac{\tilde{R}_i}{s - \lambda_i}$$

tends to zero as $\lambda_i \to \infty$. This possibility deals with an infinite eigenstructure assignment to $A - K_pC$. The possibility of assigning an infinite eigenstructure, however, does not exist in discrete systems since λ_i is restricted to $\mathbf{C}^{\odot}$ in order to guarantee the stability of the resulting closed-loop system. Given the fact that $|\lambda_i|$ cannot go to ∞, it is easy to observe that the *notions of exact* LTR (ELTR) *and asymptotic* LTR (ALTR) *in discrete-time systems are equivalent in the sense that any target loop that is asymptotically recoverable is also exactly recoverable and vice versa.* Because of this, throughout this paper, whenever we talk about recovery, we mean both exact and asymptotic recovery. For example, whenever we say that an admissible target loop is recoverable, we mean by it that the specified target loop is exactly as well as asymptotically recoverable as stated in definition 4. Nevertheless, we will in Part 2 of this paper, distinguish between ELTR and ALTR. This is because, as we mentioned in the introduction, some optimization based design methods such as H_∞ norm minimization methods some times lead to suboptimal designs that correspond to asymptotic recovery. To be specific, in optimization methods, one normally generates a sequence of observer gains by solving parameterized algebraic Riccati equations. As the parameter tends to a certain value, the corresponding sequence of H_∞ norms of the resulting recovery matrices tends to a limit which is the infimum of the H_∞ norm of the recovery matrix over the set of all possible observer gains. A suboptimal solution is obtained when one selects an observer gain corresponding to a particular value of the parameter. On the other hand, in eigenstructure assignment methods, the required observer gain is obtained without solving any parameterized equations. Thus some times the observer gain K_p is designed as a function of a parameter and some other times independent of it.

The following lemma answers the question of how many residues $\tilde{R}_i$ can be rendered zero by an appropriate finite eigenstructure assignment of $A - K_pC$.

Lemma 2. *Let λ_i and V_i be an eigenvalue and the corresponding left eigenvector of $A - K_pC$ for any gain K_p such that $A - K_pC$ is asymptotically stable. Then the maximum possible number of $\lambda_i \in \mathbf{C}^{\odot}$ which satisfy the condition $V_i^H[B - K_pD] = 0$ is $n_a^- + n_b$. A total of n_a^- of these λ_i coincide with the system invariant zeros which are in $\mathbf{C}^{\odot}$ (the so called stable or minimum phase invariant zeros) and the remaining n_b eigenvalues can be assigned arbitrarily to any locations in $\mathbf{C}^{\odot}$. All the eigenvectors V_i that correspond to these $n_a^- + n_b$ eigenvalues span the subspace $\mathbf{R}^n/S^-(\Sigma)$. Moreover, the n_a^- eigenvectors V_i which correspond to the eigenvalues which* *coincide with the system invariant zeros in* $\mathbf{C}^{\odot}$ *coincide with the corresponding left state zero directions and span the subspace* $\mathcal{V}^*(\Sigma)/\mathcal{V}^+(\Sigma)$.

Proof : See Appendix B of [6]. ∎

Remark 3. *Instead of rendering the* $n_a^- + n_b$ *residues* $\tilde{R}_i$ *mentioned in Lemma 2 exactly zero, if one prefers, they can be rendered asymptotically zero as certain parameter tends to a particular limit. In that case* n_a^- *eigenvalues coincide asymptotically with the* n_a^- *minimum phase invariant zeros while the corresponding eigenvectors in the limit coincide with the corresponding left state zero directions and span the subspace* $\mathcal{V}^*(\Sigma)/\mathcal{V}^+(\Sigma)$. *As stated earlier, in this part of paper, we will not distinguish between such exact and asymptotic assignments.*

Lemma 2 points out that there are altogether $n_a^- + n_b$ eigenvalues which can be assigned inside $\mathbf{C}^{\odot}$ so that the corresponding terms of $\tilde{M}_p(z)$ in its dyadic expansion (40) are zero. This fact leads to some structural conditions on Σ so that any arbitrary admissible target loop can be recovered. This is explored in the following theorem.

Theorem 2. *Consider a stabilizable and detectable system* Σ. *Depending upon the controller used, we have the following characterization of* Σ *so that any arbitrary admissible target loop can be recovered.*

1. Prediction estimator based controller: *Any arbitrary admissible target loop is recoverable, i.e.,* $\mathbf{T}_p^{\mathrm{R}}(\Sigma) = \mathbf{T}(\Sigma)$, *if and only if* Σ *is left invertible and of minimum phase with no infinite zeros (i.e., D is maximal rank).*

2. Current estimator based controller: *Any arbitrary admissible target loop is recoverable, i.e.,* $\mathbf{T}_c^{\mathrm{R}}(\Sigma) = \mathbf{T}(\Sigma)$, *if and only if* Σ *is left invertible and of minimum phase with no infinite zeros of order higher than one.*

3. Reduced order estimator based controller: *Any arbitrary admissible target loop is recoverable, i.e.,* $\mathbf{T}_r^{\mathrm{R}}(\Sigma) = \mathbf{T}(\Sigma)$, *if and only if* Σ *is left invertible and of minimum phase with no infinite zeros of order higher than one.*

Proof : Let us take the case of a prediction estimator based controller. The fact that Σ is left invertible and of minimum phase with no infinite zeros implies that $n_a^+ = n_c = n_f = 0$. Thus $n_a^- + n_b = n$. Hence the result follows from (32) and Lemma 2. Conversely, it is simple to see that the recoverability of all the admissible target loops implies that $V_i^H(B-K_pD) = 0$, $i = 1, \cdots, n$. Then by Lemma 2, we know that this is possible only when $n_a^- + n_b = n$. Hence, $n_a^+ = n_c = n_f = 0$, and thus Σ is left invertible and of minimum phase with no infinite zeros. Now, for the case of current and reduced order observer based controllers, in view of Propositions 2 (i.e., item 8) and 3 (i.e., item 8), we note that $n_a^+ + n_c + n_f$ corresponding to both Σ_c and Σ_r is equal to zero if and only if Σ is left invertible and of minimum phase with no infinite zeros of order higher than one. ∎

We have the following interesting special case of Theorem 2.

Corollary 1. *Let Σ be left invertible and of minimum phase with $D = 0$ and CB of maximal rank. Then all infinite zeros of Σ are of order one, and hence it follows from Theorem 2 that $\mathbf{T}_c^R(\Sigma) = \mathbf{T}_r^R(\Sigma) = \mathbf{T}(\Sigma)$, i.e., all the admissible target loops are recoverable by appropriate current and reduced order estimator based controllers; but not in general by a prediction estimator based controller.*

Proof : It is obvious. ∎

The above result was obtained earlier by Maciejowski [15] and Zhang and Freudenberg [35] for the case of a current estimator based controller.

As is evident by Theorem 2, the required structural conditions for recovery of any arbitrary admissible target loop are very stringent, and call for $n_a^- + n_b$ to be equal to the dimension n of Σ. To see what is and what is not feasible when $n_a^- + n_b \neq n$, and to emphasize explicitly the behavior of each term of $\tilde{M}_p(z)$, let us partition the dyadic expansion (40) of $\tilde{M}_p(z)$ into three parts, each part having a particular type of characteristics,

$$\tilde{M}_p(z) = \tilde{M}_p^-(z) + \tilde{M}_p^b(z) + \tilde{M}_p^e(z), \tag{43}$$

where

$$\tilde{M}_p^-(z) = \sum_{i=1}^{n_a^-} \frac{\tilde{R}_i^-}{z - \lambda_i^-}, \quad \tilde{M}_p^b(z) = \sum_{i=1}^{n_b} \frac{\tilde{R}_i^b}{z - \lambda_i^b}$$

and

$$\tilde{M}_p^e(z) = \sum_{i=1}^{n_a^+ + n_c + n_f} \frac{\tilde{R}_i^e}{z - \lambda_i^e}.$$

In the above partition, appropriate superscripts $-$, b, and e are added to $\tilde{R}_i$ and λ_i in order to associate them respectively with $\tilde{M}_p^-(z)$, $\tilde{M}_p^b(z)$, and $\tilde{M}_p^e(z)$. Next, define the following sets where $n^e = n_a^+ + n_c + n_f$:

$$\Lambda^- = \{\lambda_i^-\,;\, i=1 \text{ to } n_a^-\},\ V^- = \{V_i^-\,;\, i=1 \text{ to } n_a^-\},\ W^- = \{W_i^-\,;\, i=1 \text{ to } n_a^-\}$$

$$\Lambda^b = \{\lambda_i^b\,;\, i=1 \text{ to } n_b\},\ \ V^b = \{V_i^b\,;\, i=1 \text{ to } n_b\},\ \ W^b = \{W_i^b\,;\, i=1 \text{ to } n_b\}$$

$$\Lambda^e = \{\lambda_i^e\,;\, i=1 \text{ to } n_e\},\ \ V^e = \{V_i^e\,;\, i=1 \text{ to } n_e\},\ \ W^e = \{W_i^e\,;\, i=1 \text{ to } n_e\}.$$

We now note that various parts of $\tilde{M}_p(z)$ have the following interpretation:

1. $\tilde{M}_p^-(z)$ contains n_a^- terms. The n_a^- eigenvalues of $A - K_pC$ represented in it form a set Λ^-. In accordance with Lemma 2, there exists a gain K_p such that $\tilde{M}_p^-(z)$ can be rendered identically zero by assigning the elements of Λ^- to coincide with the system minimum phase invariant zeros while the corresponding set of left eigenvectors V^- coincides with the the corresponding set of left state zero directions.

2. $\tilde{M}_p^b(z)$ contains n_b terms. The n_b eigenvalues of $A - K_pC$ represented in it form a set Λ^b. In accordance with Lemma 2, there exists a gain K_p such that $\tilde{M}_p^b(z)$ can be rendered zero by assigning the elements of Λ^b to arbitrary locations in $\mathbf{C}^{\odot}$.

3. $\tilde{M}_p^e(z)$ contains $n^e = n_a^+ + n_c + n_f$ terms. The n_e eigenvalues of $A - K_pC$ represented in $\tilde{M}_p^e(z)$ form a set Λ^e. In view of Lemma 2, $\tilde{M}_p^e(z)$ cannot in general be rendered zero by any assignment of Λ^e and the associated sets of right and left eigenvectors W^e and V^e.

Since both $\tilde{M}_p^-(z)$ and $\tilde{M}_p^b(z)$ can be rendered zero, for future use, we can combine them into one term,

$$\tilde{M}_p^0(z) = \tilde{M}_p^-(z) + \tilde{M}_p^b(z).$$

We define likewise, $\Lambda^0 = \Lambda^- \cup \Lambda^b$, $W^0 = W^- \cup W^b$, $V^0 = V^- \cup V^b$. Thus $\tilde{M}_p(z)$ can be rewritten as

$$\tilde{M}_p(z) = \tilde{M}_p^0(z) + \tilde{M}_p^e(z). \tag{44}$$

Since $\tilde{M}_p^e(z)$ cannot in general be rendered zero, it can be termed as recovery error matrix.

As the above discussion indicates, Lemma 2 forms the heart of the underlying mechanism of discrete-time LTR. It shows clearly what is and what is not feasible under what conditions. Although it does not directly provide methods of obtaining the gain K_p, it provides structural guide lines as to how certain eigenvalues and eigenvectors are to be assigned while indicating a multitude of ways in which freedom exists in assigning the other eigenvalues and eigenvectors of $A - K_pC$. These guidelines, in turn, can appropriately be channeled to come up with a design method to obtain an appropriate gain K_p. Leaving aside now the methods of design which will be discussed systematically in Part 2 of the paper, let us at this stage simply define the following sets of gains:

Definition 7. *Consider the system* Σ. *Let* $\mathcal{K}_p^*(\Sigma)$ *be a set of gains* $K_p \in \mathbb{R}^{n\times p}$ *such that* (1) $A - K_pC$ *is asymptotically stable, and* (2) $\tilde{M}_p^0(z)$ *is zero. In a similar manner, define* $\mathcal{K}_c^*(\Sigma_c)$ *and* $\mathcal{K}_r^*(\Sigma_r)$ *for systems* Σ_c *and* Σ_r.

As mentioned earlier, we do not parameterize here the gain K_p in terms of a tunable parameter σ. We deal only with a fixed gain K_p. In case if one deals with asymptotic recovery and thus with a sequence of controller gains $K_p(\sigma)$ for different values of σ, the set of recoverable gains is also parameterized and hence can be written as $\mathcal{K}_p^*(\Sigma, \sigma)$. In that case, one defines $\mathcal{K}_p^*(\Sigma, \sigma)$ as a set of gains $K_p(\sigma) \in \mathbb{R}^{n\times p}$ such that (1) $A - K_p(\sigma)C$ is asymptotically stable for all $\sigma > \sigma^*$ where $0 \le \sigma^* < \infty$, (2) the limits, as $\sigma \to \infty$, of all the eigenvalues of $A - K_p(\sigma)C$ remain in $\mathbb{C}^{\odot}$, and (3) $\tilde{M}_p^0(z)$ is either identically zero or asymptotically zero. Similarly, $\mathcal{K}_c^*(\Sigma_c, \sigma)$ and $\mathcal{K}_r^*(\Sigma_r, \sigma)$ are defined for systems Σ_c and Σ_r. Such a characterization is useful in design methods dealt with in Part 2 of the paper.

It is obvious that the sets of gains defined above are nonempty. We note also that whenever K_p is chosen as an element of $\mathcal{K}_p^*(\Sigma)$, the resulting error in the recovery matrix $M_p(z)$ is $M_p^e(z) = F\tilde{M}_p^e(z)$. As such $M_p^e(z)$ is called hereafter as the *'recovery error matrix'*.

Theorem 3 given below characterizes the achieved loop transfer function as well as sensitivity and complementary sensitivity functions.

Theorem 3. *Let the given system Σ be stabilisable and detectable. Also, let $L_t(z) = F\Phi B$ be an admissible target loop, i.e., $L_t(z) \in \mathbf{T}(\Sigma)$. Then for a prediction estimator based controller with estimator gain $K_p \in \mathcal{K}_p^*(\Sigma)$, we have*

$$E_p(z) = M_p^e(z)[I_m + M_p^e(z)]^{-1}[I_m + L_t(z)], \tag{45}$$

$$S_p(z) = S_t(z)[I_m + M_p^e(z)], \tag{46}$$

$$T_p(z) = T_t(z) - S_t(z)M_p^e(z), \tag{47}$$

and

$$\frac{|\,\sigma_i[S_p^*(j\omega)] - \sigma_i[S_t^*(j\omega)]\,|}{\sigma_{max}[S_t^*(j\omega)]} \leq \sigma_{max}[M_p^{e*}(j\omega)], \tag{48}$$

$$\frac{|\,\sigma_i[T_p^*(j\omega)] - \sigma_i[T_t^*(j\omega)]\,|}{\sigma_{max}[S_t^*(j\omega)]} \leq \sigma_{max}[M_p^{e*}(j\omega)]. \tag{49}$$

The above results are true for current and reduced order estimator based controllers as well provided the subscript p is changed to c and r, and quadruple (A, B, C, D) is changed to (A, B, C_c, D_c) and (A_r, B_r, C_r, D_r) respectively. Also, in the case of reduced order estimator based controller, F in (45) to (49) is to be replaced by F_2.

Proof : It follows from Lemma 2. ∎

Remark 4. *Theorem 2 is a special case of the above theorem. To see this, let us examine first the case when a prediction estimator based controller is used. If the given system Σ is left invertible and of minimum phase with no infinite zeros, then the recovery error matrix $\tilde{M}_p^e(z)$ is nonexistent and hence $E_p(z)$ can be rendered zero for all $z \in \mathbb{C}$. Similarly, if Σ is left invertible and of minimum phase with no infinite zeros of order higher than one, then $\tilde{M}_c^e(z)$ and $\tilde{M}_r^e(z)$ are nonexistent and hence the exact recovery is achievable by using either current or reduced order estimator based controllers. Thus, for the special cases considered in Theorem 2, the results of Theorem 3 are reduced to those of Theorem 2.*

Remark 5. *Theorem 3 also holds if we use the estimator gain $K_p(\sigma) \in \mathcal{K}_p^*(\Sigma, \sigma)$. However, in this case, the equalities in (45) to (47) should be replaced by pointwise convergences in z as $\sigma \to \infty$.*

As implied by Theorem 3, the recovery error matrix $M_p^e(z)$ plays a dominant role in the recovery process and hence it should be shaped to yield as best as possible the desired results. Shaping $M_p^e(z)$ involves selecting the set of eigenvalues Λ^e represented in $M_p^e(z)$ and the associated set of right and left eigenvectors W^e and V^e. Such a selection can be done in a number of ways subject to the constraints imposed in selecting the eigenvectors [16]. Since there is an ample amount of freedom in selecting Λ^e, W^e and V^e, there exists a set of admissible recovery error matrices, and such a set can be denoted as $\mathcal{M}_p^e(z)$. Similarly, sets $\mathcal{M}_c^e(z)$ and $\mathcal{M}_r^e(z)$ can be formulated for the current and reduced order estimator based controllers. Among the design methods to be discussed in Part 2 of the paper, the eigenstructure assignment methods make use of the available flexibility, and are capable of attaining any given $M_p^e(z) \in \mathcal{M}_p^e(z)$ while making $F\tilde{M}_p^0(z) = 0$. On the other hand, in optimization design methods, the optimal solution would render $F\tilde{M}_p^0(z) = 0$ while minimizing the H_∞ or H_2 norm of the recovery error matrix $M_p^e(z)$. Thus, $F\tilde{M}_p^0(z)$ is always rendered zero but the attained $M_p^e(z)$ varies from one design method to the other.

In multivariable systems, one interesting aspect of Theorem 3 is that there could exist a subspace of the control space in which $\tilde{M}_p^e(z)$ can be rendered zero. To pinpoint this, let

$$e_i = [B - K_p D]' V_i \;\;,\;\; V_i \in V^e, \tag{50}$$

and let $\mathcal{E}^e$ be the subspace of $\mathbf{R}^m$,

$$\mathcal{E}^e = \text{Span}\{e_i \mid V_i \in V^e\}. \tag{51}$$

Let the dimension of $\mathcal{E}^e$ be m_e. Now let

$$\mathcal{S}^e = \text{orthogonal complement of } \mathcal{E}^e \text{ in } \mathbf{R}^m. \tag{52}$$

Let P^e be the orthogonal projection matrix onto $\mathcal{S}^e$. Then the following theorem pinpoints the directional behavior of $\tilde{M}_p(z)$ and consequently the behavior of $S(z)$ and $T(z)$.

Theorem 4. *Let Σ be stabilisable and detectable, and $L_t(z)$ be a member of the set of admissible target loops $\mathrm{T}(\Sigma)$. Then for any $K_p \in \mathcal{K}_p^*(\Sigma)$, the corresponding prediction estimator based controller satisfies*

$$\tilde{M}_p(z) P^e = 0,$$

$$S_p(z) P^e = S_t(z) P^e$$

$$T_p(z)P^s = T_t(z)P^s,$$

where P^s is the orthogonal projection onto $\mathcal{S}^s \in \mathbb{R}^m$ as given in (52). The above results are true for current and reduced order estimator based controllers as well provided the subscript $_p$ is changed to $_c$ and $_r$, and quadruple (A,B,C,D) is changed to (A,B,C_c,D_c) and (A_r,B_r,C_r,D_r) respectively.

Proof : In view of the definitions of the matrix P^s and the subspaces $\mathcal{E}^s$ and $\mathcal{S}^s$, Theorem 3 implies the results of Theorem 4. ∎

It is interesting to note that although the projections of $S_p(z)$ and $S_t(z)$ and hence those of $T_p(z)$ and $T_t(z)$ onto $\mathcal{S}^s$ are equivalent, it need not be true that the projections of achieved and target loops, $\mathbf{C}_p(z)P(z)$ and $L_t(z)$, onto $\mathcal{S}^s$ are equivalent. That is, in general, $\mathbf{C}_p(z)P(z)P^s \neq L_t(z)P^s$. This is illustrated by the following example.

Example 1 : Consider a non-strictly proper discrete-time system characterized by

$$A = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}, \quad B = C = D = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

which is invertible with two nonminimum phase invariant zeros at $z = 1$ and $z = 2$. Let the target loop $L_t(z)$ and target sensitivity function $S_t(z)$ be specified by

$$F = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}.$$

Let the prediction estimator gain be given by

$$K_p = \begin{bmatrix} 5 & -4 \\ 1 & 0 \end{bmatrix}.$$

Then, it is easy to calculate that

$$L_t(z) = F(zI - A)^{-1}B = \frac{\begin{bmatrix} 3z-6 & 0 \\ 0 & 2z-6 \end{bmatrix}}{z^2 - 5z + 6},$$

$$S_t(z) = [I + L_t(z)]^{-1} = \frac{\begin{bmatrix} z-3 & 0 \\ 0 & z-2 \end{bmatrix}}{z},$$

$$\mathbf{C}_p(z)P(z) = \frac{\begin{bmatrix} 15z^3 - 72z^2 + 108z - 48 & -12z^3 + 48z^2 - 36z \\ 2z^3 - 8z^2 + 8z & -16z^2 + 64z - 48 \end{bmatrix}}{z^4 - 15z^3 + 64z^2 - 100z + 48}$$

and

$$S_p(z) = \frac{\begin{bmatrix} z^3 - 15z^2 + 48z - 36 & 12z^2 - 48z + 36 \\ -2z^2 + 8z - 8 & z^3 - 8z + 8 \end{bmatrix}}{z^3}.$$

Now consider a subspace $\mathcal{S}^s$ having the orthogonal projection matrix P^s as

$$P^s = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}.$$

It is now straightforward to verify that

$$S_t(z)P^s = S_p(z)P^s = \frac{\begin{bmatrix} z-3 & z-3 \\ z-2 & z-2 \end{bmatrix}}{2z}.$$

This implies that the projections of the target and the achieved sensitivity functions on to the subspace $\mathcal{S}^s$ are equal to one another. On the other hand, we have

$$L_t(z)P^s = \frac{\begin{bmatrix} 3(z-2) & 3(z-2) \\ 2(z-3) & 2(z-3) \end{bmatrix}}{2(z^2 - 5z + 6)}$$

and

$$\mathbf{C}_p(z)P(z)P^s = \frac{\begin{bmatrix} 3z^3 - 24z^2 + 72z - 48 & 3z^3 - 24z^2 + 72z - 48 \\ 2z^3 - 24z^2 + 72z - 48 & 2z^3 - 24z^2 + 72z - 48 \end{bmatrix}}{2(z^4 - 15z^3 + 64z^2 - 100z + 48)}.$$

Obviously, $L_t(z)P^s \neq \mathbf{C}_p(z)P(z)P^s$. That is, the projections of the target and the achieved loop transfer functions on to the subspace $\mathcal{S}^s$ do not match each other. □

In view of the directional behavior of the recovery error matrix $\tilde{M}_p^e(z)$ as given by Theorem 4, one could try to shape it in a particular way so as to obtain the recovery of sensitivity and complementary sensitivity functions in certain desired directions or one could try to shape $\tilde{M}_p^e(z)$ so that the subspace $\mathcal{S}^s$ has as large a dimension as possible, i.e., the subspace $\mathcal{E}^e$ has as small a dimension as possible. In this regard, we note that we have already selected Λ^0 and the corresponding sets of eigenvectors V^0 so that $\tilde{M}_p^0(z)$ is zero. We also note that although all the $n_a^+ + n_c + n_f$ vectors $V_i \in V^e$ can be selected to be linearly independent, the corresponding $e_i \equiv [B - K_p D]' V_i$ need not be linearly independent. In fact for a given $e \neq 0$, the equation

$$e = [B - K_p D]' V,$$

has $n - m + 1$ linearly independent solutions for V. Of course, not all such $n - m + 1$ vectors could be admissible eigenvectors of $A - K_p C$ for different eigenvalues of it inside $\mathbf{C}^{\odot}$, and moreover some or all of these $n - m + 1$ vectors could also be linearly dependent on already selected eigenvectors in the set V^0. Thus the problem of shaping $\mathcal{E}^e$ is to find an admissible set of eigenvalues λ_i and vectors e_i, $i = 1$ to $n_a^+ + n_c + n_f$, which are not necessarily linearly independent, but the associated eigenvectors V_i of $A - K_p C$ satisfying $e_i = [B - K_p D]' V_i$, $i = 1$ to $n_a^+ + n_c + n_f$, together with the vectors in the set V^0 form n linearly independent vectors. This problem of selecting an admissible set (λ_i, e_i) is very much related to the traditional problem of distributing the modes of a closed-loop system to various output components by an appropriate selection of the closed-loop eigenstructure. This traditional problem of 'shaping the output response characteristics' of a closed-loop system has been studied for continuous systems first by Moore [16] and Shaked [30] and more recently by Sogaard-Andersen [33] although to this date there exists no systematic design procedure.

The above discussion focuses on how to shape the subspace $\mathcal{S}^e$ in which $\tilde{M}_p(z)$, $S_t(z)$ and $T_t(z)$ are recovered. A practical problem of interest could be to achieve recovery of $\tilde{M}_p(z)$, $S_t(z)$ and $T_t(z)$ in a prescribed subspace $\mathcal{S}^e$. We will discuss this aspect of the problem in Subsection C.

We will next examine the open-loop eigenvalues of the prediction estimator based controller $\mathbf{C}_p(z)$ and the mechanism of pole zero cancellation between the controller eigenvalues and the input or output decoupling zeros [22] of the system. It is important to know the eigenvalues of $\mathbf{C}_p(z)$ as they are included among the invariant zeros of the closed-loop system and hence affect the performance of it, e.g., command following. The controller transfer function is given by (16) while the eigenvalues of it are

$$\lambda[A - K_p C - BF + K_p DF].$$

To study the nature of these eigenvalues, let

$$\det[zI_n - A + K_p C] = \phi^0(z)\phi^s(z)$$

where $\phi^0(z)$ and $\phi^e(z)$ are polynomials in z whose zeros are the eigenvalues of $A - K_pC$ that belong to the sets Λ^0 and Λ^e respectively. Also, let

$$F\tilde{M}_p^e(z) = \frac{R^e(z)}{\phi^e(z)} \tag{53}$$

where $R^e(z)$ is a polynomial matrix in z. Now consider the following when $K_p \in \mathcal{K}_p^*(\Sigma)$:

$$\begin{aligned}
\det[zI_n &- A + K_pC + BF - K_pDF] \\
&= \det[zI_n - A + K_pC] \det[I_n + (zI_n - A + K_pC)^{-1}(B - K_pD)F] \\
&= \phi^0(z)\phi^e(z) \det[I_m + F(zI_n - A + K_pC)^{-1}(B - K_pD)] \\
&= \phi^0(z)\phi^e(z) \det[I_m + F\tilde{M}_p(z)] \\
&= \phi^0(z)\phi^e(z) \det[I_m + F\tilde{M}_p^e(z)] \\
&= \phi^0(z)\phi^e(z) \det[I_m + \frac{R^e(z)}{\phi^e(z)}] \\
&= \phi^0(z) \frac{\det[I_m\phi^e(z) + R^e(z)]}{[\phi^e(z)]^{m-1}}.
\end{aligned} \tag{54}$$

We note that the observer can be designed such that $\phi^0(z)$ and $\phi^e(z)$ are coprime. Thus the open-loop eigenvalues of the controller (16) are the zeros of $\phi^0(z)$ and

$$\frac{\det[I_m\phi^e(z) + R^e(z)]}{[\phi^e(z)]^{m-1}}.$$

Thus Λ^0 is contained among the eigenvalues of the controller. Although Λ^0 is in $\mathbf{C}^{\odot}$, there is no guarantee that the zeros of

$$\frac{\det[I_m\phi^e(z) + R^e(z)]}{[\phi^e(z)]^{m-1}}$$

are in $\mathbf{C}^{\odot}$. Hence the controller may or may not be open-loop stable. However, it is obvious to see that if recovery is achieved, i.e., if $F\tilde{M}_p^e(z) = 0$ and thus $R^e(z) = 0$, then the eigenvalues of the prediction estimator based controller are given by the roots of $\phi^0(z)\phi^e(z) = 0$. As the roots of $\phi^0(z)\phi^e(z) = 0$ are inside the unit circle, whenever recovery is achieved, the controller is open-loop stable. This is also apparent from equation (36). Note that in this case $\mathbf{C}_p(z) = N_p(z)$ where $N_p(z)$ is as in (38).

In general, the loop transfer function $\mathbf{C}_p(z)P(z)$ has $2n$ eigenvalues, n of them coming from the given system Σ and the other n from the controller. However, there are several cancellations among the input or output decoupling zeros [22] of $\mathbf{C}_p(z)P(z)$ and the controller eigenvalues. The following Lemma 3 which is a generalization of a similar one in Goodman [12], explores such a cancellation.

Lemma 3. *Let λ be an eigenvalue of $A - K_pC$ with the corresponding left eigenvector V such that $V^H[B - K_pD] = 0$. Then λ is an eigenvalue of $A - K_pC - BF + K_pDF$ with the corresponding left eigenvector as V. Moreover, λ cancels an input decoupling zero of $\mathbf{C}_p(z)P(z)$.*

Proof : See Appendix C. ∎

Thus, in view of Lemma 2, the above lemma implies that whatever may be the matrix F, if the controller is appropriately designed, there are $n_a^- + n_b$ cancellations among the eigenvalues of the controller and the input decoupling zeros of $\mathbf{C}_p(z)P(z)$. As will be seen in the next subsection, there may be additional cancellations if F satisfies certain properties.

Remark 6. *Equation (54) and Lemma 3 are equally true for current as well as reduced order estimator based controllers. In these cases, notationally the quadruple (A, B, C, D), F and $\mathbf{C}_p(z)$ are to be replaced respectively by (A, B, C_c, D_c), F and $\mathbf{C}_c(z)$ for a current estimator based controller, and by (A_r, B_r, C_r, D_r), F_2 and $\mathbf{C}_r(z)$ for a reduced order estimator based controller.*

B. Analysis For Recoverable Target Loops

In the previous subsection, loop transfer recovery analysis is conducted without taking into account any knowledge of F. It involves essentially the study of the matrix $\tilde{M}_p(z)$ or $M_p(z)$ to ascertain when it can or cannot be rendered zero. This subsection complements the analysis of Subsection A by taking into account the knowledge of F. Obviously then, the analysis of this subsection is a study of $M_p(z) = F\tilde{M}_p(z)$. One of the important questions that needs to be answered here is as follows. What class of target loops can be recovered for the given system? As it forms a coupling between the analysis and design, characterization of $L_t(z)$ to determine whether it can be recovered or not for the given system, plays an extremely important role. That is, although the physical tasks of designing F and K_p are separable, one can benefit enormously by knowing ahead what kind of target loops are recoverable. The necessary and sufficient conditions developed here on $L_t(z)$ for its recoverability, turn out to be constraints on the finite and infinite zero structure of $L_t(z)$ as related to the corresponding structure of Σ. An interpretation of these conditions reveals that recovery of $L_t(z)$ for general nonminimum phase systems is possible under a variety of conditions.

Another important question that arises before one undertakes formulating any target loop transfer function $L_t(z)$ for a given system Σ is as follows. What are the necessary and sufficient conditions on Σ so that it has at least one recoverable target loop? An answer to this question obviously helps a designer to remodel the given plant if necessary by appropriately modifying the number or type of plant inputs or/and outputs. To answer the question posed, we develop here an auxiliary system Σ^{au} of Σ for each one of the three controllers considered here, and show that the set of recoverable target loops is nonempty if and only if Σ^{au} is stabilizable by a static output feedback controller. A close look at this condition reveals a surprising necessary condition, namely, strong stabilizability of Σ is necessary for it to have at least one recoverable target loop.

Finally, another aspect of the analysis given here shows the mechanism of pole-zero cancellation between the controller eigenvalues and the input or output decoupling zeros of Σ for the case when F is known.

We proceed now to give the following result regarding the recoverability of a target loop transfer function $L_t(z) = F\Phi B$ for the given system Σ.

Theorem 5. *Consider a stabilizable and detectable discrete-time system Σ characterized by a matrix quadruple (A, B, C, D), which is not necessarily left invertible and not necessarily of minimum phase. Then, an admissible target loop transfer function $L_t(z)$ of Σ, i.e., $L_t(z) \in \mathrm{T}(\Sigma)$, is recoverable if and only if the following condition is satisfied, depending on the controller used.*

1. *For a prediction estimator based controller, the condition is that $\mathcal{S}^-(\Sigma) \subseteq \mathrm{Ker}\,(F)$.*

2. *For a current estimator based controller, the condition is that $\mathcal{S}^-(\Sigma) \cap \{x \mid Cx \in \mathrm{Im}\,(D)\} \subseteq \mathrm{Ker}\,(F)$.*

3. *For a reduced order estimator based controller, the condition is that $\mathcal{S}^-(\Sigma) \cap \{x \mid Cx \in \mathrm{Im}\,(D)\} \subseteq \mathrm{Ker}\,(F)$.*

Thus the set of recoverable target loops under each controller is characterised as follows:

1. Prediction estimator based controller :

$$\mathrm{T}_p^{\mathrm{R}}(\Sigma) = \{L_t(z) \in \mathrm{T}(\Sigma) \mid \mathcal{S}^-(\Sigma) \subseteq \mathrm{Ker}\,(F)\}.$$

2. Current estimator based controller :

$$\mathrm{T}_c^{\mathrm{R}}(\Sigma) = \{L_t(z) \in \mathrm{T}(\Sigma) \mid \mathcal{S}^-(\Sigma) \cap \{x \mid Cx \in \mathrm{Im}\,(D)\} \subseteq \mathrm{Ker}\,(F)\}.$$

3. Reduced order estimator based controller :

$$\mathrm{T}_r^{\mathrm{R}}(\Sigma) = \{L_t(z) \in \mathrm{T}(\Sigma) \mid \mathcal{S}^-(\Sigma) \cap \{x \mid Cx \in \mathrm{Im}\,(D)\} \subseteq \mathrm{Ker}\,(F)\}.$$

Proof : For the case of a prediction estimator based controller, the result follows easily from Theorem 5.1 of [6] or equivalently from Theorem 3.3 of [24] with obvious notational changes. For the current and reduced order estimator based controllers, the results follow from Propositions 2 and 3 and Theorem 5.1 of [6]. ∎

Remark 7. *We note that* $\mathrm{T}_p^{\mathrm{R}}(\Sigma) \subseteq \mathrm{T}_c^{\mathrm{R}}(\Sigma) = \mathrm{T}_r^{\mathrm{R}}(\Sigma)$.

Remark 8. *Let Σ be a strictly proper minimum phase system having infinite zeros of order one, i.e., CB of maximal rank. Then, it is shown in Goodman [12] that a target loop $L_t(z) = F\Phi B$ is recoverable by a prediction estimator based controller if $FB = 0$. This result can easily be deduced from Theorem 5.*

Several interpretations emerge from the recoverability conditions on the target loops given in Theorem 5. In fact the constraints given in Theorem 5 are nothing more than constraints on the finite and infinite zero structure and invertibility properties of $L_t(z)$. Some interesting interpretations in this regard can easily be exemplified as follows.

1. If Σ is not left invertible, any recoverable $L_t(z)$ is not left invertible. On the other hand, left invertibility of Σ does not necessarily imply that a recoverable $L_t(z)$ is left invertible. That is, whenever Σ is left invertible, a recoverable $L_t(z)$ could be either left invertible or not left invertible.

2. Any left invertible and recoverable $L_t(z)$ must contain the unstable invariant zero structure of Σ. A recoverable but not left invertible $L_t(z)$ does not necessarily contain the unstable invariant zero structure of Σ (see Example 2).

Example 2 : Consider a non-strictly proper discrete-time system with sampling period $T = 1$, and characterized by

$$A = \begin{bmatrix} -1 & -1 & -1 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix},$$

$$C = \begin{bmatrix} -1 & -1 & -1 \\ 1 & 1 & -1 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

This system is invertible and is of nonminimum phase as its invariant zeros are at $\{0,\ 0,\ 2\}$. Let the target loop $L_t(z)$ and the target sensitivity function $S_t(z)$ be specified by

$$F = \begin{bmatrix} 0.5 & 0.5 & 0 \\ -0.5 & -0.5 & 0 \end{bmatrix}.$$

The triple (A, B, F) forms a minimum phase and right invertible system and hence it does not contain the unstable invariant zero structure of Σ. However, for this example, it can easily be seen that

$$\mathcal{S}^-(\Sigma) = \mathcal{S}^-(\Sigma) \cap \{x \mid Cx \in \mathrm{Im}\,(D)\} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Since $\mathcal{S}^-(\Sigma)$ is contained in Ker (F), in accordance with Theorem 5, the given target loop is recoverable by all the three controllers considered in this paper. In fact, since Σ is invertible with D being of maximal rank, all these controllers are one and the same (see Proposition 1). Thus we can conclude that a recoverable $L_t(z)$ need not contain the unstable invariant zero structure of Σ. The following is the prediction estimator gain which achieves ELTR,

$$K_p = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ -1 & 0 \end{bmatrix}.$$

□

Our aim next is to develop the conditions on Σ so that the set of recoverable target loops is nonempty. We have the following theorem.

Theorem 6. *Consider a stabilisable and detectable system Σ characterised by a matrix quadruple (A, B, C, D), which is not necessarily of minimum phase and which is not necessarily left invertible. Let $\overline{C}_p$ and $\overline{C}_c$ be any full rank matrices of dimensions $(n_a^- + n_b) \times n$ and $(n_a^- + n_b + m_f) \times n$, respectively, such that*

1. $\text{Ker}\,(\overline{C}_p) = \mathcal{V}^+(\Sigma)$, *and*
2. $\text{Ker}\,(\overline{C}_c) = \mathcal{S}^-(\Sigma) \cap \{x \mid Cx \in \text{Im}(D)\}$.

Also, let $\overline{C}_r = \overline{C}_c$. Define three auxiliary systems:

1. Σ_p^{au} *characterised by the matrix triple $(A, B, \overline{C}_p)$,*
2. Σ_c^{au} *characterised by the matrix triple $(A, B, \overline{C}_c)$, and*
3. Σ_r^{au} *characterised by the matrix triple $(A, B, \overline{C}_r)$.*

Then we have the following results depending upon the controller used :

1. Prediction estimator based controller: $\mathbf{T}_p^{\mathrm{R}}(\Sigma)$ *is nonempty if and only if Σ_p^{au} is stabilisable by a static output feedback controller.*
2. Current estimator based controller: $\mathbf{T}_c^{\mathrm{R}}(\Sigma)$ *is nonempty if and only if Σ_c^{au} is stabilisable by a static output feedback controller.*
3. Reduced order estimator based controller: $\mathbf{T}_r^{\mathrm{R}}(\Sigma)$ *is nonempty if and only if Σ_r^{au} is stabilisable by a static output feedback controller.*

Proof : See Appendix D. ∎

Theorem 6 gives the necessary and sufficient conditions under which the set of recoverable target loops for each controller is nonempty. However, the conditions given there are not conducive to any intuitive feelings. The following corollary gives a necessary condition which is surprising as well as intuitively appealing.

Corollary 2. *The strong stabilisability of the given system Σ is a necessary condition for it to have at least one recoverable target loop under any of the three controllers discussed here.*

Proof : See Appendix E. ∎

Corollary 2 tells us that any given system Σ must be strongly stabilisable in order to have at least one recoverable target loop. On the other hand, as seen from Theorem 6, strong stabilisability of Σ alone is not sufficient for $\mathbf{T}^{\mathrm{R}}(\Sigma)$ to be nonempty. The following example illustrates this.

Example 3 : Consider a non-strictly proper discrete-time system Σ characterised by

$$A=\begin{bmatrix}1&0\\0&1\end{bmatrix},\quad B=\begin{bmatrix}1&0\\0&1\end{bmatrix},\quad C=\begin{bmatrix}0&1\\-1&0\end{bmatrix},\quad D=\begin{bmatrix}1&0\\0&1\end{bmatrix}.$$

This system is invertible with two unstable invariant zeros at $\{\,1+i,\ 1-i\,\}$. Also, this system is strongly stabilisable as it can be stabilized by the following stable output feedback compensator,

$$\mathbf{C}(z)=\frac{1}{z^2+z+0.5}\begin{bmatrix}-0.25&-0.5(z+0.5)\\0.5(z+0.5)&-0.25\end{bmatrix}.$$

However, it is simple to verify that for this system,

$$\mathcal{S}^-(\Sigma)=\mathcal{S}^-(\Sigma)\cap\{x\mid Cx\in\operatorname{Im}(D)\}=\mathbf{R}^2.$$

Hence, it follows from Theorem 5 that any recoverable target loop $L_t(z)$ must have the following form of F,

$$F=\begin{bmatrix}0&0\\0&0\end{bmatrix}.$$

But it is trivial to see that such a target loop is not admissible. Thus, this system has no recoverable target loop although it is strongly stabilisable. □

We now proceed to discuss the possible pole-zero cancellations for the recoverable target loops. First, we need the following lemma which is a generalisation of a similar one in Goodman [12].

Lemma 4. *Let λ be an eigenvalue of $A-K_pC$ with the corresponding right eigenvector W such that $FW=0$. Then λ is an eigenvalue of $A-K_pC-BF+K_pDF$ with the corresponding right eigenvector as W. Moreover, λ cancels an output decoupling zero of $\mathbf{C}_p(z)$.*

We have the following theorem.

Theorem 7. *If $E_p^*(j\omega) = 0$ for all $0 \leq |\omega| < \infty$, i.e., ELTR is achieved, then every eigenvalue of $A - K_pC - BF + K_pDF$ cancels either an output decoupling zero of $\mathbf{C}_p(z)$ or an input decoupling zero of $\mathbf{C}_p(z)P(z)$.*

Proof : We note that $E^*(j\omega) = 0$ for all $0 \leq |\omega| < \infty$ if and only if either $FW_i = 0$ or $V_i^H[B - K_pD] = 0$ or both. Hence the result follows from Lemmas 3 and 4. ∎

Remark 9. *Lemma 4 and Theorem 7 are equally true for current and reduced order estimator based controllers. In these cases, notationally the quadruple (A, B, C, D), F and $\mathbf{C}_p(z)$ are to be replaced respectively by (A, B, C_c, D_c), F and $\mathbf{C}_c(z)$ for a current estimator based controller, and by (A_r, B_r, C_r, D_r), F_2 and $\mathbf{C}_r(z)$ for a reduced order estimator based controller.*

In view of Lemmas 3 and 4, and Theorem 7, whenever recovery of a given target loop occurs, there are n cancellations among the eigenvalues of the controller and the output decoupling zeros of $\mathbf{C}_p(z)$ or the input decoupling zeros of $\mathbf{C}_p(z)P(z)$.

C. Recovery Analysis in A Given Subspace

In the last two subsections, we discussed recovery of a target loop transfer function $L_t(z) = F\Phi B$ when the recovery is required over the entire control space $\mathbf{R}^m$ and when the knowledge of state feedback gain F is either unknown or known. The traditional LTR problem as treated in there, concentrates on recovering a open-loop transfer function $L_t(z)$ which has been formed to take into account the given design specifications. Actually, design specifications are normally formulated in terms of certain required closed-loop sensitivity and complementary sensitivity functions, $S_t(z) = [I_m + L_t(z)]^{-1}$ and $T_t(z) = I_m - S_t(z)$. In LQG/LTR design philosophy, these given specifications are reflected in formulating an open-loop transfer function called the target loop transfer function. As discussed earlier, this aspect of determining a target loop transfer function is a first step in LQG/LTR design and falls in the category of loop shaping. Generating a target loop transfer function $L_t(z)$ at the present time is an engineering art and often involves the use of linear quadratic design in which the cost matrices are used as free design parameters to obtain the state feedback gain F and thus to obtain $L_t(z) = F\Phi B$ and $S_t(z) = [I_m + F\Phi B]^{-1}$. In the second step of design, the so called loop transfer recovery (LTR) design, $L_t(z)$ is recovered using a measurement feedback controller. Obviously, in the traditional LTR design where recovery is required over the entire control space $\mathbf{R}^m$, the recovery of $L_t(z)$ implies the recovery of the corresponding sensitivity function $S_t(z)$ and hence the recovery of the complementary sensitivity function $T_t(z)$. Conversely, in a similar manner, the recovery of $S_t(z)$ or equivalently that of $T_t(z)$, implies the recovery of $L_t(z)$. In other words, when recovery is required over the entire control space $\mathbf{R}^m$, recovering a certain target loop transfer function is equivalent to recovering a certain target sensitivity function. Thus, without loss of any freedom, historically, recovery of a target loop transfer function has been sought.

As seen in earlier sections, loop transfer recovery in the entire control space $\mathbf{R}^m$ is not possible in general. This may force a designer to seek recovery only in a chosen subspace $\mathcal{S}$ of the control space $\mathbf{R}^m$. In that case, it is natural to think of recovering the projections of both the target loop $L_t(z)$ and the sensitivity function $S_t(z)$ onto $\mathcal{S}$. However, as seen in Example 3, one may obtain the projections of achieved and target sensitivity functions onto $\mathcal{S}$ matching each other, but the projection of the correspondingly achieved loop transfer function may or may not match that of the target loop. This implies that the designer may have to choose between matching the projections onto $\mathcal{S}$ of (1) the achieved and the target sensitivity functions, and (2) the achieved and the target loop transfer functions. Since, most often design specifications are given in terms of sensitivity functions, it is natural to choose matching the projections onto $\mathcal{S}$ of the achieved and the target sensitivity functions. In view of this, in this section, we focus on the recovery of sensitivity functions over a subspace. For the case when $\mathcal{S}$ equals $\mathbf{R}^m$, obviously the sensitivity recovery formulation of this section coincides with the conventional LTR formulation. Thus this section can indeed be viewed as a generalization of the notion of traditional LTR to cover recovery over either the entire or any specified subspace $\mathcal{S}$ of the control space $\mathbf{R}^m$.

A brief outline of this subsection is as follows. At first, precise definitions dealing with the sensitivity recovery problem are given. Then, Lemma 5 is developed generalizing Lemma 1. It formulates the condition for the recoverability of a sensitivity function in $\mathcal{S}$ in terms of a matrix $M^s(z)$. Next, Theorem 8 specifies the required conditions on Σ so that sensitivity recovery in $\mathcal{S}$ is possible for any arbitrarily specified target sensitivity function $S_t(z)$. Similarly, Theorem 9 specifies the necessary and sufficient conditions for the recoverability of a sensitivity function when the knowledge of F is known. On the other hand, Theorem 10 establishes the necessary and sufficient conditions so that the sets of recoverable sensitivity functions of the given system Σ for a specified subspace $\mathcal{S}$, is nonempty. An important aspect of recovery analysis in a subspace is to determine the maximum possible dimension of a recoverable subspace $\mathcal{S}$. This is discussed at the end of this section.

We have the following formal definitions.

Definition 8. *The set of admissible target sensitivity functions* $\mathbf{S}(\Sigma)$ *for a given system* Σ *is defined as follows:*

$$\mathbf{S}(\Sigma) := \{\, S_t(z) \in \mathcal{M}^{m\times m}(\mathcal{R}_p) \mid S_t(z) = [I_m + L_t(z)]^{-1},\ L_t(z) \in \mathbf{T}(\Sigma)\,\}.$$

Definition 9. *Given* $S_t(z) \in \mathbf{S}(\Sigma)$ *and a subspace* $\mathcal{S} \in \mathbf{R}^m$, *we say* $S_t(z)$ *is recoverable in the subspace* $\mathcal{S}$ *if there exists a controller having the transfer function* $\mathbf{C}(z)$ *such that (i) the closed-loop system comprising of the controller and the plant is asymptotically stable, and (ii)* $S(z)P^s = S_t(z)P^s$, *where* $S(z)$ *is the achieved sensitivity function and* P^s *is the orthogonal projection matrix onto* $\mathcal{S}$.

Definition 10. *The set of recoverable* $S_t(z) \in \mathbf{S}(\Sigma)$ *in the given subspace* $\mathcal{S}$ *is denoted by* $\mathbf{S}^{\mathrm{R}}(\Sigma, \mathcal{S})$.

As usual, subscripts p, c and r are respectively used to distinguish the above sets ($\mathbf{S}(\Sigma)$ and $\mathbf{S}^{\mathrm{R}}(\Sigma, \mathcal{S})$) for prediction, current and reduced order estimator based controllers. Also, we note that the above definitions 8 to 10 are natural extensions of the corresponding definitions given earlier. In fact, the definitions 8 to 10 generalize the concept of recovery to a subspace and enable us to reanalyze all the results of the previous two subsections to cover the recovery in a given subspace $\mathcal{S}$.

The following lemma is analogous to Lemma 1.

Lemma 5. *Let the given system* Σ *be stabilisable and detectable. Also, let* $L_t(z) = F\Phi B$ *be an admissible target loop, i.e.,* $L_t(z) \in \mathbf{T}(\Sigma)$. *Then* $E^s(z)$, *the projection onto a given subspace* $\mathcal{S} \in \mathbf{R}^m$ *of the error between* *the achieved sensitivity function $S(z)$ and the target sensitivity function $S_t(z)$, is given by*

$$E^s(z) = [I_m + F\Phi B]^{-1} M^s(z) \tag{55}$$

where

$$M^s(z) = M(z)P^s. \tag{56}$$

Furthermore, for all $\omega \in \Omega$,

$$E^{s*}(j\omega) = 0 \quad \text{if and only if} \quad M^{s*}(j\omega) = 0,$$

where Ω is the set of all $0 \le |\omega| \le \pi/T$ for which $S_t^*(j\omega)$ and $S^*(j\omega)$ are well defined (i.e., all the required inverses exist). Here the expression for $M(z)$ depends on the controller used and in particular for each one of the three controllers considered in this paper, the needed expressions are as in (33), (34), and (35) with an appropriate suffix added to $M(z)$.

Proof : It is obvious. ∎

To proceed with the recovery analysis, let V^s be a matrix whose columns form an orthogonal basis of $\mathcal{S} \in \mathbf{R}^m$. Assume that the columns of V^s are scaled so that the norm of each column is unity. Let $P^s = V^s(V^s)'$ be the unique orthogonal projection matrix onto $\mathcal{S}$. Then, define three auxiliary systems Σ_p^s, Σ_c^s and Σ_r^s characterized, respectively, by the quadruples (A, BV^s, C, DV^s), (A, BV^s, C_c, D_cV^s) and $(A_r, B_rV^s, C_r, D_rV^s)$. Now treating each auxiliary system as the given system, one can re-discuss here mutatis mutandis all the results of Subsections A and B. In particular, we have the following theorems.

Theorem 8. *Consider a stabilisable and detectable discrete-time system Σ characterised by a matrix quadruple (A, B, C, D), which is not necessarily of minimum phase and which is not necessarily left invertible. Let V^s be a matrix whose columns form an orthogonal basis of a given subspace $\mathcal{S} \in \mathbf{R}^m$. Then any arbitrary admissible sensitivity function $S_t(z)$ of Σ, i.e., $S_t(z) \in \mathbf{S}(\Sigma)$ is recoverable in $\mathcal{S}$ if and only if the following condition is satisfied depending upon the controller used.*

1. Prediction estimator based controller: *Any arbitrary admissible sensitivity function $S_t(z)$ of Σ is recoverable if and only if the auxiliary system Σ_p^s is left invertible and of minimum phase with no infinite zeros (i.e., DV^s is of maximal rank).*

2. Current estimator based controller: *Any arbitrary admissible sensitivity function $S_t(z)$ of Σ is recoverable if and only if the auxiliary system Σ_c^s is left invertible and of minimum phase with no infinite zeros (i.e., $D_c V^s$ is of maximal rank).*

3. Reduced order estimator based controller: *Any arbitrary admissible sensitivity function $S_t(z)$ of Σ is recoverable if and only if the auxiliary system Σ_r^s is left invertible and of minimum phase with no infinite zeros (i.e., $D_r V^s$ is of maximal rank).*

Proof : The results are obvious in view of Theorem 2 and Lemma 5. ∎

Theorem 8 is concerned with the recovery analysis when F is arbitrary or unknown. As in Subsection B, one can formulate the recovery conditions for a known F as follows.

Theorem 9. *Consider a stabilizable and detectable system Σ characterised by a matrix quadruple (A, B, C, D), which is not necessarily of minimum phase and which is not necessarily left invertible. Let V^s be a matrix whose columns form an orthogonal basis of a given subspace $\mathcal{S} \in \mathbf{R}^m$. Then an admissible sensitivity function $S_t(z)$ of Σ, i.e., $S_t(z) \in \mathbf{S}(\Sigma)$, is recoverable in $\mathcal{S}$ if and only if the following condition is satisfied depending on the controller used.*

1. *For a prediction estimator based controller, the condition is that*

$$\mathcal{S}^-(\Sigma_p^s) \subseteq \mathrm{Ker}\,(F).$$

2. *For a current estimator based controller, the condition is that*

$$\mathcal{S}^-(\Sigma_c^s) \subseteq \mathrm{Ker}\,(F).$$

3. *For a reduced order estimator based controller, the condition is that*

$$\begin{pmatrix} 0 \\ I \end{pmatrix} \mathcal{S}^-(\Sigma_r^s) \subseteq \mathrm{Ker}\,(F).$$

Thus the set of recoverable sensitivity functions in the given subspace $\mathcal{S}$ is characterised as follows:

1. **Prediction estimator based controller:**

$$\mathbf{S}_p^{\mathrm{R}}(\Sigma, \mathcal{S}) = \{\, S_t(z) \in \mathbf{S}(\Sigma) \mid \mathcal{S}^-(\Sigma_p^s) \subseteq \mathrm{Ker}\,(F) \,\}.$$

2. **Current estimator based controller:**

$$\mathbf{S}_c^{\mathrm{R}}(\Sigma, \mathcal{S}) = \{\, S_t(z) \in \mathbf{S}(\Sigma) \mid \mathcal{S}^-(\Sigma_c^s) \subseteq \mathrm{Ker}\,(F) \,\}.$$

3. **Reduced order estimator based controller:**

$$\mathbf{S}_r^{\mathrm{R}}(\Sigma, \mathcal{S}) = \left\{\, S_t(z) \in \mathbf{S}(\Sigma) \mid \begin{pmatrix} 0 \\ I \end{pmatrix} \mathcal{S}^-(\Sigma_r^s) \subseteq \mathrm{Ker}\,(F) \,\right\}.$$

Proof : The proof is a consequence of Theorem 5. ∎

Remark 10. *If the given system Σ is strictly proper, i.e., $D = 0$, then it is simple to verify that*

$$\mathcal{S}^-(\Sigma_c^s) = \begin{pmatrix} 0 \\ I \end{pmatrix} \mathcal{S}^-(\Sigma_r^s) = \mathcal{S}^-(\Sigma_p^s) \cap \{\, x \mid Cx \in \mathrm{Im}\,(DV^s) \,\}.$$

This is not true in general for non-strictly proper systems.

In what follows, we give a necessary and sufficient condition under which $\mathbf{S}^{\mathrm{R}}(\Sigma, \mathcal{S})$ is non-empty for the given subspace $\mathcal{S} \in \mathbf{R}^m$. We have the following theorem.

Theorem 10. *Consider a stabilisable and detectable system Σ characterised by a matrix quadruple (A, B, C, D), which is not necessarily of minimum phase and which is not necessarily left invertible. Let V^s be a matrix whose columns form an orthogonal basis of a given subspace $\mathcal{S} \in \mathbf{R}^m$. Let $\overline{C}_p^s$, $\overline{C}_c^s$ and $\overline{C}_r^s$ be any full rank matrices such that*

1. $\mathrm{Ker}\,(\overline{C}_p^s) = \mathcal{S}^-(\Sigma_p^s)$,
2. $\mathrm{Ker}\,(\overline{C}_c^s) = \mathcal{S}^-(\Sigma_c^s)$, *and*
3. $\mathrm{Ker}\,(\overline{C}_r^s) = \begin{pmatrix} 0 \\ I \end{pmatrix} \mathcal{S}^-(\Sigma_r^s)$.

Define three auxiliary systems:

1. Σ_p^{aux} *characterised by the matrix triple* $(A, B, \overline{C}_p^s)$,

2. Σ_c^{aus} *characterised by the matrix triple* $(A, B, \overline{C}_c^s)$, *and*

3. Σ_r^{aus} *characterised by the matrix triple* $(A, B, \overline{C}_r^s)$.

Then we have the following results depending upon the controller used :

1. Prediction estimator based controller: $\mathbf{S}_p^R(\Sigma, \mathcal{S})$ *is nonempty if and only if* Σ_p^{aus} *is stabilisable by a static output feedback controller.*

2. Current estimator based controller: $\mathbf{S}_c^R(\Sigma, \mathcal{S})$ *is nonempty if and only if* Σ_c^{aus} *is stabilisable by a static output feedback controller.*

3. Reduced order estimator based controller: $\mathbf{S}_r^R(\Sigma, \mathcal{S})$ *is nonempty if and only if* Σ_r^{aus} *is stabilisable by a static output feedback controller.*

Proof : The results are a consequence of Theorem 6. ∎

An important aspect that arises when one is interested in the recovery analysis in a subspace is to determine the maximum possible dimension of a recoverable subspace $\mathcal{S}$. In this regard, our goal in what follows, analogous to continuous systems, is to prove that whatever may be the given target loop transfer function and whatever may be the number of unstable invariant zeros, there exists at least one $m-1$ dimensional subspace $\mathcal{S}$ of $\mathbf{R}^m$ which is always recoverable provided that the given system Σ satisfies some conditions. In what follows, for simplicity of presentation, we will make a technical assumption that all the unstable invariant zeros of Σ have geometric multiplicity equal to unity. We have the following theorem.

Theorem 11. *Let the given system* Σ *be left invertible with unstable invariant zeros having geometric multiplicity equal to unity. Then, there exists at least one* $m-1$ *dimensional subspace* $\mathcal{S}$ *of* $\mathbf{R}^m$ *such that any admissible target sensitivity function* $S_t(z)$ *of* Σ, *i.e.,* $S_t(z) \in \mathbf{S}(\Sigma)$, *is recoverable in* $\mathcal{S}$ *provided the following condition is satisfied depending on the controller used.*

1. *For a prediction estimator based controller, the condition is that* D *be of maximal rank, i.e.,* Σ *has no infinite zeros.*

2. *For a current estimator based controller, the condition is that* Σ *has no infinite zeros of order higher than one.*

3. *For a reduced order estimator based controller, the condition is that Σ has no infinite zeros of order higher than one.*

Proof : See Appendix F. ∎

VI. DUALITY OF LTR BETWEEN THE INPUT AND OUTPUT BREAK POINTS

The target open-loop transfer functions can be designed when the loop is broken at either the input or the output point of the plant depending upon the given specifications. We have analysed so far LTR recovery at the input point (LTRI), using any one of prediction, current or reduced order estimator based controllers. Now we like to consider LTR recovery when the loop is broken at the output point (LTRO). For continuous systems, such a method was introduced earlier by Kwakernaak [14] in connection with sensitivity recovery. LTRO is used when the designer specifications and the modelling of uncertainties are reflected at the output point of the plant. In the literature, it is commonly said that LTR recovery at the input and output points (LTRI and LTRO) are dual to one another. This duality is well understood in the case of prediction estimator based controllers. That is, in the case of LTRO, the first step is to design a prediction estimator based controller, via loop shaping techniques, whose loop transfer function meets the design specifications. The next step is to recover the transfer function of prediction estimator based controller via LTR technique. However, this kind of duality is not well understood when current or reduced order estimator based controllers are used. The confusion in the literature [15,35] arises because the duality is sought between the plant and the controller. The given plant Σ and the controller are not necessarily dual to one another whenever any controller other than prediction estimator based is used. An appropriate subsystem of Σ, such as Σ_c or Σ_r, has to be constructed, and then the duality has to be sought between the controller and the subsystem Σ_c or Σ_r rather than between the controller and the given plant Σ. That is, duality has to be sought in the loop transfer recovery analysis or controller design methodology rather than between the given plant and the controller. In order to avoid any confusion, we give below a formal step by step algorithm to show how duality arises for LTR recovery at the input and output points.

1. **Let a plant Σ be characterised by the quadruple (A, B, C, D). Also, let $P(z)$ be the transfer function of Σ,**

$$P(z) = C(zI_n - A)^{-1}B + D.$$

Let $L_t(z) = C(zI_n - A)^{-1}K$ be an admissible target open-loop transfer function, i.e., $\lambda(A - KC) \in \mathbb{C}^{\odot}$, when the loop is broken at the output point of the given plant. Then, in the configuration of Figure 1, we are seeking a controller $\mathbf{C}(z)$ such that the closed-loop system is asymptotically stable and

$$E^o(z) := L_t(z) - P(z)\mathbf{C}(z) \equiv 0 \quad \text{for all } z.$$

Here the controller $\mathbf{C}(z)$ could be of any type. In particular, it could be either a prediction, or a current, or a reduced order estimator based controller.

2. **Define a dual system Σ_d characterized by (A_d, B_d, C_d, D_d) where**

$$A_d := A', \quad B_d := C', \quad C_d := B', \quad D_d := D'. \tag{57}$$

Note that $P_d(z)$, the transfer function of the dual plant Σ_d is $P'(z)$. Let $L_d(z)$ be defined as

$$L_d(z) := L_t'(s) = F_d(zI_n - A_d)^{-1}B_d \;\text{ where }\; F_d := K'. \tag{58}$$

Let $L_d(z)$ be considered as a target loop transfer function for Σ_d when the loop is broken at the input point of Σ_d. Let a measurement feedback controller $\mathbf{C}_d(z)$ be used for Σ_d. Here the controller $\mathbf{C}_d(z)$ could be any one of the three controllers, depending upon how $\mathbf{C}(z)$ is chosen. Then, it is simple to verify that the loop recovery error $E^{id}(z)$ at the input point of Σ_d is

$$E^{id}(z) = L_d(z) - \mathbf{C}_d(z)P_d(z) = [E^o(z)]'$$

provided $\mathbf{C}_d(z) = \mathbf{C}'(z)$.

3. **For the purpose of analysis or design alone, consider the fictitious plant Σ_d and the fictitious target loop transfer function $L_d(z)$ as given in Step 2. Then, it is straightforward to verify that the target loop transfer function $L_t(z)$ is recoverable at the output point of the given**

plant Σ if and only if $L_d(z)$ is recoverable at the input point of the fictitious plant Σ_d. To be specific, construct prediction, current, and reduced order estimator based controllers, namely $\mathbf{C}_{pd}(z)$, $\mathbf{C}_{cd}(z)$, and $\mathbf{C}_{rd}(z)$ respectively as in (16), (20) and (29) while using the parameters A_d, B_d, C_d, D_d and F_d instead of A, B, C, D and F. Let $E_p^{id}(z)$, or $E_c^{id}(z)$ or $E_r^{id}(z)$ be the resulting loop recovery error at the input point of Σ_d when $\mathbf{C}_{pd}(z)$, or $\mathbf{C}_{cd}(z)$, or $\mathbf{C}_{rd}(z)$ is used respectively as a controller for Σ_d. Let

$$\mathbf{C}_p(z) = \mathbf{C}'_{pd}(z)\ ,\ \mathbf{C}_c(z) = \mathbf{C}'_{cd}(z) \ \text{ and } \ \mathbf{C}_r(z) = \mathbf{C}'_{rd}(z).$$

Let $E_p^o(z)$ or $E_c^o(z)$ or $E_r^o(z)$ be the loop recovery error at the output point of Σ when $\mathbf{C}_p(z)$, or $\mathbf{C}_c(z)$, or $\mathbf{C}_r(z)$ is used respectively as a controller for Σ. It is then straightforward to show that

$$E_p^o(z) = (\,E_p^{id}(z)\,)'\ ,\ E_c^o(z) = (\,E_c^{id}(z)\,)' \ \text{ and } \ E_r^o(z) = (\,E_r^{id}(z)\,)'.$$

Thus all the loop transfer recovery analysis at the input point of Σ_d as done in this chapter can easily be interpreted as the loop transfer recovery analysis at the output point of Σ. In other words, the above step by step discussion clearly shows how duality arises between the loop transfer recovery at the input and the output points whatever may be the type of controller used.

VII. CONCLUSIONS

Here we deal with issues concerning the analysis of loop transfer recovery problem using observer based controllers for general non-strictly proper not necessarily minimum phase discrete time systems. Three different observer based controllers, namely, 'prediction estimator', and full or reduced order type 'current estimator' based controllers , are used. As in our earlier work, all the analysis given here is independent of the methodology by which these controllers are designed. Moreover, the analysis corresponding to all these three controllers is unified into a single mathematical frame work. A fundamental difference between continuous time and discrete time systems is this. In the discrete case, as is well known, in order to preserve stability, all the closed-loop eigenvalues must be restricted to lie within the unit circle in complex plane. This implies that unlike continuous case which permits both finite as well as asymptotically infinite eigenvalue assignment, in the discrete case one is restricted to only finite eigenvalue assignment. Thus, in the continuous case, there exists target loops which are not exactly recoverable, but are asymptotically recoverable by appropriate infinite eigenstructure assignment. On the other hand, in discrete systems, since both asymptotic as well as exact recovery involves only finite eigenstructure assignment, every asymptotically recoverable target loop is also exactly recoverable and vice versa.

There are several fundamental results given here. At first, based on the structural properties of the given system, we decompose the recovery matrix between the target loop transfer function and that that can be achieved by any one of the controllers, into two distinct parts. The first part of recovery matrix can be rendered exactly (or asymptotically, if one chooses so) zero by an appropriate finite eigenstructure assignment of the controller dynamic matrix, while the second part cannot be rendered zero, by any means, although there exists a multitude of ways to shape it. Such a decomposition helps us to discover when and under what conditions, an arbitrarily specified target loop is recoverable by using any one of the three controllers considered. Also, it helps to characterize the recovery error and its singular value bounds, whenever a target loop is not recoverable. Thus it shows the limitations of the given system in recovering the target loop transfer functions as a consequence of its structural properties, namely finite and infinite zero structure and invertibility. The next issue of our analysis concentrates on characterizing the required necessary and sufficient conditions on the target loop transfer functions so that they are recoverable by the considered controller for the given system. As in the case of continuous systems, the conditions developed here on a target loop transfer function for its recoverability, turn out to be constraints on its finite and infinite zero structure as related to the corresponding structure of the given system. Next, necessary and sufficient conditions on the given system are established such that it has at least one recoverable target loop. In this regard, we show that strong stabilizability of the given system is necessary for it to have at least one recoverable target loop by any one of the three controllers considered here. Since recovery in all control loops in general is not feasible, our analysis next, focuses in developing the necessary or/and sufficient conditions under which recovery of target sensitivity and complementary sensitivity functions is possible in any specified subspace of the control space. Inherent in all the issues discussed here is the characterisation of the resulting controller eigenvalues and possible pole zero cancellations. Such an investigation is important in view of the fact, controller eigenvalues become the invariant zeros of the closed-loop system and thus affect the performance with respect to command following and other design objectives.

To summarise, the analysis presented here adds a considerable amount of flexibility to the process of design and helps a designer to set meaningful goals at the onset of design. In other words, although the actual physical tasks of first designing a target loop and then designing an observer based controller are separable, one can link these two tasks philosophically by knowing ahead what is feasible and how. In a sequel to this paper, for each one of the controllers considered here, we will present design methodologies which are capable of utilising the complete freedom a design can have as is discovered here.

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Appendices

A. Proof of Lemma 1

To simplify and to unify our proof of Lemma 1, we first examine the following Luenberger estimator based controller:

$$\begin{cases} v(k+1) &= Lv(k) + Gu(k) + Hy(k), \\ -u(k) &= F\hat{x}(k) = Pv(k) + Vy(k), \end{cases} \tag{59}$$

where $v \in \mathbf{R}^r$ with r being the order of the controller. It is well known (see e.g., [20]) that $\hat{x}$ is an asymptotic estimate of the state x provided that (a) L is an asymptotically stable matrix and (b) there exists a matrix $T \in \mathbf{R}^{r \times n}$ satisfying the following conditions:

1. $TA - LT = HC$,

2. $G = TB - HD$,
3. $F = PT + VC$, and
4. $VD = 0$.

Then, following the procedure given in [18], it is simple to show that the loop recovery error realised by such a Luenberger estimator based controller is

$$E(z) = M(z)[I_m + M(z)]^{-1}(I_m + F\Phi B), \tag{60}$$

where

$$M(z) = P(zI - L)^{-1}G. \tag{61}$$

Next, it is straightforward to verify that the prediction estimator based controller of (15) is a special case of Luenberger estimator based controller in (59) with

$$\begin{cases} L = A - K_pC, & G = B - K_pD, & H = K_p, \\ P = F, & V = 0, & T = I. \end{cases}$$

Hence, equation (33) of Lemma 1 follows trivially from (60) and (61).

Similarly, in order to prove that (34) is valid for the current estimator based controller, let us first partition the gain matrix $K_c = [K_0, K_1]$ in conformity with $y_0(k)$ and $y_1(k)$. Also, define a new variable

$$v(k) := \hat{x}(k) - K_1y_1(k).$$

Then it is easy to rewrite the current estimator based controller (18) in terms of the new variable $v(k)$,

$$\begin{cases} v(k+1) = (A - K_cC_c)v(k) + (B - K_cD_c)u(k) \\ \qquad\qquad\qquad\qquad + [K_0, (A - K_cC_c)K_1]y(k), \\ -u(k) = F\hat{x} = Fv(k) + [0, FK_1]y(k). \end{cases} \tag{62}$$

Again, it is straightforward to verify that the current estimator based controller (62) is a special case of Luenberger estimator based controller in (59) with

$$\begin{cases} L = A - K_cC_c, & G = B - K_cD_c, & H = [K_0, (A - K_cC_c)K_1], \\ P = F, & V = [0, FK_1], & T = I - K_1C_1. \end{cases}$$

Then, (34) follows from (60) and (61).

As expected, the reduced order estimator based controller (29) is also a special case of (59) with

$$\begin{cases} L = A_r - K_r C_r, & G = B_r - K_r D_r, \qquad H = G_r, \\ P = F_2, & V = [0,\, F_1 + F_2 K_{r1}], \quad T = [-K_{r1},\, I]. \end{cases}$$

Then, once again in view of (60) and (61), we get (35). This completes the proof of Lemma 1. ∎

B. Proof of Proposition 2

Without loss of generality but for simplicity of presentation, we assume that the given system Σ is in the form of s.c.b, i.e., it is characterized by the quadruple $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ as in Theorem 1. Let us partition $x_f = [(x_{f1})', (x_{f0})', (x_{f2})']'$, where x_{f1} is the part of output associated with infinite zeros of order one, x_{f0} is the rest of output associated with infinite zeros of order higher than one and x_{f2} consists of the state variables corresponding to the rest of infinite zeros. Then it is simple to verify that C_f and B_f have the following forms,

$$C_f = \begin{bmatrix} I_{m_1} & 0 & 0 \\ 0 & I & 0 \end{bmatrix}, \quad B_f = \begin{bmatrix} I_{m_1} & 0 \\ 0 & 0 \\ 0 & B_{f2} \end{bmatrix}.$$

Now, by appropriate permutation transformation of the state variables, we can partition the given system as follows,

$$x = \left[(x_{f1})', (x_{f0})', (x_b)', (x_a^-)', (x_a^+)', (x_c)', (x_{f2})'\right]'$$

and $\tilde{A} = A - B_0 C_0 =$

$$\begin{bmatrix} E_{f11} & E_{f10} & E_{b1} & E_{a1}^- & E_{a1}^+ & E_{c1} & E_{f12} \\ L_{f01} & L_{f00} & 0 & 0 & 0 & 0 & C_{f2} \\ L_{bf1} & L_{bf0} & A_{bb} & 0 & 0 & 0 & 0 \\ L_{af1}^- & L_{af0}^- & L_{ab}^- C_b & A_{aa}^- & 0 & 0 & 0 \\ L_{af1}^+ & L_{af0}^+ & L_{ab}^+ C_b & 0 & A_{aa}^+ & 0 & 0 \\ L_{cf1} & L_{cf0} & B_c E_{cb} & B_c E_{ca}^- & B_c E_{ca}^+ & A_{cc} & 0 \\ A_{f21} & A_{f20} & B_{f2} E_{b2} & B_{f2} E_{a2}^- & B_{f2} E_{a2}^+ & B_{f2} E_{c2} & A_{f22} + B_{f2} E_{f22} \end{bmatrix},$$

$$\tilde{B} = [B_0 \quad B_1] = \begin{bmatrix} B_{f01} & I_{m_1} & 0 & 0 \\ B_{f00} & 0 & 0 & 0 \\ B_{b0} & 0 & 0 & 0 \\ B_{a0}^{-} & 0 & 0 & 0 \\ B_{a0}^{+} & 0 & 0 & 0 \\ B_{c0} & 0 & 0 & B_c \\ B_{f02} & 0 & B_{f2} & 0 \end{bmatrix},$$

$$\tilde{C} = \begin{bmatrix} C_0 \\ \\ C_1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & C_{0a}^{-} & C_{0a}^{+} & C_{0c} & C_{0f2} \\ I_{m_1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & C_b & 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$\tilde{D} = \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Let us define

$$C_c = \begin{bmatrix} C_0 \\ C_1(A - B_0 C_0) \end{bmatrix} \quad \text{and} \quad D_c = \begin{bmatrix} I_{m_0} & 0 \\ 0 & C_1 B_1 \end{bmatrix}.$$

We note that $C_c = \Gamma C_c$ and $D_c = \Gamma D_c$, where Γ is a nonsingular matrix,

$$\Gamma = \begin{bmatrix} I_{m_0} & 0 \\ -C_0 B_1 & I \end{bmatrix}.$$

Thus, establishing the required properties for a system characterized by $(\tilde{A}, \tilde{B}, C_c, D_c)$ is equivalent to doing the same for a system characterized by $(\tilde{A}, \tilde{B}, C_c, D_c)$. We next rewrite C_c and D_c in the form,

$$C_c = \begin{bmatrix} 0 & 0 & 0 & C_{0a}^{-} & C_{0a}^{+} & C_{0c} & C_{0f2} \\ E_{f11} & E_{f10} & E_{b1} & E_{a1}^{-} & E_{a1}^{+} & E_{c1} & E_{f12} \\ L_{f01} & L_{f00} & 0 & 0 & 0 & 0 & C_{f2} \\ C_b L_{bf11} & C_b L_{bf10} & C_b A_{bb} & 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$D_c = \begin{bmatrix} I_{m_0} & 0 & 0 & 0 \\ 0 & I_{m_1} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

It is trivial then to verify that system $(\tilde{A}, \tilde{B}, C_c, D_c)$ has the same finite and infinite zero structure, and invertibility property as the system (A_1, B_1, C_1)

does, where $A_1 =$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ L_{f01} & L_{f00} & 0 & 0 & 0 & 0 & C_{f2} \\ L_{bf1} & L_{bf0} & A_{bb} & 0 & 0 & 0 & 0 \\ L_{af1}^- & L_{af0}^- & L_{ab}^- C_b & A_{aa}^- & 0 & 0 & 0 \\ L_{af1}^+ & L_{af0}^+ & L_{ab}^+ C_b & 0 & A_{aa}^+ & 0 & 0 \\ L_{cf1} & L_{cf0} & B_c E_{cb} & B_c E_{ca}^- & B_c E_{ca}^+ & A_{cc} & 0 \\ A_{f21} & A_{f20} & B_{f2} E_{b2} & B_{f2} E_{a2}^- & B_{f2} E_{a2}^+ & B_{f2} E_{c2} & A_{f22} + B_{f2} E_{f22} \end{bmatrix}$$

$$B_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & B_c \\ B_{f2} & 0 \end{bmatrix},$$

and

$$C_1 = \begin{bmatrix} L_{f01} & L_{f00} & 0 & 0 & 0 & 0 & C_{f2} \\ C_b L_{bf1} & C_b L_{bf0} & C_b A_{bb} & 0 & 0 & 0 & 0 \end{bmatrix}.$$

We note here that $C_{f2} C_{f2}' = I$.

Next, we define a dual system,

$$\bar{x}(k+1) = \bar{A}\bar{x}(k) + \bar{B}\bar{u}(k), \quad \bar{y}(k) = \bar{C}\bar{x}(k),$$

where

$$\bar{A} = A_1' \ , \ \bar{B} = C_1' \ , \ \bar{C} = B_1' ,$$

$$\bar{x} = \left[(\bar{x}_0)', (\bar{x}_a^-)', (\bar{x}_a^+)', (\bar{x}_b)', (\bar{x}_f)' \right]' ,$$

$$\bar{u} = [(\bar{u}_f)', (\bar{u}_c)']' \quad \text{and} \quad \bar{y} = [(\bar{y}_f)', (\bar{y}_b)']' .$$

The dynamic equations of the above dual system are given by

$$\bar{x}_0(k+1) = \bar{A}_{00}\bar{x}_0(k) + \bar{A}_{10}\bar{x}_a^-(k) + \bar{A}_{20}\bar{x}_a^+(k) + \bar{A}_{b0}\bar{x}_b(k) + \bar{A}_{f0}\bar{x}_f(k) + \bar{K}_f\bar{u}_f(k) + \bar{K}_c\bar{u}_c(k)$$

$$\bar{x}_a^-(k+1) = \bar{A}_{aa}^-\bar{x}_a^-(k) + \bar{L}_{ab}^-\bar{y}_b(k) + \bar{L}_{af}^-\bar{y}_f(k)$$

$$\bar{x}_a^+(k+1) = \bar{A}_{aa}^+\bar{x}_a^+(k) + \bar{L}_{ab}^+\bar{y}_b(k) + \bar{L}_{af}^+\bar{y}_f(k)$$

$$\bar{x}_b(k+1) = \bar{A}_{bb}\bar{x}_b(k) + \bar{L}_{bf}\bar{y}_f(k), \quad \bar{y}_b = \bar{C}_b\bar{x}_b$$

$$\bar{x}_f(k+1) = \bar{A}_f\bar{x}_f(k) + \bar{L}_f\bar{y}_f(k) + \bar{B}_f[\bar{u}_f(k) + \bar{E}_0\bar{x}_0(k)], \qquad \bar{y}_f = \bar{C}_f\bar{x}_f$$

where

$$\bar{A}_{00} = \begin{bmatrix} 0 & 0 & 0 \\ L_{f01} & L_{f00} & 0 \\ L_{bf1} & L_{bf0} & A_{bb} \end{bmatrix}',$$

$$\bar{A}_{10} = [L^-_{af1},\, L^0_{af0},\, L^-_{ab}C_b]', \quad \bar{A}_{20} = [L^+_{af1},\, L^+_{af0},\, L^+_{ab}C_b]',$$

$$\bar{A}_{b0} = [L_{cf1},\, L_{cf0},\, B_cE_{cb}]', \quad \bar{A}_{f0} = [A_{f21},\, A_{f20},\, B_{f2}E_{b2}]',$$

$$\bar{A}^-_{aa} = (A^-_{aa})', \quad \bar{L}^-_{ab} = (E^-_{ca})', \quad \bar{L}^-_{af} = (E^-_{a2})', \quad \bar{A}^+_{aa} = (A^+_{aa})',$$

$$\bar{L}^+_{ab} = (E^+_{ca})', \quad \bar{L}^+_{af} = (E^+_{a2})', \quad \bar{A}_{bb} = (A_{cc})', \quad \bar{L}_{bf} = (E_{c2})',$$

$$\bar{E}_0 = [0,\, I,\, 0], \quad \bar{A}_f = (A_{ff2})', \quad \bar{C}_f = (B_{f2})', \quad \bar{C}_b = (B_c)',$$

and

$$\bar{L}_f = (E_{f22})', \quad \bar{K}_f = [L_{f01},\, L_{f00},\, 0]', \quad \bar{K}_c = [C_bL_{bf1},\, C_bL_{bf0},\, C_bA_{bb}]'.$$

Next, we will perform some transformations among the state variables in order to bring the new system $(\bar{A}, \bar{B}, \bar{C})$ into the form of s.c.b. Let us first define

$$\bar{\bar{x}}_0 = \bar{x}_0 - \bar{K}_f\bar{B}_f'\bar{x}_f.$$

As $\bar{B}_f'\bar{B}_f = C_{f2}C_{f2}' = I$, it is straightforward to verify that

$$\bar{\bar{x}}_0(k+1) = \tilde{A}_{00}\bar{\bar{x}}_0(k) + \bar{K}_c\bar{u}_c(k) + \bar{A}_{10}\bar{x}^-_a(k) + \bar{A}_{20}\bar{x}^+_a(k) + \bar{A}_{b0}\bar{x}_b(k) + A_{f0}\bar{x}_f(k)$$

where

$$\tilde{A}_{00} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ L_{bf1} & L_{bf0} & A_{bb} \end{bmatrix}'$$

and A_{f0} is some appropriate matrix. Also,

$$\bar{x}_f(k+1) = \bar{A}_f\bar{x}_f(k) + \bar{L}_f\bar{y}_f(k) + \bar{B}_f[\bar{u}_f(k) + \bar{E}_0\bar{\bar{x}}_0(k) + \bar{K}_f\bar{B}_f'\bar{x}_f(k)].$$

Then it follows from the results of Sannuti and Saberi (1987) that there exists a nonsingular transformation T such that

$$[(\tilde{x}_0)', (\bar{x}^-_a)', (\bar{x}^+_a)', (\bar{x}_b)', (\bar{x}_f)']' = T\,[(\bar{\bar{x}}_0)', (\bar{x}^-_a)', (\bar{x}^+_a)', (\bar{x}_b)', (\bar{x}_f)']'$$

and

$$\tilde{x}_0(k+1) = \tilde{A}_{00}\tilde{x}_0(k) + \bar{K}_c\bar{u}_c(k) + \bar{A}_{10}\bar{x}^-_a(k) + \bar{A}_{20}\bar{x}^+_a(k) + \bar{\bar{L}}_{b0}\bar{y}_b(k) + \bar{\bar{L}}_{f0}\bar{y}_f(k)$$

$$\bar{x}_a^-(k+1) = \bar{A}_{aa}^-\bar{x}_a^-(k) + \bar{L}_{ab}^-\bar{y}_b(k) + \bar{L}_{af}^-\bar{y}_f(k)$$

$$\bar{x}_a^+(k+1) = \bar{A}_{aa}^+\bar{x}_a^+(k) + \bar{L}_{ab}^+\bar{y}_b(k) + \bar{L}_{af}^+\bar{y}_f(k)$$

$$\bar{x}_b(k+1) = \bar{A}_{bb}\bar{x}_b(k) + \bar{L}_{bf}\bar{y}_f(k), \quad \bar{y}_b = \bar{C}_b\bar{x}_b$$

$$\bar{x}_f(k+1) = \bar{A}_f\bar{x}_f(k) + \bar{L}_f\bar{y}_f(k) + \bar{B}_f[\bar{u}_f(k) + \bar{E}_0\tilde{\bar{x}}_0(k) + \tilde{\bar{E}}_b\bar{x}_b(k) + \tilde{\bar{E}}_f\bar{x}_f(k)],$$

$$\bar{y}_f = \bar{C}_f\bar{x}_f.$$

We note that the above system is not in the standard form of s.c.b since we have not separated the new invariant zeros from $\tilde{A}_{00}$ yet. Let us next examine the pair $(\tilde{A}_{00}, \bar{K}_c)$. We have

$$\begin{aligned}
\operatorname{rank}\left[\, zI - \tilde{A}_{00} \quad \bar{K}_c \,\right] &= \operatorname{rank}\begin{bmatrix} zI - \tilde{A}_{00}' \\ \\ \bar{K}_c' \end{bmatrix} \\
&= \operatorname{rank}\begin{bmatrix} zI & 0 & 0 \\ 0 & zI & 0 \\ -L_{af1} & -L_{bf0} & zI - A_{bb} \\ C_bL_{bf1} & C_bL_{bf0} & C_bA_{bb} \end{bmatrix} \\
&= \operatorname{rank}\begin{bmatrix} zI & 0 & 0 \\ 0 & zI & 0 \\ -L_{af1} & -L_{bf0} & zI - A_{bb} \\ 0 & 0 & zC_b \end{bmatrix}. \qquad (63)
\end{aligned}$$

From (63) we know that the only possibility that causes $[\, zI - \tilde{A}_{00} \quad \bar{K}_c \,]$ to drop its rank is $z = 0$. Thus, since the pair (A_{bb}, C_b) is observable, the system $(\bar{A}, \bar{B}, \bar{C})$ has stable invariant zeros at $z = 0$. Hence, it follows from the results of Sannuti and Saberi (1987) that there exists another nonsingular transformation S such that

$$\begin{aligned}
&[(\tilde{\bar{x}}_0)', (\bar{x}_a^-)', (\bar{x}_a^+)', (\bar{x}_b)', (\bar{x}_f)']' \\
&\qquad = S\left[(\tilde{\bar{x}}_c^-)', (\tilde{\bar{x}}_a^0)', (\bar{x}_a^-)', (\bar{x}_a^+)', (\bar{x}_b)', (\bar{x}_f)'\right]'
\end{aligned}$$

and

$$\begin{aligned}
\tilde{\bar{x}}_c(k+1) = \tilde{A}_{cc}\tilde{\bar{x}}_c(k) + \tilde{B}_c[\tilde{E}_{ca}^0\tilde{\bar{x}}_a^0(k) + \tilde{E}_{ca}^-\bar{x}_a^-(k) + \tilde{E}_{ca}^+\bar{x}_a^+(k)] \\
+\tilde{L}_{cb}\bar{y}_b(k) + \tilde{L}_{cf}\bar{y}_f(k)
\end{aligned}$$

$$\tilde{\bar{x}}_a^0(k+1) = 0 \cdot \tilde{\bar{x}}_a^0(k) + \tilde{A}_{0-}\bar{x}_a^-(k) + \tilde{A}_{0+}\bar{x}_a^+(k) + \tilde{L}_{ab}^0\bar{y}_b(k) + \tilde{L}_{af}^0\bar{y}_f(k)$$

$$\bar{x}_a^-(k+1) = \bar{A}_{aa}^-\bar{x}_a^-(k) + \bar{L}_{ab}^-\bar{y}_b(k) + \bar{L}_{af}^-\bar{y}_f(k)$$

$$\bar{x}_a^+(k+1) = \bar{A}_{aa}^+\bar{x}_a^+(k) + \bar{L}_{ab}^+\bar{y}_b(k) + \bar{L}_{af}^+\bar{y}_f(k)$$
$$\bar{x}_b(k+1) = \bar{A}_{bb}\bar{x}_b(k) + \bar{L}_{bf}\bar{y}_f(k), \quad \bar{y}_b = \bar{C}_b\bar{x}_b$$

$$\bar{x}_f(k+1) = \bar{A}_f\bar{x}_f(k) + \bar{L}_f\bar{y}_f(k) + \bar{B}_f[\bar{u}_f(k) + \tilde{E}_c\tilde{\bar{x}}_c(k) + \tilde{E}_a^0\tilde{\bar{x}}_a^0(k)$$
$$+ \tilde{E}_a^-\bar{x}_a^-(k) + \tilde{E}_a^+\bar{x}_a^+(k) + \tilde{\bar{E}}_b\bar{x}_b(k) + \tilde{\bar{E}}_f\bar{x}_f(k)],$$
$$\bar{y}_f = \bar{C}_f\bar{x}_f,$$

where $(\tilde{A}_{cc}, \tilde{B}_c)$ is a controllable pair, and $\tilde{E}_{ca}^0$, $\tilde{E}_{ca}^-$, $\cdots$, $\tilde{E}_a^+$ are some constant matrices with appropriate dimensions. It is now trivial to see that the above system is in the standard form of s.c.b. Hence all the properties listed in Proposition 2 can be verified easily by the properties of s.c.b and by some simple algebra. ∎

C. Proof of Lemma 3

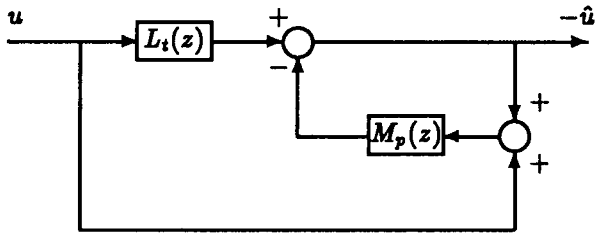
Noting from Lemma 1 that

$$E_p(z) := L_t(z) - \mathbf{C}_p(z)P(z) = M_p(z)[I + M_p(z)]^{-1}[I + L_t(z)],$$

we obtain,

$$\mathbf{C}_p(z)P(z) = L_t(z) - M_p(z)[I + M_p(z)]^{-1}[I + L_t(z)]$$
$$= [I + M_p(z)]^{-1}[L_t(z) - M_p(z)],$$

from which $\mathbf{C}_p(z)P(z)$ can be interpreted in terms of a block diagram given below.



In view of the block diagram, it is straightforward to write a state-space realization of $\mathbf{C}_p(z)P(z)$,

$$\begin{cases} \tilde{x}(k+1) = \begin{bmatrix} A & 0 \\ (B - K_pD)F & A - K_pC - BF + K_pDF \end{bmatrix} \tilde{x}(k) \\ \qquad\qquad + \begin{bmatrix} B \\ B - K_pD \end{bmatrix} u(k), \\ -\hat{u}(k) = [F, \ \ -F]\,\tilde{x}(k). \end{cases}$$

Let λ be an eigenvalue of $A - K_pC$ and the corresponding left eigenvector V be such that $V^H(B - K_pD) = 0$. It is simple then to verify that

$$[0, \quad V^H] \begin{bmatrix} \lambda I - A & 0 \\ -(B - K_pD)F & \lambda I - A + K_pC + (B - K_pD)F \end{bmatrix} = 0$$

and

$$[0, \quad V^H] \begin{bmatrix} B \\ B - K_pD \end{bmatrix} = 0.$$

This shows that λ is an input decoupling zero of $\mathbf{C}_p(z)P(z)$ and thus the result follows. ∎

D. Proof of Theorem 6

Let us first consider the case of prediction estimator based controller. Without loss of generality we assume that the given system Σ is in the form of s.c.b as in Theorem 1. Now in view of Theorem 5 a recoverable $L_t(z) = F\Phi B$ must satisfy $\mathcal{S}^-(\Sigma) \subseteq \operatorname{Ker}(F)$. This implies that $L_t(z)$ is recoverable if and only if F is of the form,

$$F = \begin{bmatrix} F_{a1}^- & 0 & F_{b1} & 0 & 0 \\ F_{a2}^- & 0 & F_{b2} & 0 & 0 \end{bmatrix}. \tag{64}$$

Thus the fact that the given system has at least one exactly recoverable target loop is equivalent to the existence of some appropriate matrices F_{a1}^-, F_{b1}, F_{a2}^- and F_{b2} such that $A - BF$ is asymptotically stable. Next, in view of the fact that $x_a^+ \oplus x_c \oplus x_f$ spans $\mathcal{S}^-(\Sigma)$, we note that $\overline{C}_p$ as defined in Theorem 6 is of the form,

$$\overline{C}_p = \Gamma \begin{bmatrix} I_{n_a^-} & 0 & 0 & 0 & 0 \\ 0 & 0 & I_{n_b} & 0 & 0 \end{bmatrix}$$

where Γ is any nonsingular matrix of dimension $(n_a^- + n_b) \times (n_a^- + n_b)$. It is now trivial to verify that the existence of a matrix F of the form in (64) such that $A - BF$ is asymptotically stable, is equivalent to the existence of a matrix G of dimension $m \times (n_a^- + n_b)$ such that $A - BG\overline{C}_p$ is asymptotically stable. This is simply due to the fact that $G\overline{C}_p$ has the same structure as F in (64).

The results for current and reduced order estimators follow from similar arguments. This completes the proof of Theorem 6. ∎

E. Proof of Corollary 2

Again, we first consider the case of prediction estimator based controller. If there exists at least one recoverable target loop, i.e., if there exists at least one target loop and a prediction estimator gain K_p such that the corresponding $\tilde{M}_p^e(z) = 0$, then we note from (54) that the eigenvalues of the prediction estimator based controller are given by $\lambda(A - K_pC) \in \mathbf{C}^{\odot}$. Hence the corresponding prediction estimator based controller is asymptotically stable and thus by definition, the given plant is strongly stabilizable. The results for current and reduced order estimators follow from similar arguments. ∎

F. Proof of Theorem 11

Again, we explicitly prove here only the case of prediction estimator based controller. Utilizing Propositions 2 and 3, the results for current and reduced order estimator based controllers can be derived in a similar way.

Let z_i, x_i and w_i, $i = 1$ to n_a^+, be respectively an unstable invariant zero and the associated right state and input zero directions of (A, B, C, D). Since (A, B, C, D) is assumed to be stabilizable and detectable, we have $w_i \neq 0$ for all $i = 1$ to n_a^+. Because if $w_i = 0$, then by definition,

$$(z_iI - A)x_i = Bw_i = 0 \quad \text{and} \quad Cx_i + Dw_i = Cx_i = 0.$$

This implies that z_i is an output decoupling zero of (A, B, C, D). But this contradicts the detectability of (A, B, C, D) as $z_i \in \mathbf{C}^{\otimes}$. Then it follows from Lemma 3.8 of [24] that there exist at least one e such that

$$e'w_i \neq 0 \text{ for all } i = 1 \text{ to } n_a^+.$$

Select e to satisfy the above equation, and then define $\mathcal{S}$ as

$$\mathcal{S} = \text{ The orthogonal complement of the subspace spanned by } e \text{ in } \mathbf{R}^m.$$

It is now trivial to see that $\mathcal{S}$ has a dimension of $m-1$ and that $w_i \notin \mathcal{S}$ for all $i = 1$ to n_a^+. Then it follows from Lemma 6.2 of [6] that the corresponding Σ_p^s is of minimum phase and left invertible. Also, it is simple to verify that Σ_p^s has no infinite zeros, i.e., DV^s is of maximal rank. This in turn implies the results of Theorem 11. ∎

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