# Closed Loop Transfer Recovery for Discrete Time Systems

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# I. INTRODUCTION

In feedback design many performance and robust stability objectives can be stated in the form of requirements placed on the singular values of particular *closed-loop* transfer functions. A well-known approach to feedback controller design is the so-called *loop shaping* approach whereby a designer specifies the closed-loop objectives in terms of requirements on the *open-loop* singular values of the compensated system. The prominent design procedure under the terminology LQG/LTR [7] is one such design methodology in multivariable systems that is based on the concept of loop shaping. This design procedure is divided into two steps. The first step involves the design of a stabilizing state-feedback law that yields a loop transfer function satisfying the design specifications. The loop properties are usually described in relation to an *open-loop* system (e.g., for a loop transfer function broken at either the control or measurement paths). Such an open-loop transfer function defines the target loop shape. The second step involves the design of an output-feedback control law (typically an observer-based compensator) such that the resulting open-loop transfer function would have either exactly or approximately the same target loop shape as the one achieved under state feedback. The procedure of designing such an output feedback control law is called loop transfer recovery (LTR). In other words, the idea of LTR is to design a compensator to recover a specific open-loop transfer function.

The recent work of Chen, Saberi and Ly [1] and [2] proposes a new concept of recovery based on the closed-loop transfer function directly, as opposed to the open-loop transfer function found in the case of a traditional LTR design for continuous-time systems. The problem can be stated as follows. Suppose that one is able to synthesize a state-feedback law that yields satisfactory closed-loop performance. And let us define the closed-loop transfer function between the external input to the controlled output under the state-feedback law to be the target closed-loop transfer function. Clearly from this definition, the closed-loop target transfer function is completely defined by the selection of a full-state feedback gain matrix. Now we would like to design an output-feedback control law with a closed-loop transfer function that matches either exactly or approximately the target closed-loop transfer function. In this respect, we are dealing with the problem of closed-loop transfer recovery (CLTR) instead of open-loop transfer recovery (LTR). In this paper, we deal with closed-loop transfer recovery for general discrete-time systems.

Our study of the mechanism in CLTR for discrete-time systems is applicable to a general class of systems and aims at three important theoretical issues:

(a) Characterization of the recovery error and the available freedom in the design of output-feedback control laws for a given system and for an arbitrarily specified target closed-loop transfer function,

(b) Development of necessary and/or sufficient conditions for a target closed-loop transfer function in order to be either exactly or asymptotically recoverable for a given system, and

(c) Development of necessary and/or sufficient conditions on a given system such that it has at least one recoverable target closed-loop transfer function.

These are some of the theoretical issues pertaining to the analysis of CLTR. Of course, one also needs to examine issues in CLTR that are related to systematic design algorithms for the recovery process. This paper concerns the analysis of the CLTR mechanism for general discrete-time systems followed by a numerical example illustrating the usefulness of the CLTR design concept. The primary objective at hand is to analyze methodically the mechanism of CLTR using an observer-based controller in its most general setting (i.e., covering the cases of prediction, current and reduced-order estimators).

The paper is organized as follows. In section II, we define precisely the problem of closed-loop transfer recovery for discrete-time systems. Recognizing the importance of finite and infinite zero structure in the LTR problem, we recall in section III a special coordinate basis (s.c.b) of [5] and [6] that clearly displays the zero structure of a given system. Section IV describes three different controller structures for the CLTR design and the recovery error matrix associated with each of these controllers. General analysis of the CLTR problem is given in section V. A numerical example illustrating the application of CLTR to the synthesis of a tip position control system for a planar flexible one-link robot arm. Conclusions are given in section VI.

Throughout the paper, $A'$ denotes the transpose of $A$, $A^H$ denotes the complex conjugate transpose of $A$, $I$ denotes an identity matrix while $I_k$ denotes the identity matrix of dimension $k \times k$. $\lambda(A)$ and $\mathrm{Re}[\lambda(A)]$ respectively denote the set of eigenvalues and real parts of eigenvalues of $A$. Similarly, $\sigma_{max}[A]$ and $\sigma_{min}[A]$ respectively denote the maximum and minimum singular values of $A$. Ker $[V]$ and Im $[V]$ denote respectively the kernel and the image of $V$. $\mathbf{C}^{\odot}$ denotes the set of complex numbers inside the open unit circle while $\mathbf{C}^{\otimes}$ is the complementary set of $\mathbf{C}^{\odot}$. Also, $\mathcal{R}_p$ denotes the sub-ring of all proper rational functions of $z$ while the set of matrices of dimension $l \times q$ whose elements belong to $\mathcal{R}_p$ is denoted by $\mathcal{M}^{l\times q}(\mathcal{R}_p)$. Given a discrete transfer function $G(z)$, we define the discrete frequency response $G^*(j\omega)$ as $G(e^{j\omega T})$ where $T$ is the sampling period of the discrete-time system. An asymptotically stable matrix is one whose eigenvalues are all in $\mathbf{C}^{\odot}$.

## II. PROBLEM STATEMENT

Let us consider a linear time-invariant discrete-time system $\Sigma$,

$$\Sigma : \begin{cases} x(k+1) = A\,x(k) + B_1\,w(k) + B_2\,u(k), \\ z(k) = C_1 x(k) + D_{11} w(k) + D_{12} u(k), \\ y(k) = C_2 x(k) + D_{21} w(k) + D_{22} u(k), \end{cases} \tag{1}$$

where $x \in \mathbf{R}^n$ is the state, $u \in \mathbf{R}^m$ is the control input, $w \in \mathbf{R}^k$ is the external signal or disturbance, $z \in \mathbf{R}^{\ell}$ is the controlled output and $y \in \mathbf{R}^p$ is the measurement output. For convenience, we also define $\Sigma_*$ to be the matrix quadruple

$$\Sigma_* := (A, B_1, C_2, D_{21}).$$

Let us assume that the pair $(A,\, B_2)$ is stabilizable and the pair $(A,\, C_2)$ detectable. Following the procedures of [1], it is simple to show that it is without loss of any generality to assume that the matrix $D_{22} = 0$. Hence, throughout this paper, we assume that $D_{22} = 0$ for simplicity of presentation. Let $F$ be a full-state feedback gain matrix such that under the state-feedback control

$$u(k) = -Fx(k) \tag{2}$$

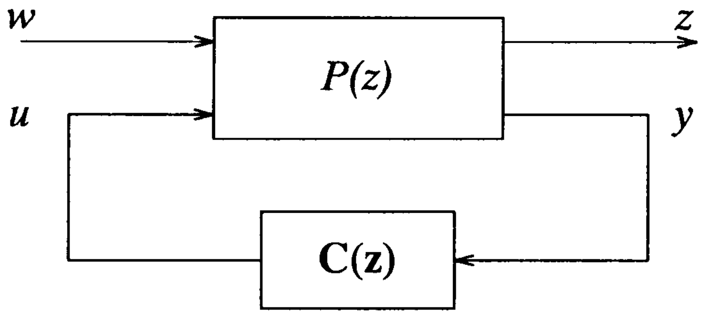



**Figure 1: Plant with an output feedback controller.**

(a) the closed-loop system is asymptotically stable, i.e., the eigenvalues of $A - B_2F$ lie in $\mathbf{C}^{\odot}$,

(b) the closed-loop transfer function from the disturbance $w$ to the controlled output $z$, denoted by $T_{zw}^{tg}(z)$, meets the given frequency dependent design specifications.

We also refer to $T_{zw}^{tg}(z)$ as the target closed-loop transfer function given by

$$T_{zw}^{tg}(z) = (C_1 - D_{12}F)(\Phi^{-1} + B_2F)^{-1}B_1 + D_{11} \tag{3}$$

where $\Phi = (zI_n - A)^{-1}$. Design of the appropriate full-state feedback gain matrix $F$ can be done, for example, via $H_2$-, $H_\infty$ theory or eigenstructure assignment. For design implementation, the next step in the design procedure is to recover the target closed-loop transfer function using only a measurement feedback (internally stabilizing) controller. This is the problem of closed-loop transfer recovery (CLTR) and the focus of this paper.

The problem can be clearly stated using the configuration shown in Figure 1. For a given system $\Sigma$ and a target closed-loop transfer function $T_{zw}^{tg}(z)$ in (3), the problem is to find an internally stabilizing proper controller $\mathbf{C}(z)$ such that the recovery error defined as

$$E(z) := T_{zw}(z) - T_{zw}^{tg}(z) \tag{4}$$

is either exactly or approximately equal to zero in the frequency region of interest. Here, $T_{zw}(z)$ represents the transfer function from $w$ to $z$ for the closed-loop system shown in Figure 1. The notion of achieving exact CLTR (ECLTR) corresponds to $E(z) = 0$ for all $z$. In the case of asymptotic recovery, one normally parameterizes the controller $\mathbf{C}(z)$ in terms of a scalar tuning parameter $\sigma$ and thus obtains a family of controllers $\mathbf{C}(z,\sigma)$. We say that asymptotic CLTR (ACLTR) is achieved if $E(z,\sigma) \to 0$ pointwise in $z$ as $\sigma \to \infty$. Achievability of ACLTR enables the designer to choose a member of the family of controllers with a particular value of $\sigma$ that yields a desired level of recovery. We now consider the following definitions in order to impart precise meanings to ECLTR and ACLTR:

**Definition 1.** *The set of admissible target closed-loop transfer functions* $\mathbf{T}(\Sigma)$ *for the plant* $\Sigma$ *is defined by*

$$\mathbf{T}(\Sigma)=\left\{T^{tg}_{zw}(z) \in \mathcal{M}^{l\times k}(\mathcal{R}_p) \mid T^{tg}_{zw}(z) \text{ is as in (3) and } \lambda(A-B_2F) \in \mathbf{C}^{\odot}\right\}.$$

**Definition 2.** $T^{tg}_{zw}(z) \in \mathbf{T}(\Sigma)$ *is said to be exactly recoverable (ECLTR) if there exists a* $\mathbf{C}(z)$ *whose transfer function belongs to* $\mathcal{M}^{m\times p}(\mathcal{R}_p)$ *such that*

1. *the closed-loop system comprising* $\mathbf{C}(z)$ *and* $\Sigma$ *as in (1) is stable,*
2. *the achieved closed-loop* $T_{zw}(z) = T^{tg}_{zw}(z)$ *for all* $z \in \mathbf{C}$.

**Definition 3.** $T_{zw}(z) \in \mathbf{T}(\Sigma)$ *is said to be asymptotically recoverable (ACLTR) if there exists a parameterized family of controllers* $\mathbf{C}(z,\sigma)$ *whose transfer functions belong to* $\mathcal{M}^{m\times p}(\mathcal{R}_p)$ *where* $\sigma$ *is a scalar parameter with positive values such that*

1. *the closed-loop system comprising* $\mathbf{C}(z,\sigma)$ *and* $\Sigma$ *as in (1) is asymptotically stable for all* $\sigma > \sigma^*$ *where* $0 \le \sigma^* < \infty$,
2. *the achieved closed-loop* $T_{zw}(z,\sigma) \to T^{tg}_{zw}(z)$ *pointwise in* $z$ *as* $\sigma \to \infty$. *Moreover, in the limit as* $\sigma \to \infty$ *the finite eigenvalues of the closed-loop system remain in* $\mathbf{C}^{\odot}$.[1]

As we will show later on, it turns out that for discrete-time systems, in contrast with the CLTR problem in continuous-time systems, every asymptotically recoverable target loop is also exactly recoverable and vice versa. One might then wonder why one needs to distinguish between ECLTR and

[1] Here we have strengthened the notion of closed-loop stability by excluding those cases where, in the limit as $\sigma \to \infty$, some finite eigenvalues of the closed-loop system would be on the unit circle. This avoids the problem of having an almost unstable behavior of the closed-loop system for large $\sigma$.

ACLTR. This is because, even for the case when ECLTR is achievable, some optimization-based design methods, such as $H_\infty$ norm minimisation, would typically produce suboptimal designs in the recovery. In this paper, we will not hereafter distinguish between the notions of exact and asymptotic recovery. Also, we will not parameterise a controller in terms of a tunable parameter $\sigma$ in an attempt to achieve whatever can be achieved asymptotically rather than exactly. We maintain that such a parameterisation can always be done if one chooses to do so. We have the following additional definitions.

**Definition 4.** *$T_{zw}^{tg}(z)$ belonging to $\mathrm{T}(\Sigma)$ is said to be recoverable if $T_{zw}^{tg}(z)$ is either exactly or asymptotically recoverable.*

**Definition 5.** *The set of recoverable target closed-loop transfer functions for the system $\Sigma$ is denoted by $\mathrm{T}_\mathrm{R}(\Sigma)$.*

**Remark 1.** *The controller $\mathrm{C}(z)$ in the above definitions is not restricted to any particular structure. However, in this paper we study the closed-loop transfer recovery for three specific structures of $\mathrm{C}(z)$; namely, prediction, current and reduced-order estimator based controllers. Furthermore, we label $\mathrm{T}_\mathrm{R}(\Sigma)$ with subscript $p$ (for prediction), $c$ (for current) and $r$ (for reduced order) as $\mathrm{T}_\mathrm{R}^p(\Sigma)$, $\mathrm{T}_\mathrm{R}^c(\Sigma)$ and $\mathrm{T}_\mathrm{R}^r(\Sigma)$ to signify results related to these particular controller structures.*

The analysis of CLTR mechanism carried out here examines three fundamental issues. The first concerns what can and what cannot be achieved for a given system and for an arbitrarily specified target closed-loop transfer function. For a given system, the second issue is to establish necessary and/or sufficient conditions on the target closed-loop transfer function so that it can be recovered. In another word, we characterise completely the set $\mathrm{T}_\mathrm{R}(\Sigma)$ of recoverable target closed-loop transfer functions. The third issue is to establish necessary and/or sufficient conditions on a given system such that it has at least one recoverable target closed-loop transfer function. That is, what are the conditions on a given system $\Sigma$ so that the set of recoverable target closed-loop transfer $\mathrm{T}_\mathrm{R}(\Sigma)$ is nonempty?

## III. PRELIMINARIES

We recall in this section a special coordinate basis (s.c.b) of a linear time-invariant system [5], [6]. Such a s.c.b has a distinct feature of explicitly displaying the finite and infinite zero structure of a given system and will play a very important role in both the analysis and the design of closed-loop transfer recovery. Consider the system characterized by

$$\Sigma_s : \begin{cases} x(k+1) = Ax(k) + Bu(k), \\ y(k) = Cx(k) + Du(k), \end{cases} \tag{5}$$

where $x \in \mathbf{R}^n$, $u \in \mathbf{R}^m$ and $y \in \mathbf{R}^p$. Without loss of generality, we assume that the matrices $[C,\ D]$ and $[B',\ D']'$ are of maximal rank. It is simple to verify that there exist non-singular transformations $U$ and $V$ such that

$$UDV = \begin{bmatrix} I_{m_0} & 0 \\ 0 & 0 \end{bmatrix}, \tag{6}$$

where $m_0$ is the rank of matrix $D$. Hence, hereafter and without loss of generality, it is assumed that matrix $D$ has the form given on the right-hand side of (6).

One can now rewrite the system of (5) as,

$$\begin{cases} x(k+1) = A\ x(k) + [B_0 \quad B_1] \begin{pmatrix} u_0(k) \\ u_1(k) \end{pmatrix}, \\ \begin{pmatrix} y_0(k) \\ y_1(k) \end{pmatrix} = \begin{bmatrix} C_0 \\ C_1 \end{bmatrix} x(k) + \begin{bmatrix} I_{m_0} & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} u_0(k) \\ u_1(k) \end{pmatrix}, \end{cases} \tag{7}$$

where the matrices $B_0$, $B_1$, $C_0$ and $C_1$ have appropriate dimensions. In what follows, whenever there is no ambiguity, in order to avoid the notational clutter, the running time index $k$ will be omitted. We have the following theorem.

**Theorem 1 (s.c.b).** *Consider the system $\Sigma_s$ characterized by the matrix quadruple $(A, B, C, D)$. There exist nonsingular transformations $\Gamma_1$, $\Gamma_2$ and $\Gamma_3$, an integer $m_f \le m - m_0$, and integer indexes $q_i$, $i = 1$ to $m_f$, such that*

$$x = \Gamma_1\,\tilde{x}, \quad y = \Gamma_2\,\tilde{y}, \quad u = \Gamma_3\,\tilde{u}$$

$$\tilde{x} = \left[x_a',\ x_b',\ x_c',\ x_f'\right]', \quad x_a = \left[(x_a^-)',\ (x_a^+)'\right]'$$

$$\tilde{y} = \left[ y_0', y_f', y_b' \right]', \quad \tilde{u} = \left[ u_0', u_f', u_c' \right]'$$

*and*

$$\tilde{A} := \Gamma_1^{-1}(A - B_0C_0)\Gamma_1 = \begin{bmatrix} A_{aa}^- & 0 & L_{ab}^-C_b & 0 & L_{af}^-C_f \\ 0 & A_{aa}^+ & L_{ab}^+C_b & 0 & L_{af}^+C_f \\ 0 & 0 & A_{bb} & 0 & L_{bf}C_f \\ B_cE_{ca}^- & B_cE_{ca}^+ & L_{cb}C_b & A_{cc} & L_{cf}C_f \\ B_fE_a^- & B_fE_a^+ & B_fE_b & B_fE_c & A_f \end{bmatrix},$$

$$\tilde{B} := \Gamma_1^{-1}\left[ B_0 \quad B_1 \right]\Gamma_3 = \begin{bmatrix} B_{0a}^- & 0 & 0 \\ B_{0a}^+ & 0 & 0 \\ B_{0b} & 0 & 0 \\ B_{0c} & 0 & B_c \\ B_{0f} & B_f & 0 \end{bmatrix},$$

$$\tilde{C} := \Gamma_2^{-1}\begin{bmatrix} C_0 \\ \\ C_1 \end{bmatrix}\Gamma_1 = \begin{bmatrix} C_{0a}^- & C_{0a}^+ & C_{0b} & C_{0c} & C_{0f} \\ 0 & 0 & 0 & 0 & C_f \\ 0 & 0 & C_b & 0 & 0 \end{bmatrix},$$

$$\tilde{D} := \Gamma_2^{-1}D\Gamma_3 = \begin{bmatrix} I_{m_0} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

*Here the states $x_a^-$, $x_a^+$, $x_b$, $x_c$ and $x_f$ are respectively of dimension $n_a^-$, $n_a^+$, $n_b$, $n_c$ and $n_f$. Furthermore, we have $\lambda(A_{aa}^-) \in \mathbb{C}^{\odot}$, $\lambda(A_{aa}^+) \in \mathbb{C}^{\otimes}$, the pair $(A_{cc}, B_c)$ is controllable, the pair $(A_{bb}, C_b)$ is observable and the triple $(A_f, B_f, C_f)$ is invertible with no invariant zeros.*

**Proof :** This follows from Theorem 2.1 of [5] and [6]. ∎

In what follows, we state some important properties of the s.c.b which are pertinent to our present work.

**Property 1.** *The given system $\Sigma_s$ is right invertible if and only if $x_b$ and hence $y_b$ are nonexistent, left invertible if and only if $x_c$ and hence $u_c$ are nonexistent, invertible if and only if both $x_b$ and $x_c$ are nonexistent. Moreover, $\Sigma_s$ is degenerate if and only if it is neither left nor right invertible.*

**Property 2.** *The invariant zeros of $\Sigma_s$ are the eigenvalues of $A_{aa}$. Moreover, the minimum phase (or stable) and the nonminimum phase (or unstable) invariant zeros of $\Sigma_s$ are the eigenvalues of $A_{aa}^-$ and $A_{aa}^+$, respectively.*

If all the invariant zeros of a system $\Sigma_s$ are in $\mathbf{C}^{\odot}$, i.e., if all the invariant zeros of $\Sigma_s$ are stable, then we say $\Sigma_s$ is of minimum phase, otherwise $\Sigma_s$ is said to be of nonminimum phase.

There are interconnections between the s.c.b and various invariant and almost invariant geometric subspaces. To show these interconnections, we need the following definition.

**Definition 6.** *For the system $\Sigma_s$, we define the subspaces*

1. *$\mathcal{V}^g(\Sigma_s)$ to be the maximal subspace of $\mathbf{R}^n$ which is $(A-BF)$-invariant and contained in* $\operatorname{Ker}(C-DF)$ *such that the eigenvalues of $(A-BF)|\mathcal{V}^g$ are contained in $\mathbf{C}_g \subseteq \mathbf{C}$ for some $F$.*

2. *$\mathcal{S}^g(\Sigma_s)$ to be the minimal $(A-KC)$-invariant subspace of $\mathbf{R}^n$ contained in* $\operatorname{Im}(B-KD)$ *such that the eigenvalues of the map which is induced by $(A-KC)$ on the factor space $\mathbf{R}^n/\mathcal{S}^g$ are contained in $\mathbf{C}_g \subseteq \mathbf{C}$ for some $K$.*

*For the cases that $\mathbf{C}_g = \mathbf{C}$, $\mathbf{C}_g = \mathbf{C}^{\odot}$ and $\mathbf{C}_g = \mathbf{C}^{\otimes}$, we replace the index $g$ in $\mathcal{V}^g$ and $\mathcal{S}^g$ by $*$, $-$ and $+$, respectively.*

Various components of the state vector of s.c.b have the following geometrical interpretations.

**Property 3.**

1. $x_a^- \oplus x_a^+ \oplus x_c$ *spans* $\mathcal{V}^*(\Sigma_s)$.
2. $x_a^- \oplus x_c$ *spans* $\mathcal{V}^-(\Sigma_s)$.
3. $x_a^+ \oplus x_c$ *spans* $\mathcal{V}^+(\Sigma_s)$.
4. $x_c \oplus x_f$ *spans* $\mathcal{S}^*(\Sigma_s)$.
5. $x_a^- \oplus x_c \oplus x_f$ *spans* $\mathcal{S}^+(\Sigma_s)$.
6. $x_a^+ \oplus x_c \oplus x_f$ *spans* $\mathcal{S}^-(\Sigma_s)$.

# IV. DIFFERENT CONTROLLER STRUCTURES

In this section, we consider three different controller structures used commonly in discrete-time systems. All three controllers are observer based, but the type of observer (or state estimator) used in each one is structurally different. The estimators considered here are (i) prediction estimator, (ii) current estimator and (iii) reduced-order estimator. Both prediction estimator and current estimator are of full-order. The reduced-order estimator is a current estimator of reduced-order. The prediction estimator reconstructs the state $x(k+1)$ based on the measurements $y(k)$ up to and including the $(k)$-th instant, where as the current estimator gives estimates of $x(k+1)$ based on the measurements $y(k+1)$ up to and including the $(k+1)$-th instant. Since in the prediction estimator based controller, the current estimated value of control does not depend on the most current value of the measurement, it might not be as accurate as the current estimator based controller. However, the prediction estimator based controller could avail itself the entire sampling period to do the required computations and hence is commonly used when the needed computations are excessive. In contrast, when the needed computations can be done in a short time compared to the sampling period, current estimator based controller can easily be used. We note that prediction estimator forces an inherent time delay which otherwise is absent in the controller structure. As expected, the three different controllers have different capabilities in regards to CLTR. However, there exists a common mathematical framework in the CLTR analysis for these controllers. In the sections to follow, we will systematically do the CLTR analysis using a generic controller which could be any one of these three controllers. In such an analysis, we shall use the following notation:

$\mathrm{T}_{\mathrm{R}}(\Sigma) :=$ The set of recoverable target closed-loops for $\Sigma$.

$T_{zw}(z) :=$ The achieved closed-loop transfer function,

$M(z) :=$ The recovery matrix (to be defined later on),

$E(z) := T_{zw}(z) - T_{zw}^{tg}(z) =$ The closed-loop recovery error,

$M_e(z) :=$ A part of the recovery matrix $M(z)$ that cannot be rendered zero and hence termed as recovery error matrix,

$M_o(z) :=$ A part of the recovery matrix $M(z)$ that can be rendered zero.

The above notation applies to a generic controller; however, whenever we refer to a particular controller, we use appropriate subscripts to identify them. Superscripts $p$, $c$ and $r$ are used respectively to represent prediction, current, and reduced order estimator based controllers. For example, $T_{zw}^p(z)$, $M_e^c(z)$ and $\mathrm{T}_{\mathrm{R}}^r(\Sigma)$ denote respectively the achieved loop transfer function with a prediction estimator based controller, the recovery error matrix when a current estimator based controller is used, and the set of recoverable target loops for $\Sigma$ using a reduced-order estimator based controller. We now proceed to give the structural details of the controllers considered here.

- **Prediction estimator based controller:**

The dynamic equations of the prediction estimator based controller are

$$\begin{cases} \hat{x}(k+1) &= A\hat{x}(k) + B_2u(k) + K_p[y(k) - C_2\hat{x}(k)], \\ u(k) &= -F\hat{x}(k), \end{cases} \tag{8}$$

where $K_p$ is the gain chosen so that $A - K_pC_2$ is asymptotically stable. The transfer function of the controller is

$$\mathbf{C}_p(z) = F(zI_n - A + B_2F + K_pC_2)^{-1}K_p. \tag{9}$$

- **Current estimator based controller:**

For simplicity and without loss of generality, we assume that the matrices $C_2$ and $D_{21}$ are in the form,

$$C_2 = \begin{bmatrix} C_{2,0} \\ C_{2,1} \end{bmatrix} \quad \text{and} \quad D_{21} = \begin{bmatrix} D_{21,0} \\ 0 \end{bmatrix}, \tag{10}$$

where $D_{21,0}$ is of maximal rank. Thus, the output $y$ can be partitioned as,

$$\begin{pmatrix} y_0(k) \\ y_1(k) \end{pmatrix} = \begin{bmatrix} C_{2,0} \\ C_{2,1} \end{bmatrix} x(k) + \begin{bmatrix} D_{21,0} \\ 0 \end{bmatrix} w(k).$$

Here we use the current measurement output $y_1(k+1)$ instead of $y_1(k)$ to compute $\hat{x}(k+1)$. That is, we build our estimate based on the measurement,

$$\begin{pmatrix} y_0(k) \\ y_1(k+1) \end{pmatrix} = \begin{bmatrix} C_{2,0} \\ C_{2,1}A \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ C_{2,1}B_2 \end{bmatrix} u(k) + \begin{bmatrix} D_{21,0} \\ C_{2,1}B_1 \end{bmatrix} w(k).$$

Thus, the dynamic equation of the current estimator is

$$\begin{aligned} \hat{x}(k+1) &= A\hat{x}(k) + B_2u(k) \\ &\quad + K_c\left\{ \begin{pmatrix} y_0(k) \\ y_1(k+1) \end{pmatrix} - \begin{bmatrix} C_{2,0} \\ C_{2,1}A \end{bmatrix} \hat{x}(k) - \begin{bmatrix} 0 \\ C_{2,1}B_2 \end{bmatrix} u(k) \right\} \end{aligned} \tag{11}$$

where the gain $K_c$ is chosen so that $A - K_cC_c$ is asymptotically stable. To implement it, we partition $K_c = [\,K_{c0} \quad K_{c1}\,]$ in conformity with $y_0$ and $y_1$ and by defining the following variable $v(k)$,

$$v(k) = \hat{x}(k) - K_{c1}y_1(k). \tag{12}$$

For future use, let us define

$$C_c = \begin{bmatrix} C_{2,0} \\ C_{2,1}A \end{bmatrix} \quad \text{and} \quad D_c = \begin{bmatrix} D_{21,0} \\ C_{2,1}B_1 \end{bmatrix}. \tag{13}$$

Then the current estimator based controller is given by

$$\begin{cases} v(k+1) &= (A - K_cC_c)v(k) + [\,K_{c0} \quad (A-K_cC_c)K_{c1}\,]\,y(k) \\ & \qquad\qquad\qquad +(B_2 - K_{c1}C_{2,1}B_2)u(k), \\ \hat{x}(k) &= v(k) + K_{c1}y_1(k), \\ u(k) &= -F\hat{x}(k). \end{cases} \tag{14}$$

The transfer function of the controller is

$$\begin{aligned} \mathbf{C}_c(z) = [0,\ FK_{c1}] + F(zI_n - A + K_cC_c + B_2F - K_{c1}C_{2,1}B_2F)^{-1} \\ \times [K_{c0},\ (A-K_cC_c)K_{c1} - (B_2 - K_{c1}C_{2,1}B_2)FK_{c1}]. \end{aligned} \tag{15}$$

- **Reduced order estimator based controller:**

Again without loss of generality but for simplicity of presentation, we assume that the matrices $C_2$ and $D_{21}$ are already in the form

$$C_2 = \begin{bmatrix} 0 & C_{2,02} \\ I_{p-m_0} & 0 \end{bmatrix} \quad \text{and} \quad D_{21} = \begin{bmatrix} D_{21,0} \\ 0 \end{bmatrix}, \tag{16}$$

where $m_0$ is the rank of $D_{21}$. Then the given system $\Sigma$ can be written as,

$$\begin{cases} \begin{pmatrix} x_1(k+1) \\ x_2(k+1) \end{pmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{pmatrix} x_1(k) \\ x_2(k) \end{pmatrix} + \begin{bmatrix} B_{1,1} \\ B_{1,2} \end{bmatrix} w(k) + \begin{bmatrix} B_{2,1} \\ B_{2,2} \end{bmatrix} u(k) \\ \begin{pmatrix} y_0(k) \\ y_1(k) \end{pmatrix} = \begin{bmatrix} 0 & C_{2,02} \\ I_{p-m_0} & 0 \end{bmatrix} \begin{pmatrix} x_1(k) \\ x_2(k) \end{pmatrix} + \begin{bmatrix} D_{21,0} \\ 0 \end{bmatrix} w(k) \\ z(k) = C_1 \quad x(k) + D_{11} \quad w(k) + D_{12} \quad u(k) \end{cases} \tag{17}$$

where $[x_1', \ x_2']' = x$ and $[y_0', \ y_1']' = y$. We note that $y_1 \equiv x_1$. Thus, one needs to estimate only the state $x_2$ in the reduced-order estimator.

Then following closely the procedure given in [4], we first rewrite the state equation for $x_1$ in terms of the measured output $y_1$ and state $x_2$ as follows,

$$y_1(k+1) = A_{11}y_1(k) + A_{12}x_2(k) + B_{1,1}w(k) + B_{2,1}u(k), \tag{18}$$

where $y_1$ and $u$ are known. Observation of $x_2$ is made via $y_0$ and

$$\tilde{y}_1(k) = A_{12}x_2(k) + B_{1,1}w(k) = y_1(k+1) - A_{11}y_1(k) - B_{2,1}u(k). \tag{19}$$

A reduced-order system for the estimation of state $x_2$ is given by

$$\begin{cases} x_2(k+1) = A_r\ x_2(k) + B_r\ w(k) + [A_{21},\ B_{2,2}] \begin{pmatrix} y_1(k) \\ u(k) \end{pmatrix}, \\ \begin{pmatrix} y_0(k) \\ \tilde{y}_1(k) \end{pmatrix} = y_r(k) = C_r\ x_2(k) + D_r\ w(k), \end{cases} \tag{20}$$

where

$$A_r := A_{22}, \quad B_r := B_{1,2}, \quad C_r := \begin{bmatrix} C_{2,02} \\ A_{12} \end{bmatrix}, \quad D_r := \begin{bmatrix} D_{21,0} \\ B_{1,1} \end{bmatrix}. \tag{21}$$

Based on (20), one can construct a reduced-order observer for $x_2$ as,

$$\hat{x}_2(k+1) = A_r\hat{x}_2(k) + [A_{21},\ B_{2,2}] \begin{pmatrix} y_1(k) \\ u(k) \end{pmatrix} + K_r[y_r(k) - C_r\hat{x}_2(k)], \tag{22}$$

where $K_r$ is the observer gain matrix chosen such that $A_r - K_rC_r$ is asymptotically stable. For the purpose of implementing (22), let us partition $K_r = [K_{r0},\ K_{r1}]$ to be compatible with the dimensions of the outputs $[y_0',\ \tilde{y}_1']'$, and at the same time define a new variable,

$$v := \hat{x}_2 - K_{r1}\tilde{y}_1.$$

We then obtain the following reduced-order estimator based controller,

$$\begin{cases} v(k+1) = (A_r - K_rC_r)v(k) + (B_{2,2} - K_{r1}B_{2,1})u(k) + G_ry(k), \\ \hat{x}(k) = \begin{bmatrix} 0 \\ I_{n-p+m_0} \end{bmatrix} v(k) + \begin{bmatrix} 0 & I_{p-m_0} \\ 0 & K_{r1} \end{bmatrix} y(k), \\ u(k) = -F\hat{x}(k) = -F_1x_1(k) - F_2\hat{x}_2(k), \end{cases} \tag{23}$$

where

$$G_r = [K_{r0},\ A_{21} - K_{r1}A_{11} + (A_r - K_rC_r)K_{r1}],$$

and where $F$ is partitioned as

$$F = [F_1, \ F_2]$$

in conformity with $[x_1', \ x_2']'$. Then the transfer function from $y(k)$ to $-u(k)$ that results in using the reduced order estimator, is given by

$$\begin{aligned} \mathbf{C}_r(z) &= F_2(zI - A_r + K_r C_r + B_{2,2}F_2 - K_{r1}B_{2,1}F_2)^{-1} \\ &\quad \cdot \Big(G_r - (B_{2,2} - K_{r1}B_{2,1})[0, \ F_1 + F_2K_{r1}]\Big) + [0, \ F_1 + F_2K_{r1}]. \quad (24) \end{aligned}$$

We now proceed to do some preliminary analysis of closed-loop recovery error $E(z)$. First we express the closed-loop *recovery error* $E(z)$ in terms of a so-called *recovery matrix* $M(z)$ that is suitable for the closed-loop recovery analysis. We have the following lemma.

**Lemma 1.** *Consider the given system $\Sigma$. Assume that $(A, B_2)$ is stabilizable and $(A, C_2)$ is detectable. Also, let $T_{zw}^{tg}(z)$ be an admissible target loop, i.e., $T_{zw}^{tg}(z) \in \mathrm{T}(\Sigma)$. Then the loop recovery error $E(z)$ between the target loop transfer function $T_{zw}^{tg}(z)$ and that realized by any one of the controllers described earlier, can be written in the form,*

$$E(z) = T_{zu}(z)M(z). \tag{25}$$

*where*

$$T_{zu}(z) = (C_1 - D_{12}F)(\Phi^{-1} + B_2F)^{-1}B_2 + D_{12},$$

*is the closed-loop transfer function from $u$ to $z$ under state feedback. Furthermore, if $(A, B_2, C_1, D_{12})$ is left invertible, then*

$$E^*(j\omega) = 0 \ \text{ if and only if } \ M^*(j\omega) = 0$$

*for all $\omega \in (-\infty, \infty)$. The expression for the recovery matrix $M(z)$ depends on the controller used. In particular, for each one of the controllers considered earlier, we have the following expressions,*

$$M^p(z) = F(zI_n - A + K_pC_2)^{-1}(B_1 - K_pD_{21}), \tag{26}$$

$$M^c(z) = F\left(zI_n - A + K_cC_c\right)^{-1}\left(B_1 - K_cD_c\right), \tag{27}$$

$$M^r(z) = F_2(zI - A_r + K_rC_r)^{-1}(B_r - K_rD_r). \tag{28}$$

**Proof :** To simplify and to unify our proof, we first examine the following Luenberger estimator based controller,

$$\begin{cases} v(k+1) = Lv(k) + G_1 y(k) + G_2 u(k), \\ \hat{x}(k) \quad = Pv(k) + Jy(k), \\ u(k) \quad = -F\hat{x}(k) \end{cases} \tag{29}$$

where $v \in \mathbf{R}^r$ with $r$ being the order of the controller and $\hat{x} \in \mathbf{R}^n$. It is well known that, in the disturbance free case (i.e., $w = 0$) the variable $\hat{x}$ is an asymptotic estimate of the state $x$ provided that the matrix $L$ is a stability matrix and there exists a matrix $Q \in \mathbf{R}^{r \times n}$ satisfying the following conditions:

$$QA - LQ = G_1 C_2, \qquad G_2 = QB_2, \qquad JC_2 + PQ = I_n. \tag{30}$$

Let $T^{\ell}_{zw}(z)$ denote the closed-loop transfer function from $w$ to $z$ with a general Luenberger observer-based controller. Then following the procedure of [1], it is simple to show that the loop recovery error realized by such a Luenberger estimator based controller is

$$E^{\ell}(z) = T^{\ell}_{zw}(z) - T^{tg}_{zw}(z) = T_{zu}(z) M^{\ell}(z),$$

where

$$M^{\ell}(z) = F[P(zI - L)^{-1}(QB_1 - G_1 D_{21}) - JD_{21}]. \tag{31}$$

Next, it is straightforward to verify that the prediction estimator based controller is a special case of Luenberger estimator based controller in (29) with

$$\left. \begin{array}{l} L \ = A - K_p C_2 \\ G_1 = K_p \\ G_2 = B_2 \\ P \ = I_n \\ J \ = 0 \\ Q \ = I_n. \end{array} \right\} \tag{32}$$

Hence, (26) follows simply from (31).

Similarly, it is simple to verify that the current and reduced-order estimator based controllers are also the special cases of Luenberger estimator based controller with

$$\left.\begin{aligned} L &= A - K_c C_c \\ G_1 &= [\,K_{c0} \quad (A - K_c C_c) K_{c1}\,] \\ G_2 &= B_2 - K_{c1} C_{2,1} B_2 \\ P &= I_n \\ J &= [\,0 \quad K_{c1}\,] \\ Q &= I_n - K_{c1} C_{2,1} \end{aligned}\right\} \tag{33}$$

and

$$\left.\begin{aligned} L &= A_r - K_r C_r \\ G_1 &= [\,K_{r0} \quad A_{21} - K_{r1} A_{11} + (A_r - K_r C_r) K_{r1}\,] \\ G_2 &= B_{2,2} - K_{r1} B_{2,1} \\ P &= \begin{bmatrix} 0 \\ I_{n-p+m_0} \end{bmatrix} \\ J &= \begin{bmatrix} 0 & I_{p-m_0} \\ 0 & K_{r1} \end{bmatrix} \\ Q &= [\,-K_{r1} \quad I_{n-p+m_0}\,] \end{aligned}\right\} \tag{34}$$

respectively. Then, once again (27) and (28) follow from (31). This completes the proof of Lemma 1. ∎

The significance of Lemma 1 can be seen in two ways. It converts the CLTR analysis problem into a study of conditions under which the recovery matrix $M(z)$ can be rendered zero. Also, it unifies the study of $M(z)$ for all three controllers into a single mathematical framework. In order to further cement such a unification, we need to investigate the structural properties of $\Sigma_c$, characterized by the quadruple $(A, B, C_c, D_c)$, and $\Sigma_r$, characterized by the quadruple $(A_r, B_r, C_r, D_r)$, in terms of those of $\Sigma_*$ defined by the quadruple $(A, B_1, C_2, D_{21})$. We have the following propositions.

**Proposition 1.**

1. *$\Sigma_c$ is of (non-) minimum phase if and only if $\Sigma_*$ is of (non-) minimum phase.*
2. *$\Sigma_c$ is stabilizable and detectable if and only if $\Sigma_*$ is stabilizable and detectable.*
3. *Invariant zeros of $\Sigma_c$ contain invariant zeros of $\Sigma_*$ and $z = 0$.*
4. *Orders of infinite zeros of $\Sigma_c$ are reduced by one from those of $\Sigma_*$.*

5. *$\Sigma_c$ is left invertible if and only if $\Sigma_*$ is left invertible.*

6. *$\mathcal{V}^+(\Sigma_c) = \mathcal{V}^+(\Sigma_*)$.*

7. *$\mathcal{S}^-(\Sigma_c) = \mathcal{S}^-(\Sigma_*) \cap \{x \mid C_2 x \in \operatorname{Im}(D_{21})\}$.*

8. *$\mathcal{S}^-(\Sigma_c) = \emptyset$ if and only if $\Sigma_*$ is left invertible and of minimum phase with no infinite zeros of order higher than one.*

**Proof :** See [3]. ∎

**Proposition 2.**

1. *$\Sigma_r$ is of (non-) minimum phase if and only if $\Sigma_*$ is of (non-) minimum phase.*

2. *$\Sigma_r$ is detectable if and only if $\Sigma_*$ is detectable.*

3. *Invariant zeros of $\Sigma_r$ are the same as those of $\Sigma_*$.*

4. *Orders of infinite zeros of $\Sigma_r$ are reduced by one from those of $\Sigma_*$.*

5. *$\Sigma_r$ is left invertible if and only if $\Sigma_*$ is left invertible.*

6. $\begin{pmatrix} 0 \\ I \end{pmatrix} \mathcal{V}^+(\Sigma_r) = \mathcal{V}^+(\Sigma_*)$.

7. $\begin{pmatrix} 0 \\ I \end{pmatrix} \mathcal{S}^-(\Sigma_r) = \mathcal{S}^-(\Sigma_*) \cap \{x \mid C_2 x \in \operatorname{Im}(D_{21})\}$.

8. *$\mathcal{S}^-(\Sigma_r) = \emptyset$ if and only if $\Sigma_*$ is left invertible and of minimum phase with no infinite zeros of order higher than one.*

**Proof :** See [3]. ∎

**Remark 2.** *For a left invertible minimum phase system $\Sigma_*$ with $D_{21} = 0$, it is simple to see that*

$$\mathcal{S}^-(\Sigma_c) = \begin{pmatrix} 0 \\ I \end{pmatrix} \mathcal{S}^-(\Sigma_r) = \mathcal{S}^-(\Sigma_*) \cap \{x \mid C_2 x \in \operatorname{Im}(D_{21})\} = \emptyset$$

*if and only if $CB$ is of maximal rank. Also, for a non-strictly proper SISO system $\Sigma_*$,*

$$\mathcal{S}^-(\Sigma_*) = \mathcal{S}^-(\Sigma_c) = \mathcal{S}^-(\Sigma_r) = \mathcal{S}^-(\Sigma_*) \cap \{x \mid C_2 x \in \operatorname{Im}(D_{21})\} = \emptyset$$

*if and only if it is of minimum phase.*

# V. GENERAL CLTR ANALYSIS

This section deals with the general analysis of the CLTR mechanism using any one of the three controllers discussed in the last section. Notationally, in all of our general discussions here, we deal with the given system $\Sigma_*$ characterized by the quadruple $(A, B_1, C_2, D_{21})$ and the prediction estimator based controller in which $K_p$ is the observer gain. In view of Lemma 1, all the general discussions presented here can be particularized to current and reduced-order estimator based controllers with appropriate notational changes. In all our main theorems, we will however explicitly point out the capabilities of each controller as they could be different for each case.

As is evident from Lemma 1, the nucleus of CLTR analysis is the study of $M^p(z)$ to ascertain how and when it can or cannot be rendered zero. The required study of $M^p(z)$ can be undertaken in two ways, with or without the prior knowledge of $F$ that prescribes the target loop transfer function $T^{tg}_{zw}(z)$. Our goal in the first subsection to follow is to study $M^p(z)$ without taking into account any specific characteristics of $F$. The second subsection, devoted to CLTR analysis while taking into account appropriate characteristics of $F$, complements the analysis of the first subsection. Decomposing $M^p(z)$ as $F\tilde{M}^p(z)$, the study of $M^p(z)$ without knowing $F$ is the same one as the study of $\tilde{M}^p(z)$. A detailed study of $\tilde{M}^p(z)$ leads to a fundamental Lemma 2 involving with an eigenstructure assignment to the observer dynamic matrix $A - K_pC_2$ by an appropriate design of $K_p$. This Lemma 2 reveals the limitations of the given system as a consequence of its structural properties in recovering an arbitrary target loop transfer function via the given controller structure. Thus it leads to Theorem 2 which, for each controller, shows under what conditions on $\Sigma$ the set of recoverable target loops $\mathrm{T}_{\mathrm{R}}(\Sigma)$ is equal to the set of admissible target loops $\mathrm{T}(\Sigma)$. Also, Lemma 2 enables one to decompose $\tilde{M}^p(z)$ into two essential parts, $\tilde{M}^p_o(z)$ and $\tilde{M}^p_e(z)$. The first part $\tilde{M}^p_o(z)$ can be rendered zero by an appropriate eigenstructure assignment to $A - K_pC_2$, while the second part $\tilde{M}^p_e(z)$ in general cannot be rendered zero, by any means, although our analysis of $\tilde{M}^p_e(z)$ reveals a multitude of ways by which it can be shaped. The decomposition of $\tilde{M}^p(z)$ into two parts and the subsequent analysis of each part forms the core of the analysis given throughout this paper. In particular, it leads to several important results given in this section.

### A. Recovery Analysis For An Arbitrary Target Loop

In this subsection, we consider that the target loop transfer function $T^{tg}_{zw}(z)$ is arbitrary. That is, we do not take into account any specific characteristics of $T^{tg}_{zw}(z)$ in analyzing the CLTR mechanism. As mentioned before, we will focus our attention on the prediction estimator based controller with gain $K_p$. Then, as implied by Lemma 1, $\tilde{M}^p(z)$ as given below forms the basis of our study,

$$\tilde{M}^p(z) = (zI_n - A + K_pC_2)^{-1}(B_1 - K_pD_{21}). \tag{35}$$

It is evident that the gain $K_p$ is the only free design parameter in $\tilde{M}^p(z)$. First of all, in order to guarantee the closed-loop stability, $K_p$ must be such that $A - K_pC_2$ is an asymptotically stable matrix. The remaining freedom in choosing $K_p$ can then be used for the purpose of achieving CLTR. We note that for left invertible $\Sigma_*$, exact loop transfer recovery (ECLTR) is possible for an arbitrary $F$ if and only if

$$[\tilde{M}^p(j\omega)]^* = (e^{j\omega T}I_n - A + K_pC_2)^{-1}(B_1 - K_pD_{21}) \equiv 0.$$

However, due to the non-singularity of $(e^{j\omega T}I_n - A + K_pC_2)^{-1}$, the fact that $[\tilde{M}^p(j\omega)]^* \equiv 0$ implies that $B_1 - K_pD_{21} \equiv 0$. Hence, rendering all the parts of $[\tilde{M}^p(j\omega)]^*$ zero is possible only for a very restrictive class of systems. In general only certain parts of $[\tilde{M}^p(j\omega)]^*$ can be rendered zero. To proceed with our analysis, for clarity of presentation we will temporarily assume that $A - K_pC_2$ is non-defective. This allows us to expand $\tilde{M}^p(z)$ and hence $M^p(z)$ in a dyadic form,

$$\tilde{M}^p(z) = \sum_{i=1}^{n} \frac{\tilde{R}_i}{z - \lambda_i} \tag{36}$$

where the residue $\tilde{R}_i$ is given by

$$\tilde{R}_i = W_iV_i^{\mathrm{H}}[B - K_pD]. \tag{37}$$

Here $W_i$ and $V_i$ are respectively the right and left eigenvectors associated with an eigenvalue $\lambda_i$ of $A - K_pC_2$ and they are scaled so that $WV^{\mathrm{H}} = V^{\mathrm{H}}W = I_n$ where

$$W = [W_1,\, W_2,\, \cdots,\, W_n] \quad \text{and} \quad V = [V_1,\, V_2,\, \cdots,\, V_n]. \tag{38}$$

**Remark 3.** *The assumption that $A - K_p C_2$ be non-defective is not essential. However, it does simplify our presentation. A removal of this condition necessitates the use of generalised right and left eigenvectors of $A - K_p C_2$ instead of the right and left eigenvectors $W_i$ and $V_i$, and consequently the expansion of $\tilde{M}^p(z)$ requires a double summation in place of the single summation used in (36).*

We are looking for conditions under which the $i$-th term of $\tilde{M}^p(z)$ in (36) can be made zero for each $i = 1$ to $n$. There is only one possibility in the discrete-time CLTR to do so; namely, by assigning $\lambda_i$ to any location in $\mathbf{C}^{\odot}$ while simultaneously rendering the corresponding residue $\tilde{R}_i$ zero. In other words, such a possibility corresponds to the appropriate finite eigenstructure assignment of $A - K_p C_2$ that renders $\tilde{R}_i$ zero. In continuous-time systems, there exists an alternative approach; namely, by assigning $\lambda_i$ asymptotically large in the negative half $s$-plane so that a term of the type

$$\frac{\tilde{R}_i}{s - \lambda_i}$$

tends to zero as $\lambda_i \to \infty$. This latter approach deals with an infinite eigenstructure assignment to $A - K_p C_2$. The possibility of assigning an infinite eigenstructure, however, does not exist in discrete-time systems since $\lambda_i$ is restricted to $\mathbf{C}^{\odot}$ in order to guarantee the stability of the resulting closed-loop system. Given the fact that $|\lambda_i|$ cannot go to $\infty$, it is easy to observe that the *notions of exact* CLTR (ECLTR) *and asymptotic* CLTR (ACLTR) *in discrete-time systems are equivalent in the sense that any target loop that is asymptotically recoverable is also exactly recoverable and vice versa.* Because of this fact, throughout this paper, whenever we talk about recovery, we mean both exact and asymptotic recovery. For example, whenever we say that an admissible target loop is recoverable, we mean that the specified target loop is exactly as well as asymptotically recoverable as stated in Definition 4. This is because, as we mentioned in the introduction, some optimization-based design methods such as $H_\infty$ norm minimization methods sometimes lead to suboptimal designs that correspond to an asymptotic recovery. To be specific, in optimization-based methods, one normally generates a sequence of observer gains by solving parameterized algebraic Riccati equations. As the parameter tends to a certain value, the corresponding sequence of $H_\infty$ norms of the resulting recovery matrices tends to a limit which is the infimum of the $H_\infty$ norm of the recovery matrix over the set of all possible observer gains. A suboptimal solution is obtained when one selects an observer gain corresponding to a particular value of the parameter. On the other hand, in eigenstructure assignment methods, the required observer gain is obtained without solving any parameterized equations. Thus, in some cases the observer gains $K_p$ are designed as a function of a parameter, and in other cases they are independent of it.

The following lemma answers the question of how many residues $\tilde{R}_i$ can be rendered zero by an appropriate finite eigenstructure assignment of $A - K_pC_2$.

**Lemma 2.** *Let $\lambda_i$ and $V_i$ be an eigenvalue and the corresponding left eigenvector of $A - K_pC_2$ for any gain $K_p$ such that $A - K_pC_2$ is asymptotically stable. Then the maximum possible number of $\lambda_i \in \mathbf{C}^{\odot}$ which satisfy the condition $V_i^H[B_1 - K_pD_{21}] = 0$ is $n_a^-(\Sigma_*) + n_b(\Sigma_*)$. A total of $n_a^-(\Sigma_*)$ of these $\lambda_i$ coincide with the system invariant zeros which are in $\mathbf{C}^{\odot}$ (the so-called stable or minimum phase invariant zeros) and the remaining $n_b(\Sigma_*)$ eigenvalues can be assigned arbitrarily to any locations in $\mathbf{C}^{\odot}$. All the eigenvectors $V_i$ that correspond to these $n_a^-(\Sigma_*) + n_b(\Sigma_*)$ eigenvalues span the subspace $\mathbf{R}^n/\mathcal{S}^-(\Sigma_*)$. Moreover, the $n_a^-(\Sigma_*)$ eigenvectors $V_i$ for the eigenvalues which coincide with the system invariant zeros in $\mathbf{C}^{\odot}$ are the corresponding left state zero directions and span the subspace $\mathcal{V}^*(\Sigma_*)/\mathcal{V}^+(\Sigma_*)$.*

**Proof :** See [3]. ∎

**Remark 4.** *Instead of rendering the $n_a^-(\Sigma_*) + n_b(\Sigma_*)$ residues $\tilde{R}_i$ mentioned in Lemma 2 exactly zero, if one prefers, they can be rendered asymptotically zero as a certain parameter tends to a particular limit. In that case $n_a^-(\Sigma_*)$ eigenvalues coincide asymptotically with the $n_a^-(\Sigma_*)$ minimum phase invariant zeros while the corresponding eigenvectors approach in the limit the corresponding left state zero directions and span the subspace $\mathcal{V}^*(\Sigma_*)/\mathcal{V}^+(\Sigma_*)$. As stated earlier, in this paper, we will not distinguish between such exact and asymptotic assignments.*

Lemma 2 points out that there are altogether $n_a^-(\Sigma_*) + n_b(\Sigma_*)$ eigenvalues which can be assigned inside $\mathbf{C}^{\odot}$ so that the corresponding terms of $\tilde{M}^p(z)$ in its dyadic expansion (36) are zero. This fact leads to some structural conditions on $\Sigma$ so that any arbitrary admissible target loop can be recovered. This is explored in the following theorem.

**Theorem 2.** *Consider a given system $\Sigma$. Assume that $(A, B_2)$ is stabilizable, $(A, C_2)$ is detectable, and $(A, B_2, C_1, D_{12})$ is left invertible. Depending upon the controller used, we have the following characterization of $\Sigma_*$ so that any arbitrary admissible target loop can be recovered.*

1. Prediction estimator based controller :
   *Any arbitrary admissible target closed-loop transfer function matrix is recoverable, i.e., $\mathrm{T}^p_{\mathrm{R}}(\Sigma) = \mathrm{T}(\Sigma)$, if $\Sigma_*$ is left invertible and of minimum phase with no infinite zeros (i.e., $D_{21}$ is maximal rank).*

2. Current estimator based controller :
   *Any arbitrary admissible target closed-loop transfer function matrix is recoverable, i.e., $\mathrm{T}^c_{\mathrm{R}}(\Sigma) = \mathrm{T}(\Sigma)$, if $\Sigma_*$ is left invertible and of minimum phase with no infinite zeros of order higher than one.*

3. Reduced order estimator based controller :
   *Any arbitrary admissible target closed-loop transfer function matrix is recoverable, i.e., $\mathrm{T}^r_{\mathrm{R}}(\Sigma) = \mathrm{T}(\Sigma)$, if $\Sigma_*$ is left invertible and of minimum phase with no infinite zeros of order higher than one.*

**Proof :** Let us take the case of a prediction estimator based controller. The fact that $\Sigma$ is left invertible and of minimum phase with no infinite zeros implies that $n_a^+(\Sigma_*) = n_c(\Sigma_*) = n_f(\Sigma_*) = 0$. Thus $n_a^-(\Sigma_*) + n_b(\Sigma_*) = n$. Hence the result follows from (25) and Lemma 2. Now, for the case of current and reduced-order observer based controllers, in view of Propositions 1 (i.e., item 8) and 2 (i.e., item 8), we note that $n_a^+ + n_c + n_f$ corresponding to both $\Sigma_c$ and $\Sigma_r$ is equal to zero if $\Sigma$ is left invertible and of minimum phase with no infinite zeros of order higher than one. ∎

**Remark 5.** *As we will show later in Corollary 1, the conditions in Theorem 2 are also necessary.*

As is evident by Theorem 2, the required structural conditions for recovery of any arbitrary admissible target loop are very stringent, and call for $n_a^-(\Sigma_*) + n_b(\Sigma_*)$ to be equal to the dimension $n$ of $\Sigma_*$. To see what is and what is not feasible when $n_a^-(\Sigma_*) + n_b(\Sigma_*) \neq n$, and to emphasize explicitly the behavior of each term of $\tilde{M}^p(z)$, let us partition the dyadic expansion (36) of $\tilde{M}^p(z)$ into three parts, each part having a particular type of characteristics,

$$\tilde{M}^p(z) = \tilde{M}^p_-(z) + \tilde{M}^p_b(z) + \tilde{M}^p_e(z), \tag{39}$$

where

$$\tilde{M}^p_-(z) = \sum_{i=1}^{n_a^-(\Sigma_*)} \frac{\tilde{R}_i^-}{z - \lambda_i^-}, \quad \tilde{M}^p_b(z) = \sum_{i=1}^{n_b(\Sigma_*)} \frac{\tilde{R}_i^b}{z - \lambda_i^b}$$

and

$$\tilde{M}^p_e(z) = \sum_{i=1}^{n_a^+(\Sigma_*)+n_c(\Sigma_*)+n_f(\Sigma_*)} \frac{\tilde{R}_i^e}{z - \lambda_i^e}.$$

In the above partition, appropriate superscripts $-$, $b$, and $e$ are added to $\tilde{R}_i$ and $\lambda_i$ in order to associate them respectively with $\tilde{M}^p_-(z)$, $\tilde{M}^p_b(z)$, and $\tilde{M}^p_e(z)$. Next, define the following sets where $n^e(\Sigma_*) = n_a^+(\Sigma_*) + n_c(\Sigma_*) + n_f(\Sigma_*)$:

$$\Lambda^- = \{\lambda_i^- \,;\, i = 1 \text{ to } n_a^-(\Sigma_*)\}, \quad V^- = \{V_i^- \,;\, i = 1 \text{ to } n_a^-(\Sigma_*)\},$$

$$W^- = \{W_i^- \,;\, i = 1 \text{ to } n_a^-(\Sigma_*)\}, \quad \Lambda^b = \{\lambda_i^b \,;\, i = 1 \text{ to } n_b(\Sigma_*)\},$$

$$V^b = \{V_i^b \,;\, i = 1 \text{ to } n_b(\Sigma_*)\}, \quad W^b = \{W_i^b \,;\, i = 1 \text{ to } n_b(\Sigma_*)\}$$

$$\Lambda^e = \{\lambda_i^e \,;\, i = 1 \text{ to } n_e(\Sigma_*)\}, \quad V^e = \{V_i^e \,;\, i = 1 \text{ to } n_e(\Sigma_*)\},$$

$$W^e = \{W_i^e \,;\, i = 1 \text{ to } n_e(\Sigma_*)\}.$$

We now note that various parts of $\tilde{M}^p(z)$ have the following interpretation:

1. $\tilde{M}^p_-(z)$ contains $n_a^-(\Sigma_*)$ terms. The $n_a^-(\Sigma_*)$ eigenvalues of $A - K_pC_2$ represented in it form a set $\Lambda^-$. In accordance with Lemma 2, there exists a gain $K_p$ such that $\tilde{M}^p_-(z)$ can be rendered identically zero by assigning the elements of $\Lambda^-$ to coincide with the system minimum phase invariant zeros while the corresponding set of left eigenvectors $V^-$ coincides with the corresponding set of left state zero directions.

2. $\tilde{M}^p_b(z)$ contains $n_b(\Sigma_*)$ terms. The $n_b(\Sigma_*)$ eigenvalues of $A - K_pC_2$ represented in it form a set $\Lambda^b$. In accordance with Lemma 2, there

exists a gain $K_p$ such that $\tilde{M}_b^p(z)$ can be rendered zero by assigning the elements of $\Lambda^b$ to arbitrary locations in $\mathbf{C}^{\odot}$.

3. $\tilde{M}_e^p(z)$ contains $n^e(\Sigma_*) = n_a^+(\Sigma_*) + n_c(\Sigma_*) + n_f(\Sigma_*)$ terms. The $n_e(\Sigma_*)$ eigenvalues of $A - K_pC_2$ represented in $\tilde{M}_e^p(z)$ form a set $\Lambda^e$. In view of Lemma 2, $\tilde{M}_e^p(z)$ cannot in general be rendered zero by any assignment of $\Lambda^e$ and the associated sets of right and left eigenvectors $W^e$ and $V^e$.

Since both $\tilde{M}_-^p(z)$ and $\tilde{M}_b^p(z)$ can be rendered zero, for future use, we can combine them into one term,

$$\tilde{M}_o^p(z) = \tilde{M}_-^p(z) + \tilde{M}_b^p(z).$$

We define likewise, $\Lambda^o = \Lambda^- \cup \Lambda^b$, $W^o = W^- \cup W^b$, $V^o = V^- \cup V^b$. Thus $\tilde{M}^p(z)$ can be rewritten as

$$\tilde{M}^p(z) = \tilde{M}_o^p(z) + \tilde{M}_e^p(z). \tag{40}$$

As the above discussion indicates, Lemma 2 forms the heart of the underlying mechanism of discrete-time CLTR. It shows clearly what is and what is not feasible under what conditions. Although it does not directly provide methods of obtaining the gain $K_p$, it provides structural guidelines as to how certain eigenvalues and eigenvectors are to be assigned while indicating a multitude of ways in which freedom exists in assigning the other eigenvalues and eigenvectors of $A - K_pC_2$. These guidelines, in turn, can appropriately be channeled to come up with a design method to obtain an appropriate gain $K_p$. Leaving aside now the methods of design, let us at this stage simply define the following sets of gains:

**Definition 7.** *Consider the system $\Sigma$. Let $\mathcal{K}_p^*(\Sigma_*)$ be a set of gains $K_p \in \mathbf{R}^{n\times p}$ such that* (i) *$A - K_pC_2$ is asymptotically stable, and* (ii) *$\tilde{M}_o^p(z)$ is zero. In a similar manner, define $\mathcal{K}_c^*(\Sigma)$ and $\mathcal{K}_r^*(\Sigma)$ for $\Sigma_c$ and $\Sigma_r$, i.e., $\mathcal{K}_c^*(\Sigma) = \mathcal{K}_p^*(\Sigma_c)$ and $\mathcal{K}_r^*(\Sigma) = \mathcal{K}_p^*(\Sigma_r)$.*

As mentioned earlier, we do not parameterize here the gain $K_p$ in terms of a tunable parameter $\sigma$. We deal only with a fixed gain $K_p$. If one deals with asymptotic recovery and thus with a sequence of controller gains $K_p(\sigma)$ for different values of $\sigma$, the set of recoverable gains is also parameterized and hence can be written as $\mathcal{K}_p^*(\Sigma_*, \sigma)$. In that case, one defines $\mathcal{K}_p^*(\Sigma_*, \sigma)$ as a set of gains $K_p(\sigma) \in \mathbf{R}^{n \times p}$ such that (i) $A - K_p(\sigma)C_2$ is asymptotically stable for all $\sigma > \sigma^*$ where $0 \le \sigma^* < \infty$, (ii) the limits, as $\sigma \to \infty$, of all the eigenvalues of $A - K_p(\sigma)C_2$ remain in $\mathbf{C}^{\odot}$, and (iii) $\tilde{M}_o^p(z)$ is either identically zero or asymptotically zero. Similarly, $\mathcal{K}_c^*(\Sigma, \sigma)$ and $\mathcal{K}_r^*(\Sigma, \sigma)$ are defined for systems $\Sigma_c$ and $\Sigma_r$.

It is obvious that the sets of gains defined above are nonempty. We note also that whenever $K_p$ is chosen as an element of $\mathcal{K}_p^*(\Sigma_*)$, the resulting error in the recovery matrix $M^p(z)$ is $M_e^p(z) = F\tilde{M}_e^p(z)$. As such $M_e^p(z)$ is called hereafter as the *'recovery error matrix'*. Essentially, there are three methods that generate such estimator gain matrices: (i) ATEA method which is capable of exploiting all the available design freedom and shaping the recovery error matrix in various ways; (ii) $H_\infty$-optimization based method which minimizes the $H_\infty$ norm of $M(z)$; and (iii) $H_2$-optimization based method which minimizes the $H_2$ norm of $M(z)$. We refer the readers to [4] for all these design methods.

Theorem 3 given below characterizes the achieved loop transfer function.

**Theorem 3.** *Consider the given system $\Sigma$. Assume that $(A, B_2)$ is stabilizable, $(A, C_2)$ is detectable, and $(A, B_2, C_1, D_{12})$ is left invertible. Also, let $T_{zw}^{tg}(z)$ be an admissible target loop, i.e., $T_{zw}^{tg}(z) \in \mathbf{T}(\Sigma)$. Then*

1. *For a prediction estimator based controller with estimator gain $K_p \in \mathcal{K}_p^*(\Sigma_*)$, we have*
$$E^p(z) = T_{zu}(z) M_e^p(z). \tag{41}$$

2. *For a current estimator based controller with estimator gain $K_c \in \mathcal{K}_c^*(\Sigma_*)$, we have*
$$E^c(z) = T_{zu}(z) M_e^c(z). \tag{42}$$

3. *For a reduced order estimator based controller with estimator gain $K_r \in \mathcal{K}_r^*(\Sigma_*)$, we have*
$$E^r(z) = T_{zu}(z) M_e^r(z). \tag{43}$$

**Proof :** It follows from Lemma 2. ∎

**Remark 6.** *Theorem 2 is a special case of the above theorem. To see this, let us examine first the case when a prediction estimator based controller is used. If the given system $\Sigma_*$ is left invertible and of minimum phase with no infinite zeros, then the recovery error matrix $\tilde{M}_e^p(z)$ is nonexistent and hence $E^p(z)$ can be rendered zero for all $z \in \mathbf{C}$. Similarly, if $\Sigma_*$ is left invertible and of minimum phase with no infinite zeros of order higher than one, then $\tilde{M}_e^c(z)$ and $\tilde{M}_e^r(z)$ are nonexistent and hence exact recovery is achievable by using either current or reduced-order estimator based controllers. Thus, for the special cases considered in Theorem 2, results of Theorem 3 are reduced to those of Theorem 2.*

**Remark 7.** *Theorem 3 also holds if we use the estimator gain $K_p(\sigma) \in \mathcal{K}_p^*(\Sigma_*, \sigma)$. However, in this case, the equality in (41) should be replaced by pointwise convergence in $z$ as $\sigma \to \infty$.*

### B. Analysis For Recoverable Target Loops

In the previous subsection, closed-loop transfer recovery analysis is conducted without taking into account any knowledge of $F$. It involves essentially the study of the matrix $\tilde{M}^p(z)$ or $M^p(z)$ to ascertain when it can or cannot be rendered zero. This subsection complements the analysis of Subsection A by taking into account the knowledge of $F$. Obviously then, the analysis of this subsection is a study of $M^p(z) = F\tilde{M}^p(z)$. One of the important questions that needs to be answered here is as follows. What class of target loops can be recovered for the given system? As it forms a coupling between the analysis and design, characterization of $T_{zw}^{tg}(z)$ to determine whether it can be recovered or not for the given system, plays an extremely important role. That is, although the physical tasks of designing $F$ and $K_p$ are separable, one can benefit enormously by knowing ahead what kind of target loops are recoverable. The necessary and sufficient conditions developed here on $T_{zw}^{tg}(z)$ for its recoverability, turn out to be constraints on the finite and infinite zero structure of the corresponding system $\Sigma_*$. An interpretation of these conditions reveals that recovery of $T_{zw}^{tg}(z)$ for general nonminimum phase systems is possible under a variety of conditions.

Another important question that arises before one undertakes formulating any target loop transfer function $T_{zw}^{tg}(z)$ for a given system $\Sigma$ is as follows. What are the necessary and sufficient conditions on $\Sigma$ so that it has at least one recoverable target loop? An answer to this question obviously helps a designer to change the given plant if necessary by appropriately modifying the number (or type) of plant inputs and/or outputs.

We proceed now to give the following result regarding recoverability of a target closed-loop transfer function $T_{zw}^{tg}(z)$ for the given system $\Sigma$.

**Theorem 4.** *Consider the given system $\Sigma$. Assume that $(A, B_2)$ is stabilizable, $(A, C_2)$ is detectable, and $(A, B_2, C_1, D_{12})$ is left invertible. Then, an admissible target closed loop transfer function $T_{zw}^{tg}(z)$ of $\Sigma$, i.e., $T_{zw}^{tg} \in \mathbf{T}(\Sigma)$, is recoverable if and only if the following condition is satisfied, depending on the controller used.*

1. *For a prediction estimator based controller, the condition is that*

$$\mathcal{S}^-(\Sigma_*) \subseteq \mathrm{Ker}\,(F).$$

2. *For a current estimator based controller, the condition is that*

$$\mathcal{S}^-(\Sigma_*) \cap \{\, x \mid C_2 x \in \mathrm{Im}\,(D_{21}) \,\} \subseteq \mathrm{Ker}\,(F).$$

3. *For a reduced-order estimator based controller, the condition is that*

$$\mathcal{S}^-(\Sigma_*) \cap \{\, x \mid C_2 x \in \mathrm{Im}\,(D_{21}) \,\} \subseteq \mathrm{Ker}\,(F).$$

*Thus the set of recoverable target loops under each controller is characterized as follows:*

1. Prediction estimator based controller :

$$\mathbf{T}_{\mathrm{R}}^{p}(\Sigma) = \Big\{\, T_{zw}^{tg}(z) \in \mathbf{T}(\Sigma) \mid \mathcal{S}^-(\Sigma_*) \subseteq \mathrm{Ker}\,(F) \,\Big\}.$$

2. Current estimator based controller :

$$\mathbf{T}_{\mathrm{R}}^{c}(\Sigma) = \Big\{\, T_{zw}^{tg}(z) \in \mathbf{T}(\Sigma) \;\Big|\; \mathcal{S}^-(\Sigma_*) \cap \{\, x \mid C_2 x \in \mathrm{Im}\,(D_{21}) \,\} \subseteq \mathrm{Ker}\,(F) \,\Big\}.$$

3. Reduced order estimator based controller :

$$\mathbf{T}^{r}_{\mathrm{R}}(\Sigma) = \Big\{ T^{tg}_{zw}(z) \in \mathbf{T}(\Sigma) \;\Big|\; \mathcal{S}^{-}(\Sigma_*) \cap \{ x \,|\, C_2 x \in \mathrm{Im}\,(D_{21}) \} \subseteq \mathrm{Ker}\,(F) \Big\}.$$

**Proof :** It follows from the results of [3]. ∎

**Remark 8.** *We note that* $\mathbf{T}^{p}_{\mathrm{R}}(\Sigma) \subseteq \mathbf{T}^{c}_{\mathrm{R}}(\Sigma) = \mathbf{T}^{r}_{\mathrm{R}}(\Sigma)$.

We have the following corollary to Theorem 4.

**Corollary 1.** *Consider a given system* $\Sigma$. *Assume that* $(A, B_2)$ *is stabilizable,* $(A, C_2)$ *is detectable, and* $(A, B_2, C_1, D_{12})$ *is left invertible. Depending upon the controller used, we have the following necessary and sufficient conditions under which any arbitrary admissible target loop can be recovered.*

1. Prediction estimator based controller :
   *Any arbitrary admissible target closed-loop transfer function matrix is recoverable, i.e.,* $\mathbf{T}^{p}_{\mathrm{R}}(\Sigma) = \mathbf{T}(\Sigma)$, *if and only if* $\Sigma_*$ *is left invertible and of minimum phase with no infinite zeros (i.e.,* $D_{21}$ *is maximal rank).*

2. Current estimator based controller :
   *Any arbitrary admissible target closed-loop transfer function matrix is recoverable, i.e.,* $\mathbf{T}^{c}_{\mathrm{R}}(\Sigma) = \mathbf{T}(\Sigma)$, *if and only if* $\Sigma_*$ *is left invertible and of minimum phase with no infinite zeros of order higher than one.*

3. Reduced order estimator based controller :
   *Any arbitrary admissible target closed-loop transfer function matrix is recoverable, i.e.,* $\mathbf{T}^{r}_{\mathrm{R}}(\Sigma) = \mathbf{T}(\Sigma)$, *if and only if* $\Sigma_*$ *is left invertible and of minimum phase with no infinite zeros of order higher than one.*

**Proof :** The proof follows from Theorem 4, the properties of $\Sigma_c$ and $\Sigma_r$, and the following fact:

*Let* $V \neq 0$ *be any matrix whose columns span a subspace of* $\mathbb{R}^n$. *Then there exists an admissible state feedback gain* $F \in \mathbb{R}^{m \times n}$, *i.e.,* $A - BF$ *is stable, such that* $V \notin \mathrm{Ker}\,(F)$.

The proof of the above fact is simple and is based on the continuity of eigenvalues. Let $F_1 \in \mathbf{R}^{m\times n}$ be such that $A - BF_1$ is stable. If $F_1 V \neq 0$, then the above statement holds by letting $F = F_1$. If $F_1 V = 0$, choose $F_2 \in \mathbf{R}^{m\times n}$ such that $F_2 V \neq 0$. Then by the continuity of eigenvalues, there exists a constant, say $\delta > 0$, such that $F := F_1 + \delta F_2$ and $A - BF$ is stable. Then we have $FV = \delta F_2 V \neq 0$. This completes the proof of the above fact. ∎

Our aim next is to develop the conditions on $\Sigma$ so that the set of recoverable target loops is nonempty. We have the following theorem.

**Theorem 5.** *Consider the given system $\Sigma$. Assume that $(A, B_2)$ is stabilizable, $(A, C_2)$ is detectable, and $(A, B_2, C_1, D_{12})$ is left invertible. Let $\overline{C}_p$ and $\overline{C}_c$ be any full rank matrices such that*

1. $\mathrm{Ker}\,(\overline{C}_p) = \mathcal{V}^+(\Sigma_*)$, *and*
2. $\mathrm{Ker}\,(\overline{C}_c) = \mathcal{S}^-(\Sigma_*) \cap \{\, x \mid C_2 x \in \mathrm{Im}\,(D_{21}) \,\}$.

*Also, let $\overline{C}_r = \overline{C}_c$. Define three auxiliary systems:*

1. $\Sigma^p_{au}$ *characterized by the matrix triple* $(A, B_2, \overline{C}_p)$,
2. $\Sigma^c_{au}$ *characterized by the matrix triple* $(A, B_2, \overline{C}_c)$, *and*
3. $\Sigma^r_{au}$ *characterized by the matrix triple* $(A, B_2, \overline{C}_r)$.

*Then we have the following results depending upon the controller used :*

1. Prediction estimator based controller :
   $\mathbf{T}^p_{\mathrm{R}}(\Sigma)$ *is nonempty if and only if* $\Sigma^p_{au}$ *is stabilizable by a static output feedback controller.*

2. Current estimator based controller :
   $\mathbf{T}^c_{\mathrm{R}}(\Sigma)$ *is nonempty if and only if* $\Sigma^c_{au}$ *is stabilizable by a static output feedback controller.*

3. Reduced order estimator based controller :
   $\mathbf{T}^r_{\mathrm{R}}(\Sigma)$ *is nonempty if and only if* $\Sigma^r_{au}$ *is stabilizable by a static output feedback controller.*

**Proof :** It follows from the results of [3]. ∎

# VI. NUMERICAL EXAMPLE

In this section we apply the above discrete-time CLTR method to the development of a tip position control system for a planar flexible one-link robot arm. Typical performance objectives for such a system would include (a) the minimum bandwidth for tracking of the tip position command signal; (b) the maximum tip position overshoot, the maximum excitation of any link vibration mode, and the maximum control torque in response to a tip position step-change command; (c) zero steady-state tip position error; and (d) the maximum time constant for the decay to zero steady-state tip position error in response to a constant disturbance torque at the motor shaft. A typical stability robustness requirement would be that the closed-loop system remain stable for a specified range of frequencies in each of the link vibration modes.

The essence of this problem is represented via a fourth-order plant dynamics of the two rigid links connected by a spring and a damper as shown in Figure 2. The variable $\tau$ is the motor torque, $d$ is the disturbance torque at the motor shaft, $q$ is the tip position of the arm, and the flexibility in the link is modeled by the pin joint, rotational spring, and rotational damper at the midpoint of the link. With the summation of the tip position, $\sum q$, added to the plant state vector to achieve the required integral control action, a state-feedback control law was designed for this system. The resulting closed-loop system is represented in Figure 3.

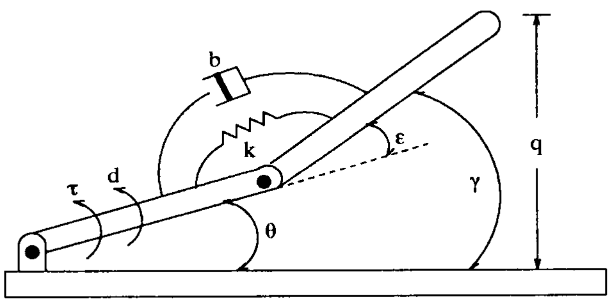



Figure 2: Flexible one-link robot arm

Consider the specific case where the link length is 1 $m$, the link mass is 1 $kg$, the rotational spring and damper values are selected to achieve

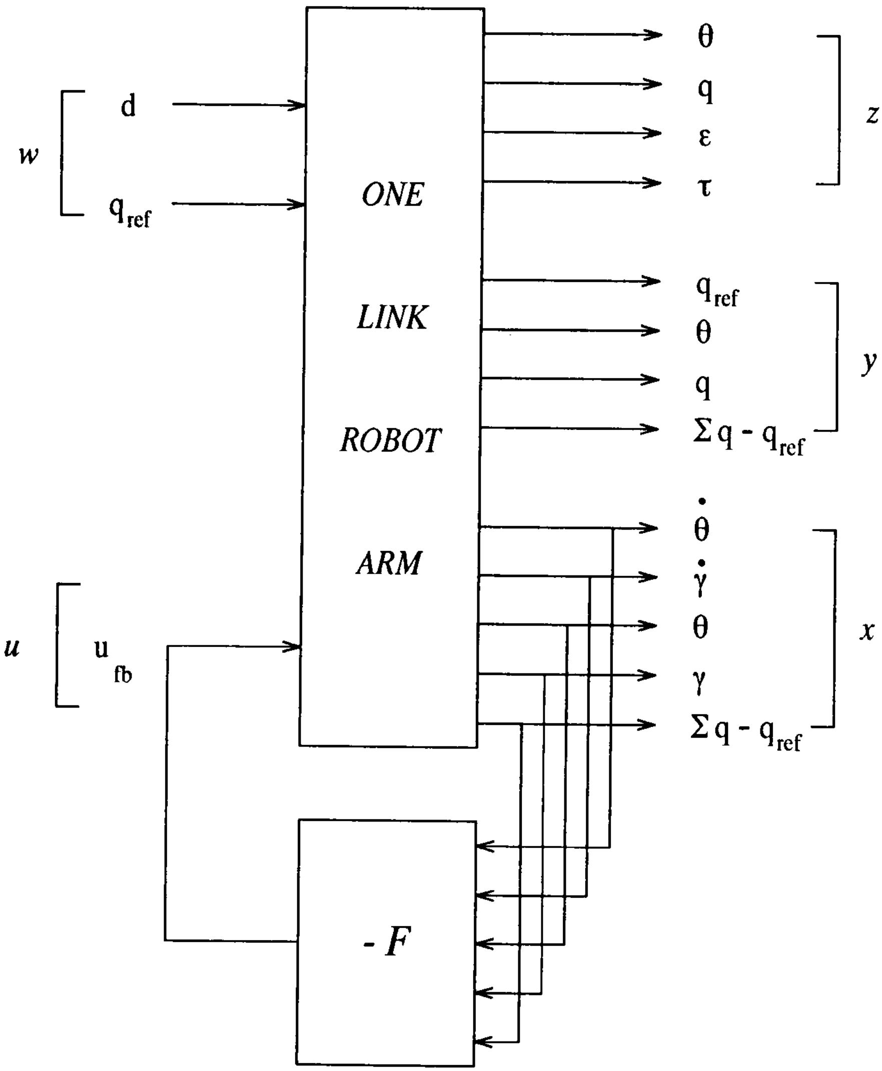



**Figure 3: Full state feedback system**

an open-loop vibration frequency of $20\pi$ rad/sec and 0.01 damping. With a sampling period of 0.02 sec, a plant state model corresponding to small perturbations in the variables $q$ and $\epsilon$ is given in the notation of (1) by,

$$A = \begin{bmatrix} 0.77984 & 0.22016 & -18.447 & 18.447 & 0 \\ 0.48435 & 0.51565 & 40.582 & -40.582 & 0 \\ 0.018420 & 0.0015797 & 0.78573 & 0.21427 & 0 \\ 0.0034752 & 0.016525 & 0.47138 & 0.5.2862 & 0 \\ 0 & 0 & 0.5 & 0.5 & 1 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 0.22013 & 1.5097 \\ -0.29228 & -2.0046 \\ 0.0024599 & 0.016871 \\ -0.003.4919 & -0.023949 \\ 0 & -1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0.22013 \\ -0.29228 \\ 0.0024599 \\ -0.003.4919 \\ 0 \end{bmatrix}$$

$$C_1 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0.5 & 0.5 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad D_{11} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 6.8584 \end{bmatrix}, \quad D_{12} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix},$$

$$C_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0.5 & 0.5 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad D_{21} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad D_{22} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

where the disturbance inputs $w$, the control input $u$, the controlled outputs $z$, the measurement outputs $y$, and the state vector $x$ are as defined in Figure 3.

A standard LQR design with weights of $10^2/\mathrm{m}^2$ on $q^2$, $10^4/\mathrm{rad}^2$ on $\epsilon^2$, $10^2/(\mathrm{m\text{-}sec})$ on $\sum q$, and $1/(N \cdot m)$ on $\tau$ yields the following state-feedback gain matrix

$$F = [2.275745 \quad -0.251072 \quad -7.050255 \quad 13.908664 \quad 0.058912]. \tag{44}$$

Here for the given data, $(A, B_2)$ stabilizable and $(A, C_2)$ is detectable. Furthermore, it is simple to verify that $(A, B_2, C_1, D_{12})$ is left invertible, and $(A, B_1, C_2, D_{21})$ is left invertible and of minimum phase with one infinite zero of order 1. Hence, by Theorem 4, any arbitrary admissible target loop for this system, and certainly the target loop specified by the state feedback gain matrix given in (44), is recoverable using a current estimator based controller and a reduced-order estimator based controller. However, it is not recoverable using a prediction estimator based controller.

Design methods for discrete-time CLTR are similar to those developed in [4]. A current estimator determined by the ATEA design method that achieves exact CLTR for this system is given by

$$\begin{cases} v(k+1) = A_{cmp,c}\, v(k) + B_{cmp,c}\, y(k), \\ -u(k) \;\; = C_{cmp,c}\, v(k) + D_{cmp,c}\, y(k), \end{cases}$$

where

$$A_{cmp,c} = \begin{bmatrix} -0.602207 & 0.247502 & -74.494142 & 6.758923 & 0.0 \\ 2.319394 & 0.479345 & 115.000852 & -25.063878 & 0.0 \\ 0.002976 & 0.001885 & 0.159394 & 0.083655 & 0.0 \\ 0.025399 & 0.016091 & 1.360471 & 0.714017 & 0.0 \\ 0.000000 & 0.000000 & 0.250000 & 0.250000 & 0.5 \end{bmatrix},$$

$$B_{cmp,c} = \begin{bmatrix} 0.0 & -160.329773 & 33.628711 & 0.00 \\ 0.0 & 288.483297 & -60.508546 & 0.00 \\ 0.0 & 0.079440 & -0.016662 & 0.00 \\ 0.0 & 0.678041 & -0.142217 & 0.00 \\ -0.5 & -0.100454 & 0.021070 & 0.25 \end{bmatrix},$$

$$C_{cmp,c} = [2.275745 \quad -0.251072 \quad -7.050255 \quad 13.908664 \quad 0.058912],$$

$$D_{cmp,c} = [0 \quad 197.971158 \quad -41.523884 \quad 0.029456].$$

Similarly, a reduced-order controller design that achieves exact CLTR for this system, is given by

$$\begin{cases} v(k+1) = A_{cmp,r}\, v(k) + B_{cmp,r}\, y(k), \\ -u(k) \;\; = C_{cmp,r}\, v(k) + D_{cmp,r}\, y(k), \end{cases}$$

where

$$A_{cmp,r} = \begin{bmatrix} -0.602207 & 0.247502 \\ 2.319394 & 0.479352 \end{bmatrix},$$

$$B_{cmp,r} = \begin{bmatrix} 0 & -161.037926 & 30.252493 & 0 \\ 0 & 284.314532 & -80.383740 & 0 \end{bmatrix},$$

$$C_{cmp,r} = [2.275745 \quad -0.251072],$$

$$D_{cmp,r} = [0 \quad 202.676685 \quad -19.090896 \quad 0.058912]. \tag{45}$$

Finally, a gain matrix for a prediction estimator that achieves some certain degree of CLTR for this system is

$$K_p = \begin{bmatrix} 1.509713 & -0.963299 & 0.202049 & 0 \\ -2.004555 & 76.880887 & -16.125546 & 0 \\ 0.016871 & 1.860387 & -0.390211 & 0 \\ -0.023949 & -1.850003 & 0.388033 & 0 \\ -1.000000 & -0.200908 & 0.042140 & 1 \end{bmatrix}. \tag{46}$$

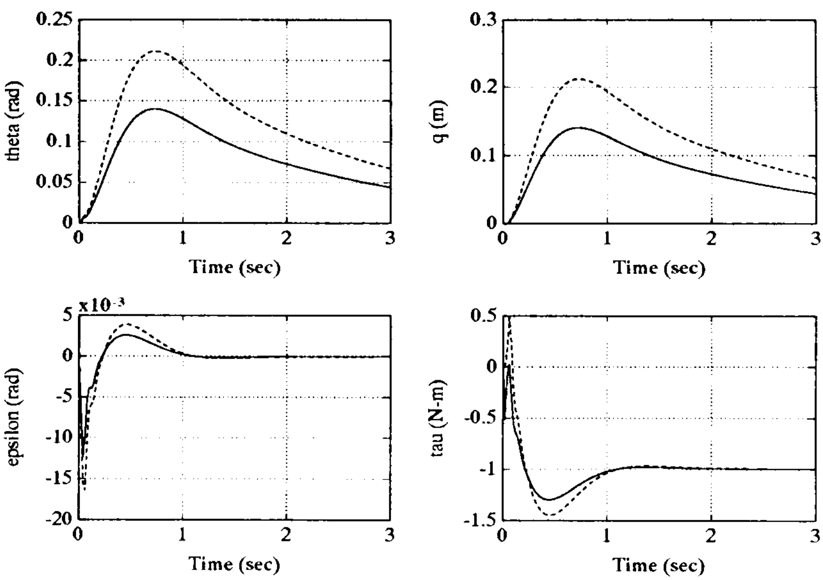
Simulated closed-loop responses to a unit torque disturbance ($d$-step) and to a unit tip position command ($q_{ref}$-step) with the full-state feedback design are compared with those achieved under the current estimator, reduced-order estimator, and prediction estimator based controllers in Figure 4. As expected, only in the case of the prediction estimator based controller do the responses differ from those of the full-state feedback design. Even with the prediction estimator based controller, the differences are only in the responses to the torque disturbance.

The maximum singular values of the closed-loop recovery errors for all the three designs are plotted in Figure 5. As expected, only the prediction estimator based controller yields a non-zero recovery error.

Finally, to compare the stability robustness of the three controllers to variation in the robot arm's open-loop vibration frequency, Figure 6 shows the maximum percent of variations in the vibration frequency for stability due to changes in the arm rotational spring constant $k$ with the above three observer based controllers. Note that the full-state feedback system is stable for all (positive) $k$ values. In particular, Figure 6 indicates substantial stability robustness achieved under the reduced-order estimator based controller.

## VII. CONCLUSION

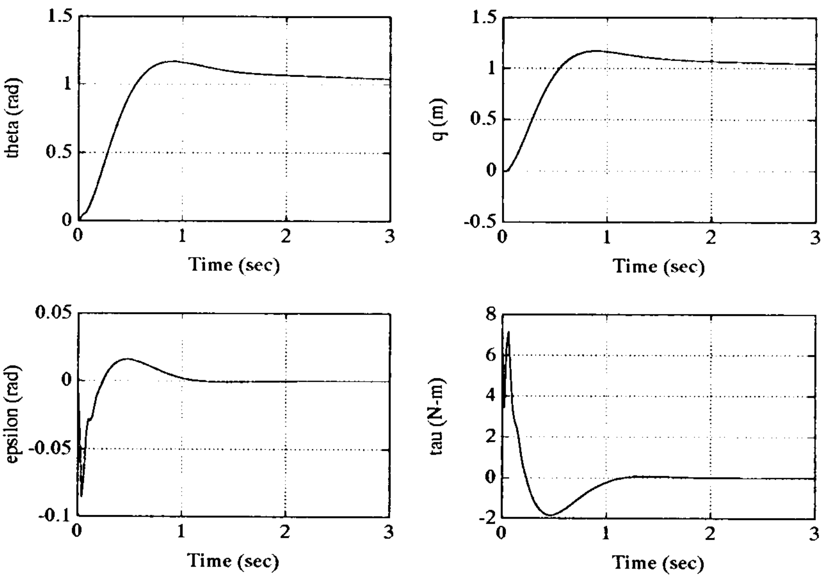
Presented in this paper is the complete analysis of closed-loop transfer recovery in discrete-time systems using observer-based controllers. Fundamental results are based on the structural properties of the system as they influence the recoverability of the target closed-loop transfer function. Conditions for recoverability have been developed pinpointing exactly when a system is recoverable for an arbitrary target loop transfer function and also specific conditions under which a given target loop transfer function




——— State feedback; Current & reduced order estimator based;

- - - - Prediction estimator based controller

**a) Responses to a unit step in $d$.**




——— State feedback & all three estimator based controllers

**b) Responses to a unit step in $q_{ref}$.**

**Figure 4: Robot arm responses to step inputs in $d$ and $q_{ref}$.**

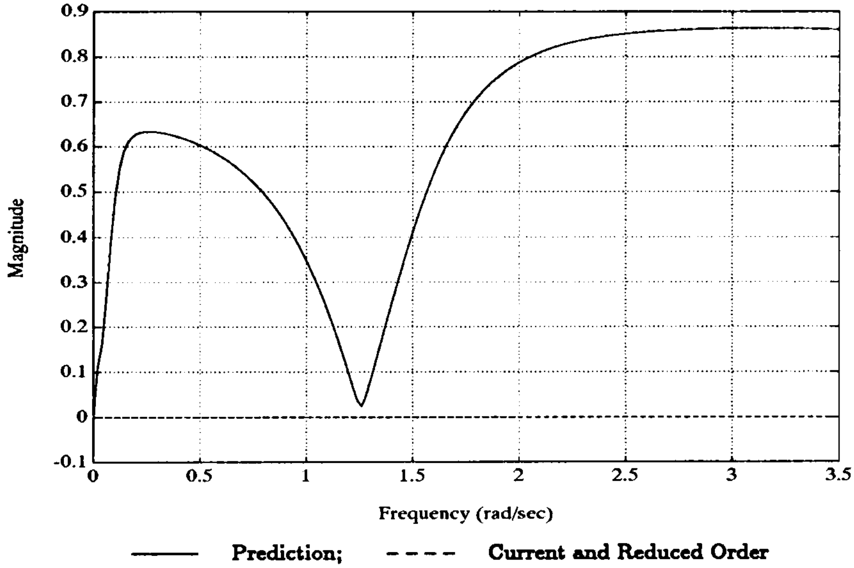



**Figure 5: Maximum singular values of the closed-loop recovery errors.**

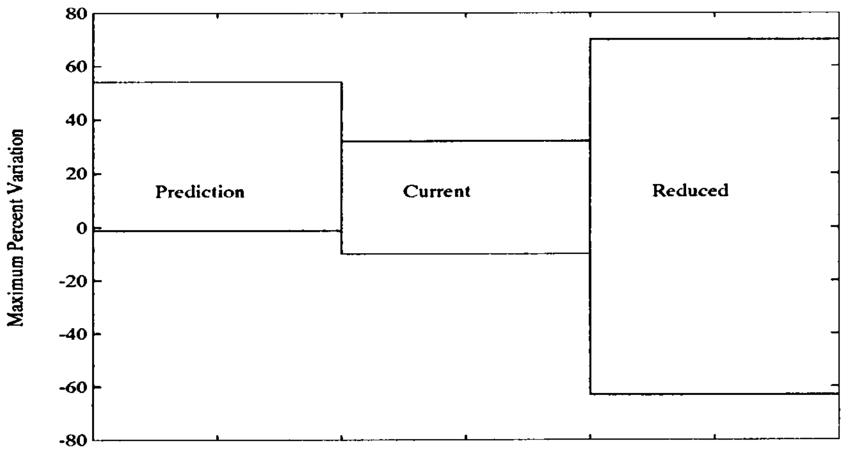



**Figure 6: Robustness to variations in robot arm vibration frequency.**

**is recoverable. Finally, results of the CLTR analysis has been applied and demonstrated in the design of a pointing control system for a two-link robot arm.**

## Acknowledgement

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