Closed-Loop Transfer Recovery with Observer-Based Controllers

Part 1: Analysis

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I. INTRODUCTION

In feedback design many performance and robust stability objectives can be stated in the form of requirements placed on the maximum singular values of particular *closed-loop* transfer functions. The underlying idea of "loop shaping" is that the magnitude (or maximum singular value) of the *closed-loop* transfer function can be directly inferred from the singular value of a corresponding *open-loop* transfer function. The

CONTROL AND DYNAMIC SYSTEMS, VOL. 51

prominent design procedure under the terminology LQG/LTR [15] is one such design methodology in multivariable systems that is based on the concept of loop shaping. This design procedure is divided into two steps. The first step involves the design of a stabilizing state-feedback law that yields a loop transfer function satisfying the design specifications. The loop properties are usually described in relation to an *open-loop* system (e.g. for a loop transfer function broken at either the control or measurement paths). Such an open-loop transfer function defines the target loop shape. The second step is to match this target loop shape using an output-feedback design following a procedure called loop transfer recovery (LTR). This step involves the design of an output-feedback control law (typically an observer-based compensator) such that the resulting open-loop transfer function would have either exactly or approximately the same target loop shape as the one achieved under state feedback. In other words, the idea of LTR is to design a compensator to recover a specific open-loop transfer function.

In this paper we examine the idea of loop recovery from a different perspective. Namely, we develop the concept of recovery based on the closed-loop transfer function directly, as opposed to the open-loop transfer function found in the case of a traditional LTR design. The problem can be stated as follows. Suppose that one is able to synthesize a state-feedback law that yields satisfactory closed-loop performance. And let's define the closed-loop transfer function between the external input to the controlled output under state-feedback law to be the target closed-loop transfer function. Clearly from this definition, the closed-loop target transfer function is completely defined by the selection of a full-state feedback gain matrix. Now we would like to design an output-feedback control law with a closed-loop transfer function that matches either exactly or approximately the target closed-loop transfer function. In this respect, we are dealing with the problem of closed-loop transfer recovery (CLTR) instead of open-loop transfer recovery (LTR).

It should be pointed out that the procedure of CLTR can further be used as an effective tool in the design of multivariable control systems. For example, one can employ the procedure of CLTR in the synthesis of H_∞-norm-based control-laws. Namely, we start with a target closed-loop transfer function achieved in H_∞-optimization under state feedback. Then we proceed to the design of a compensator (with either a full-order, reduced-order, Luenberger or generalized observer-based structure) that recovers the desired target closed-loop transfer function.

Our study of the mechanism in CLTR is applicable to a general class of systems and aims at three important theoretical issues:

(a) characterization of the recovery error and the available freedom in the design of output-feedback control-laws for a given system and for an arbitrarily specified target closed-loop transfer function,

(b) development of necessary and/or sufficient conditions for a target closed-loop transfer function to be either exactly or asymptotically recoverable in a given system, and

(c) development of necessary and/or sufficient conditions on a given system such that it has at least one recoverable (either exactly or asymptotically) target closed-loop transfer function. These are some of the theoretical issues pertaining to the analysis of CLTR. Of course, one also needs to examine issues in CLTR that are related to systematic design algorithms for the recovery process.

This paper concerns mainly with the analysis of the CLTR mechanism. A sequel to this paper will address in details the design issues. The objective at hand is however to analyze methodically the mechanism of CLTR using an observer-based controller in its most general setting (i.e covering the cases of full-order, reduced-order and generalized observers). However, in order to limit the length of this paper, results are provided only for the full-order and reduced-order observer-based controllers. The basic methodology and tools used here are akin to those in [8], [3] and [4]. We would like to point out that the formulation of the controller structure can in many ways impact the recovery process. Identifying the appropriate controller structure for the recovery task remains a research topic for future investigation.

The paper is organized as follows. In section II, we define precisely the problem of closed-loop transfer recovery. Recognizing the importance of finite- and infinite-zero structure in the LTR problem, we recall in section III a special coordinate basis (s.c.b) of [12] and [10] that clearly displays the zero structure of a given system. Section IV deals with all the fundamental analyses of CLTR. In particular, subsection IV.A deals with the analysis of CLTR via full-order observer-based control-laws while in subsection IV.B we perform the same analysis for the case of reduced-order observer-based control-laws. In section V, we illustrate our results on a lateral autopilot design for a commercial transport airplane. Finally, we draw the conclusion of our work in section VI.

Throughout the paper, A' denotes the transpose of A, A^H denotes the conjugate transpose of A, I denotes an identity matrix while I_k denotes an identity matrix of dimension $k \times k$. $\lambda(A)$ and $\mathrm{Re}[\lambda(A)]$ denote respectively the set of eigenvalues and the set of real parts of eigenvalues of A. Similarly, $\sigma_{max}[A]$ and $\sigma_{min}[A]$ denote the maximum and minimum singular values of A respectively. $\mathrm{Ker}[V]$ and $\mathrm{Im}[V]$ denote respectively the kernel and the image of V. The open left-half and the closed right-half of the s-plane are denoted respectively by $\mathcal{C}^-$ and $\mathcal{C}^+$. Also, $\mathcal{R}_p$ denotes the subring of all proper rational functions of s while the set of matrices of dimension $\ell \times q$ whose elements belong to $\mathcal{R}_p$ is denoted by $\mathcal{M}^{\ell \times q}(\mathcal{R}_p)$.

II. PROBLEM STATEMENT

Let us consider a linear time-invariant system Σ,

$$\Sigma : \begin{cases} \dot{x} = A\,x + B_1\,w + B_2\,u, \\ z = C_1 x + D_{11} w + D_{12} u, \\ y = C_2 x + D_{21} w + D_{22} u, \end{cases} \tag{1}$$

where $x \in \Re^n$ is the state, $u \in \Re^m$ is the control input, $w \in \Re^k$ is the external signal or disturbance, $z \in \Re^\ell$ is the controlled output and $y \in \Re^p$ is the measurement output. For convenience, we also define Σ_{yw} to be the matrix quadruple (A, B_1, C_2, D_{21}) and Σ_{zu} for the matrix quadruple (A, B_2, C_1, D_{12}). Let us assume that the pair $(A,\ B_2)$ is stabilizable and the pair $(A,\ C_2)$ detectable. Without loss of generality, we also assume that $[C_1,\ D_{11},\ D_{12}]$, $[C_2,\ D_{21},\ D_{22}]$, $[B_1',\ D_{11}',\ D_{21}']'$ and $[B_2',\ D_{12}',\ D_{22}']'$ are of maximal ranks. Let F be a full-state feedback gain matrix such that under the state-feedback control

$$u = -Fx \tag{2}$$

(a) the closed-loop system is asymptotically stable, i.e. the eigenvalues of $A - B_2F$ lie in the left-half s-plane,

(b) the closed-loop transfer function from the disturbance w to the controlled output z, denoted by $T_{zw}(s)$, meets the given frequency dependent design specifications.

We also refer to $T_{zw}(s)$ as the target closed-loop transfer function given by

$$T_{zw}(s) = (C_1 - D_{12}F)(\Phi^{-1} + B_2F)^{-1}B_1 + D_{11} \tag{3}$$

where $\Phi = (sI_n - A)^{-1}$. Design of the appropriate full-state feedback gain matrix F can be done, for example, via H_2-, H_∞-theory or eigenstructure assignment. For design implementation, the next step in the design procedure is to recover the target closed-loop transfer function using only a measurement feedback controller. This is the problem of closed-loop transfer recovery (CLTR) and the focus of this paper.

The problem can be clearly stated using the configuration shown in figure 1 where $P(s)$ represents the transfer function matrix of the system Σ and $\mathbf{C}(s)$ the transfer function of an output-feedback controller. For a given $P(s)$ and a target closed-loop transfer function $T_{zw}(s)$ in (3), the problem is to find an internally stabilizing controller $\mathbf{C}(s)$ such that the recovery error defined as

$$E(s) := T_{zw}^o(s) - T_{zw}(s) \tag{4}$$

is either exactly or approximately equal to zero in the frequency region of interest. Here, $T_{zw}^o(s)$ represents the transfer function from w to z for the closed-loop system shown in figure 1. As we shall see, achieving

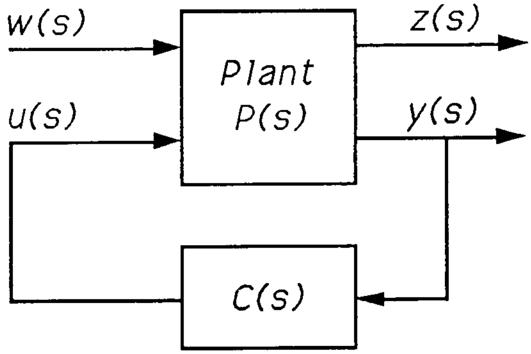

Figure 1: Plant with an Output-Feedback Controller.

exact closed-loop transfer recovery (ECLTR) is in general not possible. Hence, it is more appropriate to examine situation where approximate recovery can be achieved. An approximate CLTR is tied to the notion that recovery can be achieved to any degree of accuracy. In this process, one normally parameterizes the controller $\mathbf{C}(s)$ as a function of a positive scalar parameter σ thereby generating a family of controllers $\mathbf{C}(s,\sigma)$. We say asymptotic CLTR (ACLTR) is achieved if

$$T^o_{zw}(s,\sigma) \rightarrow T_{zw}(s)$$

as $\sigma \rightarrow \infty$ pointwise in s, or equivalently

$$E(s,\sigma) \rightarrow 0$$

as $\sigma \rightarrow \infty$ pointwise in s. From the point of view of design, once the conditions of ACLTR have been verified, a controller $\mathbf{C}(s,\sigma)$ with a particular value of σ can be found that will produce the desired level of recovery. Before we proceed to the analysis of CLTR, we need to provide precise meanings to the terminologies ECLTR and ACLTR. The following are definitions that characterize precisely the notions of ECLTR and ACLTR.

Definition 1. *The set of admissible target closed-loop transfer functions* $\mathbf{T}(\Sigma)$ *for the plant* Σ *is defined by*

$$\mathbf{T}(\Sigma) = \{T_{zw}(s) \in \mathcal{M}^{l\times k}(\mathcal{R}_p) \,|\, T_{zw}(s) \text{ is as defined in (3) and } \lambda(A - B_2F) \in \mathcal{C}^-\}.$$

Definition 2. *$T_{zw}(s) \in \mathbf{T}(\Sigma)$ is said to be exactly recoverable (ECLTR) if there exists a $\mathbf{C}(s) \in \mathcal{M}^{m \times p}(\mathcal{R}_p)$ such that*

(i) the closed-loop system comprising of $\mathbf{C}(s)$ and Σ as in (1) is asymptotically stable,

(ii) $T^o_{zw}(s) = T_{zw}(s)$.

Definition 3. *$T_{zw}(s) \in \mathbf{T}(\Sigma)$ is said to be asymptotically recoverable (ACLTR) if there exists a parameterized family of controllers $\mathbf{C}(s, \sigma) \in \mathcal{M}^{m \times p}(\mathcal{R}_p)$ where σ is a scalar parameter with positive values such that*

(i) the closed-loop system comprising of $\mathbf{C}(s, \sigma)$ and Σ as in (1) is asymptotically stable for all $\sigma > \sigma^*$ where $0 \leq \sigma^* < \infty$,

(ii) $T^o_{zw}(s, \sigma) \to T_{zw}(s)$ pointwise in s as $\sigma \to \infty$. Moreover, in the limits as $\sigma \to \infty$ the finite eigenvalues of the closed-loop system should remain in $\mathcal{C}^-$.[1]

Definition 4. *$T_{zw}(s)$ belonging to $\mathbf{T}(\Sigma)$ is said to be recoverable if $T_{zw}(s)$ is either exactly or asymptotically recoverable.*

Definition 5.

1. *The set of exactly recoverable target closed-loop transfer functions for the system Σ is denoted by $\mathbf{T}_{\mathrm{ER}}(\Sigma)$.*

2. *The set of recoverable (either exactly or asymptotically) target closed-loop transfer functions for the system Σ is denoted by $\mathbf{T}_{\mathrm{R}}(\Sigma)$.*

3. *The set of target closed-loop transfer functions which are only asymptotically recoverable but not exactly recoverable for the system Σ is denoted by $\mathbf{T}_{\mathrm{AR}}(\Sigma)$.*

Obviously, $\mathbf{T}_{\mathrm{R}}(\Sigma) = \mathbf{T}_{\mathrm{ER}}(\Sigma) \cup \mathbf{T}_{\mathrm{AR}}(\Sigma)$.

Remark 1. *The control-law $\mathbf{C}(s)$ in the above definitions is not restricted to any particular structure. However, in this paper we study*

[1] Here we have strengthened the notion of closed-loop stability by excluding those cases where, in the limits as $\sigma \to \infty$, some finite eigenvalues of the closed-loop system would be on the $j\omega$ axis. This avoids the problem of having an almost unstable behavior of the closed-loop system for large σ.

the closed-loop transfer recovery for two specific structures of $\mathbf{C}(s)$*; namely, full-order and reduced-order observer-based controllers. Furthermore, we label*

$$\{\mathbf{T}_{\mathrm{R}}(\Sigma),\ \mathbf{T}_{\mathrm{ER}}(\Sigma),\ \mathbf{T}_{\mathrm{AR}}(\Sigma)\}$$

with superscript f *(for full-order) and* r *(for reduced-order) as*

$$\{\mathbf{T}^f_{\mathrm{R}}(\Sigma),\ \mathbf{T}^f_{\mathrm{ER}}(\Sigma),\ \mathbf{T}^f_{\mathrm{AR}}(\Sigma)\}$$

and

$$\{\mathbf{T}^r_{\mathrm{R}}(\Sigma),\ \mathbf{T}^r_{\mathrm{ER}}(\Sigma),\ \mathbf{T}^r_{\mathrm{AR}}(\Sigma)\}$$

to signify results related to these particular controller structures.

The analysis of CLTR mechanism carried out here examines three fundamental issues. The first issue concerns with what can and what cannot be achieved for a given system and for an arbitrarily specified target closed-loop transfer function. For a given system, the second issue is to establish necessary and/or sufficient conditions on the target closed-loop transfer function so that it can be either exactly or asymptotically recovered. In another word, we characterize completely the set $\mathbf{T}_{\mathrm{R}}(\Sigma)$ of recoverable target closed-loop transfer functions. The third issue is to establish necessary and/or sufficient conditions on a given system such that it has at least one recoverable (either exactly or asymptotically) target closed-loop transfer function. That is, what are the conditions on a given system Σ so that the set of recoverable target closed-loop transfer $\mathbf{T}_{\mathrm{R}}(\Sigma)$ is nonempty?

III. PRELIMINARIES

We recall in this section a special coordinate basis (s.c.b) of a linear time-invariant system [12], [10]. Such a s.c.b has a distinct feature of explicitly displaying the finite and infinite zero structure of a given system and will play a very important role in both the analysis and the design of closed-loop transfer recovery. Consider the system characterized by

$$\tilde{\Sigma}: \begin{cases} \dot{\tilde{x}} = A\tilde{x} + B\tilde{u}, \\ \tilde{y} = C\tilde{x} + D\tilde{u}, \end{cases} \tag{5}$$

where $\tilde{x} \in \Re^n$, $\tilde{u} \in \Re^m$ and $\tilde{y} \in \Re^p$. Without loss of generality, we assume that matrices $[C, D]$ and $[B', D']'$ are of maximal rank. It is simple to verify that there exist non-singular transformations U and V such that

$$UDV = \begin{bmatrix} I_{m_0} & 0 \\ 0 & 0 \end{bmatrix}, \tag{6}$$

where m_0 is the rank of matrix D. Hence, hereafter and without loss of generality, it is assumed that matrix D has the form given on the right-hand side of (6). One can now rewrite the system of (5) as

$$\begin{cases} \dot{\tilde{x}} = A\tilde{x} + [B_0 \quad B_1] \begin{bmatrix} \tilde{u}_0 \\ \tilde{u}_1 \end{bmatrix}, \\ \begin{bmatrix} \tilde{y}_0 \\ \tilde{y}_1 \end{bmatrix} = \begin{bmatrix} C_0 \\ C_1 \end{bmatrix} \tilde{x} + \begin{bmatrix} I_{m_0} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{u}_0 \\ \tilde{u}_1 \end{bmatrix}, \end{cases} \tag{7}$$

where the matrices B_0, B_1, C_0 and C_1 have appropriate dimensions. We have the following theorem.

Theorem 1 (s.c.b). *Consider the system $\tilde{\Sigma}$ characterized by (A, B, C, D). There exist nonsingular transformations Γ_1, Γ_2 and Γ_3, an integer $m_f \leq m - m_0$ and integer indexes q_i, $i = 1$ to m_f, such that*

$$\tilde{x} = \Gamma_1 x \quad , \quad \tilde{y} = \Gamma_2 y \quad , \quad \tilde{u} = \Gamma_3 u$$

$$x = [x_a', \, x_b', \, x_c', \, x_f']' \quad , \quad x_a = [(x_a^-)', (x_a^+)']'$$

$$x_f = [x_{f_1}', \, x_{f_2}', \, \cdots, \, x_{f_{m_f}}']'$$

$$y = [y_0', \, y_f', \, y_b']' \quad , \quad y_f = [y_1, \, y_2, \, \cdots, y_{m_f}]'$$

$$u = [u_0', \, u_f', \, u_c']' \quad , \quad u_f = [u_1, \, u_2, \, \cdots, u_{m_f}]',$$

we have the following system of equations:

$$\dot{x}_a^- = A_{aa}^- x_a^- + B_{a0}^- y_0 + L_{af}^- y_f + L_{ab}^- y_b \tag{8}$$

$$\dot{x}_a^+ = A_{aa}^+ x_a^+ + B_{a0}^+ y_0 + L_{af}^+ y_f + L_{ab}^+ y_b \tag{9}$$

$$\dot{x}_b = A_{bb}x_b + B_{b0}y_0 + L_{bf}y_f \quad , \quad y_b = C_b x_b \tag{10}$$

$$\dot{x}_c = A_{cc}x_c + B_{c0}y_0 + L_{cb}y_b + L_{cf}y_f + B_c[E_{ca}^- x_a^- + E_{ca}^+ x_a^+] + B_c u_c \tag{11}$$

$$y_0 = C_{0a}^- x_a^- + C_{0a}^+ x_a^+ + C_{0b}x_b + C_{0c}x_c + C_{0f}x_f + u_0 \tag{12}$$

and for each $i = 1$ to m_f,

$$\dot{x}_{f_i} = A_{f_i}x_{f_i} + L_{i0}y_0 + L_{if}y_f + B_{f_i}[u_i + E_{ia}x_a + E_{ib}x_b + E_{ic}x_c + \sum_{j=1}^{m_f} E_{ij}x_{f_j}] \tag{13}$$

$$y_i = C_{f_i}x_{f_i} \quad , \quad y_f = C_f x_f . \tag{14}$$

Here the states x_a^-, x_a^+, x_b, x_c and x_f are respectively of dimension n_a^-, n_a^+, n_b, n_c and

$$n_f = \sum_{i=1}^{m_f} q_i$$

while x_i is of dimension q_i for each $i = 1$ to m_f. The control vectors u_0, u_f and u_c are respectively of dimension m_0, m_f and $m_c = m - m_0 - m_f$ while the output vectors y_0, y_f and y_b are respectively of dimension $p_0 = m_0$, $p_f = m_f$ and $p_b = p - p_0 - p_f$. The matrices A_{f_i}, B_{f_i} and C_{f_i} have the following form:

$$A_{f_i} = \begin{bmatrix} 0 & I_{q_i-1} \\ 0 & 0 \end{bmatrix}, \quad B_{f_i} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C_{f_i} = [1, \ 0, \ \cdots, \ 0]. \tag{15}$$

(Obviously for the case when $q_i = 1$, $A_{f_i} = 0$, $B_{f_i} = 1$ and $C_{f_i} = 1$.) Furthermore, we have $\lambda(A_{aa}^-) \in \mathcal{C}^-$, $\lambda(A_{aa}^+) \in \mathcal{C}^+$, the pair (A_{cc}, B_c) is controllable and the pair (A_{bb}, C_b) is observable. Also, assuming that x_i are arranged such that $q_i \leq q_{i+1}$, the matrix L_{if} has the particular form,

$$L_{if} = [L_{i1}, L_{i2}, \cdots, L_{i\,i-1}, 0, 0, \cdots, 0].$$

Also, the last row of each L_{if} is identically zero.

Proof : This follows from theorem 2.1 of [12] and [10]. ∎

We can rewrite the s.c.b given by theorem 1 in a more compact form,

$$\Gamma_1^{-1}(A - B_0C_0)\Gamma_1 = \begin{bmatrix} A_{aa}^- & 0 & L_{ab}^-C_b & 0 & L_{af}^-C_f \\ 0 & A_{aa}^+ & L_{ab}^+C_b & 0 & L_{af}^+C_f \\ 0 & 0 & A_{bb} & 0 & L_{bf}C_f \\ B_cE_{ca}^- & B_cE_{ca}^+ & L_{cb}C_b & A_{cc} & L_{cf}C_f \\ B_fE_a^- & B_fE_a^+ & B_fE_b & B_fE_c & A_f \end{bmatrix},$$

$$\Gamma_1^{-1}\,[B_0 \quad B_1\,]\,\Gamma_3 = \begin{bmatrix} B_{a0}^- & 0 & 0 \\ B_{a0}^+ & 0 & 0 \\ B_{b0} & 0 & 0 \\ B_{c0} & 0 & B_c \\ B_{f0} & B_f & 0 \end{bmatrix},$$

$$\Gamma_2^{-1}\begin{bmatrix} C_0 \\ \\ C_1 \end{bmatrix}\Gamma_1 = \begin{bmatrix} C_{0a}^- & C_{0a}^+ & C_{0b} & C_{0c} & C_{0f} \\ 0 & 0 & 0 & 0 & C_f \\ 0 & 0 & C_b & 0 & 0 \end{bmatrix}$$

and

$$\Gamma_2^{-1}D\Gamma_3 = \begin{bmatrix} I_{m_0} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

In what follows, we state some important properties of the s.c.b which are pertinent to our present work.

Property 1. *The given system $\tilde{\Sigma}$ is right-invertible if and only if x_b and hence y_b are nonexistent ($n_b = 0$, $p_b = 0$), left-invertible if and only if x_c and hence u_c are nonexistent ($n_c = 0$, $m_c = 0$), invertible if and only if both x_b and x_c are nonexistent. Moreover, $\tilde{\Sigma}$ is degenerate if and only if it is neither left- nor right-invertible.*

Property 2. *We note that (A_{bb}, C_b) and (A_{f_i}, C_{f_i}) form observable pairs. Unobservability could arise only in the variables x_a and x_c. In fact, the system $\tilde{\Sigma}$ is observable (detectable) if and only if (A_{obs}, C_{obs}) is an observable (detectable) pair, where*

$$A_{obs} = \begin{bmatrix} A_{aa} & 0 \\ B_cE_{ca} & A_{cc} \end{bmatrix}, \quad C_{obs} = \begin{bmatrix} C_{a0} & C_{c0} \\ E_a & E_c \end{bmatrix},$$

$$A_{aa} = \begin{bmatrix} A_{aa}^- & 0 \\ 0 & A_{aa}^+ \end{bmatrix},$$

$$C_{a0} = [C_{a0}^-,\ C_{a0}^+],\quad E_a = [E_a^-,\ E_a^+],\quad E_{ca} = [E_{ca}^-,\ E_{ca}^+].$$

Similarly, $(A_{cc},\ B_c)$ and $(A_{f_i},\ B_{f_i})$ form controllable pairs. Uncontrollability could arise only in the variables x_a and x_b. In fact, $\tilde{\Sigma}$ is controllable (stabilizable) if and only if $(A_{con},\ B_{con})$ is a controllable (stabilizable) pair, where

$$A_{con} = \begin{bmatrix} A_{aa} & L_{ab}C_b \\ 0 & A_{bb} \end{bmatrix},\quad B_{con} = \begin{bmatrix} B_{a0} & L_{af} \\ B_{b0} & L_{bf} \end{bmatrix},$$

$$B_{a0} = \begin{bmatrix} B_{a0}^- \\ B_{a0}^+ \end{bmatrix},\quad L_{ab} = \begin{bmatrix} L_{ab}^- \\ L_{ab}^+ \end{bmatrix},\quad L_{af} = \begin{bmatrix} L_{af}^- \\ L_{af}^+ \end{bmatrix}.$$

Property 3. *Invariant zeros of $\tilde{\Sigma}$ are the eigenvalues of A_{aa}. Moreover, the stable and the unstable invariant zeros of $\tilde{\Sigma}$ are the eigenvalues of A_{aa}^- and A_{aa}^+, respectively.*

There are interconnections between the s.c.b and various invariant and almost invariant geometric subspaces. To show these interconnections, we define

- $\mathcal{V}^g(A,\ B,\ C,\ D)$ — the maximal subspace of $\Re^n$ which is $(A + BF)$–invariant and contained in $\mathrm{Ker}(C + DF)$ such that the eigenvalues of $(A + BF)|\mathcal{V}^g$ are contained in $\mathcal{C}_g \subseteq \mathcal{C}$ for some F.

- $\mathcal{S}^g(A,\ B,\ C,\ D)$ — the minimal $(A + KC)$–invariant subspace of $\Re^n$ containing in $\mathrm{Im}(B + KD)$ such that the eigenvalues of the map which is induced by $(A + KC)$ on the factor space $\Re^n/\mathcal{S}^g$ are contained in $\mathcal{C}_g \subseteq \mathcal{C}$ for some K.

For the cases that $\mathcal{C}_g = \mathcal{C}$, $\mathcal{C}_g = \mathcal{C}^-$ and $\mathcal{C}_g = \mathcal{C}^+$, we replace the index g in $\mathcal{V}^g$ and $\mathcal{S}^g$ by '$*$', '$-$' and '$+$', respectively. Various components of the state vector of s.c.b have the following geometrical interpretations.

Property 4.

1. $x_a^- \oplus x_a^+ \oplus x_c$ spans $\mathcal{V}^*(A,\ B,\ C,\ D)$.
2. $x_a^- \oplus x_c$ spans $\mathcal{V}^-(A,\ B,\ C,\ D)$.
3. $x_a^+ \oplus x_c$ spans $\mathcal{V}^+(A,\ B,\ C,\ D)$.

4. $x_c \oplus x_f$ *spans* $\mathcal{S}^*(A, B, C, D)$.
5. $x_a^- \oplus x_c \oplus x_f$ *spans* $\mathcal{S}^+(A, B, C, D)$.
6. $x_a^+ \oplus x_c \oplus x_f$ *spans* $\mathcal{S}^-(A, B, C, D)$.

IV. GENERAL ANALYSIS

In this section, we deal with the general analysis of closed-loop transfer recovery in the cases of full- and reduced-order observer-based controllers. We will study three fundamental issues in closed-loop transfer recovery. The first issue is concerned with what can and what cannot be achieved for a given system and for an arbitrarily target closed-loop transfer function. The second issue deals with the development of necessary and/or sufficient conditions a target closed-loop transfer function must satisfy in order for it to be recovered either exactly or asymptotically. And the third issue is concerned with the development of necessary and/or sufficient conditions on a given system such that it has at least one (either exactly or asymptotically) recoverable target closed-loop transfer function.

To begin, let us consider Luenberger observer-based controllers which include as special cases the full-order and reduced-order observer-based controllers. Lemma 1 in this section gives an explicit expression for the recovery error function, between the target closed-loop transfer function and the one realized by a Luenberger observer-based controller. Lemma 2 expresses CLTR in terms of a transfer function matrix denoted here as the recovery matrix. This formulation plays a central role in the development of our results. Without loss of generality, we assume that $D_{22} = 0$ with justification given in Appendix A. Let us now consider the following Luenberger observer-based controller,

$$\begin{cases} \dot{v} = Lv + G_1 y + G_2 u, \\ \hat{x} = Pv + Jy, \\ u = -F\hat{x} \end{cases} \tag{16}$$

where $v \in \Re^r$ with r being the order of the controller and $\hat{x} \in \Re^n$. It is well known that, in the disturbance-free case (i.e. $w = 0$) the variable

$\hat{x}$ is an asymptotic estimate of the state x provided that the matrix L is a stability matrix and there exists a matrix $Q \in \Re^{r\times n}$ satisfying the following conditions:

$$QA - LQ = G_1 C_2, \qquad G_2 = QB_2, \qquad JC_2 + PQ = I_n. \tag{17}$$

Let $T^{\ell}_{zw}(s)$ denote the closed-loop transfer function from w to z with a general Luenberger observer-based controller. Then we have the following lemmas.

Lemma 1. *With the observer defined in (16) and (17), the recovery error function between the target closed-loop transfer $T_{zw}(s)$ and the one realized by Luenberger observer-based controller $T^{\ell}_{zw}(s)$ is given by*

$$E_\ell(s) = T^{\ell}_{zw}(s) - T_{zw}(s) = T_{zu}(s) \cdot M_\ell(s) \tag{18}$$

where

$$T_{zu}(s) = (C_1 - D_{12}F)(\Phi^{-1} + B_2 F)^{-1} B_2 + D_{12}, \tag{19}$$

is the closed-loop transfer function from u to z under state feedback and

$$M_\ell(s) = F[P(sI - L)^{-1}(QB_1 - G_1 D_{21}) - JD_{21}]. \tag{20}$$

Proof : The result follows directly after some simple algebra. ∎

Lemma 2. *Given that the system Σ_{zu} is left-invertible. Then*

1. *Exact recovery takes place (i.e. $E_\ell(j\omega) = 0\ \forall \omega \in \Re$) if and only if $M_\ell(j\omega) = 0\ \forall\omega \in \Re$.*

2. *Asymptotic recovery is achievable (i.e. $\|E_\ell(j\omega)\|$ can be made arbitrarily small for all $\omega \in \Re$) if and only if $\|M_\ell(j\omega)\|$ can be made arbitrarily small for all $\omega \in \Re$.*

Proof : It is obvious. ∎

Note that the conditions given in lemma 2 are not necessary for $E_\ell(s)$ to be zero or small if the system Σ_{zu} is not left-invertible. Nonetheless, they remain as sufficient conditions for the recovery error to be zero or small. To see this, let us examine the following example.

Example 1 : Consider a system characterized by

$$\dot{x} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} x + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} u + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w,$$

$$y = [\,0 \quad 1\,]\, x,$$

$$z = [\,0 \quad 1\,]\, x.$$

Let the target closed-loop transfer function be specified by

$$F = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

and let

$$L = \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix}, \quad G_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad G_2 = B_2, \quad P = Q = I_2 \text{ and } J = 0.$$

Then it is simple to verify that

$$T_{zu}(s) = [\,\tfrac{1}{s+1} \quad 0\,], \quad M_\ell(s) = \begin{bmatrix} 0 \\ \frac{1}{s+1} \end{bmatrix}.$$

and $E_\ell(s) = T_{zu}(s) M_\ell(s) = 0$. Hence exact recovery can take place even though $M_\ell(s) \neq 0$.

As seen from the above example when the system Σ_{zu} is not left-invertible, one may be able to find an observer-based controller such that $M_\ell(s)$ is nonzero and the recovery error $E_\ell(s)$ is equal to zero. However, this situation is fairly limited and may not generally be valid for other non-left-invertible systems. Thus, a general analysis in closed-loop transfer recovery entails a detailed study of the matrix $M_\ell(s)$ which depends on both the state-feedback matrix F and the observer parameters. Since the state-feedback gain matrix F is considered given, the only degree-of-freedom left for closed-loop transfer recovery is in the selection of the observer parameters. First of all, the observer parameters must be selected such that closed-loop stability of the observer-based control system is guaranteed. The remaining degrees-of-freedom in choosing the observer parameters are then used to achieve CLTR. That is, one uses the available freedom in the observer design to make the norm of $M_\ell(j\omega)$ either zero or small over the range of frequencies of interest. Due to the significance of the matrix $M_\ell(s)$ in CLTR, we refer $M_\ell(s)$ as the recovery matrix.

In the next subsections, we focus our attention to full-order and reduced-order observer-based controllers and perform a complete analysis of CLTR for each of these cases.

A. Full-Order Observer-Based Controller

In this subsection, we consider the problem of closed-loop transfer recovery using a full-order observer-based controller design. State-space description of a full-order observer-based controller is given by

$$\begin{cases} \dot{\hat{x}} = (A - KC_2)\hat{x} + B_2 u + Ky, \\ u = -F\hat{x}, \end{cases} \tag{21}$$

where the full-order observer gain matrix K is chosen such that $A - KC_2$ is asymptotically stable. The transfer function of this full-order observer-based controller is

$$-u(s) = \mathbf{C}(s)y(s),$$

where

$$\mathbf{C}(s) = F(sI_n - A + B_2F + KC_2)^{-1}K.$$

Note that the full-order observer-based controller is a special case of the Luenberger observer-based controller in (16) and (17) with

$$\begin{cases} L = A - KC_2, & G_1 = K, & G_2 = B_2, \\ P = I_n, & J = 0, & Q = I_n. \end{cases} \tag{22}$$

From lemma 1 it follows that the recovery error and the recovery matrix, denoted here by $E_f(s)$ and $M_f(s)$ respectively, are given by

$$E_f(s) = T_{zu}(s)M_f(s), \tag{23}$$

where $T_{zu}(s)$ is as defined in (19) and

$$M_f(s) = F(\Phi^{-1} + KC_2)^{-1}(B_1 - KD_{21}). \tag{24}$$

(The subscript ℓ in (18) and (20) is replaced by f to signify the specific case of a full-order observer-based controller.)

1. Analysis For Arbitrary Target Closed-Loop Transfer Functions

Study of equations (23) and (24) will provide a clear insight into the basic mechanism of CLTR. In fact, these equations indicate that examination of the recovery error $E_f(s)$ can be done generally in terms of the study of $M_f(s)$. Lemma 1 and the expression for $M_f(s)$ given in (24) will therefore form the basis of our investigation. Since the state-feedback gain matrix F is considered given, the only degree-of-freedom for closed-loop transfer recovery is in the selection of the observer gain K. First of all, in order to guarantee the overall closed-loop stability, K must be selected such that the observer-dynamic matrix

$$A_o = A - KC_2 \tag{25}$$

is asymptotically stable (i.e. $\lambda(A_o) \in \mathcal{C}^-$). The remaining degrees-of-freedom in choosing K can then be used for the purpose of achieving CLTR. Now in view of (24) and lemma 2, exact closed-loop transfer recovery (ECLTR) is possible for an *arbitrarily* given F if $\tilde{M}_f(j\omega) \equiv 0$ where $M_f(s) = F\tilde{M}_f(s)$ and

$$\tilde{M}_f(s) = (sI_n - A_o)^{-1}(B_1 - KD_{21}). \tag{26}$$

Since the matrix $(j\omega I_n - A_o)^{-1}$ is nonsingular, $\tilde{M}(j\omega) \equiv 0$ clearly implies that $B_1 - KD_{21} \equiv 0$. The class of systems in which $B_1 - KD_{21}$ can be rendered exactly zero is rather restrictive. Hence, one would most likely resort to an approach based on asymptotic closed-loop transfer recovery (ACLTR), i.e. to render $\tilde{M}_f(j\omega)$ approximately zero in some sense. As mentioned in section 2, to analyze whether ACLTR is possible we need to parameterize the controller with a tuning parameter σ. In the case of a full-order observer-based controller, this parameterization can be simply introduced in the observer gain matrix $K(\sigma)$. The resulting family of controllers parameterized by $K(\sigma)$ is

$$\mathbf{C}(s,\sigma) = F[sI_n - A + B_2F + K(\sigma)C_2]^{-1}K(\sigma). \tag{27}$$

In this formulation $M_f(s)$ and $\tilde{M}_f(s)$ are now functions of σ, denoted respectively by $M_f(s,\sigma)$ and $\tilde{M}_f(s,\sigma)$. In our analysis and for the sake of clarity, we assume that A_o is nondefective. This allows us to expand $\tilde{M}_f(s,\sigma)$ and hence $M_f(s,\sigma)$ in a dyadic form,

$$\tilde{M}_f(s,\sigma) = \sum_{i=1}^{n} \frac{\tilde{R}_i}{s-\lambda_i} \tag{28}$$

where the residue matrix $\tilde{R}_i$ is given by

$$\tilde{R}_i = W_i V_i^H [B_1 - K(\sigma)D_{21}]. \tag{29}$$

Here W_i and V_i are respectively the right- and left-eigenvectors associated with the eigenvalue λ_i of A_o and

$$WV^H = V^H W = I_n.$$

The matrices W and V can be partitioned as follows,

$$W = [W_1, W_2, \cdots, W_n] \quad \text{and} \quad V = [V_1, V_2, \cdots, V_n]. \tag{30}$$

In general, all λ_i, V_i and W_i are functions of σ. However for economy of notation, we will omit the dependence on σ explicitly unless it is needed for clarity.

In what follows, we examine conditions under which the i-th term in the dyadic expansion of $\tilde{M}_f(s,\sigma)$ in (28) can be made zero or small. There are basically two ways to achieve this:

1. The first possibility is to assign the closed-loop eigenvalue λ_i to any finite value in C^- while simultaneously rendering the residue $\tilde{R}_i$ zero either exactly or asymptotically, i.e.

$$\tilde{R}_i = W_i(\sigma)V_i^H(\sigma)[B_1 - K(\sigma)D_{21}] = 0$$

or

$$\tilde{R}_i(\sigma) = W_i(\sigma)V_i^H(\sigma)[B_1 - K(\sigma)D_{21}] \to 0$$

as $\sigma \to \infty$. This procedure involves a finite eigenstructure assignment of A_o.

2. The other possibility is to make

$$\frac{\tilde{R}_i}{s-\lambda_i} \to 0$$

pointwise in s as $\sigma \to \infty$ by placing the eigenvalue $\lambda_i(\sigma)$ asymptotically at infinity and making sure that the residue $\tilde{R}_i(\sigma)$ is uniformly bounded as $\sigma \to \infty$. It is important to recognize that placing λ_i asymptotically at infinity alone will not give the desired result unless the residue $\tilde{R}_i$ is also bounded. This amounts to assigning $W_i(\sigma)$ and $V_i(\sigma)$ such that

$$\tilde{R}_i(\sigma) = W_i(\sigma)V_i^H(\sigma)[B_1 - K(\sigma)D_{21}]$$

remains bounded while $| \lambda_i | \to \infty$ as $\sigma \to \infty$. Thus, this procedure deals with an infinite eigenstructure assignment of A_o.

Two fundamental questions immediately arise in the application of an eigenstructure assignment technique to this problem of closed-loop transfer recovery:

(1) How many left-eigenvectors of A_o can be assigned to the null space of $[B_1 - K(\sigma)D_{21}]'$? and,

(2) How many eigenvalues of A_o can be placed at asymptotically infinite locations in $\mathcal{C}^-$ and at the same time the residues associated with these eigenvalues are also finite?

The answers to these two questions are given in the following two lemmas.

In the analysis that follows, we shall apply the s.c.b transformation developed in section 3 to the system Σ_{yw}. Let n_a^- and n_a^+ be respectively the number of stable and unstable invariant zeros of Σ_{yw} and n_f the number of infinite zeros of Σ_{yw}. Moreover, we let

$$n_c := \dim\{\mathcal{V}^*(A, B_1, C_2, D_{21}) \cap \mathcal{S}^*(A, B_1, C_2, D_{21})\}$$

and

$$n_b := n - n_a^- - n_a^+ - n_c - n_f.$$

These integers (i.e. n_a^-, n_a^+, n_b, n_c and n_f) can be readily obtained from the s.c.b of Σ_{yw}.

Lemma 3. *For any given $K(\sigma)$ such that A_o is stable, let λ_i be an eigenvalue of A_o and V_i its corresponding left-eigenvector. Then the* *maximum number of $\lambda_i \in \mathcal{C}^-$ which satisfy the condition*

$$\tilde{R}_i = W_i V_i^H [B_1 - K(\sigma) D_{21}] = 0$$

is $(n_a^- + n_b)$. A total of n_a^- of these eigenvalues λ_i coincide with the stable invariant zeros of the system Σ_{yw} and the remaining n_b of these eigenvalues can be assigned arbitrarily to any location in $\mathcal{C}^-$. The eigenvectors V_i that correspond to these $(n_a^- + n_b)$ eigenvalues span the subspace

$$\Re^n / \mathcal{S}^-(A, B_1, C_2, D_{21}).$$

Moreover, n_a^- of these eigenvectors corresponding to the eigenvalues at the stable invariant zeros are simply the left-state zero directions and span the subspace

$$\mathcal{V}^*(A, B_1, C_2, D_{21}) / \mathcal{V}^+(A, B_1, C_2, D_{21}).$$

Proof : See [3].

Lemma 4. *For any given $K(\sigma)$ such that A_o is stable, let λ_i be an eigenvalue of A_o, V_i and W_i its corresponding left- and right-eigenvectors respectively. The maximum number of eigenvalues of A_o that can be assigned arbitrarily to asymptotically infinite locations in $\mathcal{C}^-$ while at the same time the residue matrix*

$$\tilde{R}_i = W_i V_i^H [B_1 - K(\sigma) D_{21}]$$

remain bounded as $|\lambda_i| \to \infty$ is $(n_b + n_f)$. Furthermore, the left-eigenvectors V_i of these asymptotically infinite eigenvalues span the subspace

$$\Re^n / \mathcal{V}^*(A, B_1, C_2, D_{21}).$$

Proof : See [3]. ∎

Remark 2. *Consider the case when Σ_{yw} is right-invertible and D_{21} is of maximal rank. Clearly this covers the special case where Σ_{yw} is a non-strictly proper single-input and single-output system. For this case, we have $n_b + n_f = 0$ and hence there is no eigenvalue λ_i of A_o that can be assigned to an infinite location and at the same time the corresponding residue matrix $\tilde{R}_i$ is bounded.*

As implied by lemma 3, there are n_b eigenvalues where one can assign arbitrarily to any locations in $\mathcal{C}^-$ and still maintain $\tilde{R}_i \equiv 0$. These n_b eigenvalues are therefore among the $(n_b + n_f)$ eigenvalues indicated in lemma 4. That is, there is a set of n_b eigenvalues which can be placed arbitrarily at either asymptotically finite locations in $\mathcal{C}^-$ according to lemma 3 or at asymptotically infinite locations in $\mathcal{C}^-$ according to lemma 4. For practical design considerations, such as limited controller bandwidth and sensor noise reduction, one often keeps these n_b eigenvalues at stable and reasonably finite locations.

Combining the results of lemmas 3 and 4, one can deduce all the conditions under which various terms of $\tilde{M}_f(s,\sigma)$ can be made zero either exactly or asymptotically. There are altogether $(n_a^- + n_b + n_f)$ eigenvalues which can be assigned either at finite or at asymptotically infinite locations so that the corresponding terms of $\tilde{M}_f(s,\sigma)$ in its dyadic expansion (28) are either exactly or asymptotically zero. Thus, a question arises as to under what conditions $(n_a^- + n_b + n_f)$ is equal to n the system dimension. It is easy to see that $n_a^- + n_b + n_f = n$ if and only if Σ_{yw} is left-invertible ($n_c = 0$) and of minimum phase[2] ($n_a^+ = 0$). If Σ_{yw} is not left-invertible and/or of nonminimum phase, then there are exactly $n_e \equiv n_a^+ + n_c$ terms of $\tilde{M}_f(s,\sigma)$ which cannot in general be rendered zero. To fully understand the behavior of $\tilde{M}_f(s,\sigma)$, we need to partition it into four parts,

$$\tilde{M}_f(s,\sigma) = \tilde{M}_-(s,\sigma) + \tilde{M}_b(s,\sigma) + \tilde{M}_\infty(s,\sigma) + \tilde{M}_e(s,\sigma), \tag{31}$$

where

$$\tilde{M}_-(s,\sigma) = \sum_{i=1}^{n_a^-} \frac{\tilde{R}_i}{s-\lambda_i},$$

$$\tilde{M}_b(s,\sigma) = \sum_{i=n_a^-+1}^{n_a^-+n_b} \frac{\tilde{R}_i}{s-\lambda_i},$$

$$\tilde{M}_\infty(s,\sigma) = \sum_{i=n_a^-+n_b+1}^{n_a^-+n_b+n_f} \frac{\tilde{R}_i}{s-\lambda_i},$$

[2] A system is said to be of nonminimum phase if at least one of its invariant zeros is in the closed right-half plane, otherwise it is said to be of minimum phase.

and

$$\tilde{M}_e(s,\sigma) = \sum_{i=n_a^-+n_b+n_f+1}^{n} \frac{\tilde{R}_i}{s-\lambda_i}.$$

Let $\Lambda_-(\sigma)$, $\Lambda_b(\sigma)$, $\Lambda_\infty(\sigma)$ and $\Lambda_e(\sigma)$ be the sets of eigenvalues of A_o associated with the parts $\tilde{M}_-(s,\sigma)$, $\tilde{M}_b(s,\sigma)$, $\tilde{M}_\infty(s,\sigma)$ and $\tilde{M}_e(s,\sigma)$ respectively . Corresponding to each of these partitions of eigenvalues, we define the associated right- and left-eigenvectors of A_o in the sets $W_-(\sigma)$, $W_b(\sigma)$, $W_\infty(\sigma)$, $W_e(\sigma)$ and $V_-(\sigma)$, $V_b(\sigma)$, $V_\infty(\sigma)$, $V_e(\sigma)$ respectively. Also as a convenient notation, we will use an overbar on a variable to denote its limit (whenever it exists) as $\sigma \to \infty$. For example, $\overline{\tilde{M}}_e(s)$ and $\overline{W}_e$ denote respectively the limits of $\tilde{M}_e(s,\sigma)$ and $W_e(\sigma)$ as $\sigma \to \infty$. Various parts of $\tilde{M}_f(s,\sigma)$ have now the following interpretation:

1. $\tilde{M}_-(s,\sigma)$ contains n_a^- terms with eigenvalues in the set $\Lambda_-(\sigma)$. In accordance with lemma 3, there exists a gain $K(\sigma)$ such that $\tilde{M}_-(s,\sigma)$ is identically zero by placing elements of $\Lambda_-(\sigma)$ at the stable invariant zeros of Σ_{yw} and the set of left-eigenvectors $V_-(\sigma)$ to be the left-state zero directions. In fact, $K(\sigma)$ can be selected such that $\Lambda_-(\sigma)$ and $V_-(\sigma)$ approach asymptotically the set of system stable invariant zeros and their left-state zero directions as $\sigma \to \infty$. In this case, we have $\tilde{M}_-(s,\sigma) \to 0$ as $\sigma \to \infty$.

2. $\tilde{M}_b(s,\sigma)$ contains n_b terms with eigenvalues in the set $\Lambda_b(\sigma)$. In accordance with lemmas 3 and 4, there exists a gain $K(\sigma)$ such that $\tilde{M}_b(s,\sigma)$ is identically zero by assigning elements of $\Lambda_b(\sigma)$ arbitrarily at either finite or infinite locations in $\mathcal{C}^-$ asymptotically as $\sigma \to \infty$. As discussed earlier, in order to limit the controller bandwidth, we will assume hereafter that these eigenvalues are assigned to asymptotically finite locations. Also, $K(\sigma)$ can be designed so that $\tilde{M}_b(s,\sigma) \to 0$ as $\sigma \to \infty$.

3. $\tilde{M}_\infty(s,\sigma)$ contains n_f terms with eigenvalues in the set $\Lambda_\infty(\sigma)$. In accordance with lemma 4, there exists a gain $K(\sigma)$ such that $\tilde{M}_\infty(s,\sigma) \to 0$ as $\sigma \to \infty$ by assigning elements of $\Lambda_\infty(\sigma)$ arbitrarily to asymptotically infinite locations in $\mathcal{C}^-$.

4. $\tilde{M}_e(s,\sigma)$ contains the remaining $n_e \equiv n_a^+ + n_c$ terms with eigenvalues in the set $\Lambda_e(\sigma)$. This term does not exist (i.e. $n_e = 0$) when the system Σ_{yw} is left-invertible and of minimum phase. In view of lemmas 3 and 4, $\tilde{M}_e(s,\sigma)$ cannot in general be rendered zero either asymptotically or otherwise by any assignment of $\Lambda_e(\sigma)$ and the associated sets of right- and left-eigenvectors $W_e(\sigma)$ and $V_e(\sigma)$. However, as to be discussed later, selection of the full-order observer design gain $K(\sigma)$ can be done so that the error term $\tilde{M}_e(s,\sigma)$ has a particular frequency-shaped properties or some desired H_2- and H_∞-norms. Note that the eigenvalues of A_o in Λ_e can be assigned to any locations in $\mathcal{C}^-$, except of course for those corresponding to the stable but unobservable eigenvalues of A, since $(A,\ C_2)$ is assumed to be a detectable pair. These arbitrary locations can either be asymptotically finite or infinite.

As the above discussion indicates, lemmas 3 and 4 form the heart of the underlying mechanism of CLTR. They enable us to decompose $\tilde{M}_f(s,\sigma)$ and hence $M_f(s,\sigma)$ into four distinct parts exhibiting clearly conditions under which closed-loop recovery is or is not possible. Although the results presented so far do not directly provide methods for obtaining the gain $K(\sigma)$, they do however give the essential closed-loop eigenstructure and hence guidelines in the assignment of the eigenvalues and eigenvectors of A_o. These guidelines can in turn be used to formulate a systematic method for designing the full-order observer gain $K(\sigma)$. In a sequel paper [2], we will discuss in details three possible methods for designing $K(\sigma)$. They are:

(1) A method based on the minimization of the H_2-norm of $M_f(s,\sigma)$,

(2) A method based on the minimization of the H_∞-norm of $M_f(s,\sigma)$ and,

(3) Asymptotic time-scale and eigenstructure assignment (ATEA) method.

The latter method is an extension of the one given in [11] and [9] which allows designers a great flexibility in the shaping of $\tilde{M}_f(s,\sigma)$. Putting these aside, we come back to the problem of characterizing achievable closed-loop transfer recovery. To do this, we simply assume that the observer gain $K(\sigma)$ was given and has been chosen from the set $\mathcal{K}^*(A, B_1, C_2, D_{21}, \sigma)$ defined below.

Definition 6. *$\mathcal{K}^*(A, B_1, C_2, D_{21}, \sigma)$ is a set of gains $K(\sigma) \in \Re^{n\times p}$ such that*

(1) *$A_o(\sigma) = A - K(\sigma)C_2$ is stable for all $\sigma > \sigma^*$ where $0 \leq \sigma^* < \infty$,*

(2) *The finite eigenvalues of $A_o(\sigma)$ remain in $\mathcal{C}^-$ as $\sigma \to \infty$,*

(3′) *If $n_f = 0$ then $\tilde{M}_-(s,\sigma)$ and $\tilde{M}_b(s,\sigma)$ are identically zero for all σ,*

(3″) *If $n_f \neq 0$ then, as $\sigma \to \infty$, $\tilde{M}_-(s,\sigma)$ and $\tilde{M}_b(s,\sigma)$ are either identically zero or asymptotically zero. Moreover, the eigenvalues in the set $\Lambda_-(\sigma)$ and $\Lambda_b(\sigma)$ tend to finite locations in $\mathcal{C}^-$ and,*

(4) *$\tilde{M}_\infty(s,\sigma) \to 0$ as $\sigma \to \infty$.*

It is obvious that $\mathcal{K}^*(A, B_1, C_2, D_{21}, \sigma)$ is a nonempty set.

Remark 3. *In the case where the system Σ_{yw} does not have any infinite zeros (i.e. $n_f = 0$), every element $K(\sigma)$ of $\mathcal{K}^*(A, B_1, C_2, D_{21}, \sigma)$ is independent of σ and furthermore $\|K(\sigma)\| \leq \alpha < \infty$ for all σ. On the other hand, if the system Σ_{yw} has at least one infinite zero (i.e. $n_f \neq 0$), then $K(\sigma)$ of $\mathcal{K}^*(A, B_1, C_2, D_{21}, \sigma)$ must be a function of σ and $\|K(\sigma)\| \to \infty$ as $\sigma \to \infty$.*

Theorem 2 given below characterizes the asymptotic behavior of the achieved loop transfer function for $K(\sigma) \in \mathcal{K}^*(A, B_1, C_2, D_{21}, \sigma)$. We have the following theorem.

Theorem 2. *Consider the closed-loop system Σ^c comprising of the system Σ and a full-order observer-based controller. Let (A, B_2) be stabilizable and (A, C_2) be detectable. Then, given any F such that $A-BF$ is asymptotically stable and for a gain $K(\sigma) \in \mathcal{K}^*(A, B_1, C_2, D_{21}, \sigma)$, the closed-loop system Σ^c is asymptotically stable. Moreover, as $\sigma \to \infty$,*

$$E_f(s,\sigma) \to T_{zu}(s)F\overline{\tilde{M}}_e(s). \tag{32}$$

Proof : Expression (32) follows from lemmas 3 and 4, as well as the properties of $\mathcal{K}^*(A, B_1, C_2, D_{21}, \sigma)$. ∎

In view of theorem 2, $\overline{\tilde{M}_e(s)}$ can be termed as the limit of the recovery matrix. We have the following corollaries of theorem 2.

Corollary 1. *Let the system Σ_{yw} be left-invertible and of minimum phase. Then $\mathbf{T}^f_{\mathrm{R}}(\Sigma) = \mathbf{T}(\Sigma)$. Moreover, for any gain*

$$K(\sigma) \in \mathcal{K}^*(A, B_1, C_2, D_{21}, \sigma),$$

the corresponding full-order observer-based controller achieves closed-loop transfer recovery for any given $T_{zw}(s) \in \mathbf{T}(\Sigma)$.

Proof : For a left-invertible and minimum-phase system Σ_{yw}, we have $n_a^+ = 0$ and $n_c = 0$. Thus $n_a^+ + n_c = 0$ and $\tilde{M}_e(s, \sigma)$ is nonexistent. Hence, the results are obvious. ∎

Remark 4. *Results of corollary 1 are exactly those of Fujita* et al *[6].*

Corollary 2. *Let the system Σ_{yw} be left-invertible and of minimum phase. And let D_{21} be of maximal column rank. Then*

$$\mathbf{T}^f_{\mathrm{ER}}(\Sigma) = \mathbf{T}^f_{\mathrm{R}}(\Sigma) = \mathbf{T}(\Sigma) \;\; \text{and} \;\; \mathbf{T}^f_{\mathrm{AR}}(\Sigma) = \emptyset.$$

Moreover, the observer gain

$$K(\sigma) \in \mathcal{K}^*(A, B_1, C_2, D_{21}, \sigma)$$

is independent of σ and the corresponding full-order observer-based controller achieves exact closed-loop transfer recovery (ECLTR) for any given $T_{zw}(s) \in \mathbf{T}(\Sigma)$.

Proof : When the system Σ_{yw} is left-invertible and of minimum phase and D_{21} is of maximal column rank, then we have $n_a^+ = 0$, $n_c = 0$ and $n_f = 0$. Note that a left-invertible system Σ_{yw} has no infinite zeros if and only if D_{21} is of maximal rank. Thus, both $\tilde{M}_e(s, \sigma)$ and $\tilde{M}_\infty(s, \sigma)$ are nonexistent. Hence the results of corollary 2 are obvious. ∎

Remark 5. *If the system Σ_{yw} is a non-strictly proper and of minimum-phase single-input single-output system, then $\mathbf{T}^f_{\mathrm{ER}}(\Sigma) = \mathbf{T}^f_{\mathrm{R}}(\Sigma)$ and $\mathbf{T}^f_{\mathrm{AR}}(\Sigma) = \emptyset$.*

Remark 6. *Whenever ECLTR is feasible, the corresponding full-order observer-gain matrix $K(\sigma) \in \mathcal{K}^*(A, B_1, C_2, D_{21}, \sigma)$ is finite and constant for all σ and hence $\mathbf{C}(s, \sigma) = \mathbf{C}(s)$.*

2. Analysis For Recoverable Target Closed-Loop Transfer Functions

In the previous subsection, the analysis of closed-loop transfer recovery does not take into account any knowledge of the state-feedback gain matrix F. It is essentially a study of the matrix $\tilde{M}_f(s)$ or $\tilde{M}_f(s,\sigma)$ as to when it can or cannot be rendered zero using a full-order observer-based controller. This section complements the analysis of the previous subsection by taking directly into account the knowledge of F. Obviously then, the analysis of this section is a study of $M_f(s) = F\tilde{M}_f(s)$ or $M_f(s,\sigma) = F\tilde{M}_f(s,\sigma)$. Two basic issues have been addressed:

(1) What class of target closed-loops can be recovered exactly (or asymptotically) for a given system Σ? Or equivalently, what are the necessary and sufficient conditions a target closed-loop transfer function $T_{zw}(s)$ must satisfy so that it can exactly (or asymptotically) be recoverable for the given system? and,

(2) What are the necessary and sufficient conditions on the system Σ so that it has at least one recoverable target loop?

Answers to these questions would enable designers to identify the appropriate number and type of control inputs and measurement outputs in the plant model needed in the CLTR task. To answer the questions, we introduce an auxiliary system Σ_E where now the condition for the set of exactly recoverable target loops $\mathbf{T}^f_{\mathrm{ER}}(\Sigma)$ to be nonempty is equivalent to the auxiliary system Σ_E being stabilizable by a static output-feedback control. Similarly, another auxiliary system Σ_A can be introduced for the ACLTR case. Here, it will be shown that the set of recoverable target loops $\mathbf{T}^f_{\mathrm{AR}}(\Sigma)$ is nonempty if and only if Σ_A is stabilizable by a static output-feedback control.

In what follows, we derive conditions for ECLTR and ACLTR in terms of geometric properties. We also give the necessary and sufficient conditions for the sets $\mathbf{T}^f_{\mathrm{ER}}(\Sigma)$ and $\mathbf{T}^f_{\mathrm{AR}}(\Sigma)$ to be non-empty. We have the following theorems.

Theorem 3. *Let the system Σ_{zu} be left-invertible and stabilizable and Σ_{yw} be detectable. Then, an admissible target closed-loop transfer function $T_{zw}(s)$ of Σ (i.e. $T_{zw}(s) \in \mathbf{T}(\Sigma)$) is exactly recoverable by a* *full-order observer-based controller if and only if*

$$\mathcal{S}^-(A, B_1, C_2, D_{21}) \subseteq Ker(F).$$

That is,

$$\mathbf{T}^f_{\mathrm{ER}}(\Sigma) = \{\, T_{zw}(s) \in \mathbf{T}(\Sigma) \mid \mathcal{S}^-(A, B_1, C_2, D_{21}) \subseteq Ker(F) \,\}.$$

Proof : See Appendix B. ∎

The following theorem characterizes the non-emptiness of $\mathbf{T}^f_{\mathrm{ER}}(\Sigma)$.

Theorem 4. *Let the system Σ_{zu} be left-invertible and stabilizable and Σ_{yw} be detectable. Let $\overline{C}_E$ be any full-rank matrix of dimension $(n_a^- + n_b) \times n$ such that*

$$Ker(\overline{C}_E) = \mathcal{S}^-(A, B_1, C_2, D_{21}).$$

Then, the system Σ has at least one exactly recoverable target closed-loop transfer function (i.e. $\mathbf{T}^f_{\mathrm{ER}}(\Sigma)$) is nonempty if and only if the auxiliary system Σ_E characterized by the matrix triple $(A, B_2, \overline{C}_E)$ is stabilizable by a static output-feedback controller.

Proof : See Appendix C. ∎

Theorems 5 and 6 below state the results for the case of ACLTR.

Theorem 5. *Let the system Σ_{zu} be left-invertible and stabilizable and Σ_{yw} be detectable. Then, an admissible target closed-loop transfer function $T_{zw}(s)$ of Σ (i.e. $T_{zw}(s) \in \mathbf{T}(\Sigma)$) is recoverable (either exactly or asymptotically) by a full-order observer-based controller if and only if*

$$\mathcal{V}^+(A, B_1, C_2, D_{21}) \subseteq Ker(F).$$

That is,

$$\mathbf{T}^f_{\mathrm{R}}(\Sigma) = \{\, T_{zw}(s) \in \mathbf{T}(\Sigma) \mid \mathcal{V}^+(A, B_1, C_2, D_{21}) \subseteq Ker(F) \,\}.$$

Proof : See Appendix D. ∎

As in theorem 4, theorem 6 below characterizes the non-emptiness of $\mathbf{T}^f_{\mathrm{R}}(\Sigma)$.

Theorem 6. *Let the system Σ_{zu} be left-invertible and stabilizable and Σ_{yw} be detectable. Let $\overline{C}_A$ be any full-rank matrix of dimension $(n_a^- + n_b + n_f) \times n$ such that*

$$Ker(\overline{C}_A) = \mathcal{V}^+(A, B_1, C_2, D_{21}).$$

Then, the system Σ has at least one asymptotically recoverable target closed-loop transfer function (i.e. $\mathbf{T}_{\mathrm{R}}^f(\Sigma)$ is nonempty) if and only if the auxiliary system Σ_A characterized by the matrix triple $(A, B_2, \overline{C}_A)$ is stabilizable by a static output-feedback controller.

Proof : It follows along the same lines as in theorem 4. ∎

B. Reduced-Order Observer-Based Controller

In this section, let us consider the problem of closed-loop transfer recovery using reduced-order observer-based controllers. Without loss of generality and for simplicity of presentation, we assume that the matrices C_2 and D_{21} are already in the form

$$C_2 = \begin{bmatrix} 0 & C_{2,02} \\ I_{p-m_0} & 0 \end{bmatrix} \quad \text{and} \quad D_{21} = \begin{bmatrix} D_{21,0} \\ 0 \end{bmatrix}, \tag{33}$$

where m_0 is the rank of D_{21}. The system Σ can be rewritten as,

$$\begin{cases} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_{1,1} \\ B_{1,2} \end{bmatrix} w + \begin{bmatrix} B_{2,1} \\ B_{2,2} \end{bmatrix} u, \\ \begin{bmatrix} y_0 \\ y_1 \end{bmatrix} = \begin{bmatrix} 0 & C_{2,02} \\ I_{p-m_0} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} D_{21,0} \\ 0 \end{bmatrix} w, \\ z = C_1 \; x + D_{11} \; w + D_{12} \; u, \end{cases} \tag{34}$$

where $[x_1', \; x_2']' = x$ and $[y_0', \; y_1']' = y$. We note that $y_1 \equiv x_1$. Thus, one needs to estimate only the state x_2 in the reduced-order observer design. The procedure follows closely the development given in [7] and [1]. We first rewrite the state equation for x_1 in terms of the measured output y_1 and state x_2 as follows,

$$\dot{y}_1 = A_{11} y_1 + A_{12} x_2 + B_{1,1} w + B_{2,1} u, \tag{35}$$

where y_1 and u are known. Observation of x_2 is made via y_0 and

$$\tilde{y}_1 = A_{12} x_2 + B_{1,1} w = \dot{y}_1 - A_{11} y_1 - B_{2,1} u. \tag{36}$$

A reduced-order system for the estimation of the remaining state x_2 is given by

$$\begin{cases} \dot{x}_2 = A_{22} x_2 + B_{1,2} w + [A_{21}, B_{2,2}] \begin{bmatrix} y_1 \\ u \end{bmatrix}, \\ \begin{bmatrix} y_0 \\ \tilde{y}_1 \end{bmatrix} = \begin{bmatrix} C_{2,02} \\ A_{12} \end{bmatrix} x_2 + \begin{bmatrix} D_{21,0} \\ B_{1,1} \end{bmatrix} w. \end{cases} \tag{37}$$

Based on equation (37), we can construct a reduced-order observer for the state x_2 as follows,

$$\begin{aligned} \dot{\hat{x}}_2 = A_{22}\hat{x}_2 + [A_{21}, B_{2,2}] \begin{bmatrix} y_1 \\ u \end{bmatrix} \\ + K_r \left(\begin{bmatrix} y_0 \\ \dot{y}_1 - A_{11} y_1 - B_{2,1} u \end{bmatrix} - \begin{bmatrix} C_{2,02} \\ A_{12} \end{bmatrix} \hat{x}_2 \right), \end{aligned} \tag{38}$$

where K_r is the observer gain matrix for the reduced-order system. It is chosen such that

$$A_{or} = A_{22} - K_r \begin{bmatrix} C_{2,02} \\ A_{12} \end{bmatrix}$$

is asymptotically stable. In order to remove the dependency on $\dot{y}_1$, let us partition $K_r = [K_{r0}, K_{r1}]$ to be compatible with the dimensions of the outputs $[y_0', \tilde{y}_1']'$ and at the same time define a new variable $v := \hat{x}_2 - K_{r1}\tilde{y}_1$. We obtain the following reduced-order observer-based controller,

$$\begin{cases} \dot{v} = A_{or} v + (B_{2,2} - K_{r1} B_{2,1}) u \\ \qquad + [K_{r0}, A_{21} - K_{r1} A_{11} + A_{or} K_{r1}] \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}, \\ \hat{x} = \begin{bmatrix} 0 \\ I_{n-p+m_0} \end{bmatrix} v + \begin{bmatrix} 0 & I_{p-m_0} \\ 0 & K_{r1} \end{bmatrix} y, \\ u = -F\hat{x}, \end{cases} \tag{39}$$

We further note that the reduced-order observer-based controller given above is a special case of Luenberger observer-based controller of (16) with the following parameters,

$$\begin{cases} L = A_{or}, \quad G_1 = [K_{r0}, A_{21} - K_{r1} A_{11} + A_{or} K_{r1}], \\ P = \begin{bmatrix} 0 \\ I_{n-p+m_0} \end{bmatrix}, \quad J = \begin{bmatrix} 0 & I_{p-m_0} \\ 0 & K_{r1} \end{bmatrix}, \\ G_2 = B_{2,2} - K_{r1} B_{2,1}, \quad Q = [-K_{r1}, I_{n-p+m_0}]. \end{cases} \tag{40}$$

Now, let us partition F as

$$F = [F_1,\ F_2]$$

in conformity with $[x_1',\ x_2']'$. It follows from lemma 1 that the recovery error and the recovery matrix, denoted here by $E_r(s)$ and $M_r(s)$ respectively, are given by

$$E_r(s) = T_{zu}(s) \cdot M_r(s), \tag{41}$$

where $T_{zu}(s)$ is as defined in (19) and

$$M_r(s) = F_2(sI - A_r + K_rC_r)^{-1}(B_r - K_rD_r), \tag{42}$$

with

$$A_r = A_{22},\quad B_r = B_{1,2},\quad C_r = \begin{bmatrix} C_{2,02} \\ A_{12} \end{bmatrix},\quad D_r = \begin{bmatrix} D_{21,0} \\ B_{1,1} \end{bmatrix}.$$

Remark 7. *The expression for $M_r(s)$ is identical to $M_f(s)$ of the full-order observer-based controller in (24), where F_2, (A_r, B_r, C_r, D_r) and K_r now take the place of F, (A, B_1, C_2, D_{21}) and K.*

We have the following important lemma regarding the properties of Σ_r characterized by (A_r, B_r, C_r, D_r).

Lemma 5.

1. *Σ_r is of (non-) minimum phase if and only if (A, B_1, C_2, D_{21}) is of (non-) minimum phase.*
2. *Σ_r is detectable if and only if Σ_{yw} is detectable.*
3. *Invariant zeros of Σ_r are the same as those of Σ_{yw}.*
4. *Orders of infinite zeros of Σ_r are reduced by one from those of Σ_{yw}.*
5. *Σ_r is left-invertible if and only if Σ_{yw} is left-invertible.*
6. $\begin{pmatrix} 0 \\ I \end{pmatrix} \mathcal{V}^+(A_r, B_r, C_r, D_r) = \mathcal{V}^+(A, B_1, C_2, D_{21})$.
7. $\begin{pmatrix} 0 \\ I \end{pmatrix} \mathcal{S}^-(A_r, B_r, C_r, D_r) = \mathcal{S}^-(A, B_1, C_2, D_{21}) \cap \mho$, *where* $\mho := \{x \mid C_2x \in ImD_{21}\}$.

Proof See [5]. ∎

1. Analysis For Arbitrary Target Closed-Loop Transfer Functions

We note that with lemma 5 the analysis and design of full-order and reduced-order observer-based controllers for CLTR have been placed into the same framework. Now, let $\mathcal{K}^*(A_r, B_r, C_r, D_r, \sigma)$ be defined in a similar way as in definition 6. We have the following results which are analogous to the case of a full-order observer-based controller.

Theorem 7. *Consider the closed-loop system Σ^c comprising of the given system Σ and a reduced-order observer-based controller. Let (A, B_2) be stabilizable and (A, C_2) be detectable. Then, for any F such that $A - B_2F$ is asymptotically stable and for any gain*

$$K(\sigma) \in \mathcal{K}^*(A_r, B_r, C_r, D_r, \sigma),$$

the closed-loop system Σ^c is asymptotically stable. Moreover, as $\sigma \to \infty$,

$$E_r(s, \sigma) \to T_{zu}(s) F_2 \overline{\tilde{M}}_{re}(s), \tag{43}$$

where $\overline{\tilde{M}}_{re}(s)$ is for the reduced-order system Σ_r and can be derived following the procedure given in Section IV.A.1.

Proof : The proof follows along the same lines as in theorem 2 and the properties of Σ_r in lemma 5. ∎

In view of theorem 7, $\overline{\tilde{M}}_{re}(s)$ can also be termed as the limit of the recovery matrix for the case of a reduced-order observer-based controller. We have the following corollaries of theorem 7.

Corollary 3. *Let Σ_{yw} be left-invertible and of minimum phase. Then $\mathbf{T}^r_R(\Sigma) = \mathbf{T}(\Sigma)$. Furthermore, for any gain $K(\sigma) \in \mathcal{K}^*(A_r, B_r, C_r, D_r, \sigma)$, the corresponding reduced-order observer-based controller achieves closed loop transfer recovery for any given $T_{zw}(s) \in \mathbf{T}(\Sigma)$.*

Proof : The proof follows along the same lines as in corollary 1 and the properties of Σ_r in lemma 5. ∎

Corollary 4. *Let Σ_{yw} be left-invertible and of minimum phase with no infinite zero of order higher than one which implies that D_r is of* *maximal column rank. Then*

$$\mathbf{T}^r_{\mathrm{R}}(\Sigma) = \mathbf{T}^r_{\mathrm{ER}}(\Sigma) = \mathbf{T}(\Sigma)$$

and $\mathbf{T}^r_{\mathrm{AR}}(\Sigma) = \emptyset$*. Moreover, any gain*

$$K(\sigma) \in \mathcal{K}^*(A_r, B_r, C_r, D_r, \sigma)$$

is independent of σ *and the corresponding reduced-order observer-based controller achieves exact closed-loop transfer recovery (ECLTR) for any given* $T_{zw}(s) \in \mathbf{T}(\Sigma)$*.*

Proof : The proof follows along the same lines as in corollary 2 and the properties of Σ_r in lemma 5. ∎

2. Analysis For Recoverable Target Closed-Loop Transfer Functions

In what follows, we state in terms of geometric properties conditions under which ECLTR and ACLTR can be achieved using a reduced-order observer-based controller. As in the case of full-order observer-based controller, necessary and sufficient conditions are given that characterize the non-empty sets $\mathbf{T}^r_{\mathrm{ER}}(\Sigma)$ and $\mathbf{T}^r_{\mathrm{R}}(\Sigma)$. We have the following theorems.

Theorem 8. *Let the system* Σ_{zu} *be left-invertible and stabilizable and* Σ_{yw} *be detectable. Then an admissible target closed-loop transfer function* $T_{zu}(s)$ *of* Σ *(i.e.* $T_{zw}(s) \in \mathbf{T}(\Sigma)$*) is exactly recoverable by a reduced-order observer-based controller if and only if*

$$\mathcal{S}^-(A, B_1, C_2, D_{21}) \cap \mho \subseteq Ker(F).$$

That is,

$$\mathbf{T}^r_{\mathrm{ER}}(\Sigma) = \{\, T_{zw}(s) \in \mathbf{T}(\Sigma) \mid \mathcal{S}^-(A, B_1, C_2, D_{21}) \cap \mho \subseteq Ker(F)\}.$$

Proof : In view of lemma 5, we note that

$$\mathcal{S}^-(A, B_1, C_2, D_{21}) \cap \mho \subseteq Ker(F)$$

is equivalent to

$$\mathcal{S}^-(A_r, B_r, C_r, D_r) \subseteq Ker(F_2).$$

Hence the proof follows along the same lines as in theorem 3. ∎

Remark 8. *It is simple to observe from theorems 3 and 8 as well lemma 5 that* $\mathbf{T}^f_{\mathrm{ER}}(\Sigma) \subseteq \mathbf{T}^r_{\mathrm{ER}}(\Sigma)$. *That is, if a target closed-loop transfer function is exactly recoverable by a full-order observer-based controller, then it is also exactly recoverable by a reduced-order observer-based controller. But the reverse is not true in general.*

The following theorem characterizes the non-emptiness of $\mathbf{T}^r_{\mathrm{ER}}(\Sigma)$ for reduced-order observer-based controllers.

Theorem 9. *Let the system* Σ_{zu} *be left-invertible and stabilizable and* Σ_{yw} *be detectable. Let* $\overline{C}_{re}$ *be any maximal rank matrix such that*

$$Ker(\overline{C}_{re}) = \mathcal{S}^-(A, B_1, C_2, D_{21}) \cap \mho.$$

Then the given system Σ *has at least one exactly recoverable target closed-loop transfer function using a reduced-order observer-based controller (i.e.* $\mathbf{T}^r_{\mathrm{ER}}(\Sigma)$ *is nonempty) if and only if the auxiliary system* Σ_{re} *characterized by the matrix triple* $(A, B_2, \overline{C}_{re})$ *is stabilizable by a static output-feedback controller.*

Proof : It follows along the same lines as in theorem 4. ∎

Theorem 10 given below deals with ACLTR for reduced-order observer based controller.

Theorem 10. *Let the system* Σ_{zu} *be left-invertible and stabilizable and* Σ_{yw} *be detectable. Then an admissible target closed-loop transfer function* $T_{zw}(s)$ *of* Σ *(i.e.* $T_{zw}(s) \in \mathbf{T}(\Sigma)$*) is recoverable (either exactly or asymptotically) by a reduced-order observer-based controller if and only if*

$$\mathcal{V}^+(A, B_1, C_2, D_{21}) \subseteq Ker(F).$$

That is,

$$\mathbf{T}^r_{\mathrm{R}}(\Sigma) = \{\, T_{zw}(s) \in \mathbf{T}(\Sigma) \mid \mathcal{V}^+(A, B_1, C_2, D_{21}) \subseteq Ker(F) \,\}.$$

Proof : In view of lemma 5, we note that

$$\mathcal{V}^+(A, B_1, C_2, D_{21}) \subseteq Ker(F)$$

is equivalent to

$$\mathcal{V}^+(A_r, B_r, C_r, D_r) \subseteq Ker(F_2).$$

Hence the proof follows along the same lines as in theorem 5. ∎

Remark 9. *It is trivial to see from theorems 5 and 10 that* $\mathbf{T}^f_{\mathrm{R}}(\Sigma) = \mathbf{T}^r_{\mathrm{R}}(\Sigma)$. *That is, if a target closed-loop transfer function is recoverable by a full-order observer-based controller, then it is also recoverable by a reduced-order observer-based controller. And the reverse is also true. Hence, it is obvious that the nonemptiness of* $\mathbf{T}^r_{\mathrm{AR}}(\Sigma)$ *is characterized by the same condition as in theorem 6.*

V. NUMERICAL EXAMPLE

The above analysis of CLTR is applied to the development of a localizer capture and track-hold design of a commercial transport. This numerical example is not intended to provide a complete illustration of all the analysis results discussed in the previous sections. The main reason for using this design problem is that it provides a realistic design situation where asymptotic and exact closed-loop transfer recovery using full-order and reduced-order observer-based controllers are applicable. For completeness, we provide a brief overview of the design procedure used in the synthesis of the chosen state-feedback gain F which defines the target closed-loop transfer function $T_{zw}(s)$ for closed-loop transfer recovery. Detailed description and design requirements for such a system have been extensively covered in literature (see for example the Special Issues in IEEE Control System Magazine [14]). It should be emphasized here that the analysis provided in previous sections are applicable to arbitrary state-feedback laws, regardless of the procedures from which these state-feedback laws are derived (i.e. H_2-, H_∞-norm based design methods, eigenstructure assignment or others).

Design model used in this example consists of the basic 4^{th}-order lateral aircraft dynamics augmented with appropriate kinematic equations for the heading ψ and lateral track distance y_{track} along with a state for the integral of lateral track error $\int(y_{track} - y_c)dt$. State matrices describing the synthesis model without actuation dynamics in the notations of (1) are given below for a typical landing approach condition,

$$A = \begin{bmatrix} -0.2093 & 0.00077518 & 0.1003 & -0.991 & 0 & 0 & 0 \\ -7.492 & -3.44 & -0.0052035 & 0.9783 & 0 & 0 & 0 \\ 0 & 1 & 0 & -0.0012877 & 0 & 0 & 0 \\ 2.031 & -0.1696 & -0.0050925 & -0.2089 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5.597 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \tag{44}$$

$$B_1 = \begin{bmatrix} 0.026222 & -0.0033036 \\ 9.2044 & 0.069525 \\ 0 & 0 \\ 0.10099 & 0.086836 \\ 0 & 0 \\ 0 & 0 \\ 0 & -1 \end{bmatrix},$$

$$B_2 = \begin{bmatrix} 0.0013217 & 0.063301 \\ 2.011 & 2.012 \\ 0 & 0 \\ 0.1304 & -1.393 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} -19.856 & 0.07354 & 9.5153 & -94.015 & 0 & 0 & 0 \\ 0 & 0 & 14.142 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 20 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$D_{11} = \begin{bmatrix} 0 & 0 \\ -14.142 & 0 \\ 0 & 0 \\ 0 & -1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad D_{12} = \begin{bmatrix} 0.12539 & 6.0053 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 10 & 0 \\ 0 & 31.623 \end{bmatrix},$$

$$C_2 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$D_{21} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad D_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

The state variables are

$$x = [\beta, p, \phi, r, \psi, y_{track}, \int (y_{track} - y_c)dt]',$$

where β is the sideslip angle in degrees, p is the roll rate in degrees per second, ϕ is the roll angle in degrees, r is the yaw rate in degrees per second, ψ is the heading angle in degrees, y_{track} is the lateral track distance in feet and y_c is the command lateral track distance in feet. The control inputs

$$u = [\delta_{ac}, \delta_{rc}]'$$

consist of the aileron δ_{ac} and rudder δ_{rc} deflections in degrees. The disturbance inputs

$$w = [\phi_c, y_c]'$$

contain the bank angle command ϕ_c in degrees and the lateral track command y_c in feet. The measurement output variables are

$$y = [p, \phi, r, \psi, y_{track}, \int (y_{track} - y_c)dt]'.$$

The controlled output variables z shown above are made up of weighted plant outputs z_p and control variables u in the following form

$$z = [(Q^{1/2} z_p)', (R^{1/2} u)']'.$$

The performance variables z_p include sideslip acceleration $\dot{\beta}$, bank angle deviation $(\phi - \phi_c)$, yaw rate r, lateral track deviation $(y_{track} - y_c)$ and integral track error $\int (y_{track} - y_c)dt$. The control variables u are included in the controlled output vector z to ensure that the resulting state-feedback design does not have excessive control gain and bandwidth. These control variables are scaled by a diagonal weighting matrix R. Note that in the design trade-offs, loop shapings are tuned on the performance variables z_p using a diagonal weighting matrix Q. Final selection of the diagonal weighting matrices Q and R is made after numerous design iterations that involve at each time closed-loop stability analysis, frequency responses of the transfer function $y_{track}(s)/y_c(s)$, time simulation to a lateral track command. It is observed in the design iterations that increasing penalty on the sideslip acceleration $\dot{\beta}$ will improve the aircraft turn-coordination, but at the expense of slower responses to bank angle and lateral track command inputs. In order to achieve good tracking performance and turn-coordination, responses of the controlled output vector z to the command inputs w must be kept as small as possible. State matrices for the desired controlled output vector $z(t)$ are given in equation (44).

Control-law synthesis is performed at one particular landing approach condition. A state-feedback law that yields satisfactory stability and closed-loop responses to a lateral track command y_c is obtained from the following H_∞-norm bound solution, i.e $||T_{zw}(s)||_{H_\infty} < 45$. An acceptable state-feedback gain matrix F is given below,

$$F = \begin{bmatrix} 0.10939 & 0.92375 & 4.2514 & 2.3791 & 3.9476 & 0.088639 & 0.0048466 \\ -1.1237 & 0.081467 & 0.32548 & -2.9659 & -1.6125 & -0.05404 & -0.0035325 \end{bmatrix} \tag{45}$$

Analysis of closed-loop transfer recovery for the above localizer capture and track-hold design proceeds as follows. First of all, we examine whether conditions for exact and asymptotic closed-loop recovery can be achieved with the given set of measurements and using an output-feedback observer-based control-law. It is simple to verify that the system Σ_{zu} is left-invertible. Thus according to Lemma 2, conditions for exact and asymptotic closed-loop recovery are governed completely by the existence of solutions that make the recovery error zero or arbitrarily small. Next we observe that, with the given measurement output $y(t)$ and disturbances $w(t)$, the system Σ_{yw} is left-invertible and has no invariant zeros. Thus from Corollary 1, a full-order observer-based controller can be used to achieve closed-loop transfer recovery for *any* closed-loop $T_{zw}(s)$ under state feedback and obviously the full-state feedback design defined in equation (45). Furthermore, from Corollary 3, the transfer function $T_{zw}(s)$ is also recoverable using a reduced-order observer-based controller. It can be determined from the s.c.b transformation that the system Σ_{yw} has two infinite-zeros of order 1. This result indicates that recovery using a full-order observer-based controller can only be achieved asymptotically. An acceptable observer-gain design K is given below and it is obtained using the ATEA design method [2].

$$K = \begin{bmatrix} 28.529 & 0.099657 & -3.6288 & 0 & -0.016212 & 34.789 \\ 9998.8 & 0.00010319 & 108.9 & 0 & 0 & 9.263 \\ 1 & 0.5 & -0.0012877 & 0 & 0 & 0 \\ 108.72 & -0.0094053 & 75.315 & 0 & 0 & -854.23 \\ 0 & 0 & 1 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 5.597 & 0.5 & 0 \\ 9.8159 & -0.00080954 & -854.25 & 0 & 0 & 9926.5 \end{bmatrix} \quad (46)$$

Development of different design methods for CLTR will be covered in a sequel paper [2]. It should be noted that in theory the observer gain matrix must be large in order to recover asymptotically the closed-loop performance (i.e. the case of ACLTR). In this particular design example, we notice that reducing the recovery error at low frequency does indeed involve a high observer gain design synthesized using either the ATEA or ARE-based methods. Singular value plots of the closed-loop transfer function $T_{zw}(j\omega)$ are shown in figure 2. The observer gain of equation (46) seems to provide reasonably small recovery error at low frequency and at the same time does not lead to excessive control responses to lateral track commands. Performance evaluation based on transient responses is depicted in figure 3 corresponding to the time simulation of system responses to a lateral track command

$$y_c = 1000(1 - e^{-0.065t})(\text{feet}).$$

This figure shows time responses of the full-state feedback design. Results corresponding to the above full-order observer design are the same within the resolution of the graph as those shown in figure 3.

Now we proceed to the problem of closed-loop transfer recovery using a reduced-order observer-based controller. It turns out that, for the above localizer capture and track-hold problem, one can actually achieved *exact* closed-loop recovery using a reduced-order observer-based controller. This result comes directly from Lemma 5 and the fact that the system Σ_{yw} is left-invertible, has no invariant zeros and has only infinite zeros of order 1. The ensuing reduced system Σ_r as defined in section IV.B has no infinite zeros. Hence, in this case, ECLTR

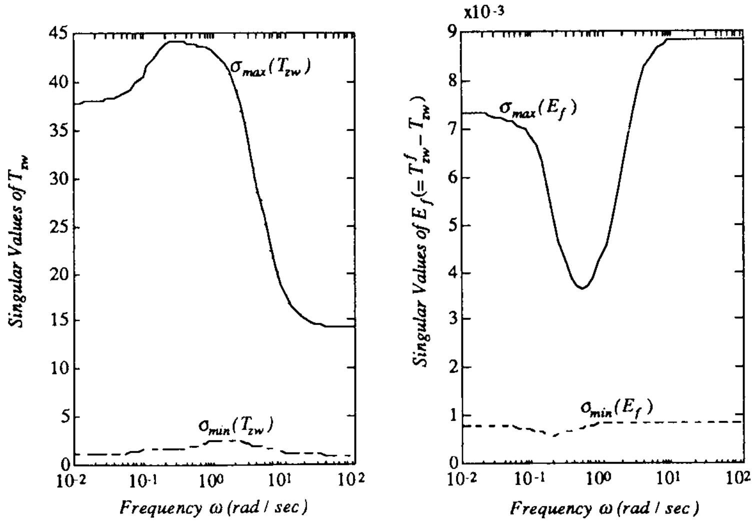

Figure 2: Singular Value Plots of $T_{zw}(j\omega)$ and $E_f(j\omega)$.

is possible. Again using the ATEA design method, we have obtained a reduced-order observer-based controller that yields exactly the same closed-loop transfer function $T_{zw}(s)$. State matrices of this controller are given below.

$$\left\{ \begin{aligned} \dot{v} &= A_{cmp} v + B_{cmp} y, \\ -u &= C_{cmp} v + D_{cmp} y, \end{aligned} \right. \tag{47}$$

where

$$A_{cmp} = -0.12265$$

$$B_{cmp} = [0.0095756 \quad 0.10031 \quad -0.81378 \quad 0.1095 \quad 0 \quad -0.0002034]$$

$$C_{cmp} = \begin{bmatrix} 0.10939 \\ -1.1237 \end{bmatrix}$$

$$D_{cmp} = \begin{bmatrix} 0.92406 & 4.2514 & 2.3791 & 3.9476 & 0.088639 & 0.0052271 \\ 0.078262 & 0.32548 & -2.9656 & -1.6125 & -0.05404 & -0.0074414 \end{bmatrix}.$$

Performance of this reduced-order controller is identical to that of the full-state feedback case (see Figures 2 and 3). The design is an output-feedback controller of first-order and having a controller pole at $s = -0.12$ *rad/sec*. Hence, the design concept of CLTR has enabled us to synthesize a low-order implementable output-feedback design for a

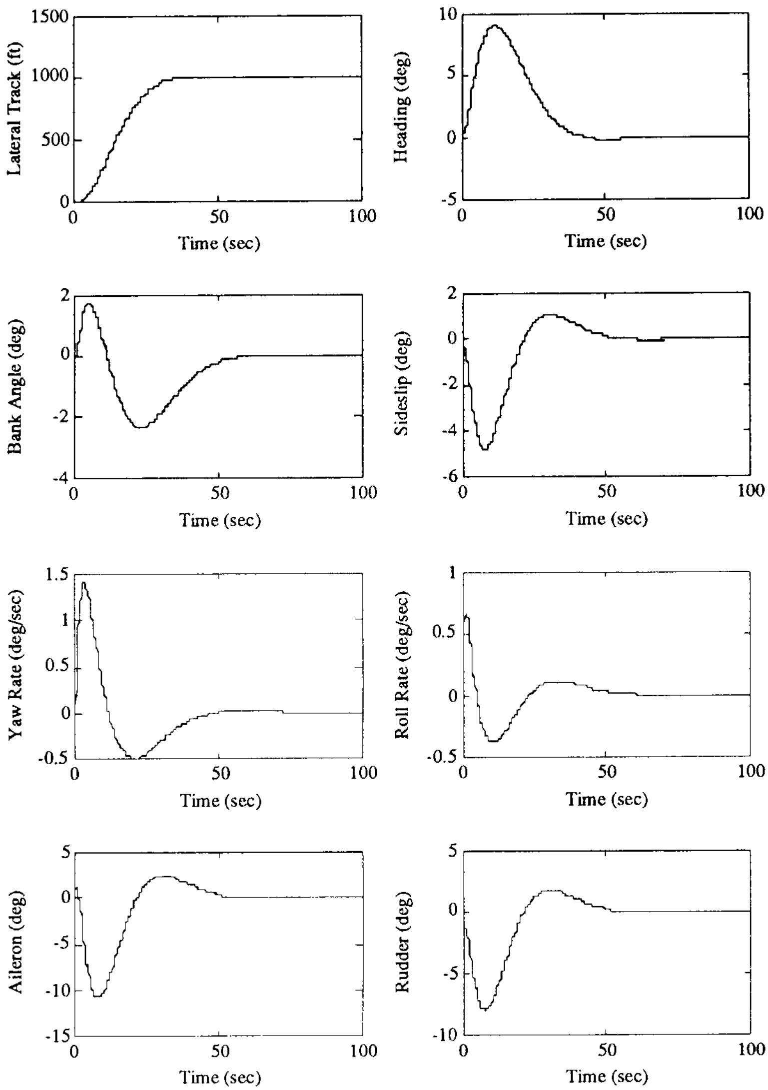

Figure 3: Aircraft Responses to a Lateral Track Command of 1000ft.

typical localizer capture and lateral track hold system starting from a satisfactory state-feedback control law. It should be pointed out that if actuator dynamics have been included into the design model, then exact closed-loop transfer recovery is no longer possible, even with a reduced-order observer-based controller since the infinite zeros of Σ_{yw} are no longer of order 1. However, the system is still asymptotically recoverable .

VI. CONCLUSIONS

In this paper, we deal with issues concerning the analysis of closed-loop transfer recovery using full-order and reduced-order observer-based controllers. There are several fundamental results given here. Based on the structural properties of the given system, we decompose the recovery matrix in the recovery error between the target closed-loop transfer function and that achieved by observer-based controllers, into three distinct parts for any arbitrarily specified admissible target closed-loop transfer function. The first part of recovery matrix can be rendered exactly zero by an appropriate finite eigenstructure assignment of observer dynamic matrix, while the second part can be rendered arbitrarily close to zero by an appropriate infinite eigenstructure assignment. The third part in general cannot be rendered zero, either exactly or asymptotically, by any means although there exists a multitude of ways to shape it.

The above analysis is general and applies to any arbitrarily specified target closed-loop transfer function. Results of the analysis enable designers to identify limitations of the given system in recovering the target closed-loop transfer function as a consequence of its structural properties, namely finite and infinite zero structures and invertibility. The next issue of our analysis concentrates on characterizing the required necessary and/or sufficient conditions on the target closed-loop transfer functions so that they are either exactly or asymptotically recoverable by means of observer-based controllers for the given system. Conditions developed here for a target closed-loop transfer function to be recoverable turn out to be constraints in its finite and infinite zero structures inherent of the system under consideration. The last issue covered in our analysis is to find the necessary and/or sufficient conditions on the given system such that it has at least one recoverable target closed-loop transfer function.

In a sequel, we will present design issues concerning the closed-loop transfer recovery.

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A. APPENDIX A — JUSTIFICATION OF $D_{22} = 0$

The justification of the assumption $D_{22} = 0$ is as follows: Let us define a new auxiliary measurement output y_{new} as

$$y_{new} = y - D_{22}u = C_2x + D_{21}w.$$

Then we will show that a compensator

$$u(s) = K(s)y_{new}(s)$$

is equivalent to the compensator

$$u(s) = K(s)[I + D_{22}K(s)]^{-1}y(s)$$

under assumption that the closed-loop system is well-posed, i.e. the inverse of $I + D_{22}K(s)$ exists for almost all $s \in \mathcal{C}$. Let us consider the following relation,

$$\begin{aligned} u(s) &= K(s)y_{new}(s) \\ &= K(s)[y(s) - D_{22}u(s)] \\ &= K(s)y(s) - K(s)D_{22}u(s) \end{aligned}$$

This implies that

$$[I + K(s)D_{22}]u(s) = K(s)y(s).$$

Hence,

$$u(s) = [I + K(s)D_{22}]^{-1}K(s)y(s) = K(s)[I + D_{22}K(s)]^{-1}y(s). \quad (48)$$

Thus, whenever D_{22} is nonzero, one can define a new set of measurement output, namely y_{new}, and design a controller $K(s)$. Then the controller in (48) will yield the same performance when it is applied to the original system. ∎

B. APPENDIX B — PROOF OF THEOREM 3

Under the assumption that Σ_{zu} is left-invertible, it follows from Lemma 2 that an admissible target loop $T_{zw}(s)$ is exactly recoverable by a full-order observer-based controller (i.e. $E_f(s) = 0$) if and only if there exists an observer gain K such that $A - KC_2$ is asymptotically stable and the corresponding $M_f(s) = 0$. Thus, it is sufficient to show that $M_f(s) = 0$ if and only if

$$S^-(A, B_1, C_2, D_{21}) \subseteq Ker(F).$$

To show this, let us consider an auxiliary system characterized by

$$\Sigma_{au} : \begin{cases} \dot{x} = A'x + C_2'u + F'w, \\ z = B_1'x + D_{21}'u. \end{cases} \quad (49)$$

Also, with a state-feedback law

$$u = -K'x,$$

the closed-loop transfer function from w to z, denoted here by $T_{zw}^{au}(s)$, is simply

$$T_{zw}^{au}(s) = M_f'(s).$$

Hence, the problem of finding an observer gain matrix such that $A - KC_2$ is asymptotically stable and $M_f(s) = 0$ is equivalent to the well-known disturbance decoupling problem. Then it follows from Stoorvogel [16] that the disturbance decoupling problem with internal stability is solvable to Σ_{au} in (49) if and only if

$$\mathcal{S}^-(A, B_1, C_2, D_{21}) \subseteq Ker(F).$$

This completes the proof of theorem 3. ∎

C. APPENDIX C — PROOF OF THEOREM 4

Without loss of generality, we assume that (A, B_1, C_2, D_{21}) is in the form of s.c.b as in theorem 1. Now in view of theorem 3, an exactly recoverable target closed-loop transfer function $T_{zw}(s)$ must satisfy the condition $\mathcal{S}^-(A, B_1, C_2, D_{21}) \subseteq Ker(F)$. This implies that $T_{zw}(s)$ is recoverable if and only if F is the form,

$$F = \begin{bmatrix} F_{a1}^- & 0 & F_{b1} & 0 & 0 \\ F_{a2}^- & 0 & F_{b2} & 0 & 0 \end{bmatrix}. \tag{50}$$

Thus condition that the given system has at least one exactly recoverable target closed-loop is simply equivalent to the existence of some appropriate matrix F_{a1}^-, F_{a2}^-, F_{b1} and F_{b2} such that $A - B_2F$ is asymptotically stable. Next in view of the properties of s.c.b, we note that $\overline{C}_E$ as defined in theorem 4 is of the form,

$$\overline{C}_E = \Gamma \begin{bmatrix} I_{n_a^-} & 0 & 0 & 0 & 0 \\ 0 & 0 & I_{n_b} & 0 & 0 \end{bmatrix},$$

where Γ is any non-singular matrix of dimension $(n_a^- + n_b) \times (n_a^- + n_b)$. It is now trivial to verify that existence of a matrix F of the form in (50) such that $A - B_2F$ is asymptotically stable, is equivalent to existence of a matrix G such that $A - B_2G\overline{C}_E$ is asymptotically stable. This is simply due to the fact $G\overline{C}_E$ has the same structure as F in (50). This completes the proof of theorem 4. ∎

D. APPENDIX D — PROOF OF THEOREM 5

Under the assumption that Σ_{zu} is left-invertible, it follows from Lemma 2 that an admissible target loop $T_{zw}(s)$ is asymptotically recoverable by a full-order observer-based controller if and only if there exists an observer gain $K(\sigma)$ such that $A - K(\sigma)C_2$ is asymptotically stable and the corresponding $M_f(s,\sigma) \to 0$ pointwise in s as $\sigma \to \infty$. Following the proof of theorem 3 in Appendix B, it is simple to see that such a problem is equivalent to the well-known almost disturbance decoupling problem with internal stability (ADDPS) and it is shown in Scherer [13] that ADDPS is solvable to Σ_{au} in (49) if and only if

$$\mathcal{V}^+(A, B_1, C_2, D_{21}) \subseteq Ker(F),$$

and we adhere to the notion of closed-loop stability by excluding those cases where, in the limits as $\sigma \to \infty$, the finite eigenvalues of the closed-loop system are on the $j\omega$ axis. This completes the proof of theorem 5. ∎

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